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# **SYLLABUS**

## **OPERATIONS RESEARCH**

### **SECTION-A**

#### **1. LINEAR PROGRAMMING**

Mathematical formulation of the problem, Graphical solution, The simplex method, Concept of duality, Dual simplex method.

#### **2. TRANSPORTATION PROBLEMS**

Basic feasible solutions by different methods, Finding optimal solutions, Degeneracy in transportation problems, Unbalanced transportation problems.

### **SECTION-B**

#### **3. ASSIGNMENT PROBLEMS**

Balanced and Unbalanced assignments, Assignments to given schedules.

#### **4. QUEUEING THEORY**

Queueing systems and their characteristics, The M/M/1/FIFO/queueing systems.

### **SECTION-C**

#### **5. INVENTORY CONTROL**

Notations, Models I-IV, Probabilistic Models.

#### **6. SIMULATION**

Basic Concepts, Methods and Softwares of Simulation and its applications.

### **SECTION-D**

#### **7. NETWORK SCHEDULING BY CPM/PERT**

CPM/PERT, Time calculations and elements of crashing a network.

#### **8. GAME THEORY**

Definitions, Two person Zero-sum Game with pure and Mixed strategies, Graphical and Linear Programming method for Games, and Dominance Rules.

**CHAPTER 1 LINEAR PROGRAMMING**

NOTES

**★ STRUCTURE ★**

- Introduction to Operations Research
- Introduction to Linear Programming Problems (LPP)
- Graphical Method
- Simplex Method
- Big M Method
- Two Phase Method
- Formulation Problems
- Revised Simplex Method (RSM)
- Introduction and Formulation
- Duality Theorems
- Duality of Simplex Method
- The Dual Simplex Method
- Economic Interpretation of Dual Variable
- Summary
- Problems

**INTRODUCTION TO OPERATIONS RESEARCH**

The roots of Operations Research (O.R.) can be traced many decades ago. First this term was coined by Mc Closky and Trefthen of United Kingdom in 1940 and it came in existence during world war II when the allocations of scarce resources were done to the various military operations. Since then the field has developed very rapidly. Some chronological events are listed below :

- 1952 – Operations Research Society of America (ORSA).
- 1957 – Operations Research Society of India (ORSI)  
– International Federation of O.R. Societies
- 1959 – First Conference of ORSI
- 1963 – Opsearch (the journal of O.R. by ORSI).

However the term 'Operations Research' has a number of different meaning. The Operational Research Society of Great Britain has adopted the following illaborate definition :

"Operational Research is the application of the methods of science to complex problems arising in the direction and management of large systems of men, machines, materials and money in industry, business, government and defence. The distinctive approach is to develop a scientific method of the system, incorporating measurements of factors such as chance and risk, with which to predict and compare the outcomes

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of alternative decisions strategies and controls. The purpose is to help management to determine its policy and actions scientifically.”

Whereas ORSA has offered the following shorter definition :

“Operations Research is concerned with scientifically deciding how to best design and operate man-machine systems, usually under conditions requiring the allocation of scarce resources”.

Many individuals have described O.R. according to their own view. Only three are quoted below :

“O.R. is the art of giving bad answers to problems which otherwise have worse answers”—T.L. Saaty

“O.R. is a scientific approach to problems solving for executive management”—H.M. Wagner

“O.R. is a scientific knowledge through interdisciplinary team effort for the purpose of determining the best utilization of limited resources”.—H.A. Taha

An abbreviated list of applications of O.R. techniques are given below :

- |    |                                  |   |   |
|----|----------------------------------|---|---|
| 1. | Manufacturing                    | : | Production scheduling<br>Inventory control<br>Product mix<br>Replacement policies                 |
| 2. | Marketing                        | : | Advertising budget allocation<br>Supply chain management  |
| 3. | Organizational behaviour         | : | Personnel planning<br>Scheduling of training programs<br>Recruitment policies                     |
| 4. | Facility planning                | : | Factory location<br>Hospital planning<br>Telecommunication network planning<br>Warehouse location |
| 5. | Finance                          | : | Investment analysis<br>Portfolio analysis   |
| 6. | Construction                     | : | Allocation of resources to projects<br>Project scheduling   |
| 7. | Military                         |   |   |
| 8. | Different fields of engineering. |   |   |

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### INTRODUCTION TO LINEAR PROGRAMMING PROBLEMS (LPP) :

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1. When a problem is identified then the attempt is to make an mathematical model. In decision making all the decisions are taken through some variables which are known as decision variables. In engineering design, these variables are known as design vectors. So in the formation of mathematical model the following three phases are carried out :

- (i) Identify the decision variables.
- (ii) Identify the objective using the decision variables and
- (iii) Identify the constraints or restrictions using the decision variables.

Let there be  $n$  decision variable  $x_1, x_2, \dots, x_n$  and the general form of the mathematical model which is called as Mathematical programming problem under decision-making can be stated as follows :

Maximize/Minimize  $z = f(x_1, x_2, \dots, x_n)$   
 Subject to,  $g_i(x_1, x_2, \dots, x_n) \{ \leq, \geq \text{ or } = \} b_i$   
 $i = 1, 2, \dots, m.$

and the type of the decisions i.e.  $x_j \geq 0$   
 or  $x_j \leq 0$  or  $x_j$ 's are unrestricted  
 or combination types decisions.

In the above, if the functions  $f$  and  $g_i$  ( $i = 1, 2, \dots, m$ ) are all linear, then the model is called "Linear Programming Problem (LPP)". If any one function is non-linear then the model is called "Non-linear Programming Problem (NLPP)".

II. We define some basic aspects of LPP in the following :

(a) **Convex set** : A set  $X$  is said to be convex if

$$x_1, x_2 \in X, \text{ then for } 0 \leq \lambda \leq 1,$$

$$x_3 = \lambda x_1 + (1 - \lambda)x_2 \in X$$

Some examples of convex sets are :

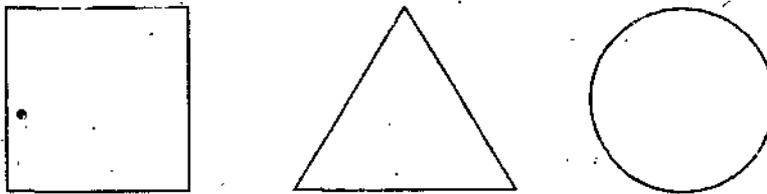


Fig. 1.1 Convex sets

Some examples of non-convex sets are :

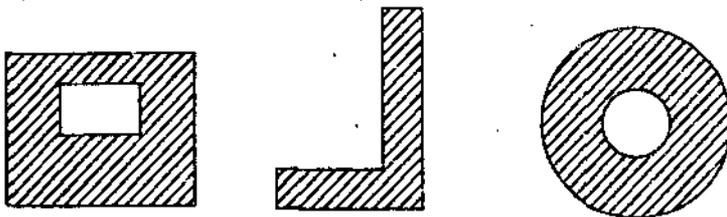


Fig. 1.2 Non-convex sets

Basically if all the points on a line segment forming by two points lies inside the set/geometric figure then it is called convex.

(b) **Extreme point or vertex or corner point of a convex set** : It is a point in the convex set which can not be expressed as  $\lambda x_1 + (1 - \lambda)x_2$  where  $x_1$  and  $x_2$  are any two points in the convex set.

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For a triangle, there are three vertices, for a rectangle there are four vertices and for a circle there are infinite number of vertices.

(c) Let  $Ax = b$  be the constraints of an LPP. The set  $X = \{x \mid Ax = b, x \geq 0\}$  is a convex set.

**NOTES**

**Feasible Solution** : A solution which satisfies all the constraints in LPP is called feasible solution.

**Basic Solution** : Let  $m =$  no. of constraints and  $n =$  no. of variables and  $m < n$ . Then the solution from the system  $Ax = b$  is called basic solution. In this system there are  ${}^nC_m$  number of basic solutions. By setting  $(n - m)$  variables to zero at a time, the basic solutions are obtained. The variables which is set to zero are known as 'non-basic' variables. Other variables are called basic variables.

**Basic Feasible Solution (BFS)** : A solution which is basic as well as feasible is called basic feasible solution.

**Degenerate BFS** : If a basic variable takes the value zero in a BFS, then the solution is said to be degenerate.

**Optimal BFS** : The BFS which optimizes the objective function is called optimal BFS.

**GRAPHICAL METHOD**

Let us consider the constraint  $x_1 + x_2 = 1$ . The feasible region of this constraint comprises the set of points on the straight line  $x_1 + x_2 = 1$ .

If the constraint is  $x_1 + x_2 \geq 1$ , then the feasible region comprises not only the set of points on the straight line  $x_1 + x_2 = 1$  but also the points above the line. Here above means away from origin.

If the constraint is  $x_1 + x_2 \leq 1$ , then the feasible region comprises not only the set of points on the straight line  $x_1 + x_2 = 1$  but also the points below the line. Here below means towards the origin.

The above three cases depicted below :

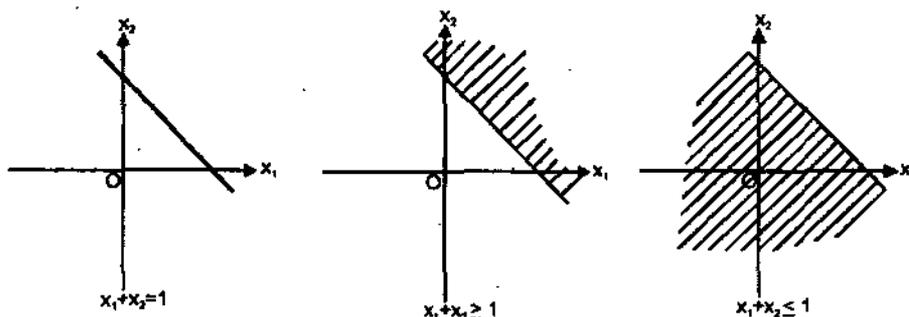


Fig. 1.3

For the constraints  $x_1 \geq 1, x_1 \leq 1, x_2 \geq 1, x_2 \leq 1$  the feasible regions are depicted below :

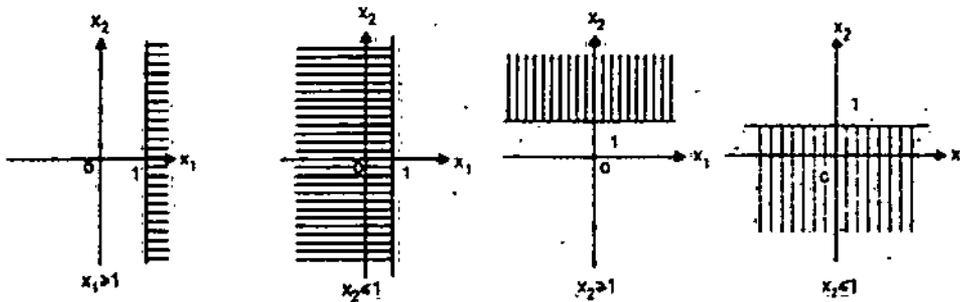


Fig. 1.4

For the constraints  $x_1 - x_2 = 0$ ,  $x_1 - x_2 \geq 0$  and  $x_1 - x_2 \leq 0$  the feasible regions are depicted in Fig. 1.5.

The steps of graphical method can be stated as follows :

(i) Plot all the constraints and identify the individual feasible regions.

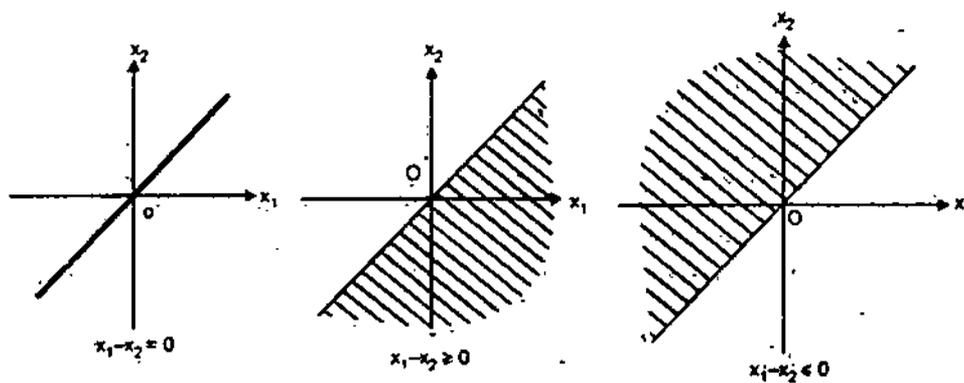


Fig. 1.5

(ii) Identify the common feasible region and identify the corner points i.e. vertices of the common feasible region.

(iii) Identify the optimal solution at the corner points if exists.

Example 1. Using graphical method solve the following LPP :

$$\text{Maximize } z = 5x_1 + 3x_2$$

$$\text{Subject to. } 2x_1 + 5x_2 \leq 10.$$

$$5x_1 + 2x_2 \leq 10.$$

$$2x_1 + 3x_2 \geq 6.$$

$$x_1 \geq 0, x_2 \geq 0.$$

Solution. Let us present all the constraints in intercept form i.e.,

$$\frac{x_1}{5} + \frac{x_2}{2} \leq 1 \quad \dots(I)$$

$$\frac{x_1}{2} + \frac{x_2}{5} \leq 1 \quad \dots(II)$$

$$\frac{x_1}{3} + \frac{x_2}{2} \geq 1 \quad \dots(III)$$

The common feasible region ABC is shown in Fig. 1.6 and the individual regions are indicated by arrows. (Due to non-negativity constraints i.e.,  $x_1 \geq 0, x_2 \geq 0$ , the common feasible region is obtained in the first quadrant).

NOTES

The corner points are  $A\left(\frac{18}{11}, \frac{10}{11}\right)$ ,  $B\left(\frac{10}{7}, \frac{10}{7}\right)$  and  $C(0, 2)$ . The value of the objective function at the corner points are  $z_A = \frac{120}{11} = 10.91$ ,  $z_B = \frac{80}{7} = 11.43$  and  $z_C = 6$ .

Here the common feasible region is bounded and the maximum has occurred at the corner point B. Hence the optimal solution is

$$x_1^* = \frac{10}{7}, \quad x_2^* = \frac{10}{7} \quad \text{and} \quad z^* = \frac{80}{7} = 11.43.$$

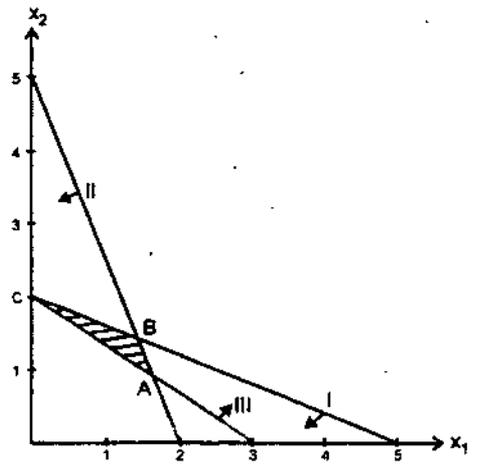


Fig. 1.6

**Example 2.** Using graphical method solve the following LPP :

$$\text{Minimize } z = 3x_1 + 10x_2$$

$$\text{Subject to, } 3x_1 + 2x_2 \geq 6,$$

$$4x_1 + x_2 \geq 4,$$

$$2x_1 + 3x_2 \geq 6,$$

$$x_1 \geq 0, \quad x_2 \geq 0.$$

**Solution.** Let us present all the constraints in intercept form i.e.,

$$\frac{x_1}{2} + \frac{x_2}{3} \geq 1 \quad \dots\text{(I)}$$

$$\frac{x_1}{1} + \frac{x_2}{4} \geq 1 \quad \dots\text{(II)}$$

$$\frac{x_1}{3} + \frac{x_2}{2} \geq 1 \quad \dots\text{(III)}$$

Due to the non-negativity constraints i.e.,  $x_1 \geq 0$  and  $x_2 \geq 0$  the feasible region will be in the first quadrant.

The common feasible region is shown in Fig. 1.7 where the individual feasible regions are shown by arrows. Here the common feasible region is unbounded.

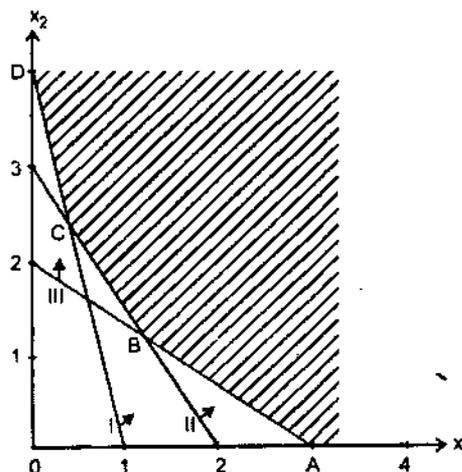


Fig. 1.7

i.e., open with the corner points  $A(3, 0)$ ,  $B\left(\frac{3}{5}, \frac{8}{5}\right)$ ,  $C\left(\frac{2}{5}, \frac{12}{5}\right)$  and  $D(0, 4)$ . The value of the objective function at the corner points are  $z_A = 9$ ,  $z_B = \frac{89}{5} = 17.8$ ,  $z_C = \frac{126}{5} = 25.2$ , and  $z_D = 40$ .

Here the maximum has occurred at A and there is no other point in the feasible region at which the objective function value is lower than 9. Hence the optimal solution is

$$x_1^* = 3, x_2^* = 0 \text{ and } z^* = 9$$

**Example 3.** Solve the following LPP by graphical method :

$$\begin{aligned} \text{Maximize } z &= 3x_1 - 15x_2 \\ \text{Subject to, } x_1 + x_2 &\leq 8, \\ x_1 - 4x_2 &\leq 8, \\ x_1 &\geq 0, x_2 \text{ unrestricted in sign.} \end{aligned}$$

**Solution.** Since  $x_2$  is unrestricted in sign this means  $x_2$  may be  $\geq 0$  or  $\leq 0$ . Also  $x_1 \geq 0$ . Then the common feasible region will be in the first and fourth quadrant. Let us present all the constraints in intercept forms i.e.,

$$\frac{x_1}{8} + \frac{x_2}{8} \leq 1 \quad \dots(I)$$

$$\frac{x_1}{8} - \frac{x_2}{2} \leq 1 \quad \dots(II)$$

The common feasible region is shown in Fig. 1.8 where the individual feasible regions are shown by arrows.

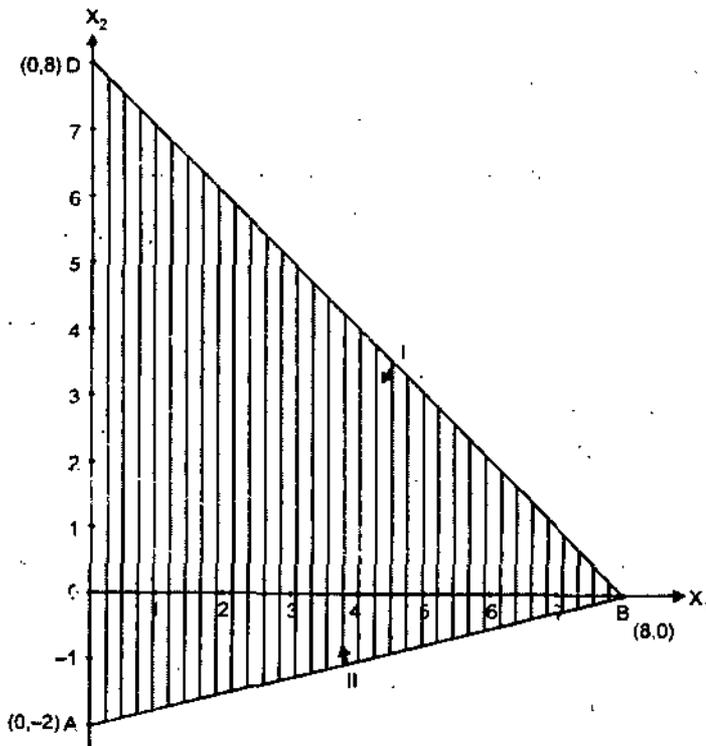


Fig. 1.8

NOTES

The value of the objective function at the corner points are  $z_A = 30$ ,  $z_B = 24$  and  $z_C = -120$ . Since the common feasible region is bounded and the maximum has occurred at A, the optimal solution is

$$x_1^* = 0, x_2^* = -2 \text{ and } z^* = 30.$$

**NOTES**

**Exceptional Cases in Graphical Method**

There are three cases may arise. When the value of the objective function is maximum/minimum at more than one corner points then 'multiple optima' solutions are obtained.

Sometimes the optimum solution is obtained at infinity, then the solution is called 'unbounded solution'. Generally, this type of solution is obtained when the common feasible region is unbounded and the type of the objective function leads to unbounded solution.

When there does not exist any common feasible region, then there does not exist any solution. Then the given LPP is called *infeasible* i.e., *having no solution*. For example, consider the LPP which is infeasible

$$\begin{aligned} &\text{Maximize } z = 5x_1 + 10x_2 \\ &\text{Subject to, } x_1 + x_2 \leq 2, \\ &\quad \quad \quad x_1 + x_2 \geq 3, \\ &\quad \quad \quad x_1, x_2 \geq 0. \end{aligned}$$

**Example 4.** Solve the following LPP using graphical method :

$$\begin{aligned} &\text{Maximize} && z = x_1 + \frac{3}{5}x_2 \\ &\text{Subject to,} && 5x_1 + 3x_2 \leq 15, \\ &&& 3x_1 + 4x_2 \leq 12, \\ &&& x_1, x_2 \geq 0. \end{aligned}$$

**Solution.** Let us present all the constraints in intercept forms i.e.,

$$\begin{aligned} \frac{x_1}{3} + \frac{x_2}{5} &\leq 1 && \dots(I) \\ \frac{x_1}{4} + \frac{x_2}{3} &\leq 1 && \dots(II) \end{aligned}$$

Due to non-negativity constraints i.e.,  $x_1 \geq 0, x_2 \geq 0$  the common feasible region is obtained in the first quadrant as shown in Fig. 1.9 and the individual feasible regions are shown by arrows:

The corner points are  $O(0, 0)$ ,  $A(3, 0)$ ,  $B\left(\frac{24}{11}, \frac{15}{11}\right)$  and  $C(0, 3)$ . The values of the objective function at the corner points are obtained as  $z_O = 0$ ,  $z_A = 3$ ,  $z_B = 3$ ,  $z_C = \frac{9}{4}$ .

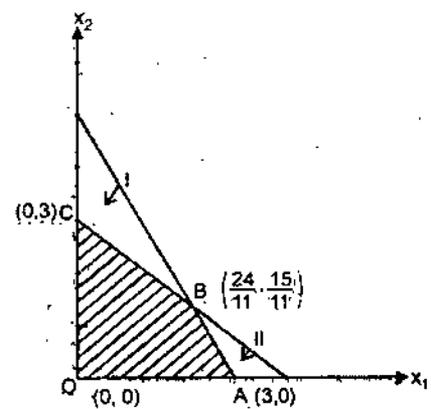


Fig. 1.9

Since the common feasible region is bounded and the maximum has occurred at two corner points i.e., at A and B respectively, these solutions are called multiple optima. So the solutions are

$$x_1^* = 3, x_2^* = 0 \quad \text{and} \quad x_1^* = \frac{15}{11}, x_2^* = \frac{24}{11} \quad \text{and} \quad z^* = 3.$$

**Example 5.** Using graphical method show that the following LPP is unbounded.

$$\text{Maximize } z = 10x_1 + 3x_2$$

$$\text{Subject to, } -2x_1 + 3x_2 \leq 6,$$

$$x_1 + 2x_2 \geq 4,$$

$$-x_1, x_2 \geq 0.$$

**Solution.** Due to the non-negativity constraints i.e.,  $x_1 \geq 0$  and  $x_2 \geq 0$  the common feasible region will be obtained in the first quadrant. Let us present the constraints in the intercept forms i.e.,

$$\frac{x_1}{-3} + \frac{x_2}{2} \leq 1 \quad \dots \text{(I)}$$

$$\frac{x_1}{4} + \frac{x_2}{2} \geq 1 \quad \dots \text{(II)}$$

The common feasible region is shown in Fig. 1.10 which is unbounded i.e., open region.

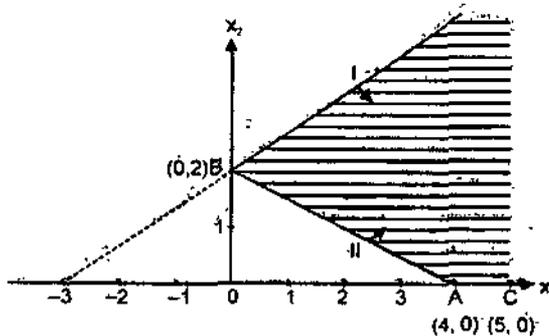


Fig. 1.10

There are two corner points A(4, 0) and B(0, 2). The objective function values are  $z_A = 40$  and  $z_B = 6$ . Here the maximum is 40. Since the region is open, let us examine some other points.

Consider the point C(5, 0) and the value of the objective function is  $z_C = 50$  which is greater than  $z_A$ . Therefore  $z_A$  is no longer optimal. If we move along x-axis, we observe that the next value is higher than the previous value and we reach to infinity for optimum value. Hence the problem is unbounded.

**Note.** For the same problem minimum exists which is the point B.

**PROBLEMS**

Using graphical method solve the following LPP :

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1. Maximize  $z = 13x_1 + 117x_2$   
 Subject to,  $x_1 + x_2 \leq 12$ ,  
 $x_1 - x_2 \geq 0$ ,  
 $4x_1 + 9x_2 \leq 36$ ,  
 $0 \leq x_1 \leq 2$  and  $0 \leq x_2 \leq 10$ .
2. Maximize  $z = 3x_1 + 15x_2$   
 Subject to,  $4x_1 + 5x_2 \leq 20$ ,  
 $x_2 - x_1 \leq 1$ ,  
 $0 \leq x_1 \leq 4$  and  $0 \leq x_2 \leq 3$ .
3. Maximize  $z = 5x_1 + 7x_2$   
 Subject to,  $3x_1 + 8x_2 \leq 12$ ,  
 $x_1 + x_2 \leq 2$ ,  
 $2x_1 \leq 3$ ,  
 $x_1, x_2 \geq 0$ .
4. Minimize  $z = 2x_1 + 3x_2$   
 Subject to,  $x_2 - x_1 \geq 2$ ,  
 $5x_1 + 3x_2 \leq 15$ ,  
 $2x_1 \geq 1$ ,  
 $x_2 \leq 4$ ,  
 $x_1, x_2 \geq 0$ .
5. Minimize  $z = 10x_1 + 9x_2$   
 Subject to,  $x_1 + 2x_2 \leq 10$ ,  
 $x_1 - x_2 \leq 0$ ,  
 $x_1 \leq 0, x_2 \geq 0$ .
6. Minimize  $z = 4x_1 + 3x_2$   
 Subject to,  $2x_1 + 3x_2 \leq 12$ ,  
 $3x_1 - 2x_2 \leq 12$ ,  
 $x_1$  unrestricted in sign,  $x_2 \geq 0$ .
7. Maximize  $z = 10x_1 + 11x_2$   
 Subject to,  $x_1 + x_2 \geq 4$ ,  
 $0 \leq x_2 \leq 3$ ,  
 $x_1 \geq 2$ ,  
 $x_1 \geq 0$ .
8. Minimize  $z = -x_1 + 2x_2$   
 Subject to,  $x_1 - x_2 \geq 1$ ,  
 $x_1 + x_2 \geq 5$ ,  
 $x_1, x_2 \geq 0$ .
9. Maximize  $z = 4x_1 + 5x_2$

Subject to,  $4x_1 - 5x_2 \leq 20$ ,

$$x_2 - x_1 \leq 1,$$

$$0 \leq x_2 \leq 3$$

$$0 \leq x_1 \leq 4.$$

10.

Maximize  $z = -3x_1 + 4x_2$

Subject to,  $-3x_1 + 4x_2 \leq 12$ ,

$$2x_1 - x_2 \geq -2,$$

$$x_1 \leq 4.$$

$$x_1 \geq 0, x_2 \geq 0.$$

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## ANSWERS

1.  $x_1 = 2, x_2 = 2, z^* = 260$

2.  $x_1 = \frac{5}{3}, x_2 = \frac{8}{3}, z^* = 45$

3.  $x_1 = \frac{4}{5}, x_2 = \frac{6}{5}, z^* = \frac{62}{5}$

4.  $x_1 = \frac{1}{2}, x_2 = \frac{5}{2}, z^* = \frac{17}{2}$

5.  $x_1 = 0, x_2 = 0, z^* = 0$

6.  $x_1 = 4, x_2 = 0, z^* = 16$

7. Unbounded solution

8. Unbounded solution

9. Multiple optima :

$$x_1 = \frac{5}{3}, x_2 = \frac{8}{3}, \text{ and } x_1 = 4, x_2 = \frac{4}{5} \text{ and } z^* = 20$$

10. Multiple optima :

$$x_1 = \frac{4}{5}, x_2 = \frac{18}{5} \text{ and } x_1 = 4, x_2 = 6 \text{ and } z^* = 12$$

## SIMPLEX METHOD

The algorithm is discussed below with the help of a numerical example *i.e.*, consider

$$\text{Maximize } z = 4x_1 + 8x_2 + 5x_3$$

$$\text{Subject to, } x_1 + 2x_2 + 3x_3 \leq 18,$$

$$2x_1 + 6x_2 + 4x_3 \leq 15,$$

$$x_1 + 4x_2 + x_3 \leq 6,$$

$$x_1, x_2, x_3 \geq 0.$$

**Step 1.** If the problem is in minimization, then convert it to maximization as

$$\text{Min } z = - \text{Max } (-z).$$

**Step 2.** All the right side constants must be positive. Multiply by  $-1$  both sides for negative constants. All the variables must be non-negative.

**Step 3.** Make standard form by adding slack variables for ' $\leq$ ' type constraints, surplus variables for ' $\geq$ ' type constraints and incorporate these variables in the objective function with zero coefficients.

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For example, 
$$\text{Maximum } z = 4x_1 + 8x_2 + 5x_3 + 0s_1 + 0s_2 + 0s_3$$

$$\text{Subject to, } x_1 + 2x_2 + 3x_3 + s_1 = 18,$$

$$2x_1 + 6x_2 + 4x_3 + s_2 = 15;$$

$$x_1 + 4x_2 + x_3 + s_3 = 6,$$

$$x_1, x_2, x_3 \geq 0, s_1, s_2, s_3 \geq 0$$

Note that an unit matrix due to  $s_1, s_2$  and  $s_3$  variables is present in the coefficient matrix which is the key requirement for simplex method.

**Step 4.** Simplex method is an iterative method. Calculations are done in a table which is called simplex table. For each constraint there will be a row and for each variable there will be a column. Objective function coefficients  $c_j$  are kept on the top of the table.  $x_B$  stands for basis column in which the variables are called 'basic variables'. Solution column gives the solution, but in iteration 1, the right side constants are kept. At the bottom  $z_j - c_j$  row is called 'net evaluation' row.

In each iteration one variable departs from the basis and is called departing variable and in that place one variable enter which is called entering variable to improve the value of the objective function.

Minimum ratio column determines the departing variable.

**Iteration 1.**

		$c_j$	4	8	5	0	0	0	Min.
$c_B$	$x_B$	soln.	$x_1$	$x_2$	$x_3$	$s_1$	$s_2$	$s_3$	ratio
0	$s_1$	18	1	2	3	1	0	0	
0	$s_2$	15	2	6	4	0	1	0	
0	$s_3$	6	1	4	1	0	0	1	
	$z_j - c_j$								

Note: Variables which are forming the columns of the unit matrix enter into the basis column. In this table the solution is  $s_1 = 18, s_2 = 15, s_3 = 6, x_1 = 0, x_2 = 0, x_3 = 0$  and  $z = 0$ .

To test optimality we have to calculate  $z_j - c_j$  for each column as follows :

$$z_j - c_j = c_B^T \cdot [x_j] - c_j$$

For first column,  $(0, 0, 0) \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = 4 = -4$

For second column,  $(0, 0, 0) \begin{pmatrix} 2 \\ 6 \\ 4 \end{pmatrix} = 8 = -8$  and so on.

These are displayed in the following table :

		$c_j$	4	8	5	0	0	0	Min.
$c_B$	$x_B$	soln.	$x_1$	$x_2$	$x_3$	$s_1$	$s_2$	$s_3$	ratio
0	$s_1$	18	1	2	3	1	0	0	
0	$s_2$	15	2	6	4	0	1	0	
0	$s_3$	6	1	4	1	0	0	1	
	$z_j - c_j$		-4	-8	-5	0	0	0	



**Decisions :** If all  $z_j - c_j \geq 0$ . Then the current solution is optimal and stop. Else, Select the negative most value from  $z_j - c_j$  and the variable corresponding to this value will be the entering variable and that column is called 'key column'. Indicate this column with an upward arrow symbol.

In the given problem '- 8' is the most negative and variable  $x_2$  is the entering variable. If there is a tie in the most negative, break it arbitrarily.

To determine the *departing variable*, we have to use minimum ratio. Each ratio is calculated as  $\frac{[\text{soln.}]}{[\text{key column}]}$ , componentwise division only for positive elements (*i.e.*,  $> 0$ ) of the key column. In this example,

$$\min. \left\{ \frac{18}{2}, \frac{15}{6}, \frac{6}{4} \right\} = \min. \{9, 2.5, 1.5\} = 1.5$$

The element corresponding to the min. ratio *i.e.*, here  $s_3$  will be the departing variable and the corresponding row is called 'key row' and indicate this row by an outward arrow symbol. The intersection element of the key row and key column is called key element. In the present example, 4 is the key element which is highlighted. This is the end of this iteration, the final table is displayed below :

**Iteration 1:**

		$c_j$	4	8	5	0	0	0	Min.
$c_B$	$x_B$	soln.	$x_1$	$x_2$	$x_3$	$s_1$	$s_2$	$s_3$	ratio
0	$s_1$	18	1	2	3	1	0	0	$\frac{18}{2} = 9$
0	$s_2$	15	2	6	4	0	1	0	$\frac{15}{6} = 2.5$ →
0	$s_3$	6	1	4	1	0	0	1	$\frac{6}{4} = 1.5$
	$z_j - c_j$		-4	-8	-5	0	0	0	



**Step 5.** For the construction of the next iteration (new) table the following calculations are to be made :

NOTES

- (a) Update the  $x_B$  column and the  $c_B$  column.
- (b) Divide the key row by the key element.
- (c) Other elements are obtained by the following formula :

NOTES

$$\left( \begin{array}{c} \text{new} \\ \text{element} \end{array} \right) = \left( \begin{array}{c} \text{old} \\ \text{element} \end{array} \right) - \frac{\left( \begin{array}{c} \text{element} \\ \text{corresponding to} \\ \text{key row} \end{array} \right) \left( \begin{array}{c} \text{element} \\ \text{corresponding to} \\ \text{key column} \end{array} \right)}{\text{key element}}$$

- (d) Then go to step 4.

Iteration 2.

		$c_j$	4	8	5	0	0	0	Min.
$c_B$	$x_B$	soln.	$x_1$	$x_2$	$x_3$	$s_1$	$s_2$	$s_3$	ratio
0	$s_1$	15	$\frac{1}{2}$	0	$\frac{5}{2}$	1	0	$-\frac{1}{2}$	$15 \times \frac{3}{5} = 6$
0	$s_2$	6	$\frac{1}{2}$	0	$\frac{5}{2}$	0	1	$-\frac{3}{2}$	$6 \times \frac{2}{5} = 2.4 \rightarrow$
8	$x_2$	$\frac{3}{2}$	$\frac{1}{4}$	1	$\frac{1}{4}$	0	0	$\frac{1}{4}$	$\frac{3}{4} \times 4 = 3$
	$z_j - c_j$		-2	0	-3	0	0	2	

↑

Iteration 3.

		$c_j$	4	8	5	0	0	0	Min.
$c_B$	$x_B$	soln.	$x_1$	$x_2$	$x_3$	$s_1$	$s_2$	$s_3$	ratio
0	$s_1$	9	0	0	0	1	-1	1	-
5	$x_3$	$\frac{12}{5}$	$\frac{1}{5}$	0	1	0	$\frac{2}{5}$	$-\frac{3}{5}$	$\frac{12}{5} \times \frac{5}{1} = 12$
8	$x_2$	$\frac{9}{10}$	$\frac{1}{5}$	1	0	0	$-\frac{1}{10}$	$\frac{2}{5}$	$\frac{9}{10} \times \frac{5}{1} = 4.5 \rightarrow$
	$z_j - c_j$		$-\frac{7}{5}$	0	0	0	$\frac{6}{5}$	$\frac{1}{5}$	

↑

Iteration 4.

		$c_j$	4	8	5	0	0	0	Min.
$c_B$	$x_B$	soln.	$x_1$	$x_2$	$x_3$	$s_1$	$s_2$	$s_3$	ratio
0	$s_1$	9	0	0	0	1	-1	1	

NOTES

5	$x_3$	$\frac{3}{2}$	0	-1	1	0	$\frac{1}{2}$	-1	
4	$x_1$	$\frac{9}{2}$	1	5	0	0	$-\frac{1}{2}$	2	
	$z_j - c_j$		0	7	0	0	$\frac{1}{2}$	3	

Since all  $z_j - c_j \geq 0$ , the current solution is optimal.

$$x_1^* = \frac{9}{2}, x_2^* = 0, x_3^* = \frac{3}{2}, z^* = \frac{51}{2}$$

**Note (exceptional cases).**

- (a) If in the key column, all the elements are non-positive i.e., zero or negative, then min. ratio cannot be calculated and the problem is said to be unbounded.
- (b) In the net evaluation of the optimal table all the basic variables will give the value zero. If any non-basic variable give zero net evaluation then it indicates that there is an alternative optimal solution. To obtain that solution, consider the corresponding column as key column and apply one simplex iteration.
- (c) For negative variables,  $x \leq 0$ , set  $x = -x'$ ,  $x' \geq 0$ .

For unrestricted variables set  $x = x' - x''$  where  $x', x'' \geq 0$ .

**Example 6.** Solve the following by simplex method :

$$\text{Maximize } z = x_1 + 3x_2$$

$$\text{Subject to, } -x_1 + 2x_2 \leq 2, x_1 - 2x_2 \leq 2, x_1, x_2 \geq 0.$$

**Solution.** Standard form of the given LPP can be written as follows :

$$\text{Maximum } z = x_1 + 3x_2 + 0.s_1 + 0.s_2$$

$$\text{Subject to, } -x_1 + 2x_2 + s_1 = 2, x_1 - 2x_2 + s_2 = 2.$$

$$x_1, x_2 \geq 0, s_1, s_2 \text{ slacks } \geq 0.$$

**Iteration 1.**

		$c_j$	1	3	0	0	Min.
$c_B$	$x_B$	soln.	$x_1$	$x_2$	$s_1$	$s_2$	ratio
0	$s_1$	2	-1	2	1	0	$\frac{2}{2} = 1 \rightarrow$
0	$s_2$	2	1	-2	0	1	-
	$z_j - c_j$		-1	-3	0	0	

Iteration 2.

NOTES

		$c_j$	1	3	0	0	Min.
$c_B$	$x_B$	soln.	$x_1$	$x_2$	$s_1$	$s_2$	ratio
3	$x_2$	1	$-\frac{1}{2}$	1	$\frac{1}{2}$	0	
0	$s_2$	4	0	0	1	1	
	$\bar{z}_j - c_j$		$-\frac{5}{2}$	0	$\frac{3}{2}$	0	

↑

Since all the elements in the key column are non-positive, we cannot calculate min. ratio. Hence the given LPP is said to be unbounded.

**PROBLEMS**

Solve the following LPP by simplex method:

- Maximize  $z = 3x_1 + 2x_2$   
 S/t.  $5x_1 + x_2 \leq 10, 4x_1 + 5x_2 \leq 60; x_1, x_2 \geq 0$
- Maximize  $z = 5x_1 + 4x_2 + x_3$   
 S/t.  $6x_1 + x_2 + 2x_3 \leq 12, 8x_1 + 2x_2 + x_3 \leq 30,$   
 $4x_1 + x_2 + 2x_3 \leq 16, x_1, x_2, x_3 \geq 0$
- Maximize  $z = 3x_1 + 2x_2$   
 S/t.  $3x_1 + 4x_2 \leq 12, 2x_1 + 5x_2 \leq 10, x_1, x_2 \geq 0$
- Maximize  $z = 3x_1 + 2x_2 + x_3$   
 S/t.  $3x_1 + x_2 + 2x_3 \leq 20, x_1 + 3x_2 + 4x_3 \leq 16, x_1, x_2, x_3 \geq 0$
- Maximize  $z = 4x_1 - 2x_2 - x_3$   
 S/t.  $x_1 + x_2 + x_3 \leq 3, 2x_1 + 2x_2 + x_3 \leq 4, x_1 - x_2 \leq 0, x_1,$   
 $x_2, x_3 \geq 0$
- Maximize  $z = 5x_1 + 3x_2 + 3x_3$   
 S/t.  $4x_1 + 4x_2 + 3x_3 \leq 12000, 0.4x_1 + 0.5x_2 + 0.3x_3 \leq 1800,$   
 $0.2x_1 + 0.2x_2 + 0.1x_3 \leq 960, x_1, x_2, x_3 \geq 0$
- Maximize  $z = 3x_1 + 2x_2 + 2x_3$   
 S/t.  $2x_1 - x_2 + 3x_3 \leq 18, x_1 + x_2 + 2x_3 \leq 12, x_1, x_2, x_3 \geq 0$
- Maximize  $z = 3x_1 + x_2 + x_3 + x_4$   
 S/t.  $-2x_1 + 2x_2 + x_3 = 4, 3x_1 + x_2 + x_4 = 6, x_i \geq 0$  for all  $i$
- Maximize  $z = x_1 + x_2$   
 S/t.  $x_1 - 2x_2 \leq 2, -x_1 + 2x_2 \leq 2, x_1, x_2 \geq 0$
- Find all the optimal BFS to the following :

Maximize  $z = x_1 + x_2 + x_3 + x_4$   
 S/t.  $x_1 + x_2 \leq 2, x_3 + x_4 \leq 5, x_1, x_2, x_3, x_4 \geq 0$

**ANSWERS**

1.  $x_1 = 0, x_2 = 10, z^* = 20$  (It 3)
2.  $x_1 = 0, x_2 = 12, x_3 = 0, z^* = 48$  (It 3)
3.  $x_1 = 4, x_2 = 0, z^* = 12$  (It 2)
4.  $x_1 = \frac{11}{2}, x_2 = \frac{7}{2}, x_3 = 0, z^* = \frac{47}{2}$  (It 3)
5.  $x_1 = 1, x_2 = 1, x_3 = 0, z^* = 2$  (It 3)
6.  $x_1 = 3000, x_2 = 0, x_3 = 0, z^* = 15000$  (It 2)
7.  $x_1 = 10, x_2 = 2, x_3 = 0, z^* = 34$  (It 3)
8. Solution 1 :  $x_1 = 1, x_2 = 3, x_3 = x_4 = 0$  (It 2)  
Solution 2 :  $x_1 = 0, x_2 = 2, x_3 = 0, x_4 = 4, z^* = 6$
9. Unbounded solution (It 2)
10. (2, 0, 5, 0), (0, 2, 5, 0), (0, 2, 0, 5), (2, 0, 0, 5).

**NOTES****BIG M METHOD**

The method is also known as 'penalty method' due to Charnes. If there is ' $\geq$ ' type constraint, we add surplus variable and if there is '=' type, then the constraint is in equilibrium. Generally, in these cases there may not be any unit matrix in the standard form of the coefficient matrix.

To bring unit matrix we take help of another type of variable, known as 'artificial variable'. The addition of artificial variable creates infeasibility in the system which was already in equilibrium. To overcome this, we give a very large number denoted as M to the coefficient of the artificial variable in the objective function. For maximization problem, we add " $-M$ . (artificial variable)" in the objective function so that the profit comes down. For minimization problem we add " $M$ . (artificial variable)" in the objective function so that the cost goes up. Therefore the simplex method tries to reduce the artificial variable to the zero level so that the feasibility is restored and the objective function is optimized.

The only **drawback** of the big M method is that the value of M is not known but it is a very large number. Therefore we cannot develop computer program for this method.

**Note.** (a) Once the artificial variable departs from the basis, it will never again enter in the subsequent iterations due to big M. Due to this we drop the artificial variable column in the subsequent iterations once the variable departs from the basis.

(b) If in the optimal table, the artificial variable remains with non-zero value, then the problem is said to be '*infeasible*'.

If the artificial variable remains in the optimal table with zero value, then the solution is said to be '*pseudo optimal*'.

(c) The rule for 'multiple solution' and 'unbounded solution' are same as given by simplex method. The big-M method is a simple variation of simplex method.

**Example 7.** Using Big-M method solve the following LPP :

$$\text{Minimize } z = 10x_1 + 3x_2$$

$$\text{s.t. } x_1 + 2x_2 \geq 3, x_1 + 4x_2 \geq 4; x_1, x_2 \geq 0$$

**Solution.** Standard form of the given LPP is

$$\text{Min. } z = - \text{Max. } (-2 = -10x_1 - 3x_2 + 0.s_1 + 0.s_2 - Ma_1 - Ma_2)$$

$$\text{S/t. } x_1 + 2x_2 - s_1 + a_1 = 3$$

$$x_1 + 4x_2 - s_2 + a_2 = 4$$

$$x_1, x_2 \geq 0, s_1, s_2 \text{ surplus} \geq 0, a_1, a_2 \text{ artificial} \geq 0$$

**NOTES**

**Iteration 1.**

			$c_j$	-10	-3	0	0	-M	-M	Min.
$c_B$	$x_B$	soln.	$x_1$	$x_2$	$s_1$	$s_2$	$a_1$	$a_2$		ratio
-M	$a_1$	3	1	2	-1	0	1	0		$\frac{3}{2} = 1.5$
-M	$a_2$	4	1	4	0	-1	0	1		$\frac{4}{4} = 1 \rightarrow$
$z_j - c_j$			8	-2M + 10	-6M + 3	M	M	0	0	

↑

**Iteration 2.**

			$c_j$	-10	-3	0	0	-M	Min.
$c_B$	$x_B$	soln.	$x_1$	$x_2$	$s_1$	$s_2$	$a_1$		ratio
-M	$a_1$	1	$\frac{1}{2}$	0	-1	$\frac{1}{2}$	1		$\frac{1}{1/2} = 2 \rightarrow$
-3	$x_2$	1	$\frac{1}{4}$	1	0	$-\frac{1}{4}$	0		$\frac{1}{1/4} = 4$
$z_j - c_j$			$-\frac{M}{2} + \frac{37}{4}$	0	M	$-\frac{M}{2} + \frac{3}{4}$	0		

↑

**Iteration 3.**

			$c_j$	-10	-3	0	0	Min.
$c_B$	$x_B$	soln.	$x_1$	$x_2$	$s_1$	$s_2$		ratio
-10	$x_1$	2	1	0	-2	1		$\frac{2}{1} = 2 \rightarrow$
-3	$x_2$	$\frac{1}{2}$	0	1	$\frac{1}{2}$	$-\frac{1}{2}$		-
$z_j - c_j$			0	0	$\frac{37}{2}$	$-\frac{17}{2}$		

↑

**Iteration 4. (Optimal)**

$c_j$			-10	-3	0	0	Min.
$c_B$	$x_B$	soln.	$x_1$	$x_2$	$s_1$	$s_2$	ratio
0	$s_2$	2	1	0	-2	1	
-3	$x_2$	$\frac{3}{2}$	$\frac{1}{2}$	1	$-\frac{1}{2}$	0	
$z_j - c_j$			$\frac{17}{2}$	0	$\frac{3}{2}$	0	

NOTES

Since all  $z_j - c_j \geq 0$ , the current solution is optimal.

$$x_1^* = 0, x_2^* = \frac{3}{2}, z^* = \frac{9}{2}$$

**Example 8.** Solve the following LPP by Big-M method :

$$\text{Minimize } z = 2x_1 + x_2 + 3x_3$$

$$\text{S/t, } -3x_1 + x_2 - 2x_3 \geq 1, x_1 - 2x_2 + x_3 \geq 2; x_1, x_2, x_3 \geq 0.$$

**Solution.** The standard form of the given problem can be written as follows :

$$\text{Min. } z = - \text{Max. } (-z = -2x_1 - x_2 - 3x_3 + 0.s_1 + 0.s_2 - M.a_1 - M.a_2)$$

$$\text{S/t, } -3x_1 + x_2 - 2x_3 - s_1 + a_1 = 1,$$

$$x_1 - 2x_2 + x_3 - s_2 + a_2 = 2,$$

$$x_1, x_2, x_3 \geq 0, s_1, s_2 \text{ surplus } s \geq 0, a_1, a_2 \text{ artificial variables } \geq 0.$$

**Iteration 1.**

$c_j$			-2	-1	-3	0	0	-M	-M	Min.
$c_B$	$x_B$	soln.	$x_1$	$x_2$	$x_3$	$s_1$	$s_2$	$a_1$	$a_2$	ratio
-M	$a_1$	1	-3	1	-2	-1	0	1	0	
-M	$a_2$	2	1	-2	1	0	-1	0	1	
$z_j - c_j$			$2M + 2$	$M + 1$	$M + 3$	M	M	0	0	

Since all  $z_j - c_j \geq 0$ , the first iteration itself give optimal solution. But in solution i.e.,  $a_1 = 1, a_2 = 2$  present with non-zero value. Hence the given problem does not possess any feasible solution.

**TWO PHASE METHOD**

To overcome the drawback of Big-M method, two phase method has been framed. In the first phase an auxiliary LP Problem is formulated as follows :

$$\text{Minimize } T = \text{Sum of artificial variables}$$

$$\text{S/t, original constraints}$$

NOTES

which is solved by simplex method. Here artificial variables act as decision variables. So Big-M is not required in the objective function. If  $T^* = 0$ , then go to phase two calculations. else ( $T^* \neq 0$ ) write the problem is infeasible. In phase two, the optimal table of phase one is considered with the following modifications :

Delete the artificial variable's columns and incorporate the original objective function and also update the  $c_B$  values. Calculate  $z_j - c_j$  values. If all  $z_j - c_j \geq 0$ , the current solution is optimal else go to the next iteration.

Note. (a) Multiple solutions, if it exists, can be detected from the optimal table of phase two.

(b) In phase I, the problem is always minimization type irrespective of the type of the original given objective function.

**Example 9.** Using two phase method solve the following LPP :

$$\text{Minimize } z = 10x_1 + 3x_2$$

$$\text{S/t. } x_1 + 2x_2 \geq 3, x_1 + 4x_2 \geq 4; x_1, x_2 \geq 0$$

**Solution.** Standard form of the given LPP is

$$\text{Min. } z = - \text{Max. } (-z = -10x_1 - 3x_2 + 0.s_1 + 0.s_2 - Ma_1 - Ma_2)$$

$$\text{S/t. } x_1 + 2x_2 - s_1 + a_1 = 3,$$

$$x_1 + 4x_2 - s_2 + a_2 = 4,$$

$$x_1, x_2 \geq 0, s_1, s_2, \text{ surplus} \geq 0, a_1, a_2 \text{ artificial} \geq 0$$

**Phase I** Min  $T = a_1 + a_2 = - \text{Max}$

$$(-T = 0, x_1 + 0.x_2 + 0.s_1 + 0.s_2 - a_1 - a_2)$$

$$\text{S/t. } x_1 + 2x_2 - s_1 + a_1 = 3; x_1 + 4x_2 - s_2 + a_2 = 4,$$

$$x_1, x_2, s_1, s_2, a_1, a_2 \geq 0$$

**Iteration 1.**

		$c_j$	0	0	0	0	-1	-1	Min.
$c_B$	$x_B$	soln.	$x_1$	$x_2$	$s_1$	$s_2$	$a_1$	$a_2$	ratio
-1	$a_1$	3	1	2	-1	0	1	0	$\frac{3}{2} = 1.5$
-1	$a_2$	4	1	4	0	-1	0	1	$\frac{4}{4} = 1 \rightarrow$
		$z_j - c_j$	-2	-6	1	1	0	0	

↑

**Iteration 2.**

		$c_j$	0	0	0	0	-1	-1	Min.
$c_B$	$x_B$	soln.	$x_1$	$x_2$	$s_1$	$s_2$	$a_1$	$a_2$	ratio
-1	$a_1$	1	$\frac{1}{2}$	0	-1	$\frac{1}{2}$	1	$-\frac{1}{2}$	$\frac{1}{1/2} = 2 \rightarrow$

NOTES

0	$x_2$	1	$\frac{1}{4}$	1	0	$-\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{1/4} = 4$
$z_j - c_j$			$-\frac{1}{2}$	0	1	$-\frac{1}{2}$	0	$\frac{3}{2}$	

↑

Iteration 3.

			$c_j$	0	0	0	0	-1	-1	Min.
$c_B$	$x_B$	soln.	$x_1$	$x_2$	$s_1$	$s_2$	$a_1$	$a_2$	ratio	
0	$x_1$	2	1	0	-2	1	2	-1		
0	$x_2$	$\frac{1}{2}$	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$		
$z_j - c_j$			0	0	0	0	1	1		

Since all  $z_j - c_j \geq 0$ , the solution is optimal  $a_1^* = 0$ ,  $a_2^* = 0$  and  $T^* = 0$ . Therefore we go to phase II calculations.

Phase II

Iteration 1.

			$c_j$	-10	-3	0	0	Min.
$c_B$	$x_B$	soln.	$x_1$	$x_2$	$s_1$	$s_2$	ratio	
-10	$x_1$	2	1	0	-2	1	$\frac{2}{1} = 2$	→
-3	$x_2$	$\frac{1}{2}$	0	1	$\frac{1}{2}$	$-\frac{1}{2}$		
$z_j - c_j$			0	0	$\frac{37}{2}$	$-\frac{17}{2}$		

Iteration 2.

			$c_j$	-10	-3	0	0	Min.
$c_B$	$x_B$	soln.	$x_1$	$x_2$	$s_1$	$s_2$	ratio	
0	$s_2$	2	1	0	-2	1		
3	$x_2$	$\frac{3}{2}$	$\frac{1}{2}$	1	$-\frac{1}{2}$	0		
$z_j - c_j$			$\frac{17}{2}$	0	$\frac{3}{2}$	0		

Since all  $z_j - c_j \geq 0$ , the current solution is optimal.

$$\therefore x_1^* = 0, x_2^* = \frac{3}{2}, z^* = \frac{9}{2}$$

**Example 10.** Solve the following LPP by two phase method.

$$\text{Maximize } z = 2x_1 + x_2 - 3x_3$$

$$\text{S/t. } x_1 + 2x_2 + 2x_3 \geq 12, 3x_1 - 2x_2 + 4x_3 \leq 10$$

$$x_1 \geq 0, x_2 \leq 0, x_3 \geq 0.$$

**Solution.** Set  $x_2 = -x'_2, x'_2 \geq 0$ .

The standard form of the given LPP is

$$\text{Maximize } z = 2x_1 - x'_2 - 3x_3 + 0.s_1 + 0.s_2 - M.a_1$$

$$\text{S/t. } x_1 - 2x'_2 + 2x_3 - s_1 + a_1 = 12,$$

$$3x_1 + 2x'_2 + 4x_3 + s_2 = 10,$$

$$x_1, x'_2, x_3 \geq 0, s_1 \text{ (surplus)} \geq 0, s_2 \text{ (slack)} \geq 0 \text{ and } a_1 \text{ (artificial)} \geq 0.$$

**Phase I.**

$$\text{Minimize } T = a_1 = - \text{Max. } (-T = 0.x_1 + 0.x'_2 + 0.x_3 + 0.s_1 + 0.s_2 - a_1)$$

$$\text{S/t. } x_1 - 2x'_2 + 2x_3 - s_1 + a_1 = 12$$

$$3x_1 + 2x'_2 + 4x_3 + s_2 = 10$$

$$x_1, x'_2, x_3, s_1, s_2, a_1 \geq 0.$$

**Iteration 1.**

		$c_j$	0	0	0	0	0	-1	Min.
$c_B$	$x_B$	soln.	$x_1$	$x'_2$	$x_3$	$s_1$	$s_2$	$a_1$	ratio
-1	$a_1$	12	1	-2	2	-1	0	1	$\frac{12}{2} = 6$
0	$s_1$	10	3	2	4	0	1	0	$\frac{10}{4} = 2.5 \rightarrow$
		$z_j - c_j$	-1	2	-2	i	0	0	

↑

**Iteration 2.**

		$c_j$	0	0	0	0	0	-1	Min.
$c_B$	$x_B$	soln.	$x_1$	$x'_2$	$x_3$	$s_1$	$s_2$	$a_1$	ratio
-1	$a_1$	7	$-\frac{1}{2}$	-3	0	-1	$-\frac{1}{2}$	1	
0	$x_3$	$\frac{5}{2}$	$\frac{3}{4}$	$\frac{1}{2}$	1	0	$\frac{1}{4}$	0	
		$z_j - c_j$	$\frac{1}{2}$	3	0	1	$\frac{1}{2}$	0	

Since all  $z_j - c_j \geq 0$ , the current solution is optimal and  $T^* = 7 \neq 0$ .

This implies that there does not exist any feasible solution to the given LPP.

**Note.** In simplex, Big-M and two phase methods, if there is a tie in min. ratio or in negative most value of net evaluation, the optimal feasible solution will lead to degenerate solution.

NOTES

## PROBLEMS

Solve the following LPP using *Big-M* method and *Two phase* method :

1. Minimize  $z = 2x_1 + 3x_2$   
S/t,  $2x_1 + x_2 \geq 1$ ,  $x_1 + 2x_2 \geq 1$ ;  $x_1, x_2 \geq 0$ .
2. Maximize  $z = -5x_1 + 3x_2$   
S/t,  $2x_1 - 4x_2 \leq 16$ ,  $3x_1 + 4x_2 \geq 12$ ;  $x_1, x_2 \geq 0$ .
3. Maximize  $z = 2x_1 + 3x_2 + 2x_3$   
S/t,  $3x_1 + 2x_2 + 2x_3 = 16$ ,  $2x_1 + 4x_2 + x_3 = 20$ ,  
 $x_1 \geq 0$ ,  $x_2$  unrestricted in sign,  $x_3 \geq 0$ .
4. Maximize  $z = 2x_1 + 3x_2 + x_3$   
S/t,  $3x_1 + 2x_2 + x_3 = 15$ ,  $x_1 + 4x_2 = 10$ ,  
 $x_1$  unrestricted in sign,  $x_2, x_3 \geq 0$ .
5. Maximize  $z = 2x_1 + 2x_2 + 3x_3$   
S/t,  $x_1 - 2x_2 + x_3 \leq 8$ ,  $3x_1 + 4x_2 + 2x_3 \geq 2$ ,  
 $x_1 \geq 0$ ,  $x_2 \leq 0$ ,  $x_3 \geq 0$ .
6. Maximize  $z = 3x_1 + 2x_2 + x_3 - x_4$   
S/t,  $x_1 + 2x_2 + 3x_3 = 16$ ,  $3x_1 + x_2 + 2x_3 = 20$ ,  
 $2x_1 + x_2 + x_3 + x_4 = 12$ ;  $x_1, x_2, x_3, x_4 \geq 0$ .
7. Minimize  $z = x_1 + 2x_2$   
S/t,  $2x_1 + x_2 = 4$ ,  $3x_1 + 4x_2 \geq 5$ ,  $x_1 + x_2 \leq 4$ ,  
 $x_1, x_2 \geq 0$ .
8. Minimize  $z = x_1 + 3x_2 + 5x_3$   
S/t,  $2x_1 + 5x_2 + x_3 \geq 12$ ,  $x_1 + 2x_2 + 3x_3 \geq 10$ ,  
 $x_1, x_2, x_3 \geq 0$ .
9. Minimize  $z = 5x_1 + x_2$   
S/t,  $2x_1 + x_2 \leq 2$ ,  $3x_1 + 4x_2 \geq 12$ ;  $x_1, x_2 \geq 0$ .
10. Maximize  $z = x_1 - 3x_2$   
S/t,  $-x_1 + 2x_2 \leq 15$ ,  $x_1 + 3x_2 = 10$ ;  $x_1 \geq 0$ ,  $x_2 \leq 0$ .
11. Find a BFS of the following system :  
 $x_1 + x_2 \geq 1$ ,  $-2x_1 + x_2 \geq 2$ ,  $2x_1 + 3x_2 \leq 6$ ;  $x_1, x_2 \geq 0$ .

## ANSWERS

1.  $x_1 = \frac{1}{3}$ ,  $x_2 = \frac{1}{3}$ ,  $z^* = \frac{5}{3}$  (Big-M 3 It)
2. Unbounded solution. (Big-M 4 It)

NOTES

- 3.  $x_1 = 0, x_2 = 4, x_3 = 4, z^* = 20$  (Big-M 4 It)
- 4. Unbounded solution (Big-M 2 It)
- 5.  $x_1 = 0, x_2 = 0, x_3 = 8, z^* = 24$ . (Big-M 5 It)
- 6.  $x_1 = 4, x_2 = 0, x_3 = 4, x_4 = 0, z^* = 16$ . (Big-M 4 It)
- 7.  $x_1 = 2, x_2 = 0, z^* = 2$ . (Big-M 3 It)
- 8.  $x_1 = 10, x_2 = 0, x_3 = 0, z^* = 10$ . (Big-M 5 It)
- 9. Infeasible solution.
- 10. Unbounded solution.
- 11.  $x_1 = 0, x_2 = 2$  (use Phase-I) (3 It).

**FORMULATION PROBLEMS**

**Example 11.** A manufacturer produces two types of machines  $M_1$  and  $M_2$ . Each  $M_1$  requires 4 hrs. of grinding and 2 hrs. of polishing whereas each  $M_2$  model requires 2 hrs. of grinding and 4 hrs. of polishing. Manufacturer has 2 grinders and 3 polishers. Each grinder works for 40 hrs a week and each polisher works 40 hrs a week. Profit on an  $M_1$  model is Rs. 3 and on an  $M_2$  model is Rs. 4. Whatever is produced is sold in the market. How should the manufacturer allocate his production capacity to two types of models so that he may make the maximum profit in a week. Formulate the LPP and solve graphically.

**Solution.** Let  $x_1$  = No. of  $M_1$  machines, and  
 $x_2$  = No. of  $M_2$  machines to be produced in a week.

The above data is summarized as follows :

	$M_1$	$M_2$
Grinding	4 hrs.	2 hrs.
Polishing	2 hrs.	4 hrs.
Profit	Rs. 3	Rs. 4

	Time available per week
2 Grinding	80 hrs.
3 Polishing	120 hrs.

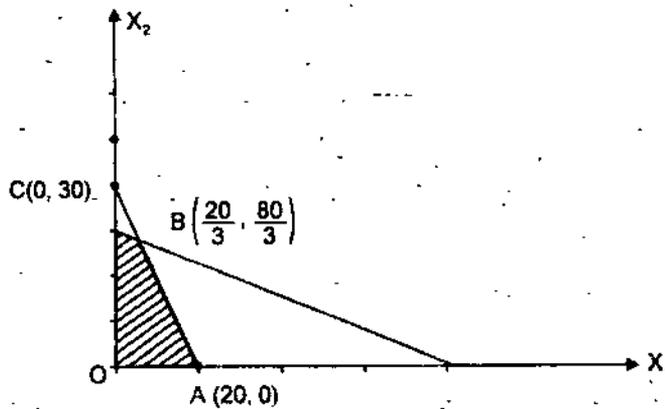
Therefore the LPP can be formulated as follows :

$$\begin{aligned} &\text{Maximize profit} = 3x_1 + 4x_2 \\ &\text{S/t, } 4x_1 + 2x_2 \leq 80 \quad (\text{grinding}) \\ &\quad 2x_1 + 4x_2 \leq 120 \quad (\text{polishing}) \\ &\quad x_1, x_2 \geq 0 \end{aligned}$$

The graphical region is shown below.

- Profit at A = 60
- Profit at B = 126.67
- Profit at C = 120

NOTES



∴ The optimal solution is  $x_1 = \frac{20}{3}$ ,  $x_2 = \frac{80}{3}$ , and max. profit = Rs. 126.67.

**Example 12.** A firm can produce three types of woolen clothes, say, A, B and C using three kinds of wool, say red wool, green wool and blue wool. One unit of length of type A cloth needs 2 yards of red wool and 3 yards of blue wool; one unit length of type B cloth needs 3 yards of red wool, 2 yards of green wool and 2 yards of blue wool; and one unit length of type C cloth needs 5 yards of green wool and 4 yards of blue wool. The firm has only a stock of 8 yards of red wool, 10 yards of green wool and 15 yards of blue wool. It is assumed that income obtained from one unit length of type A cloth is Rs. 3, of type B cloth is Rs. 5 and that of type C cloth is Rs. 4. Formulate the above problem as a LP problem.

**Solution.** The given data is summarized below :

Wool	Garment type			Stock available
	A	B	C	
Red	2	3	-	8
Green	-	2	5	10
Blue	3	2	4	15
Income (Rs.)	3	5	4	

Suppose that he produces  $x_1$ ,  $x_2$  and  $x_3$  unit lengths of A, B and C clothes respectively. Then the LPP is :

$$\begin{aligned} \text{Maximize income} &= 3x_1 + 5x_2 + 4x_3 \\ \text{S/t, } 2x_1 + 3x_2 &\leq 8 && \text{(Red wool)} \\ 2x_2 + 5x_3 &\leq 10 && \text{(Green wool)} \\ 3x_1 + 2x_2 + 5x_3 &\leq 15 && \text{(Blue wool)} \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

The solution is obtained as  $x_1^* = 1.67$ ,  $x_2^* = 1.56$ ,  $x_3^* = 1.38$ ,  $z^* = 18.29$  (It 4, Simplex).

**PROBLEMS**

**NOTES**

1. A manufacturer of furniture makes only chair and tables. A chair requires two hours on m/c A and six hours on m/c B. A table requires five hours on m/c A and two hours on m/c B. 16 hours are available on m/c A and 22 hours on m/c B per day. Profits for a chair and table be Rs. 1 and Rs. 5 respectively. Formulate the LPP of finding daily production of these items for maximum profit and solve graphically.
2. A tailor has 80 sq. m. of cotton material and 120 sq. m. of woolen material. A suit requires 1 sq. m. of cotton and 3 sq. m. of woolen material and a dress requires 2 sq. m. of each. A suit sells for Rs. 500 and a dress for Rs. 400. Pose a LPP in terms of maximizing the income.
3. A company owns two mines : mine A produces 1 tonne of high grade ore, 3 tonnes of medium grade ore and 5 tonnes of low grade ore each day; and mine B produces 2 tonnes of each of the three grades of ore each day. The company needs 80 tonnes of high grade ore, 160 tonnes of medium grade ore and 200 tonnes of low grade ore. If it costs Rs. 200 per day to work each mine, find the number of days each mine has to be operated for producing the required output with minimum total cost.
4. A company manufactures two products A and B. The profit per unit sale of A and B is Rs. 10 and Rs. 15 respectively. The company can manufacture at most 40 units of A and 20 units of B in a month. The total sale must not be below Rs. 400 per month. If the market demand of the two items be 40 units in all, write the problem of finding the optimum number of items to be manufactured for maximum profit, as a problem of LP. Solve the problem graphically or otherwise.
5. A company is considering two types of buses—ordinary and semideluxe for transportation. Ordinary bus can carry 40 passengers and requires 2 mechanics for servicing. Semideluxe bus can carry 60 passengers and requires 3 mechanics for servicing. The company can transport at least 300 persons daily and not more than 12 mechanics can be employed. The cost of purchasing buses is to be minimized, given that the ordinary bus costs Rs. 1,20,000 and semideluxe bus costs Rs. 1,50,000. Formulate this problem as a LPP.
6. A pharmaceutical company has 100 kg. of ingredient A, 180 kg. of ingredient B and 120 kg. of ingredient C available per month. They can use these ingredients to make three basic pharmaceutical products namely 5-10-5; 5-5-10 and 20-5-10; where the numbers in each case represent the percentage by weight of A, B and C respectively in each of the products. The cost of these ingredients are given below :

<i>Ingredient</i>	<i>Cost per kg. (Rs.)</i>
A	80
B	20
C	50
Inert ingredients	20

Selling price of these products are Rs. 40.5, Rs. 43 and Rs. 45 per kg. respectively. There is a capacity restriction of the company for the product 5-10-5, so they cannot produce more than 30 kg. per month. Determine how much of each of the products they should produce in order to maximize their monthly profit.

7. A fruit squash manufacturing company manufactures three types of squashes. The basic formula are :  
 5 litre lemonade : 2 oz. lemons, 2 kg. of sugar, 2 oz. citric acid and water.

5 litre grape fruits : 11/2 kg of grape fruit, 11/2 kg. of sugar, 11/2 oz. citric acid and water.

5 litre orangeade : 11/2 dozen oranges, 11/2 kg. of sugar, 1 oz. citric acid and water.

The squashes sell at

Lemonade : Rs. 37.50 per 5 litre;

Grape fruit : Rs. 40.00 per 5 litre;

Orangeade : Rs. 42.50 per 5 litre.

In the last week of the season they have in stock 2500 dozen lemons, 2000 kgs. grape fruit, 750 dozen oranges, 5000 kgs. of sugar and 3000 ozs. citric acid. What should be their manufacturing quantities in the week to maximize the turnover ?

8. A farmer is raising cows in his farm. He wishes to determine the qualities of the available types of feed that should be given to each cow to meet certain nutritional requirements at a minimum cost. The numbers of each type of basic nutritional ingredient contained within a kg. of each feed type is given in the following table, along with the daily nutritional requirements and feed costs.

Nutritional ingredient	kg. of corn	kg. of tankage	kg. of green grass	Min. daily requirement
Carbohydrates	9	2	4	20
Proteins	3	8	6	18
Vitamins	1	2	6	15
Cost	7	6	5	

Formulate a linear programming model for this problem so as to determine the optimal mix of feeds.

## ANSWERS

- No chairs and 3.2 tables to be produced for max. profit of Rs. 16.
- Max. sells =  $500x_1 + 400x_2$   
S/t,  $x_1 + 2x_2 \leq 80$ ,  $3x_1 + 2x_2 \leq 120$ ,  $x_1 = \text{no. of suits} \geq 0$  and  $x_2 = \text{no. of dresses} \geq 0$ .
- Mine A to be operated for 40 days and mine B to be operated for 20 days and min. cost = Rs. 12000.
- Max. profit =  $10x_1 + 15x_2$   
S/t,  $x_1 \leq 40$ ,  $x_2 \leq 20$ ,  $x_1 + x_2 \geq 40$ ,  $10x_1 + 15x_2 \geq 400$ ,  $x_1, x_2 \geq 0$  and  $x_1^* = 40$ ,  $x_2^* = 20$ , max. profit = Rs. 700.
- Min. cost =  $1,20,000x_1 + 1,50,000x_2$   
S/t,  $40x_1 + 60x_2 \geq 300$ ,  $2x_1 + 3x_2 \leq 12$ ,  $x_1, x_2 \geq 0$ .
- Let  $x_1, x_2, x_3$  be three products in kg. to be manufactured.  
Max. profit =  $16x_1 + 17x_2 + 10x_3$   
S/t,  $x_1 + x_2 + 4x_3 \leq 2000$ ,  $2x_1 + x_2 + x_3 \leq 3600$ ,  
 $x_1 + 2x_2 + 2x_3 \geq 2400$ ,  $x_1 \leq 30$ ,  $x_1 + x_2 + x_3 \geq 0$   
Solution.  $x_1 = 30$ ,  $x_2 = 1185$ ,  $x_3 = 0$ , Profit = Rs. 20,625. (Simplex 3It)
- Let  $5x_1, 5x_2, 5x_3$  litre be the lemonade, grape fruit and orangeade to be manufactured per week.

NOTES

NOTES

- Max. profit =  $37.5x_1 + 40x_2 + 42.5x_3$   
 S/t,  $2x_1 \leq 2500$ ,  $3x_2 \leq 4000$ ,  $3x_3 \leq 1500$ ,  
 $4x_1 + 3x_2 + 3x_3 \leq 10,000$ ,  $4x_1 + 3x_2 + 2x_3 \leq 6000$ ,  $x_1, x_2, x_3 \geq 0$ .
8. Min. cost =  $7x_1 + 6x_2 + 5x_3$   
 S/t,  $9x_1 + 2x_2 + 4x_3 \geq 20$ ,  
 $3x_1 + 8x_2 + 6x_3 \geq 18$   
 $x_1 + 2x_2 + 6x_3 \geq 15$   
 $x_1, x_2, x_3 \geq 0$ .

**REVISED SIMPLEX METHOD (RSM)**

**I. Algorithm**

**Step 1.** Write the standard form of the given LPP and convert it into maximization type if it is in minimization type *i.e.*,

$$\text{Max. } z = cx$$

$$\text{S/t, } Ax = b, x \geq 0.$$

Use the following notations :

$$c^T = [c_1, c_2, \dots, c_n] \text{ Profit coefficients.}$$

Columns of A as  $A_1, A_2, \dots, A_m$ .

$$\pi = (\pi_1, \pi_2, \dots) \text{ Simplex multipliers}$$

$$x_B = \text{Basis vector}$$

$$c_B^T = \text{Profit coefficient in the basis}$$

$$B = \text{Basis matrix, } B^{-1} = \text{Basis inverse}$$

$$\bar{c}_j = \text{Net evaluations,}$$

$$j = \text{Index of non-basic variables}$$

$$\bar{b} = \text{Current BFS}$$

**Step 2:** For iteration 1

$$B = I, B^{-1} = I$$

else for other iterations

Find  $B = [x_B] = [A_{x_B}]$  and hence find  $B^{-1}$ .

**Step 3.** Calculate

$$\pi = c_B^T B^{-1} \text{ and } \bar{b} = B^{-1} \cdot b \text{ (current solution)}$$

$$\bar{c}_j = \pi A_j - c_j$$

*Decisions :* If all  $\bar{c}_j \geq 0$  then the current BFS is optimal, else select the negative most of  $\bar{c}_j$ , say  $\bar{c}_k$ . Then  $x_k$  will be the 'Entering Variable' and  $\bar{A}_k = \text{key column} = B^{-1} \cdot A_k$ .

**Step 4.** Produce the following revised simplex table :

$x_B$	$B^{-1}$	$\bar{b}$	Entering variable	Key column

Encircle the key element obtained from the min. ratio  $\left\{ \frac{\bar{b}}{[\text{Key column}]} \right\}$ .

Element corresponding to the key element will depart from  $[x_B]$ .

Step 5. Go to step 2.

Repeat the procedure until optimal BFS is obtained.

Note. (a) If, in step 4, all the elements in the key column are non-positive, then the given problem is unbounded.

(b) If, in the optimal BFS, artificial variables (if any) take zero value then the solution is degenerate else, for non-zero value, the given problem is said to be infeasible.

## NOTES

## II. Advantages

In computational point of view, the Revised Simplex Method is superior than ordinary simplex method. Due to selected column calculations in revised simplex method, less memory is required in computer. Whereas the ordinary simplex method requires more memory space in computer.

**Example 13.** Using revised simplex method solve the following LPP :

$$\begin{aligned} \text{Maximize } z &= 5x_1 + 2x_2 + 3x_3 \\ \text{S/t, } x_1 + 2x_2 + 2x_3 &\leq 8 \\ 3x_1 + 4x_2 + x_3 &\leq 7 \\ x_1, x_2, x_3 &\geq 0. \end{aligned}$$

**Solution.** Standard form of the given LPP is

$$\begin{aligned} \text{Max. } z &= 5x_1 + 2x_2 + 3x_3 + 0s_1 + 0s_2 \\ \text{S/t, } x_1 + 2x_2 + 2x_3 + s_1 &= 8 \\ 3x_1 + 4x_2 + x_3 + s_2 &= 7 \\ x_1, x_2, x_3 \geq 0, s_1, s_2 \text{ are slacks and } &\geq 0 \end{aligned}$$

Then,  $A_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, A_2 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}, A_3 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, A_4 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, A_5 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, b = \begin{bmatrix} 8 \\ 7 \end{bmatrix}$

Let us consider the index of the variables  $x_1$  be 1,  $x_2$  be 2,  $x_3$  be 3,  $s_1$  be 4,  $s_2$  be 5.

**Iteration 1.**

$$x_B = (s_1, s_2), B = [A_4, A_5] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I, B^{-1} = I.$$

$$c_B^T = (0, 0), \bar{b} = B^{-1} \cdot b = b, J = (1, 2, 3).$$

$$\pi = c_B^T \cdot B^{-1} = (0, 0) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = (0, 0) = (\pi_1, \pi_2).$$

Net evaluations :

$$\bar{c}_1 = \pi A_1 - c_1 = -5 \leftarrow \text{negative most and entering variable is } x_1$$

$$\bar{c}_2 = \pi A_2 - c_2 = -2$$

$$\bar{c}_3 = \pi A_3 - c_3 = -3.$$

Key column :  $B^{-1} \cdot A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

Table 1

$x_B$	$B^{-1}$	$\bar{b}$	Entering variable	Key column
$s_1$	1 0	8		1
$s_2$	0 1	7	$x_1$	3

This indicates the departing variable as  $s_2$ .

Iteration 2.

$$x_B = (s_1, x_1), B = [A_4, A_1] = \begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix}, B^{-1} = \begin{bmatrix} 1 & -1/3 \\ 0 & 1/3 \end{bmatrix}$$

$$\bar{b} = B^{-1} \cdot b = \begin{bmatrix} 1 & -1/3 \\ 0 & 1/3 \end{bmatrix} \begin{bmatrix} 8 \\ 7 \end{bmatrix} = \begin{bmatrix} 17/3 \\ 7/3 \end{bmatrix}, J = (2, 3, 5).$$

$$\pi = c_B^T \cdot B^{-1} = [0, 5] \begin{pmatrix} 1 & -1/3 \\ 0 & 1/3 \end{pmatrix} = \left(0, \frac{5}{3}\right)$$

Net evaluations :

$$\bar{c}_2 = \pi A_2 - c_2 = \left(0, \frac{5}{3}\right) \begin{pmatrix} 2 \\ 4 \end{pmatrix} - 2 = \frac{14}{3}$$

$$\bar{c}_3 = \pi A_3 - c_3 = \left(0, \frac{5}{3}\right) \begin{pmatrix} 2 \\ 1 \end{pmatrix} - 3 = -\frac{4}{3} \leftarrow \text{Entering variable } x_3$$

$$\bar{c}_5 = \pi A_5 - c_5 = \left(0, \frac{5}{3}\right) \begin{pmatrix} 0 \\ 1 \end{pmatrix} - 0 = \frac{5}{3}$$

Key column :  $B^{-1} \cdot A_3 = \begin{pmatrix} 1 & -1/3 \\ 0 & 1/3 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 5/3 \\ 1/3 \end{pmatrix}$

Table 2

$x_B$	$B^{-1}$	$\bar{b}$	Entering variable	Key column
$s_1$	1 -1/3	17/3	$x_3$	5/3
$x_2$	0 1/3	7/3		1/3

(This indicates the departing variable as  $s_1$ ).

Iteration 3.

$$x_B = (x_3, x_1), B = [A_3, A_1] = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}, B^{-1} = \begin{pmatrix} 3/5 & -1/5 \\ -1/5 & 2/5 \end{pmatrix}$$

$$J = 2, 4, 5$$

$$\bar{b} = B^{-1} \cdot b = \begin{pmatrix} 3/5 & -1/5 \\ -1/5 & 2/5 \end{pmatrix} \begin{pmatrix} 8 \\ 7 \end{pmatrix} = \begin{pmatrix} 17/5 \\ 6/5 \end{pmatrix}$$

$$\pi = c_B^T \cdot B^{-1} = (3, 5) \begin{pmatrix} 3/5 & -1/5 \\ -1/5 & 2/5 \end{pmatrix} = \left( \frac{4}{5}, \frac{7}{5} \right)$$

NOTES

Net evaluations :

$$\bar{c}_3 = \pi A_3 - c_3 = \left( \frac{4}{5}, \frac{7}{5} \right) \begin{pmatrix} 2 \\ 4 \end{pmatrix} - 2 = \frac{26}{5}$$

$$\bar{c}_4 = \pi A_4 - c_4 = \left( \frac{4}{5}, \frac{7}{5} \right) \begin{pmatrix} 1 \\ 0 \end{pmatrix} - 0 = \frac{4}{5}$$

$$\bar{c}_5 = \pi A_5 - c_5 = \left( \frac{4}{5}, \frac{7}{5} \right) \begin{pmatrix} 0 \\ 1 \end{pmatrix} - 0 = \frac{7}{5}$$

As all  $\bar{c}_j > 0 \Rightarrow$  the current  $\bar{b}$  is optimal.

$$\therefore x_1^* = \frac{6}{5}, x_2^* = 0, x_3^* = \frac{17}{5} \text{ and } z^* = \frac{81}{5}$$

**Example 14.** Solve by revised simplex method.

$$\text{Minimize } z = 12x_1 + 20x_2$$

$$\text{S/t. } 6x_1 + 8x_2 \geq 100$$

$$7x_1 + 12x_2 \geq 120$$

$$x_1, x_2 \geq 0.$$

**Solution.** Standard form :

$$\text{Min. } z = - \text{Max. } (-z = -12x_1 - 20x_2 + 0 \cdot s_1 + 0 \cdot s_2 - M a_1 - M a_2)$$

$$\text{S/t. } 6x_1 + 8x_2 - s_1 + a_1 = 100$$

$$7x_1 + 12x_2 - s_2 + a_2 = 120$$

 $x_1, x_2 \geq 0, s_1, s_2$  surplus and  $\geq 0, a_1, a_2$  artificial and  $\geq 0$ .

$$\text{Let } A_1 = \begin{bmatrix} 6 \\ 7 \end{bmatrix}, A_2 = \begin{bmatrix} 8 \\ 12 \end{bmatrix}, A_3 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, A_4 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, A_5 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, A_6 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; b = \begin{bmatrix} 100 \\ 120 \end{bmatrix}$$

Let the index of the variables  $x_1, x_2, s_1, s_2, a_1, a_2$  be 1, 2, 3, 4, 5 and 6 respectively.**Iteration 1.**

$$x_B = (a_1, a_2), B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = B^{-1}, \bar{b} = b, c_B^T = [-M, -M]$$

$$\pi = c_B^T \cdot B^{-1} = [-M, -M], J = (1, 2, 3, 4)$$

Net evaluations :

$$\bar{c}_1 = \pi A_1 - c_1 = -13M + 12$$

$$\bar{c}_2 = \pi A_2 - c_2 = -20M + 20 \leftarrow \text{Most negative and } x_2 \text{ as entering variable}$$

$$\bar{c}_3 = \pi A_3 - c_3 = M$$

$$\bar{c}_4 = \pi A_4 - c_4 = M$$

$$\text{Key column} = B^{-1} \cdot A_2 = \begin{bmatrix} 8 \\ 12 \end{bmatrix}$$

NOTES

Table 1

$x_B$	$B^{-1}$	$\bar{b}$	Entering variable	Key column
$a_1$	1 0	100	$x_2$	8
$a_2$	0 1	120		(12)

(This table indicates  $a_2$  as departing variable).

Iteration 2.

$$x_B = (a_1, x_2), B = [A_3, A_2] = \begin{bmatrix} 1 & 8 \\ 0 & 12 \end{bmatrix}, B^{-1} = \begin{bmatrix} 1 & -2/3 \\ 0 & 1/12 \end{bmatrix}$$

$$J = (1, 3, 4, 6)$$

$$\pi = c_B^T \cdot B^{-1} = (-M, -20) \begin{bmatrix} 1 & -2/3 \\ 0 & 1/12 \end{bmatrix} = \left( -M, \frac{2}{3}M - \frac{5}{3} \right)$$

$$\bar{b} = B^{-1} \cdot b = \begin{bmatrix} 1 & -2/3 \\ 0 & 1/12 \end{bmatrix} \begin{bmatrix} 100 \\ 120 \end{bmatrix} = \begin{bmatrix} 20 \\ 10 \end{bmatrix}$$

Net evaluations :

$$\bar{c}_1 = \pi A_1 - c_1 = -\frac{4}{3}M + \frac{1}{3}$$

$$\bar{c}_3 = \pi A_3 - c_3 = M$$

$$\bar{c}_4 = \pi A_4 - c_4 = -\frac{2}{3}M + \frac{5}{3}$$

$$\bar{c}_6 = \pi A_6 - c_6 = \frac{5}{3}M - \frac{5}{3}$$

$$\text{Key column} = B^{-1} \cdot A_1 = \begin{bmatrix} 4/3 \\ 7/12 \end{bmatrix}$$

Table 2

$x_B$	$B^{-1}$	$\bar{b}$	Entering variable	Key column
$a_1$	1 -2/3	20	$x_1$	(4/3)
$x_2$	0 1/12	10		7/12

(This table indicates  $a_1$  as departing variable).

## Iteration 3.

$$x_B = (x_1, x_2), B = (A_1, A_2) = \begin{pmatrix} 6 & 8 \\ 7 & 12 \end{pmatrix}, B^{-1} = \begin{pmatrix} 3/4 & -1/2 \\ -7/16 & 3/8 \end{pmatrix}$$

$$J = (3, 4, 5, 6)$$

$$\bar{\pi} = c_B^T \cdot B^{-1} = (-12, -20) \begin{pmatrix} 3/4 & -1/2 \\ -7/16 & 3/8 \end{pmatrix} = \left( -\frac{1}{4}, -\frac{3}{2} \right)$$

$$\bar{b} = B^{-1} \cdot b = \begin{pmatrix} 3/4 & -1/2 \\ -7/16 & 3/8 \end{pmatrix} \begin{pmatrix} 100 \\ 120 \end{pmatrix} = \begin{pmatrix} 15 \\ 5/4 \end{pmatrix}$$

Net evaluations :

$$\bar{c}_3 = \pi A_3 - c_3 = \frac{1}{4}$$

$$\bar{c}_4 = \pi A_4 - c_4 = \frac{3}{2}$$

$$\bar{c}_5 = \pi A_5 - c_5 = M - \frac{1}{4}$$

$$\bar{c}_6 = \pi A_6 - c_6 = M - \frac{3}{2}$$

Since all  $\bar{c}_j > 0 \Rightarrow$  the current  $\bar{b}$  is optimal.

$$\therefore x_1^* = 15, x_2^* = \frac{5}{4} \text{ and } z^* = 205.$$

## PROBLEMS

Using revised simplex method solve the following LPP :

1. Maximize  $z = x_1 + x_2 + 3x_3$   
S/t,  $3x_1 + 2x_2 + x_3 \leq 3, 2x_1 + x_2 + 2x_3 \leq 2; x_1, x_2, x_3 \geq 0.$
2. Maximize  $z = 3x_1 + 4x_2$   
S/t,  $x_1 - x_2 \geq 0, -x_1 + 3x_2 \leq 3; x_1, x_2 \geq 0.$
3. Minimize  $z = x_1 + x_2$   
S/t,  $2x_1 + x_2 \geq 4, x_1 + 7x_2 \geq 7; x_1, x_2 \geq 0.$
4. Minimize  $z = 2x_1 - x_2 + 2x_3$   
S/t,  $-x_1 + x_2 + x_3 = 4, -x_1 + x_2 - x_3 \leq 6,$   
 $x_1 \leq 0, x_2 \geq 0, x_3$  unrestricted in sign.
5. Maximize  $z = -x_1 + 2x_2 + 3x_3$   
S/t,  $-2x_1 + x_2 + 3x_3 = 2, 2x_1 + 3x_2 + 4x_3 = 1; x_1, x_2, x_3 \geq 0.$
6. Maximize  $z = 2x_1 + x_2 + 3x_3$   
S/t,  $x_1 + x_2 + 2x_3 \leq 5, 2x_1 + 3x_2 + 4x_3 = 12; x_1, x_2, x_3 \geq 0.$
7. Maximize  $z = 5x_1 + 2x_2 + 3x_3$   
S/t,  $x_1 + 2x_2 + 2x_3 \leq 8, 3x_1 + 4x_2 + x_3 \leq 7; x_1, x_2, x_3 \geq 0.$

NOTES

**ANSWERS**

## NOTES

1.  $x_1 = 0, x_2 = 0, x_3 = 1, z^* = 3.$
2. Unbounded solution.
3.  $x_1 = \frac{21}{13}, x_2 = \frac{10}{13}, z^* = \frac{31}{13}.$
4.  $x_1 = -5, x_2 = 0, x_3 = -1, z^* = -12$  (Iteration 3)
5. Infeasible solution.
6.  $x_1 = 3, x_2 = 2, x_3 = 0, z^* = 8.$
7.  $x_1 = \frac{6}{5}, x_2 = 0, x_3 = \frac{17}{5}, z^* = \frac{81}{5}$  (Iteration 3)

**INTRODUCTION AND FORMULATION**

For every LP Problem we can construct another LP problem using the same data. These two problems try to achieve two different objectives within the same data. The original problem is called *Primal* problem and the constructed problem is called *Dual*. This is illustrated through the following example :

A company makes three products X, Y, Z using three raw materials A, B and C. The raw material requirement is given below : (for 1 unit of product).

	X	Y	Z	Availability
A	1	2	1	36 units
B	2	1	4	60 units
C	2	5	1	45 units
Profit	Rs. 40	Rs. 25	Rs. 50	

Let the company decide to produce  $x_1, x_2$  and  $x_3$  units of the products X, Y and Z respectively in order to maximize the profit. We obtain the following LP problems :

$$\text{Maximize profit} = 40x_1 + 25x_2 + 50x_3$$

$$\text{Subject to, } x_1 + 2x_2 + x_3 \leq 36.$$

$$2x_1 + x_2 + 4x_3 \leq 60,$$

$$2x_1 + 5x_2 + x_3 \leq 45,$$

$$x_1, x_2, x_3 \geq 0.$$

Adding slack variables  $s_1, s_2$  and  $s_3$  to the constraints, we solve the problem by simplex method. The optimal solution is

$$x_1 = 20, x_2 = 0, x_3 = 5 \text{ and optimal profit} = \text{Rs. } 1050.$$

Suppose the company wishes to sell the three raw materials A, B and C instead of using them for production of the products X, Y and Z. Let the selling prices be Rs.  $y_1$ , Rs.  $y_2$  and Rs.  $y_3$  per unit of raw material A, B and C respectively.

The cost of the purchaser due to all raw materials is

$$36y_1 + 60y_2 + 45y_3.$$

Then the purchaser forms the following LP problem :

$$\begin{aligned} \text{Minimize } T &= 36y_1 + 60y_2 + 45y_3 \\ \text{Subject to, } y_1 + 2y_2 + 2y_3 &\geq 40, \\ 2y_1 + y_2 + 5y_3 &\geq 25, \\ y_1 + 4y_2 + y_3 &\geq 50, \\ y_1, y_2, y_3 &\geq 0. \end{aligned}$$

NOTES

The solution is obtained as :

$$y_1 = 0, y_2 = 10, y_3 = 10, \text{ Optimal cost} = \text{Rs. } 1050.$$

In the above, the company's problem is called primal problem and purchaser's problem is called dual problem. Also we can use these two terms interchangeably. In the primal problem, the company achieve a profit of Rs. 1050 by producing 20 units of X and 5 units of Z. Instead, if the company sells the raw material B with Rs. 10 per unit and C with Rs. 10 per unit then also the company achieve a sale of Rs. 1050.

**(a) Formulation**

In the above, both the problems are called symmetric problem since the objective function is maximization (minimization), all the constraints are ' $\leq$ ' type ( $\geq$  type) and non-negative decision variables.

The decision variables in the primal are called primal variables and the decision variables in the dual are called dual variables.

Let us consider the following table for formulation of the dual.

<i>Primal</i> (Maximization)	<i>Dual</i> (Minimization)
Right hand side constants	Cost vector
Cost vector	Right hand side constants.
Coefficient matrix	Transpose of coefficient matrix
' $\leq$ '	' $\geq$ '
Max. $z = cx$ S/t, $Ax \leq b$ $x \geq 0$	Min. $T = b^T y$ S/t $A^T y \geq c^T$ $y \geq 0$

**(b) Asymmetric Primal-Dual Problems**

<i>Primal (Maximization)</i>	<i>Dual (Minimization)</i>
a. Coefficient matrix	$A^T$
b. Right hand side constants	Cost vector

NOTES

$c$ . Cost vector	Right hand side constants
$i$ -th constraint	$i$ -th dual variable
$\leq$ type	$y_i \geq 0$
$\geq$ type	$y_i \leq 0$
$=$ type	$y_i$ unrestricted in sign
$j$ -th primal variable	$j$ -th dual constraint
$x_j$ unrestricted in sign	$=$ type
$x_j \leq 0$	$\leq$ type
$x_j \geq 0$	$\geq$ type

Also in (a) and (b),

No. of primal constraints = No. of dual variables.

No. of primal variables = No. of dual constraints

Note. The dual of the dual is the primal.

**Example 15.** Obtain the dual of

$$\text{Minimize } z = 8x_1 + 3x_2 + 15x_3$$

$$\text{Subject to, } 2x_1 + 4x_2 + 3x_3 \geq 28,$$

$$3x_1 + 5x_2 + 6x_3 \geq 30,$$

$$x_1, x_2, x_3 \geq 0.$$

**Solution.** Let  $y_1$  and  $y_2$  be the variables corresponding to the first and second constraints respectively. Objective function, maximize  $T = 28y_1 + 30y_2$ . There will be three dual constraints due to three primal variables. In primal

$$A = \begin{bmatrix} 2 & 4 & 3 \\ 3 & 5 & 6 \end{bmatrix}, c = [8, 3, 15]$$

$\therefore$  In dual

$$A^T \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \leq c^T$$

$\Rightarrow$

$$\begin{pmatrix} 2 & 3 \\ 4 & 5 \\ 3 & 6 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \leq \begin{pmatrix} 8 \\ 3 \\ 15 \end{pmatrix}$$

$\Rightarrow$

$$2y_1 + 3y_2 \leq 8 \text{ (due to } x_1)$$

$$4y_1 + 5y_2 \leq 3 \text{ (due to } x_2)$$

$$3y_1 + 6y_2 \leq 15 \text{ (due to } x_3)$$

Hence the dual problem is

$$\text{Maximize } T = 28y_1 + 30y_2$$

$$\text{Subject to, } 2y_1 + 3y_2 \leq 8$$

$$4y_1 + 5y_2 \leq 3$$

$$3y_1 + 6y_2 \leq 15$$

$$y_1, y_2 \geq 0.$$

**Example 16.** Find the dual of

$$\text{Maximize } z = 2x_1 + x_2 + 5x_3$$

$$\text{Subject to, } x_1 + x_2 + x_3 = 10.$$

$$4x_1 - x_2 + 2x_3 \geq 12.$$

$$3x_1 + 2x_2 - 3x_3 \leq 6.$$

$$x_1, x_2, x_3 \geq 0.$$

**Solution.** First we have to express all the constraints in ' $\leq$ ' form due to maximization problem :

The first constraint :  $x_1 + x_2 + x_3 \leq 10$

and  $x_1 + x_2 + x_3 \geq 10$

$$\Rightarrow -x_1 - x_2 - x_3 \leq -10$$

The second constraint :  $-4x_1 + x_2 - 2x_3 \leq -12$

Let  $y_1, y_2, y_3$  and  $y_4$  be four dual variables corresponding to the newly converted constraints respectively.

Objective function :  $\text{Minimize } T = 10y_1 - 10y_2 - 12y_3 + 6y_4$

Again,

$$A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ -4 & 1 & -2 \\ 3 & 2 & -3 \end{bmatrix}, c = [2, 1, 5]$$

$\therefore$  Constraints in dual :

$$A^T \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} \geq c^T.$$

Thus the dual problem is

$$\text{Minimize } T = 10y_1 - 10y_2 - 12y_3 + 6y_4.$$

$$\text{Subject to, } y_1 - y_2 - 4y_3 + 3y_4 \geq 2 \text{ (due to } x_1)$$

$$y_1 - y_2 + y_3 + 2y_4 \geq 1 \text{ (due to } x_2)$$

$$y_1 - y_2 - 2y_3 - 3y_4 \geq 5 \text{ (due to } x_3)$$

$$y_1, y_2, y_3, y_4 \geq 0.$$

Set  $w_1 = y_1 - y_2, w_2 = -y_3, w_3 = y_4 \Rightarrow w_1$  unrestricted in sign,  $w_2 \leq 0, w_3 \geq 0$ .

This conversion leads to

$$\text{Minimize } T = 10w_1 + 12w_2 + 6w_3$$

$$\text{Subject to, } w_1 + 4w_2 + 3w_3 \geq 2,$$

$$w_1 - w_2 + 2w_3 \geq 1,$$

$$w_1 + 2w_2 - 3w_3 \geq 5.$$

$w_1$  unrestricted in sign,

$$w_2 \leq 0, w_3 \geq 0.$$

NOTES

**Example 17.** Find the dual of

$$\text{Maximize } z = 5x_1 + 4x_2 - 3x_3$$

$$\text{Subject to, } 2x_1 + 4x_2 - x_3 \leq 14,$$

$$x_1 - 2x_2 + x_3 = 10,$$

$$x_1 \geq 0, x_2 \text{ unrestricted in sign, } x_3 \leq 0.$$

NOTES

**Solution.** First we have introduce non-negative variables.

$$\therefore \text{ Set } x_2 = x'_2 - x''_2; x'_2, x''_2 \geq 0 \text{ and } x_3 = -x'_3, x'_3 \geq 0.$$

The given problem reduces to

$$\text{Maximize } z = 5x_1 + 4x'_2 - 4x''_2 + 3x'_3$$

$$\text{Subject to, } 2x_1 + 4x'_2 - 4x''_2 + x'_3 \leq 14$$

$$x_1 - 2x'_2 + 2x''_2 - x'_3 \leq 10$$

$$x_1, x'_2, x''_2, x'_3 \geq 0$$

The second constraint is expressed as

$$x_1 - 2x'_2 + 2x''_2 - x'_3 \leq 10$$

$$\text{and } -x_1 + 2x'_2 - 2x''_2 + x'_3 \leq -10$$

Let  $y_1, y_2, y_3$  be the three dual variables corresponding to the three constraints respectively. Then the symmetric dual is

$$\text{Minimize } T = 14y_1 + 10y_2 - 10y_3$$

$$\text{Subject to, } 2y_1 + y_2 - y_3 \geq 5 \text{ (due to } x_1)$$

$$4y_1 - 2y_2 + 2y_3 \geq 4 \text{ (due to } x'_2)$$

$$-4y_1 + 2y_2 - 2y_3 \geq -4 \text{ (due to } x''_2)$$

$$y_1 - y_2 + y_3 \geq -3 \text{ (due to } x'_3)$$

$$y_1, y_2, y_3 \geq 0$$

$$\text{Set } w_1 = y_1, w_2 = y_2 - y_3 \Rightarrow w_1 \geq 0 \text{ and } w_2 \text{ unrestricted.}$$

Also the second and third constraint reduces to

$$4y_1 - 2y_2 + 2y_3 = 4$$

Therefore the dual is

$$\text{Minimize } T = 14w_1 + 10w_2$$

$$\text{Subject to, } 2w_1 + w_2 \geq 5,$$

$$4w_1 - 2w_2 = 4,$$

$$-w_1 + w_2 \leq 3$$

$$w_1 \geq 0 \text{ and } w_2 \text{ unrestricted in sign.}$$

## DUALITY THEOREMS

### Theorem 1. (Weak Duality)

Consider the symmetric primal (max. type) and Dual (min. type). The value of the objective function of the (dual) minimum problem for any feasible solution is always greater than or equal to that of the maximum problem (primal) for any feasible solution.

**Proof.** Let  $x^0$  be a feasible solution to the primal.

Then  $Ax^0 \leq b$ ,  $x^0 \geq 0$  and  $z = cx^0$ .

Let  $y^0$  be a feasible solution to the dual.

Then  $A^T y^0 \geq c^T$ ,  $y^0 \geq 0$  and  $T = b^T y^0$ .

Taking transpose on both sides, we have

$$\begin{aligned} & c \leq (y^0)^T A \\ \Rightarrow & cx^0 \leq (y^0)^T Ax^0 \\ \Rightarrow & cx^0 \leq (y^0)^T b \\ \Rightarrow & cx^0 \leq b^T y^0 \quad (\because (y^0)^T b = b^T y^0) \end{aligned}$$

Hence proved.

### Theorem 2.

Let  $x^0$  and  $y^0$  be the feasible solutions to the corresponding primal and dual problem such that  $cx^0 = b^T y^0$ , then  $x^0$  and  $y^0$  are optimal solutions to the respective problems.

**Proof.** Let  $x^*$  be any other feasible solution to the primal problem.

Then by Theorem 1,  $cx^* \leq b^T y^0$

$$\Rightarrow cx^* \leq cx^0$$

Hence  $x^0$  is an optimal solution to the primal problem because the primal problem is a maximization problem.

Similarly, we can prove that  $y^0$  is an optimal solution for the dual problem.

### Theorem 3. (Fundamental Theorem of Duality)

If both the primal and dual problems are feasible and both have optimal solutions then the optimal values of the objective functions of both the problems are equal.

### Theorem 4. (Complementary Slackness Conditions (CSC))

Let  $x^0$  and  $y^0$  be the feasible solutions for the primal and dual problems respectively. Let  $u$  be the slack variables of the primal and  $v$  be the surplus variables of the dual. Then  $x^0$  and  $y^0$  are optimal solutions to the respective primal and dual problems respectively iff

$$(x^0)^T v = 0 \text{ and } (y^0)^T u = 0$$

NOTES

## Results on Feasibility

NOTES

		Primal (Max. z)	
		Feasible Solution	Infeasible solution
Dual (Min. T)	Feasible solution	Max. z = Min T	Dual unbounded (Min. T $\rightarrow -\infty$ )
	Infeasible solution	Primal unbounded (Max. z $\rightarrow \infty$ )	May occur.

Let the primal as : Minimize  $z = -x_1 - x_2$

$$\text{S/t. } x_1 - x_2 = 3,$$

$$x_1 - x_2 = -3.$$

$$x_1 \geq 0, x_2 \geq 0.$$

Then the dual can be written as

$$\text{Maximize } T = 3y_1 - 3y_2$$

$$\text{S/t. } y_1 + y_2 \leq -1.$$

$$-y_1 - y_2 \leq -1.$$

$y_1, y_2$  unrestricted in sign.

Here both the primal and the dual are inconsistent and hence no feasible solutions.

**Example 18.** Using the C.S.C. find the optimal solution of the following primal.

$$\text{Minimize } z = 2x_1 + 3x_2 + 5x_3 + 3x_4 + 2x_5$$

$$\text{S/t. } x_1 + x_2 + 2x_3 + 3x_4 + x_5 \geq 4,$$

$$2x_1 - 2x_2 + 3x_3 + x_4 + x_5 \geq 3,$$

$$x_1, x_2, x_3, x_4, x_5 \geq 0.$$

**Solution.** The dual is

$$\text{Maximize } T = 4y_1 + 3y_2$$

$$\text{S/t. } y_1 + 2y_2 \leq 2$$

$$y_1 - 2y_2 \leq 3$$

$$2y_1 + 3y_2 \leq 5$$

$$3y_1 + y_2 \leq 3$$

$$y_1 + y_2 \leq 2$$

$$y_1, y_2 \geq 0.$$

The solution of this dual, by graphically is  $y_1 = \frac{4}{5}, y_2 = \frac{3}{5}, T^* = 5$ . Let  $u_1, u_2, u_3, u_4$  and  $u_5$  be the slack variables of the dual and  $v_1, v_2$  be the surplus variables of the primal. Then by C.S.C., we have

$$x_1 u_1 = 0, x_2 u_2 = 0, x_3 u_3 = 0,$$

$$x_4 u_4 = 0, x_5 u_5 = 0, y_1 v_1 = 0, y_2 v_2 = 0.$$

Since  $y_1^*$  and  $y_2^*$  are non-zero  $\Rightarrow v_1 = v_2 = 0$ .

It is also seen that at optimality, the two constraints  $y_1 + 2y_2 \leq 2$  and  $3y_1 + y_2 \leq 3$  are satisfying in-equality sense which mean  $u_1^* = 0$  and  $u_4^* = 0$ .

For the remaining constraints,  $u_2^*$ ,  $u_3^*$  and  $u_5^*$  are non-zero i.e., by C.S.C.,  $x_2^* = 0$ ,  $x_3^* = 0$  and  $x_5^* = 0$ .

Then the primal constraints reduces to

$$x_1^* + 3x_4^* = 4$$

$$2x_1^* + x_4^* = 3.$$

Solving we get

$$x_1^* = 1 \text{ and } x_4^* = 1.$$

Hence the optimal solution of the primal is

$$x_1^* = 1, x_2^* = 0, x_3^* = 0, x_4^* = 1, x_5^* = 0 \text{ and } z^* = 5.$$

NOTES

## DUALITY OF SIMPLEX METHOD

The fundamental theorem of duality helps to obtain the optimal solution of the dual from optimal table of the primal and vice-versa. Using C.S.C., the correspondence between the primal (dual) variables and slack and/or surplus variables of the dual (primal) to be identified. Then the optimal solution of the dual (primal) can be read off from the net evaluation row of the primal (dual) of the simplex table.

For example, if the primal variable corresponds to a slack variable of the dual, then the net evaluation of the slack variable in the optimal table will give the optimal solution of the primal variable.

**Example 15.** Using the principle of duality solve the following problem :

$$\text{Minimize } z = 4x_1 + 14x_2 + 3x_3$$

$$\text{S/t, } -x_1 + 3x_2 + x_3 \geq 3,$$

$$2x_1 + 2x_2 - x_3 \geq 2,$$

$$x_1, x_2, x_3 \geq 0.$$

**Solution.** The dual problem is

$$\text{Maximize } T = 3y_1 + 2y_2$$

$$\text{S/t, } -y_1 + 2y_2 \leq 4$$

$$3y_1 + 2y_2 \leq 14$$

$$y_1 - y_2 \leq 3$$

$$y_1, y_2 \geq 0$$

Standard form :

$$\text{Maximize } T = 3y_1 + 2y_2 + 0.u_1 + 0.u_2 + 0.u_3$$

$$\text{S/t, } -y_1 + 2y_2 + u_1 = 4$$

$$3y_1 + 2y_2 + u_2 = 14$$

$$y_1 - y_2 + u_3 = 3$$

$y_1, y_2 \geq 0$ ,  $u_1, u_2, u_3$  are slacks and  $\geq 0$ .

Let the surplus variables of the dual  $v_1$  and  $v_2$ .

Then by C.S.C.,

$$y_1 v_1 = 0, y_2 v_2 = 0,$$

$$x_1 u_1 = 0, x_2 u_2 = 0, x_3 u_3 = 0.$$

**NOTES**

Let us solve the dual by simplex method and the optimal table is given below (Iteration 3) :

			$c_j$				
			3	2	0	0	0
$c_B$	$x_B$	Soln.	$y_1$	$y_2$	$u_1$	$u_2$	$u_3$
0	$u_1$	6	0	0	1	$-\frac{1}{5}$	$\frac{3}{5}$
2	$y_2$	1	0	1	0	$\frac{1}{5}$	$-\frac{3}{5}$
3	$y_1$	4	1	0	0	$\frac{1}{5}$	$\frac{2}{5}$
$z_j - c_j$			0	0	0	1	0

The optimal solution of the dual is  $y_1^* = 4, y_2^* = 1, T^* = 14$ .

The optimal solution of the primal can be read off from the  $(z_j - c_j)$ -row. Since  $x_1, x_2, x_3$  corresponds to  $u_1, u_2, u_3$  respectively, then

$$x_1^* = 0, x_2^* = 1, x_3^* = 0, \text{ and } z^* = 14.$$

**THE DUAL SIMPLEX METHOD**

**Step 1.** Convert the minimization LP problem into an symmetric maximization LP problem (i.e., all constraints are  $\leq$  type) if it is in the minimization form.

**Step 2.** Introduce the slack variables and obtain the first iteration dual simplex table.

			$c_j$	
$c_B$	$x_B$	Soln.	$(x)$	
	$[x_B]$			
$z_j - c_j$				
Max. ratio				

**Step 3.(a)** If all  $z_j - c_j$  and  $x_B$  are non-negative, then an optimal basic feasible solution has been attained.

**(b)** If all  $z_j - c_j \geq 0$  and at least one of  $x_B$  is negative then go to step 4.

**(c)** If at least one  $(z_j - c_j)$  is negative, the method is not applicable.

**Step 4.** Select the most negative of  $x_B$ 's and that basic variable will leave the basis and the corresponding row is called 'key-row'.

**Step 5.(a)** If all the elements of the key row is positive, then the problem is infeasible.

**(b)** If at least one element is negative then calculate the maximum ratios as follows :

$$\text{Max} \left\{ \frac{(z_j - c_j) \text{ value}}{\text{Negative element of the key row}} \right\}$$

**NOTES**

The maximum ratio column is called 'key column' and the intersection element of key row and key column is called 'key element'.

**Step 6.** Obtain the next table which is the same procedure as of simplex method.

**Step 7.** Go to step 3.

**Note. 1.** Difference between simplex method and dual-simplex method : In simplex method, we move from a feasible non-optimal solution to feasible optimal solution. Whereas in dual simplex method, we move from an infeasible optimal solution to feasible optimal solution.

**2.** The term 'dual' is used in dual simplex method because the rules for leaving and entering variables are derived from the dual problem but are used in the primal problem.

**Example 20.** Using dual simplex method solve the following LP problem.

$$\text{Minimize } z = 4x_1 + 2x_2$$

$$\text{S/t, } x_1 + 2x_2 \geq 2, 3x_1 + x_2 \geq 3.$$

$$4x_1 + 3x_2 \geq 6, x_1, x_2 \geq 0.$$

**Solution.** Min.  $z = - \text{Max. } (-z) = - \text{Max. } (-z = -4x_1 - 2x_2).$

Multiply  $-1$  to all the  $\geq$  constraints to make  $\leq$  type.

Then the standard form is obtained as follows :

$$\text{Max } -z = -4x_1 - 2x_2 + 0.s_1 + 0.s_2 + 0.s_3$$

$$\text{S/t, } -x_1 - 2x_2 + s_1 = -2$$

$$-3x_1 - x_2 + s_2 = -3$$

$$-4x_1 - 3x_2 + s_3 = -6$$

$$x_1, x_2 \geq 0, s_1, s_2, s_3 \text{ are slacks and } \geq 0.$$

**Iteration 1.**

$c_j$			-4	-2	0	0	0
$c_B$	$x_B$	Soln.	$x_1$	$x_2$	$s_1$	$s_2$	$s_3$
0	$s_1$	-2	-1	-2	1	0	0
0	$s_2$	-3	-3	-1	0	1	0
0	$s_3$	-6	-4	-3	0	0	1
$z_j - c_j$			4	2	0	0	0
Max. ratio			$\frac{4}{-4}$	$\frac{2}{-3}$			

↑  
Key column

**Iteration 2.**

NOTES

			$c_j$	- 4	- 2	0	0	0
$c_B$	$x_B$	Soln.	$x_1$	$x_2$	$s_1$	$s_2$	$s_3$	
0	$s_1$	2	5/3	0	1	0	- 2/3	
0	$s_2$	- 1	- 5/3	0	0	1	- 1/3	→ Key row
- 2	$x_2$	2	4/3	1	0	0	- 1/3	
$z_j - c_j$			4/3	0	0	0	2/3	
Max. ratio			- 4/5	-	-	-	- 2	



Key column

**Iteration 3.**

			$c_j$	- 4	- 2	0	0	0
$c_B$	$x_B$	Soln.	$x_1$	$x_2$	$s_1$	$s_2$	$s_3$	
0	$s_1$	1	0	0	1	1	- 1	
- 4	$x_1$	3/5	1	0	0	- 3/5	1/5	
- 2	$x_2$	6/5	0	1	0	4/5	- 3/5	
$z_j - c_j$			0	0	0	4/5	2/5	

Hence optimal feasible solution is  $x_1^* = \frac{3}{5}$ ,  $x_2^* = \frac{6}{5}$  and  $z^* = \frac{24}{5}$ .

**Example 21.** Use dual simplex method to solve the following LP problem.

$$\text{Maximize } z = - 4x_1 - 3x_2$$

$$\text{Subject to, } x_1 + x_2 \leq 1, x_2 \geq 1, - x_1 + 2x_2 \leq 1, x_1, x_2 \geq 0.$$

**Solution.** The constraint  $x_2 \geq 1$  is rewritten as  $- x_2 \leq - 1$ . Adding slack variables, the standard form is

$$\text{Maximize } z = - 4x_1 - 3x_2 + 0.s_1 + 0.s_2 + 0.s_3$$

$$\text{S/t. } x_1 + x_2 + s_1 = 1, - x_2 + s_2 = - 1, - x_1 + 2x_2 + s_3 = 1$$

$x_1, x_2 \geq 0, s_1, s_2, s_3$  are slacks and  $\geq 0$ .

NOTES

Iteration 1.

			$c_j$	-4	-3	0	0	0
$c_B$	$x_B$	Soln.	$x_1$	$x_2$	$s_1$	$s_2$	$s_3$	
0	$s_1$	1	1	1	1	0	0	
0	$s_2$	-1	0	-1	0	1	0	→ Key row
0	$s_3$	1	-1	2	0	0	1	
$z_j - c_j$			4	3	0	0	0	
Max. ratio			-	-3	-	-	-	

↑  
Key column

Iteration 2.

			$c_j$	-4	-3	0	0	0
$c_B$	$x_B$	Soln.	$x_1$	$x_2$	$s_1$	$s_2$	$s_3$	
0	$s_1$	0	1	0	1	-1	0	
-3	$x_2$	1	0	1	0	-1	0	
0	$s_3$	-1	-1	0	0	2	1	→ Key row
$z_j - c_j$			4	0	0	3	0	
Max. ratio			-4	-	-	-	-	

↑  
Key column

Iteration 3.

			$c_j$	-4	-3	0	0	0
$c_B$	$x_B$	Soln.	$x_1$	$x_2$	$s_1$	$s_2$	$s_3$	
0	$s_1$	-1	0	0	1	3	1	→ Key row
-3	$x_2$	1	0	1	0	-1	0	
-4	$x_1$	1	1	0	0	-2	-1	
$z_j - c_j$			0	0	0	11	4	
Max. ratio			-	-	-	-	-	

Since all the elements in the key row are positive, the given problem is infeasible.

## ECONOMIC INTERPRETATION OF DUAL VARIABLE

## NOTES

Let  $x^*$  and  $y^*$  be the optimal solutions of the respective primal and dual problems respectively and objective function values are same i.e.,  $z^* = T^*$ . In the primal, the small change in the resources (i.e., right hand side constants) gives the small change in  $z^*$ . Consequently, the  $y^*$  value for each primal constraint gives the net change in the optimal value of the objective function for unit increase in right hand side constants. Hence the dual variables are called 'shadow prices'.

### SUMMARY

- "Operational Research is the application of the methods of science to complex problems arising in the direction and management of large systems of men, machines, materials and money in industry, business, government and defence.
- In decision-making all the decisions are taken through some variables which are known as decision variables. In engineering design, these variables are known as design vectors.
- A solution which satisfies all the constraints in LPP is called feasible solution.
- A solution which is basic as well as feasible is called basic feasible solution.
- If a basic variable takes the value zero in a BFS, then the solution is said to be degenerate.
- The BFS which optimizes the objective function is called optimal BFS.

### PROBLEMS

1. Obtain the dual of the following LP problems :

(a) Maximize  $z = 4x_1 + 2x_2 + x_3 + 6x_4$

Subject to,  $6x_1 - 3x_2 + x_3 + 5x_4 \leq 15,$

$x_1 - x_2 + 6x_3 + 2x_4 \geq 8,$

$x_1, x_2, x_3, x_4 \geq 0$

(b) Maximize  $z = 2x_1 + x_2$

Subject to,  $2x_1 + 3x_2 \geq 4,$

$3x_1 + 4x_2 \leq 10,$

$x_1 + 5x_2 = 9,$

$x_1 \geq 0, x_2 \geq 0.$

(c) Minimize  $z = 3x_1 + 4x_2 - x_3$

Subject to,  $2x_1 + 3x_2 + 5x_3 \geq 10,$

$3x_1 + 10x_3 \leq 14,$

$x_1 \geq 0, x_2 \leq 0, x_3 \geq 0.$

(d) Minimize  $z = 10x_1 + 15x_2$

Subject to,  $3x_1 + 2x_2 = 15,$

$5x_1 + 4x_2 = 20,$

$x_1, x_2$  unrestricted in sign.

- (e) Maximize  $z = x_1 - 2x_2 + 3x_3$   
 Subject to,  $2x_1 + 5x_2 \leq 16$ ,  
 $5x_2 + 4x_3 \geq 8$ ,  
 $x_1 + x_2 + x_3 = 10$ ,  
 $x_1 \geq 0, x_2 \leq 0, x_3$  unrestricted in sign.
2. Use principle of duality to solve the following LP problems :
- (a) Minimize  $z = 4x_1 + 3x_2$   
 S/t,  $2x_1 + x_2 \geq 40, x_1 + 2x_2 \geq 50, x_1 + x_2 \geq 35$   
 $x_1, x_2 \geq 0$
- (b) Maximize  $z = 2x_1 + x_2$   
 S/t,  $x_1 - 2x_2 \leq 10, x_1 + x_2 \leq 6, x_1 - x_2 \leq 2, x_1 - 2x_2 \leq 1$   
 $x_1, x_2 \geq 0$
- (c) Minimize  $z = 6x_1 + x_2$   
 S/t,  $2x_1 + x_2 \geq 3, x_1 - x_2 \geq 0, x_1, x_2 \geq 0$
- (d) Minimize  $z = 30x_1 + 30x_2 + 10x_3$   
 S/t,  $2x_1 + x_2 + x_3 \geq 6, x_1 + x_2 + 2x_3 \leq 8, x_1, x_2, x_3 \geq 0$
- (e) Maximize  $z = 5x_1 + 2x_2$   
 S/t,  $x_1 - x_2 \leq 1, x_1 + x_2 \geq 4, x_1 - 3x_2 \leq 3, x_1, x_2 \geq 0$
3. Using the complementary slackness condition solve the following LP problem :
- Maximize  $z = 2x_1 + 3x_2 + 6x_3$   
 S/t,  $x_1 + 3x_2 + 4x_3 \leq 4, 2x_1 + x_2 + 3x_3 \leq 2, x_1, x_2, x_3 \geq 0$ .
4. With the help of the following example, verify that the dual of the dual is the primal.
- Maximize  $z = 3x_1 + 2x_2 + 5x_3$   
 S/t,  $4x_1 + 3x_2 - x_3 \leq 20, 3x_1 + 2x_2 + 5x_3 = 18$ ,  
 $0 \leq x_1 \leq 4, x_2 \geq 0, x_3 \leq 0$ .
5. Verify the fundamental theorem of duality using the following LP problems :
- (a) Maximize  $z = 2x_1 + 10x_2$   
 S/t,  $2x_1 + 5x_2 \leq 16, 6x_1 \leq 30, x_1, x_2 \geq 0$ .
- (b) Minimize  $z = 2x_1 - x_2$   
 S/t,  $x_1 + x_2 \leq 5, x_1 + 2x_2 \geq 8, x_1, x_2 \geq 0$ .
6. Use dual-simplex method to solve the following LP problems :
- (a) Minimize  $z = x_1 + 3x_2$   
 S/t,  $2x_1 + x_2 \geq 4, 3x_1 + 2x_2 \geq 5, x_1, x_2 \geq 0$ .
- (b) Minimize  $z = 2x_1 + x_2$   
 S/t,  $x_1 + x_2 \geq 2, 3x_1 + 2x_2 \geq 4, x_1, x_2 \geq 0$ .
- (c) Minimize  $z = 2x_1 + 3x_2 + 10x_3$   
 S/t,  $2x_1 - 5x_2 + 4x_3 \geq 30$ ,  
 $3x_1 + 2x_2 - 5x_3 \geq 25$ ,  
 $x_1 + 3x_2 + x_3 \leq 30$ ,  
 $x_1, x_2, x_3 \geq 0$ .
- (d) Maximize  $z = -2x_1 - x_2 - 3x_3$   
 S/t,  $-3x_1 + x_2 - 2x_3 - x_4 = 1, x_1 - 2x_2 + x_3 - x_5 = 2, x_i \geq 0 \forall i$

## NOTES

- (e) Minimize  $z = 2x_1 + 3x_2 + 4x_3$   
 S/t,  $3x_1 + 10x_2 + 5x_3 \geq 3$ ,  $3x_1 - 10x_2 + 9x_3 \leq 30$ ,  
 $x_1 + 2x_2 + x_3 \geq 4$ ,  $x_1, x_2, x_3 \geq 0$ .
- (f) Minimize  $z = 6x_1 + 2x_2 + 5x_3 + 3x_4$   
 S/t,  $3x_1 + 2x_2 - 3x_3 + 5x_4 \geq 10$ ,  $4x_2 + 3x_3 - 5x_4 \geq 12$ ,  
 $5x_1 - 4x_2 + x_3 + x_4 \geq 10$ ,  $x_1, x_2, x_3, x_4 \geq 0$ .
- (g) Maximize  $z = -x_1 - 2x_2 - 3x_3$   
 S/t,  $2x_1 - x_2 - x_3 \geq 4$ ,  $x_1 - x_2 + 2x_3 \leq 8$ ,  $x_1, x_2, x_3 \geq 0$ .

7. One unit of product A requires 3 units of raw material and 2 hours of labour and contributes the profit of Rs. 7. One unit of product B requires one unit of raw material and one hour of labour and contributes the profit of Rs. 5. There are 40 units of raw material and 40 hours of labour available. The objective is to maximize the profit. Calculate the shadow prices of the raw material and labour.

**ANSWERS**

1. (a) Minimize  $T = 15y_1 - 8y_2$   
 Subject to,  $6y_1 - y_2 \geq 4$   
 $-3y_1 + y_2 \geq 2$   
 $y_1 - 6y_2 \geq 1$   
 $5y_1 - 2y_2 \geq 6$   
 $y_1, y_2 \geq 0$ , and then set  $w_1 = y_1$  and  $w_2 = -y_2$
- (b) Minimize  $T = 4w_1 + 10w_2 + 9w_3$   
 S/t,  $2w_1 + 3w_2 + w_3 \geq 2$   
 $3w_1 + 4w_2 + 5w_3 \geq 1$   
 $w_1 \leq 0, w_2 \geq 0, w_3$  unrestricted.
- (c) Maximize  $T = -10w_1 - 14w_2$   
 S/t,  $-2w_1 - 3w_2 \leq 3$   
 $-w_1 \leq 4$   
 $5w_1 + 10w_2 \geq 1$   
 $w_1 \leq 0, w_2 \geq 0$ .
- (d) Maximize  $T = 15w_1 + 20w_2$   
 S/t,  $3w_1 + 5w_2 = 10$ ,  $2w_1 + 4w_2 = 15$ ,  
 $w_1, w_2$  unrestricted.
- (e) Minimize  $T = 16w_1 + 8w_2 + 10w_3$   
 S/t,  $2w_1 + w_3 \geq 1$ ,  $5w_2 + w_3 \leq -2$ ,  $5w_1 + 4w_2 + w_3 = 3$   
 $w_1 \geq 0, w_2 \leq 0, w_3$  unrestricted.
2. (a)  $x_1 = 5, x_2 = 30, z^* = 110$ .  
 (b)  $x_1 = 4, x_2 = 2, z^* = 10$ .  
 (c)  $x_1 = 1, x_2 = 1, z^* = 7$ .  
 (d)  $x_1 = \frac{4}{3}, x_2 = 0, x_3 = \frac{10}{3}, z^* = \frac{220}{3}$ .  
 (e) Unbounded solution.

3.  $x_1 = 0, x_2 = \frac{4}{5}, x_3 = \frac{2}{5}, z^* = 4.8.$
5. (a) Max.  $z = 32 =$  Min. T.  
 (b) Min.  $z = -5 =$  Max. T.
6. (a)  $x_1 = 2, x_2 = 0, z^* = 2$  (It - 3).  
 (b)  $x_1 = 0, x_2 = 2, z^* = 2$  (It - 2)  
 (c)  $x_1 = 15, x_2 = 0, x_3 = 0, z^* = 30$  (It - 2)  
 (d) Infeasible solution (It - 3)  
 (e)  $x_1 = 0, x_2 = 2, x_3 = 0, z^* = 6$  (It - 2)  
 (f)  $x_1 = 0, x_2 = \frac{11}{3}, x_3 = 0, x_4 = \frac{8}{15}, z^* = \frac{134}{15}$  (It - 3)  
 (g)  $x_1 = 2, x_2 = 0, x_3 = 0, z^* = -2$  (It - 2)
7. Shadow prices for raw material is zero and for labour is five.

NOTES

## CHAPTER 2 TRANSPORTATION PROBLEMS

## NOTES

## ★ STRUCTURE ★

- Introduction and Mathematical Formulation
- Finding Initial Basic Feasible Solution
- UV-Method/Modi Method
- Degeneracy in T.P.
- Max-type T.P.
- Unbalanced T.P.
- Summary
- Problems

### INTRODUCTION AND MATHEMATICAL FORMULATION

Transportation problem (T.P.) is generally concerned with the distribution of a certain commodity/product from several origins/sources to several destinations with minimum total cost through single mode of transportation. If different modes of transportation considered then the problem is called 'solid T.P'. In this chapter we shall deal with simple T.P.

Suppose there are  $m$  factories where a certain product is produced and  $n$  markets where it is needed. Let the supply from the factories be  $a_1, a_2, \dots, a_m$  units and demands at the markets be  $b_1, b_2, \dots, b_n$  units.

Also consider

$c_{ij}$  = Unit of cost of shipping from factory  $i$  to market  $j$ .

$x_{ij}$  = Quantity shipped from factory  $i$  to market  $j$ .

Then the LP formulation can be started as follows :

Minimize  $z$  = Total cost of transportation

$$= \sum_{i=1}^m \sum_{j=1}^n c_{ij} x_{ij}$$

Subject to,  $\sum_{j=1}^n x_{ij} \leq a_i, i = 1, 2, \dots, m.$

(Total amount shipped from any factory does not exceed its capacity)

$$\sum_{i=1}^m x_{ij} \geq b_j, j = 1, 2, \dots, n.$$

(Total amount shipped to a market meets the demand of the market)

$$x_{ij} \geq 0 \text{ for all } i \text{ and } j.$$

Here the market demand can be met if

$$\sum_{i=1}^m a_i \geq \sum_{j=1}^n b_j.$$

If  $\sum_{i=1}^m a_i = \sum_{j=1}^n b_j$ , i.e., total supply = total demand, the problem is said to be "Balanced T.P." and all the constraints are replaced by equality sign.

$$\text{Minimize } z = \sum \sum c_{ij} x_{ij}$$

$$\text{Subject to, } \sum_{j=1}^n x_{ij} = a_i, \quad i = 1, 2, \dots, m.$$

$$\sum_{i=1}^m x_{ij} = b_j, \quad j = 1, 2, \dots, n.$$

$$x_{ij} \geq 0 \text{ for all } i \text{ and } j.$$

(Total  $m + n$  constraints and  $mn$  variables)

The T.P. can be represented by table form as given below :

	$M_1$	$M_2$	.....	$M_n$	
$F_1$	$x_{11}$	$x_{12}$		$x_{1n}$	$a_1$
	$c_{11}$	$c_{12}$		$c_{1n}$	
$F_2$	$x_{21}$	$x_{22}$		$x_{2n}$	$a_2$
	$c_{21}$	$c_{22}$		$c_{2n}$	
.....	.....	.....	.....	.....	.....
$F_m$	$x_{m1}$	$x_{m2}$		$x_{mn}$	$a_m$
	$c_{m1}$	$c_{m2}$		$c_{mn}$	
	$b_1$	$b_2$	.....	$b_n$	

Demand

Supply

In the above, each cell consists of decision variable  $x_{ij}$  and per unit transportation cost  $c_{ij}$ .

**Theorem 1.** A necessary and sufficient condition for the existence of a feasible solution to a T.P. is that the T.P. is balanced.

**Proof. (Necessary part)**

Total supply from an origin  $\sum_{j=1}^n x_{ij} = a_i, \quad i = 1, 2, \dots, m.$

Overall supply,  $\sum_{i=1}^m \sum_{j=1}^n x_{ij} = \sum_{i=1}^m a_i$

Total demand met of a destination

$$\sum_{i=1}^m x_{ij} = b_j, \quad j = 1, 2, \dots, n.$$

NOTES

NOTES

Overall demand, 
$$\sum_{j=1}^n \sum_{i=1}^m x_{ij} = \sum_{j=1}^n b_j$$

Since overall supply exactly met the overall demand.

$$\sum_i \sum_j x_{ij} = \sum_j \sum_i x_{ij}$$

$\Rightarrow$  
$$\sum_{i=1}^m a_i = \sum_{j=1}^n b_j.$$

(Sufficient part) Let  $\sum_i a_i = \sum_j b_j = l$  and  $x_{ij} = a_i b_j / l$  for all  $i$  and  $j$ .

Then 
$$\sum_{j=1}^n x_{ij} = \sum_{j=1}^n (a_i b_j) / l = a_i \left( \sum_{j=1}^n b_j \right) / l = a_i = 1, 2, \dots, m.$$

$$\sum_{i=1}^m x_{ij} = \sum_{i=1}^m (a_i b_j) / l = b_j \left( \sum_{i=1}^m a_i \right) / l = b_j, j = 1, 2, \dots, n.$$

$x_{ij} \geq 0$  since  $a_i$  and  $b_j$  are non-negative.

Therefore  $x_{ij}$  satisfies all the constraints and hence  $x_{ij}$  is a feasible solution.

**Theorem 2.** The number of basic variables in the basic feasible solution of an  $m \times n$  T.P. is  $m + n - 1$ .

**Proof.** This is due to the fact that the one of the constraints is redundant in balanced T.P.

We have overall supply, 
$$\sum_{i=1}^m \sum_{j=1}^n x_{ij} = \sum_{i=1}^m a_i$$

and overall demand 
$$\sum_{j=1}^n \sum_{i=1}^m x_{ij} = \sum_{j=1}^n b_j$$

Since  $\sum_i a_i = \sum_j b_j$ , the above two equations are identical and we have only  $m + n - 1$  independent constraints. Hence the theorem is proved.

**Note. 1.** If any basic variable takes the value zero then the basic feasible solution (BFS) is said to be degenerate. Like LPP, all non-basic variables take the value zero.

**2.** If a basic variable takes either positive value or zero, then the corresponding cell is called 'Basic cell' or 'Occupied cell'. For non-basic variable the corresponding cell is called 'Non-basic cell' or 'Non-occupied cell' or 'Non-allocated cell'.

**Loop.** This means a closed circuit in a transportation table connecting the occupied (or allocated) cells satisfying the following :

- (i) It consists of vertical and horizontal lines connecting the occupied (or allocated) cells.
- (ii) Each line connects only two occupied (or allocated) cells.
- (iii) Number of connected cells is even.

- (iv) Lines can skip the middle cell of three adjacent cells to satisfy the condition (ii).

The following are the examples of loops.

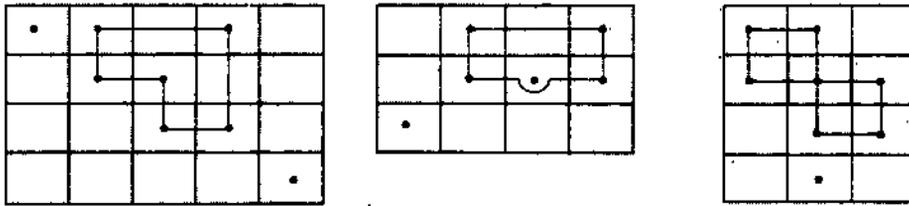


Fig. 2.1

Note. A solution of a T.P. is said to be basic if it does not consist of any loop.

NOTES

**FINDING INITIAL BASIC FEASIBLE SOLUTION**

In this section three methods are to be discussed to find initial BFS of a T.P. In advance, it can be noted that the above three methods may give different initial BFS to the same T.P. Also allocation = minimum (supply, demand).

**(a) North-West Corner Rule (NWC)**

- (i) Select the north west corner cell of the transportation table.
- (ii) Allocate the min (supply, demand) in that cell as the value of the variable.

If supply happens to be minimum, cross-off the row for further consideration and adjust the demand.

If demand happens to be minimum, cross-off the column for further consideration and adjust the supply.

- (iii) The table is reduced and go to step (i) and continue the allocation until all the supplies are exhausted and the demands are met.

Example 1. Find the initial BFS of the following T.P. using NWC rule.

		To				
		M <sub>1</sub>	M <sub>2</sub>	M <sub>3</sub>	M <sub>4</sub>	
From	F <sub>1</sub>	3	2	4	1	20
	F <sub>2</sub>	2	4	5	3	15
	F <sub>3</sub>	3	5	2	6	25
	F <sub>4</sub>	4	3	1	4	40
		30	20	25	25	Supply
		Demand				

Solution. Here, total supply = 100 = total demand. So the problem is balanced T.P. The north-west corner cell is (1, 1) cell. So allocate min. (20, 30) = 20 in that cell. Supply exhausted. So cross-off the first row and demand is reduced to 10. The reduced table is

NOTES

	M <sub>1</sub>	M <sub>2</sub>	M <sub>3</sub>	M <sub>4</sub>	
F <sub>2</sub>	2	4	5	3	15
F <sub>3</sub>	3	5	2	6	25
F <sub>4</sub>	4	3	1	4	40
	10	20	25	25	

Here the north-west corner cell is (2, 1) cell. So allocate min. (15, 10) = 10 in that cell. Demand met. So cross-off the first column and supply is reduced to 5. The reduced table is

	M <sub>2</sub>	M <sub>3</sub>	M <sub>4</sub>	
F <sub>2</sub>	4	5	3	5
F <sub>3</sub>	5	2	6	25
F <sub>4</sub>	3	1	4	40
	20	25	25	

Here the north-west corner cell is (2, 2) cell. So allocate min. (5, 20) = 5 in that cell. Supply exhausted. So cross-off the second row (due to F<sub>2</sub>) and demand is reduced to 15. The reduced table is

	M <sub>2</sub>	M <sub>3</sub>	M <sub>4</sub>	
F <sub>3</sub>	5	2	6	25
F <sub>4</sub>	3	1	4	40
	15	25	25	

Here the north-west corner cell is (3, 2) cell. So allocate min. (25, 15) = 15 in that cell. Demand met. So cross-off the second column (due to M<sub>2</sub>) and supply is reduced to 10. The reduced table is

	M <sub>3</sub>	M <sub>4</sub>	
F <sub>3</sub>	2	6	10
F <sub>4</sub>	1	4	40
	25	25	

Here the north-west corner cell is (3, 3) cell. So allocate min. (10, 25) = 10 in that cell. Supply exhausted. So cross-off the third row (due to F<sub>3</sub>) and demand is reduced to 15. The reduced table is

	M <sub>3</sub>	M <sub>4</sub>	
F <sub>4</sub>	1	4	40
	25	25	

continuing we obtain the allocation 15 to (4, 3) cell and 25 to (4, 4) cell so that supply exhausted and demand met. The complete allocation is shown below:

	M <sub>1</sub>	M <sub>2</sub>	M <sub>3</sub>	M <sub>4</sub>	
F <sub>1</sub>	20				20
F <sub>2</sub>	10	5			15
F <sub>3</sub>		15	10		25
F <sub>4</sub>			15	25	40
	30	20	25	25	

NOTES

Thus, the initial BFS is

$$x_{11} = 20, x_{21} = 10, x_{22} = 5, x_{32} = 15, x_{33} = 10, x_{43} = 15, x_{44} = 25.$$

The transportation cost

$$= 20 \times 3 + 10 \times 2 + 5 \times 4 + 15 \times 5 + 10 \times 2 + 5 \times 1 + 25 \times 4$$

$$= \text{Rs. } 310.$$

**(b) Least Cost Entry Method (LCM) (or Matrix Minimum Method)**

(i) Find the least cost from transportation table. If the least value is unique, then go for allocation.

If the least value is not unique then select the cell for allocation for which the contributed cost is minimum.

(ii) If the supply is exhausted cross-off the row and adjust the demand.

If the demand is met cross-off the column and adjust the supply.

Thus the matrix is reduced.

(iii) Go to step (i) and continue until all the supplies are exhausted and all the demands are met.

**Example 2.** Find the initial BFS of Example 1 using least cost entry method

	M <sub>1</sub>	M <sub>2</sub>	M <sub>3</sub>	M <sub>4</sub>	
F <sub>1</sub>	3	2	4	1	20
F <sub>2</sub>	2	4	5	3	15
F <sub>3</sub>	3	5	2	6	25
F <sub>4</sub>	4	3	1	4	40
	30	20	25	25	

**Solution.** Here the least value is 1 and occurs in two cells (1, 4) and (4, 3). But the contributed cost due to cell (1, 4) is  $1 \times \min(20, 25)$  i.e., 20 and due to cell (4, 3) is  $1 \times \min(40, 25)$  i.e., 25. So we selected the cell (1, 4) and allocate 20. Cross-off the first row since supply exhausted and adjust the demand to 5. The reduced table is given below :

NOTES

2	4	5	3	15
3	5	2	6	25
4	3	1	4	40
30	20	25	5	

The least value is 1 and unique. So allocate  $\min. (40, 25) = 25$  in that cell. Cross-off the third column (due to  $M_3$ ) since the demand is met and adjust the supply to 15. The reduced table is given below :

2	4	3	15
3	5	6	25
4	3	4	15
30	20	5	

The least value is 2 and unique. So allocate  $\min. (15, 30) = 15$  in that cell. Cross-off the second row (due to  $F_2$ ) since the supply exhausted and adjust the demand to 15. The reduced table is given below :

3	5	6	25
4	3	4	15
15	20	5	

The least value is 3 and occurs in two cells (3, 1) and (4, 2). The contributed cost due to cell (3, 1) is  $3 \times \min. (25, 15) = 45$  and due to cell (4, 2) is  $3 \times \min. (15, 20) = 45$ . Let us select the (3, 1) cell for allocation and allocate 15. Cross-off the first column (due to  $M_1$ ) since demand is met and adjust the supply to 10. The reduced table is given below :

5	6	10
3	4	15
20	5	

Continuing the above method and we obtain the allocations in the cell (4, 2) as 15, in the cell (3, 2) as 5 and in the cell (3, 4) as 5. The complete allocation is shown below :

	$M_1$	$M_2$	$M_3$	$M_4$	
$F_1$				20	1
$F_2$	15				
$F_3$	15	5		5	
$F_4$		15	25		
	4	3	1		4

The initial BFS is

$$x_{14} = 20, x_{21} = 15, x_{31} = 15, x_{32} = 5, x_{34} = 5, x_{42} = 15, x_{43} = 25.$$

The transportation cost

$$= 20 \times 1 + 15 \times 2 + 15 \times 3 + 5 \times 5 + 5 \times 6 + 15 \times 3 + 25 \times 1$$

$$= \text{Rs. } 220.$$

Note. If the least cost is only selected columnwise then it is called 'column minima' method. If the least cost is only selected row wise then it is called 'row minima' method.

NOTES

**(c) Vogel's Approximation Method (VAM)**

- (i) Calculate the row penalties and column penalties by taking the difference between the lowest and the next lowest costs of every row and of every column respectively.
- (ii) Select the largest penalty by encircling it. For tie cases, it can be broken arbitrarily or by analyzing the contributed costs.
- (iii) Allocate in the least cost cell of the row/column due to largest penalty.
- (iv) If the demand is met, cross off the corresponding column and adjust the supply.

If the supply is exhausted, cross-off the corresponding row and adjust the demand.

Thus the transportation table is reduced.

- (v) Go to Step (i) and continue until all the supplies exhausted and all the demands are met.

**Example 3.** Find the initial BFS of example 1 using Vogel's approximation method.

**Solution.**

	M <sub>1</sub>	M <sub>2</sub>	M <sub>3</sub>	M <sub>4</sub>	Row penalties
F <sub>1</sub>	3	2	4	20 1	20 (1)
F <sub>2</sub>	2	4	5	3	15 (1)
F <sub>3</sub>	3	5	2	6	25 (1)
F <sub>4</sub>	4	3	1	4	40 (2)
Column penalties	30 (1)	20 (1)	25 (1)	25 (2)	

Since there is a tie in penalties, let us break the tie by considering the contributed costs. Due to M<sub>4</sub>, the contributed cost is 1 × min. (20, 25) = 20. While due to F<sub>4</sub>, the contributed cost is 1 × min. (40, 25) = 25. So select the column due to M<sub>4</sub> for allocation and we allocate min. (20, 25) i.e., 20 in (1, 4) cell. Then cross-off the first row as supply is exhausted and adjust the corresponding demand as 5. The reduced table is

NOTES

	M <sub>1</sub>	M <sub>2</sub>	M <sub>3</sub>	M <sub>4</sub>	Row penalties
F <sub>2</sub>	2	4	5	3	15 (1)
F <sub>3</sub>	3	5	2	6	25 (1)
F <sub>4</sub>	4	3	1	4	40 (2)
Column penalties	30 (1)	20 (1)	25 (1)	5 (1)	

Here the largest penalty is 2 which is due to F<sub>4</sub>. Allocate in (4, 3) cell as min. (40, 25) = 25. Cross-off the third column due to M<sub>3</sub>, since demand is met and adjust the corresponding supply to 15. The reduced table is

	M <sub>1</sub>	M <sub>2</sub>	M <sub>4</sub>	Row penalties
F <sub>2</sub>	2	4	3	15 (1)
F <sub>3</sub>	3	5	6	25 (2)
F <sub>4</sub>	4	3	4	15 (1)
Column penalties	30 (1)	20 (1)	5 (1)	

Here the largest penalty is 2 which is due to F<sub>3</sub>. Allocate in (3, 1) cell as min. (25, 30) = 25. Cross-off the third row due to F<sub>3</sub> since supply is exhausted and adjust the corresponding demand to 5. The reduced table is

	M <sub>1</sub>	M <sub>2</sub>	M <sub>4</sub>	Row penalties
F <sub>2</sub>	2	4	3	15 (1)
F <sub>4</sub>	4	3	4	15 (1)
Column penalties	5 (2)	20 (1)	5 (1)	

Here the largest penalty is 2 which is due to M<sub>1</sub>. Allocate in (2, 1) cell as min. (15, 5) = 5. Cross-off the first column due to M<sub>1</sub> since demand is met and adjust the supply to 10. The reduced table is

	M <sub>2</sub>	M <sub>4</sub>	Row penalties
F <sub>2</sub>	4	3	10 (1)
F <sub>4</sub>	3	4	15 (1)
Column penalties	20 (1)	5 (1)	

Here tie has occurred. The contributed cost is minimum due to (2, 4) cell which is 3 × min. (10, 5) = 15. So allocate min. (10, 5) = 5 in (2, 4) cell. Cross-off the fourth column which is due to M<sub>4</sub> since demand is met and adjust the corresponding supply to 5. On continuation we obtain the allocation of 5 in (2, 2) cell and 15 in (4, 2) cell. The complete allocation is shown below :

	$M_1$	$M_2$	$M_3$	$M_4$	
$F_1$				20	
	3	2	4		1
$F_2$	5	5		5	
	2	4	5		3
$F_3$	25				
	3	5	2		6
$F_4$		15	25		
	4	3	1		4

The initial BFS is

$$x_{14} = 20, x_{21} = 5, x_{22} = 5, x_{24} = 5, x_{31} = 25, x_{42} = 15, x_{43} = 25.$$

The transportation cost

$$\begin{aligned} &= 1 \times 20 + 2 \times 5 + 4 \times 5 + 3 \times 5 + 3 \times 25 + 3 \times 15 + 1 \times 25 \\ &= \text{Rs. } 210. \end{aligned}$$

NOTES

### UV-METHOD/MODI METHOD

Taking the initial BFS by any method discussed above, this method find the optimal solution to the transportation problem. The steps are given below :

(i) For each row consider a variable  $u_i$  and for each column consider another variable  $v_j$ .

Find  $u_i$  and  $v_j$  such that

$$u_i + v_j = c_{ij} \text{ for every basic cells.}$$

(ii) For every non-basic cells, calculate the net evaluations as follows :

$$\bar{c}_{ij} = u_i + v_j - c_{ij}$$

If all  $\bar{c}_{ij}$  are non-positive, the current solution is optimal.

If at least one  $\bar{c}_{ij} > 0$ , select the variable having the largest positive net evaluation to enter the basis.

(iii) Let the variable  $x_{rc}$  enter the basis. Allocate an unknown quantity  $\theta$  to the cell  $(r, c)$ .

Identify a loop that starts and ends in the cell  $(r, c)$ .

Subtract and add  $\theta$  to the corner points of the loop clockwise/anticlockwise.

(iv) Assign a minimum value of  $\theta$  in such a way that one basic variable becomes zero and other basic variables remain non-negative. The basic cell which reduces to zero leaves the basis and the cell with  $\theta$  enters into the basis.

If more than one basic variables become zero due to the minimum value of  $\theta$ , then only one basic cell leaves the basis and the solution is called degenerate.

(v) Go to step (i) until an optimal BFS has been obtained.

NOTES

Note. In step (ii), if all  $\bar{c}_{ij} < 0$ , then the optimal solution is unique. If at least one  $\bar{c}_{ij} < 0$ , then we can obtain alternative solution. Assign  $\theta$  in that cell and repeat one iteration (from step (iii)).

**Example 4.** Consider the initial BFS by LCM of Example 2, find the optimal solution of the T.P.

**Solution. Iteration 1.**

	M <sub>1</sub>	M <sub>2</sub>	M <sub>3</sub>	M <sub>4</sub>	
F <sub>1</sub>				20	u <sub>1</sub> = -5
	3	2	4	1	
F <sub>2</sub>	15				u <sub>2</sub> = -1
	2	4	5	3	
F <sub>3</sub>	15	5		5	u <sub>3</sub> = 0 (Let)
	3	5	2	6	
F <sub>4</sub>		15	25		u <sub>4</sub> = -2
	4	3	1	4	
	V <sub>1</sub> =3	V <sub>2</sub> =5	V <sub>3</sub> =3	V <sub>4</sub> =6	

For non-basic cells :  $\bar{c}_{ij} = u_i + v_j - c_{ij}$

$$\bar{c}_{11} = -5, \bar{c}_{12} = -2, \bar{c}_{13} = -6, \bar{c}_{22} = 0, \bar{c}_{23} = -3, \bar{c}_{24} = 2, \bar{c}_{33} = 1, \bar{c}_{41} = -3, \bar{c}_{44} = 0.$$

Since all  $\bar{c}_{ij}$  are not non-positive, the current solution is not optimal.

Select the cell (2, 4) due to largest positive value and assign an unknown quantity  $\theta$  in that cell. Identify a loop and subtract and add  $\theta$  to the corner points of the loop which is shown below :

			20	
	3	2	4	1
15 - $\theta$				
	2	4	5	3
15 + $\theta$	5		5 - $\theta$	
	3	5	2	6
	4	15	25	
		3	1	4

Select  $\theta = \min(5, 15) = 5$ . The cell (3, 4) leaves the basis and the cell (2, 4) enters into the basis. Thus the current solution is updated.

**Iteration 2.**

	M <sub>1</sub>	M <sub>2</sub>	M <sub>3</sub>	M <sub>4</sub>	
				20	u <sub>1</sub> = -2
	3	2	4	1	
10				5	u <sub>2</sub> = 0 (Let)
	2	4	5	3	
20	5				u <sub>3</sub> = 1
	3	5	2	6	
		15	25		u <sub>4</sub> = -1
	4	3	1	4	
	V <sub>1</sub> =2	V <sub>2</sub> =4	V <sub>3</sub> =2	V <sub>4</sub> =3	

For non-basic cells :  $\bar{c}_{ij} = u_i + v_j - c_{ij}$

$$\bar{c}_{11} = -3, \bar{c}_{12} = 0, \bar{c}_{13} = -4, \bar{c}_{22} = 0, \bar{c}_{23} = -3, \bar{c}_{34} = -2, \bar{c}_{41} = -3, \bar{c}_{44} = -2.$$

Since all  $\bar{c}_{ij}$  are not non-positive, the current solution is not optimal.

Select the cell (3, 3) due to largest positive value and assign an unknown quantity  $\theta$  in that cell. Identify a loop and subtract and add  $\theta$  to the corner points of the loop which is shown below :

			20	
	3	2	4	1
10				5
	2	4	5	3
20		5 - $\theta$	$\theta$	
	3		5	2
		15 + $\theta$	25 - $\theta$	
	4	3	1	4

Select  $\theta = \min. (5, 25) = 5$ . The cell (3, 2) leaves the basis and the cell (3, 3) enters into the basis. Thus the current solution is updated.

**Iteration 3.**

			20	
	3	2	4	
10				5
	2	4	5	3
20		5		
	3	5	2	6
		20	20	
	4	3	1	4

$u_1 = -2$   
 $u_2 = 0$  (Let)  
 $u_3 = 1$   
 $u_4 = 0$   
 $V_1 = 2 \quad V_2 = 3 \quad V_3 = 1 \quad V_4 = 3$

For non-basic cells :  $\bar{c}_{ij} = u_i + v_j - c_{ij}$

$\bar{c}_{11} = -3, \bar{c}_{12} = -1, \bar{c}_{13} = -5, \bar{c}_{22} = -1, \bar{c}_{23} = -5, \bar{c}_{32} = -1, \bar{c}_{34} = -2, \bar{c}_{41} = -2, \bar{c}_{44} = -1.$

Since all  $\bar{c}_{ij}$  are non-positive, the current solution is optimal. Thus, the optimal solution is

$x_{14} = 20, x_{21} = 10, x_{24} = 5, x_{31} = 20, x_{33} = 5, x_{42} = 20, x_{43} = 20.$

The optimal transportation cost

$= 1 \times 20 + 2 \times 10 + 3 \times 5 + 3 \times 20 + 2 \times 5 + 3 \times 20 + 1 \times 20 = \text{Rs. } 205.$

**Example 5.** Consider the initial BFS by VAM of Example 3, find the optimal solution of the T.P.

**Solution. Iteration 1.**

			20	
	3	2	4	1
5		5		5
	2	4	5	3
25				
	3	5	2	6
		15	25	
	4	3	1	4

$u_1 = -2$   
 $u_2 = 0$  (Let)  
 $u_3 = 1$   
 $u_4 = -1$   
 $V_1 = 2 \quad V_2 = 4 \quad V_3 = 2 \quad V_4 = 3$

NOTES

For non-basic cells :  $\bar{c}_{ij} = u_i + v_j - c_{ij}$

$$\bar{c}_{11} = -3, \bar{c}_{12} = 0, \bar{c}_{13} = -4, \bar{c}_{23} = -3, \bar{c}_{32} = 0, \bar{c}_{33} = 1, \bar{c}_{34} = -2, \bar{c}_{41} = -3, \bar{c}_{44} = -2.$$

Since all  $\bar{c}_{ij}$  are not non-positive, the current solution is not optimal.

**NOTES**

Select the cell (3, 3) due to largest positive value and assign an unknown quantity  $\theta$  in that cell. Identify a loop and subtract and add  $\theta$  to the corner points of the loop which is shown below :

	3	2	4	20	1
5+ $\theta$		5- $\theta$		5	
	2		4	5	3
25- $\theta$			$\theta$		
	3		5		2
		15+ $\theta$		25- $\theta$	
4		3	1		4

Select  $\theta = \min. (5, 25, 25) = 5$ . The cell (2, 2) leaves the basis and the cell (3, 3) enters into the basis. Thus the current solution is updated.

**Iteration 2.**

	3	2	4	20	1	$u_1 = -2$
10				5		$u_2 = 0$ (Let)
	2	4		5		$u_3 = 1$
20			5			$u_4 = 0$
	3		5		2	
		20		20		
4		3		1		

$V_1 = 2 \quad V_2 = 3 \quad V_3 = 1 \quad V_4 = 3$

For non-basic cells :  $\bar{c}_{ij} = u_i + v_j - c_{ij}$

$$\bar{c}_{11} = -3, \bar{c}_{12} = -1, \bar{c}_{13} = -5, \bar{c}_{22} = -1, \bar{c}_{23} = -5, \bar{c}_{32} = -1, \bar{c}_{34} = -2, \bar{c}_{41} = -2, \bar{c}_{44} = -1.$$

Since all  $\bar{c}_{ij}$  are non-positive, the current solution is optimal. Thus the optimal solution is

$$x_{14} = 20, x_{21} = 10, x_{24} = 5, x_{31} = 20, x_{33} = 5, x_{42} = 20, x_{43} = 20.$$

The optimal transportation cost = Rs. 205.

**Note.** To find optimal solution to a T.P., the number of iterations by uv-method is always more if we consider the initial BFS by NWC.

**DEGENERACY IN T.P.**

A BFS of a T.P. is said to be degenerate if one or more basic variables assume a zero value. This degeneracy may occur in initial BFS or in the subsequent iterations of uv-method. An initial BFS could become degenerate when the supply and demand in the intermediate stages of any one method (NWC/LCM/VAM) are equal corresponding to a selected cell for allocation. In uv-method it is identified only when more than one corner points in a loop vanishes due to minimum value of  $\theta$ .

For the degeneracy in initial BFS, arbitrarily we can delete the row due to supply adjusting the demand to zero or delete the column due to demand adjusting the supply to zero whenever there is a tie in demand and supply.

For the degeneracy in *uv*-method, arbitrarily we can make one corner as non-basic cell and put zero in the other corner.

**Example 6.** Find the optimal solution to the following T.P.:

Source	Destination			Available
	1	2	3	
1	50	30	190	10
2	80	45	150	30
3	220	180	50	40
Requirement	40	20	20	80

**Solution.** Let us find the initial BFS using VAM :

	1	2	3	Row penalties
1	50	30	190	10 (20)
2	80	45	150	30 (35)
3	220	180	50	40 (130)
Column penalties	40 (30)	20 (15)	20 (100)	

Select (3, 3) cell for allocation and allocate  $\min(40, 20) = 20$  in that cell. Cross-off the third column as the requirement is met and adjust the availability to 20. The reduced table is given below :

	1	2	Row penalties
1	50	30	10 (20)
2	80	45	30 (35)
3	220	180	20 (40)
Column penalties	40 (30)	20 (15)	

Select (3, 2) cell for allocation. Now there is a tie in allocation. Let us allocate 20 in (3, 2) cell and cross-off the second column and adjust the availability to zero. The reduced table is given below :

	1	
1	50	10
2	80	30
3	220	0
	40	

NOTES

On continuation we obtain the remaining allocations as 0 in (3, 1) cell, 30 in (2, 1) cell and 10 in (1, 1) cell. The complete initial BFS is given below and let us apply the first iteration of uv-method :

Iteration 1.

NOTES

10	50	30	190	$u_1 = -170$ $u_2 = -140$ $u_3 = 0$ (Let)
30	80	45	150	
0	20	20	50	
$V_1 = 220 \quad V_2 = 180 \quad V_3 = 50$				

For non-basic cells :

$$\bar{c}_{ij} = u_i + v_j - c_{ij}$$

$$\bar{c}_{12} = -20, \quad \bar{c}_{13} = -310, \quad \bar{c}_{22} = -5, \quad \bar{c}_{23} = -240.$$

Since all  $\bar{c}_{ij} < 0$ , the current solution is optimal. Hence, the optimal solution is

$$x_{11} = 10, \quad x_{21} = 30, \quad x_{31} = 0, \quad x_{32} = 20, \quad x_{33} = 20.$$

The transportation cost

$$= 50 \times 10 + 80 \times 30 + 0 + 180 \times 20 + 50 \times 20$$

$$= \text{Rs. } 7500.$$

### MAX-TYPE T.P.

Instead of unit cost in transportation table, unit profit is considered then the objective of the T.P. changes to maximize the total profits subject to supply and demand restrictions. Then this problem is called 'max-type' T.P.

To obtain optimal solution, we consider

$$\text{Loss} = - \text{Profit}$$

and convert the max type transportation matrix to a loss matrix. Then all the methods described in the previous sections can be applied. Thus the optimal BFS obtained for the loss matrix will be the optimal BFS for the max-type T.P.

**Example 7.** A company has three plants at locations A, B and C, which supply to four markets D, E, F and G. Monthly plant capacities are 500, 800 and 900 units respectively. Monthly demands of the markets are 600, 700, 400 and 500 units respectively. Unit profits (in rupees) due to transportation are given below :

	D	E	F	G
A	8	5	3	8
B	7	4	5	2
C	6	8	4	2

Determine an optimal distribution for the company in order to maximize the total transportation profits.

**Solution.** The given problem is balanced-max type T.P. All profits are converted to losses by multiplying  $-1$ .

	D	E	F	G	
A	-8	-5	-3	-6	500
B	-7	-4	-5	-2	800
C	-6	-8	-4	-2	900
	600	700	400	500	2200

NOTES

The initial BFS by LCM is given below

500				
	-8	-5	-3	-6
100			400	300
	-7	-4	-5	-2
		700		200
	-6	-8	-4	-2

To find optimal solution let us apply  $uv$ -method.

**Iteration 1.**

500				$\theta$	$u_1 = -1$
	-8	-5	-3	-6	
100			400	300	$u_2 = 0$
	-7	-4	-5	-2	
		700		200	$u_3 = 0$ (Let)
	-6	-8	-4	-2	

$V_1 = -7 \quad V_2 = -8 \quad V_3 = -5 \quad V_4 = -2$

For non-basic cells :  $\bar{c}_{ij} = u_i + v_j - c_{ij}$

$\bar{c}_{12} = -4, \bar{c}_{13} = -3, \bar{c}_{14} = 3, \bar{c}_{22} = -4, \bar{c}_{31} = -1, \bar{c}_{33} = -1.$

Since all  $\bar{c}_{ij}$  are not non-positive, the current solution is not optimal. Select the cell (1, 4) due to largest positive value and assign an unknown quantity  $\theta$  in that cell. Identify a loop and subtract and add  $\theta$  to the corner points of the loop which is shown above.

Select  $\theta = \min. (500, 300) = 300$ . The cell (2, 4) leaves the basis and the cell (1, 4) enters into the basis. Thus the current solution is updated.

**Iteration 2.**

200				300	$u_1 = -4$
	-8	-5	-3	-6	
400			400		$u_2 = -3$
	-7	-4	-5	-2	
$\theta$		700		200	$u_3 = 0$ (Let)
	-6	-8	-4	-2	

$V_1 = -4 \quad V_2 = -8 \quad V_3 = -2 \quad V_4 = -2$

For non-basic cells,

$\bar{c}_{12} = -7, \bar{c}_{13} = -3, \bar{c}_{22} = -7, \bar{c}_{24} = -3, \bar{c}_{31} = 2, \bar{c}_{33} = 2.$

NOTES

Since all the  $\bar{c}_{ij}$  are not non-positive, the current solution is not optimal. There is a tie in largest positive values. Let us select the cell (3, 1) and assign an unknown quantity  $\theta$  in that cell. Identify a loop and subtract and add  $\theta$  to the corner points of the loop which is shown above.

Select  $\theta = \min. (200, 200) = 200$ . Since only one cell will leave the basis, let the cell (3, 3) leaves the basis and assign a zero in the cell (1, 1). The cell (3, 1) enters into the basis. Thus the current solution is updated.

Iteration 3.

0			500
	-8	-5	-3
400		400	
	-7	-4	-5
200	700		
	-6	-8	-4

$u_1 = -2$

$u_2 = -1$

$u_3 = 0$  (Let)

$V_1 = -6 \quad V_2 = -8 \quad V_3 = -4 \quad V_4 = -4$

For non-basic cells,

$$\bar{c}_{12} = -5, \bar{c}_{13} = -3, \bar{c}_{22} = -5, \bar{c}_{24} = -3, \bar{c}_{33} = 0, \bar{c}_{34} = -4.$$

Since all the  $\bar{c}_{ij}$  are non-positive, the current solution is optimal.

Thus the optimal solution, which is degenerate, is

$$x_{11} = 0, x_{14} = 500, x_{21} = 400, x_{23} = 400, x_{31} = 200, x_{32} = 700.$$

The maximum transportation profit

$$= 0 + 3000 + 2800 + 2000 + 1200 + 5600 = \text{Rs. } 14600.$$

Since  $\bar{c}_{33} = 0$ , this indicates that there exists an alternative optimal solution. Assign an unknown quantity  $\theta$  in the cell (3, 3). Identify a loop and subtract and add  $\theta$  to the corner points of the loop which is shown below :

0			500
	-8	-5	-3
400 + $\theta$		400 - $\theta$	
	-7	-4	-5
200	700	$\theta$	
	-6	-8	-4

Select  $\theta = \min. (200, 400) = 200$ . The cell (3, 1) leaves the basis and the cell (3, 3) enters into the basis.

Iteration 4.

0			500
	-8	-5	-3
600		200	
	-7	-4	-5
	700	200	
	-6	-8	-4

$u_1 = -2$

$u_2 = -1$

$u_3 = 0$  (Let)

$V_1 = -6 \quad V_2 = -8 \quad V_3 = -4 \quad V_4 = -4$

For non-basic cells,

$$\bar{c}_{12} = -5, \bar{c}_{13} = -3, \bar{c}_{22} = -5, \bar{c}_{24} = -3, \bar{c}_{31} = 0, \bar{c}_{34} = -2.$$

Since all the  $\bar{c}_y$  are non-positive, the current solution is optimal. Thus the alternative optimal solution is

$$x_{11} = 0, x_{14} = 500, x_{21} = 600, x_{23} = 200, x_{32} = 700, x_{33} = 200.$$

and the maximum transportation profit is Rs. 14,600.

NOTES

**UNBALANCED T.P.**

If total supply  $\neq$  total demand, the problem is called unbalanced T.P.. To obtain feasible solution, the unbalanced problem should be converted to balanced problem by introducing dummy source or dummy destination, whichever is required. Suppose, (supply  $= \sum a_i > \sum b_j$  (= demand)). Then add one dummy destination with demand  $= (\sum a_i - \sum b_j)$  with either zero transportation costs or some penalties, if they are given. Suppose (supply  $= \sum a_i < \sum b_j$  (= demand)). Then add one dummy source with supply  $= (\sum b_j - \sum a_i)$  with either zero transportation costs or some penalties, if they are given.

After making it balanced the mathematical formulation is similar to the balanced T.P.

**Example 8.** A company wants to supply materials from three plants to three new projects. Project I requires 50 truck loads, project II requires 40 truck loads and project III requires 60 truck loads. Supply capacities for the plants  $P_1, P_2$  and  $P_3$  are 30, 55 and 45 truck loads. The table of transportation costs are given below :

	I	II	III
$P_1$	7	10	12
$P_2$	8	12	7
$P_3$	4	9	10

Determine the optimal distribution.

**Solution.** Here total supplies = 130 and total requirements = 150. The given problem is unbalanced T.P. To make it balanced consider a dummy plants with supply capacity of 20 truck loads and zero transportation costs to the three projects. Then the balanced T.P. is

		To			
		I	II	III	
From	$P_1$	7	10	12	30
	$P_2$	8	12	7	55
	$P_3$	4	9	10	45
	$P_4$ (Dummy)	0	0	0	20
		50	40	60	

Using VAM, we obtain the initial BFS as given below :

NOTES

5	20	5
7	10	12
8	12	7
45	4	9
0	20	0

To find optimal solution let us apply uv-method.

Iteration 1:

5	20+ $\theta$	5- $\theta$	
7		10	12
8		12	7
45			
4		9	10
0	20	$\theta$	0

$u_1 = 0$  (Let)  
 $u_2 = -5$   
 $u_3 = -3$   
 $u_4 = -10$   
 $V_1 = 7$   $V_2 = 10$   $V_3 = 12$

For non-basic cells,  $\bar{c}_{ij} = u_i + v_j - c_{ij}$

$\bar{c}_{21} = -6, \bar{c}_{22} = -7, \bar{c}_{32} = -2, \bar{c}_{33} = -1, \bar{c}_{41} = -3, \bar{c}_{43} = 2,$

Since  $\bar{c}_{43}$  is only positive value assign an unknown quantity  $\theta$  in (4, 3) cell. Identify a loop and subtract and add  $\theta$  to the corner points of the loop which is shown above.

Select  $\theta = \min. (5, 20) = 5$  so that the cell (1, 3) leaves the basis and the cell (4, 3) enters into the basis.

Iteration 2.

5	25		
7	10	12	
8	12	7	
45			
4	15	5	
0	0	0	

$u_1 = 0$  (Let)  
 $u_2 = -3$   
 $u_3 = -3$   
 $u_4 = -10$   
 $V_1 = 7$   $V_2 = 10$   $V_3 = 10$

For non-basic cells, we obtain

$\bar{c}_{13} = -2, \bar{c}_{21} = -4, \bar{c}_{22} = -5, \bar{c}_{32} = -2, \bar{c}_{33} = -3, \bar{c}_{41} = -3$

Since  $\bar{c}_{ij} < 0$ , the current solution is optimal. Thus the optimal solution is

Supply 15 truck loads from  $P_1$  to I, 25 truck loads from  $P_1$  to II, 55 truck loads from  $P_2$  to III, 45 truck loads from  $P_3$  to I. Demands of 15 truck loads for II and 5 truck loads for III will remain unsatisfied.

## SUMMARY

- Transportation problem (T.P.) is generally concerned with the distribution of a certain commodity/product from several origins/sources to several destinations with minimum total cost through single mode of transportation.
- A BFS of a T.P. is said to be degenerate if one or more basic variables assume a zero value. This degeneracy may occur in initial BFS or in the subsequent iterations of uv-method.
- For the degeneracy in initial BFS, arbitrarily we can delete the row due to supply adjusting the demand to zero or delete the column due to demand adjusting the supply to zero whenever there is a tie in demand and supply.
- If total supply  $\neq$  total demand, the problem is called unbalanced T.P.. To obtain feasible solution, the unbalanced problem should be converted to balanced problem by introducing dummy source or dummy destination, whichever is required.

### NOTES

## PROBLEMS

1. There are three sources which store a given product. The sources supply these products to four dealers. The capacities of the sources and the demands of the dealers are given. Capacities  $S_1 = 150$ ,  $S_2 = 40$ ,  $S_3 = 80$ , Demands  $D_1 = 90$ ,  $D_2 = 70$ ,  $D_3 = 50$ ,  $D_4 = 60$ . The cost matrix is given as follows :

		To			
		$D_1$	$D_2$	$D_3$	$D_4$
From	$S_1$	27	23	31	69
	$S_2$	10	45	40	32
	$S_3$	30	54	35	57

Find the minimum cost of T.P.

2. There are three factories  $F_1, F_2, F_3$  situated in different areas with supply capacities as 200, 400 and 350 units respectively. The items are shipped to five markets  $M_1, M_2, M_3, M_4$  and  $M_5$  with demands as 150, 120, 230, 200, 250 units respectively. The cost matrix is given as follows :

	$M_1$	$M_2$	$M_3$	$M_4$	$M_5$
$F_1$	2	5	6	4	7
$F_2$	4	3	5	8	8
$F_3$	4	6	2	1	5

Determine the optimal shipping cost and shipping patterns.

3. Find the initial basic feasible solution to the following T.P. using (a) NWC, (b) LCM, and (c) VAM :  
(i)

		To					
		D	E	F	G	H	
From	A	11	7	5	8	9	50
	B	10	11	8	4	5	90
	C	9	6	12	5	5	60
		20	40	20	40	80	

(ii)

NOTES

		To					
		A	B	C	D	E	
From	I	9	10	0	8	9	90
	II	11	12	5	8	3	20
	III	4	9	1	2	0	50
	IV	8	0	3	5	6	50
		80	60	20	40	10	

4. Solve the following transportation problem :

		To					
		D <sub>1</sub>	D <sub>2</sub>	D <sub>3</sub>	D <sub>4</sub>	D <sub>5</sub>	
From	S <sub>1</sub>	3	5	2	1	3	45
	S <sub>2</sub>	2	1	-	4	6	55
	S <sub>3</sub>	5	4	3	1	2	65
	S <sub>4</sub>	-	4	6	5	7	50
		27	42	51	62	33	

(Supply from S<sub>2</sub> to D<sub>3</sub> and S<sub>4</sub> to D<sub>1</sub> are restricted)

5. A transportation problem for which the costs, origin and availabilities, destinations and requirements are given below :

	D <sub>1</sub>	D <sub>2</sub>	D <sub>3</sub>	
O <sub>1</sub>	2	1	2	40
O <sub>2</sub>	9	4	7	60
O <sub>3</sub>	1	2	9	10
	40	50	20	

Check whether the following basic feasible solution  $x_{11} = 20$ ,  $x_{13} = 20$ ,  $x_{21} = 10$ ,  $x_{22} = 50$ , and  $x_{31} = 10$  is optimal. If not, find an optimal solution.

6. Goods have to be transported from sources S<sub>1</sub>, S<sub>2</sub> and S<sub>3</sub> to destinations D<sub>1</sub>, D<sub>2</sub> and D<sub>3</sub>. The T.P. cost per unit capacities of the sources and requirements of the destinations are given in the following table :

	D <sub>1</sub>	D <sub>2</sub>	D <sub>3</sub>	Capacity
S <sub>1</sub>	8	5	6	120
S <sub>2</sub>	15	10	12	80
S <sub>3</sub>	3	9	10	80
Requirement	150	80	50	

Determine a T.P. schedule so that the cost is minimized.

7. Four products are produced in four machines and their profit margins are given by the table as follows :

	P <sub>1</sub>	P <sub>2</sub>	P <sub>3</sub>	P <sub>4</sub>	Capacity
M <sub>1</sub>	10	7	8	6	40
M <sub>2</sub>	5	9	6	4	55
M <sub>3</sub>	7	4	11	5	60
M <sub>4</sub>	4	10	7	8	45
Requirement	35	42	68	55	

NOTES

Find a suitable production plan of products in machines so that the profit is maximized while the capacities and requirements are met.

8. Identical products are produced in four factories and sent to four warehouses for delivery to the customers. The costs of transportation, capacities and demands are given as below :

		Warehouses				
		W <sub>1</sub>	W <sub>2</sub>	W <sub>3</sub>	W <sub>4</sub>	
Factories	F <sub>1</sub>	9	6	11	5	200
	F <sub>2</sub>	4	5	8	5	150
	F <sub>3</sub>	7	8	4	6	350
	F <sub>4</sub>	3	3	10	10	250
Demands		260	100	340	200	Capacities

Find the optimal schedule of delivery for minimization of cost of transportation. Is there any alternative solution ? If yes, then find it.

9. Starting with LCM initial BFS, find the optimal solution to the following T.P. problem :

		To					
		5	1	2	4	3	60
From		1	4	2	3	6	55
		4	2	3	5	2	40
		3	5	6	3	7	50
	Demands	42	33	41	52	27	Supply

10. A company manufacturing air coolers has two plants located at Mumbai and Kolkata with a weekly capacity of 200 units and 100 units respectively. The company supplies air coolers to its 4 show-rooms situated at Ranchi, Delhi, Lucknow and Kanpur which have a demand of 75, 100, 100 and 30 units respectively. The cost per unit (in Rs.) is shown in the following table :

	Ranchi	Delhi	Lucknow	Kanpur
Mumbai	90	90	100	100
Kolkata	50	70	130	85

Plan the production programmes so as to minimize the total cost of transportation.

**ANSWERS****NOTES**

1.  $x_{11} = 30, x_{12} = 70, x_{13} = 50, x_{24} = 40, x_{31} = 60, x_{34} = 20$ .  
Minimum T.P. cost = Rs. 8190.
2. Solution 1:  
 $x_{11} = 150, x_{15} = 50, x_{22} = 120, x_{23} = 80, x_{25} = 200, x_{33} = 150, x_{34} = 200$ .  
Solution 2:  
 $x_{11} = 150, x_{15} = 50, x_{22} = 120, x_{23} = 230, x_{25} = 50, x_{34} = 200, x_{35} = 150$ .  
Minimum shipping cost = Rs. 3510.
3. (i) (a)  $x_{11} = 20, x_{12} = 30, x_{22} = 10, x_{23} = 20, x_{24} = 40, x_{25} = 20, x_{35} = 60$ .  
T.P. cost = Rs. 1260.  
(b)  $x_{11} = 20, x_{12} = 10, x_{13} = 20, x_{24} = 40, x_{25} = 50, x_{32} = 30, x_{35} = 30$ .  
T.P. cost = Rs. 1130.  
(c)  $x_{12} = 40, x_{13} = 10, x_{21} = 20, x_{23} = 10, x_{24} = 40, x_{25} = 20, x_{35} = 60$ .  
T.P. cost = Rs. 1170.
- (ii) (a)  $x_{11} = 80, x_{12} = 10, x_{22} = 20, x_{32} = 30, x_{33} = 20, x_{34} = 0, x_{44} = 40, x_{45} = 10$ .  
T.P. cost = Rs. 1610.  
(b)  $x_{11} = 70, x_{13} = 20, x_{21} = 10, x_{22} = 10, x_{31} = 0, x_{34} = 40, x_{35} = 10, x_{42} = 50$ .  
T.P. cost = Rs. 940.  
(c)  $x_{11} = 60, x_{12} = 10, x_{13} = 20, x_{21} = 10, x_{25} = 10, x_{31} = 10, x_{34} = 40, x_{42} = 50$ .  
T.P. cost = Rs. 900.
4. Solution 1:  
 $x_{13} = 15, x_{14} = 30, x_{21} = 27, x_{22} = 28, x_{34} = 32, x_{35} = 33, x_{42} = 14, x_{43} = 36$ .  
Solution 2:  
 $x_{13} = 45, x_{21} = 27, x_{22} = 28, x_{34} = 32, x_{35} = 33, x_{42} = 14, x_{43} = 6, x_{44} = 30$ .  
Minimum T.P. cost = Rs. 512.
5. The new optimal solution is  $x_{11} = 30, x_{13} = 10, x_{22} = 50, x_{23} = 10, x_{31} = 10$ .  
Minimum T.P. cost = Rs. 360.
6.  $x_{11} = 70, x_{12} = 0, x_{13} = 50, x_{22} = 80, x_{31} = 80$ .  
Minimum T.P. cost = Rs. 1900.
7.  $x_{11} = 35, x_{14} = 5, x_{22} = 42, x_{23} = 8, x_{24} = 5, x_{33} = 60, x_{44} = 45$ .  
Total Profit = Rs. 1846.
8. Solution 1:  
 $x_{14} = 160, x_{21} = 110, x_{24} = 40, x_{33} = 340, x_{41} = 150, x_{42} = 100$ .  
Solution 2:  
 $x_{14} = 200, x_{21} = 110, x_{33} = 340, x_{41} = 150, x_{42} = 100$ .  
Minimum T.P. cost = Rs. 3550.
9. Solution 1:  
 $x_{12} = 33, x_{13} = 27, x_{21} = 42, x_{23} = 11, x_{24} = 2, x_{33} = 3, x_{35} = 27, x_{44} = 50$ .  
Solution 2:  
 $x_{12} = 30, x_{13} = 30, x_{21} = 42, x_{23} = 11, x_{24} = 2, x_{32} = 3, x_{35} = 27, x_{44} = 50$ .  
Minimum T.P. cost = Rs. 370.
10.  $x_{12} = 75, x_{13} = 95, x_{14} = 30, x_{21} = 75, x_{22} = 25$ .  
Minimum T.P. cost = Rs. 24750.

## CHAPTER 3 ASSIGNMENT PROBLEMS

NOTES

## ★ STRUCTURE ★

- Introduction and Mathematical Formulation
- Hungarian Algorithm
- Unbalanced Assignments
- Max-type Assignment Problems
- Routing Problems
- Summary
- Problems

## INTRODUCTION AND MATHEMATICAL FORMULATION

Consider  $n$  machines  $M_1, M_2, \dots, M_n$  and  $n$  different jobs  $J_1, J_2, \dots, J_n$ . These jobs to be processed by the machines one to one basis *i.e.*, each machine will process exactly one job and each job will be assigned to only one machine. For each job the processing cost depends on the machine to which it is assigned. Now we have to determine the assignment of the jobs to the machines one to one basis such that the total processing cost is minimum. This is called an *assignment problem*.

If the number of machines is equal to the number of jobs then the above problem is called *balanced* or *standard* assignment problem. Otherwise, the problem is called *unbalanced* or *non-standard* assignment problem. Let us consider a balanced assignment problem:

For linear programming problem formulation, let us define the decision variables as

$$x_{ij} = \begin{cases} 1, & \text{if job } j \text{ is assigned to machine } i \\ 0, & \text{otherwise} \end{cases}$$

and the cost of processing job  $j$  on machine  $i$  as  $c_{ij}$ . Then we can formulate the assignment problem as follows :

$$\text{Minimize } z = \sum_{i=1}^n \sum_{j=1}^n c_{ij} x_{ij} \quad \dots(1)$$

subject to, 
$$\sum_{j=1}^n x_{ij} = 1, \quad i = 1, 2, \dots, n$$

(Each machine is assigned exactly to one job)

$$\sum_{i=1}^n x_{ij} = 1, \quad j = 1, 2, \dots, n.$$

(Each job is assigned exactly to one machine)

$$x_{ij} = 0 \text{ or } 1 \text{ for all } i \text{ and } j$$

In matrix form,

$$\text{Minimize } z = Cx$$

$$\text{subject to, } Ax = 1,$$

$$x_{ij} = 0 \text{ or } 1, i, j = 1, 2, \dots, n.$$

NOTES

where A is a  $2n \times n^2$  matrix and total unimodular i.e., the determinant of every sub square matrix formed from it has value 0 or 1. This property permits us to replace the constraint  $x_{ij} = 0 \text{ or } 1$  by the constraint  $x_{ij} \geq 0$ . Thus we obtain

$$\text{Minimize } z = Cx$$

$$\text{subject to, } Ax = 1, x \geq 0$$

The dual of (1) with the non-negativity restrictions replacing the 0-1 constraints can be written as follows :

$$\text{Maximize } W = \sum_{i=1}^n u_i + \sum_{j=1}^n v_j$$

$$\text{subject to, } u_i + v_j \leq c_{ij} \quad i, j = 1, 2, \dots, n.$$

$$u_i, v_j \text{ unrestricted in signs} \quad i, j = 1, 2, \dots, n.$$

**Example 1.** A company is facing the problem of assigning four operators to four machines. The assignment cost in rupees is given below :

		Machine			
		M <sub>1</sub>	M <sub>2</sub>	M <sub>3</sub>	M <sub>4</sub>
Operator	I	5	7	—	4
	II	7	5	3	2
	III	9	4	6	—
	IV	7	2	7	6

In the above, operators I and III can not be assigned to the machines M<sub>3</sub> and M<sub>4</sub> respectively. Formulate the above problem as a LP model.

**Solution.** Let  $x_{ij} = \begin{cases} 1, & \text{if the } i\text{th operator is assigned to } j\text{th machine} \\ 0, & \text{otherwise} \end{cases}$

$$i, j = 1, 2, 3, 4.$$

be the decision variables.

By the problem,  $x_{13} = 0$  and  $x_{34} = 0$ .

The LP model is given below :

$$\begin{aligned} \text{Minimize } z = & 5x_{11} + 7x_{12} + 4x_{14} + 7x_{21} + 5x_{22} + 3x_{23} + 2x_{24} \\ & + 9x_{31} + 4x_{32} + 6x_{33} + 7x_{41} + 2x_{42} + 7x_{43} + 6x_{44} \end{aligned}$$

subject to,

(Operator assignment constraints)

$$\begin{aligned}x_{11} + x_{12} + x_{14} &= 1 \\x_{21} + x_{22} + x_{23} + x_{24} &= 1 \\x_{31} + x_{32} + x_{33} &= 1 \\x_{41} + x_{42} + x_{43} + x_{44} &= 1\end{aligned}$$

NOTES

(Machine assignment constraints)

$$\begin{aligned}x_{11} + x_{21} + x_{31} + x_{41} &= 1 \\x_{12} + x_{22} + x_{32} + x_{42} &= 1 \\x_{23} + x_{33} + x_{43} &= 1 \\x_{14} + x_{24} + x_{44} &= 1\end{aligned}$$

$$x_{ij} \geq 0 \text{ for all } i \text{ and } j.$$

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## HUNGARIAN ALGORITHM

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This is an efficient algorithm for solving the assignment problem developed by the Hungarian mathematician König. Here the optimal assignment is not affected if a constant is added or subtracted from any row or column of the balanced assignment cost matrix. The algorithm can be started as follows :

- (a) Bring at least one zero to each row and column of the cost matrix by subtracting the minimum of the row and column respectively.
- (b) Cover all the zeros in cost matrix by *minimum* number of horizontal and vertical lines.
- (c) If number of lines = order of the matrix, then select the zeros as many as the order of the matrix in such a way that they cover all the rows and columns.

(Here  $A_{n \times n}$  means  $n$ th order matrix)

- (d) If number of lines  $\neq$  order of the matrix, then perform the following and create a new matrix :
  - (i) Select the minimum element from the uncovered elements of the cost matrix by the lines.
  - (ii) Subtract the uncovered elements from the minimum element.
  - (iii) Add the minimum element to the junction (i.e., crossing of the lines) elements.
  - (iv) Other elements on the lines remain unaltered.
  - (v) Go to Step (b).

**Example 2.** A construction company has four engineers for designing. The general manager is facing the problem of assigning four designing projects to these engineers. It is also found that Engineer 2 is not competent to design project 4. Given the time estimate required by each engineer to design a given project, find an assignment which minimizes the total time.

NOTES

		Projects			
		P1	P2	P3	P4
Engineers	E1	6	5	13	2
	E2	8	10	4	-
	E3	10	3	7	3
	E4	9	8	6	2

**Solution.** Let us first bring zeros rowwise by subtracting the respective minima from all the row elements respectively.

4	3	11	0
4	6	0	-
7	0	4	0
7	6	4	0

Let us bring zero columnwise by subtracting the respective minima from all the column elements respectively. Here the above operations is to be performed only on first column, since at least one zero has appeared in the remaining columns.

0	3	11	0
0	6	0	-
3	0	4	0
3	6	4	0

(This completes Step-a)

Now (Step-b) all the zeros are to be covered by minimum number of horizontal and vertical lines which is shown below. It is also to be noted that this covering is not unique.

It is seen that no. of lines = 4 = order of the matrix. Therefore by Step-c, we can go for assignment *i.e.*, we have to select 4 zeros such that they cover all the rows and columns which is shown below:

0	3	11	0
0	6	0	-
3	0	4	0
3	6	4	0

0	3	11	0
0	6	0	-
3	0	4	0
3	6	4	0

Therefore the optimal assignment is

$$E1 \rightarrow P1, \quad E2 \rightarrow P3, \quad E3 \rightarrow P2, \quad E4 \rightarrow P4$$

and the minimum total time required =  $6 + 4 + 3 + 2 = 15$  units.

**Example 3.** Solve the following job machine assignment problem. Cost data are given below :

NOTES

		Machines					
		1	2	3	4	5	6
Jobs	A	21	35	20	20	32	28
	B	30	31	22	25	28	30
	C	28	29	25	27	27	21
	D	30	30	26	26	31	28
	E	21	31	25	20	27	30
	F	25	29	22	25	30	21

**Solution.** Let us first bring zeros first rowwise and then columnwise by subtracting the respective minima elements from each row and each column respectively and the cost matrix, thus obtained, is as follows :

0	11	0	0	7	8
7	5	0	3	1	8
6	4	4	6	1	0
3	0	0	0	0	2
0	7	5	0	2	10
3	4	1	4	4	0

By Step-b, all the zeros are covered by minimum number of horizontal and vertical lines which is shown below :

0	11	0	0	7	8
7	5	0	3	1	8
6	4	4	6	1	0
3	0	0	0	0	2
0	7	5	0	2	10
3	4	1	4	4	0

Here no. of lines  $\neq$  order of the matrix. Hence, we have to apply Step-d. The minimum uncovered element is 1. By applying Step-d we obtain the following matrix :

NOTES

0	11	1	0	7	9
6	4	0	2	0	8
5	3	1	5	0	0
3	0	1	0	0	3
0	7	6	0	2	11
2	3	1	3	3	0

Now, by Step-b, we cover all the zeros by minimum number of horizontal and vertical straight lines.

0	11	1	0	7	9
6	4	0	2	0	8
5	3	4	5	0	0
3	0	1	0	0	3
0	7	6	0	2	11
2	3	1	3	3	0

Now, the no. of lines = order of the matrix. So we can go for assignment by Step-c. The assignment is shown below :

0	11	1	0	7	9
6	4	0	2	0	8
5	3	4	5	0	0
3	0	1	0	0	3
0	7	6	0	2	11
2	3	1	3	3	0

The optimal assignment is A→1, B→3, C→5, D→2, E→4, F→6. An alternative assignment is also obtained as A→4, B→3, C→5, D→2, E→1, F→6. For both the assignments, the minimum cost is 21 + 22 + 27 + 30 + 20 + 21 i.e., Rs. 141.

### UNBALANCED ASSIGNMENTS

For unbalanced or non-standard assignment problem no. of rows ≠ no. of columns in the assignment cost matrix i.e., we deal with a rectangular cost matrix. To find an assignment for this type of problem, we have to first convert this unbalanced

problem into a balanced problem by adding dummy rows or columns with zero costs so that the defective function will be unaltered. For machine-job problem, if no. of machines (say,  $m$ ) > no. of jobs (say,  $n$ ), then create  $m-n$  dummy jobs and the processing cost of dummy jobs as zero. When a dummy job gets assigned to a machine, that machine stays idle. Similarly the other case i.e.,  $n > m$ , is handled.

**Example 4.** Find an optimal solution to an assignment problem with the following cost matrix :

	M1	M2	M3	M4	M5
J1	13	5	20	5	6
J2	15	10	16	10	15
J3	6	12	14	10	13
J4	13	11	15	11	15
J5	15	6	16	10	6
J6	6	15	14	5	12

**Solution.** The above problem is unbalanced. We have to create a dummy machine M6 with zero processing time to make the problem as balanced assignment problem. Therefore we obtain the following :

	M1	M2	M3	M4	M5	M6 (dummy)
J1	13	5	20	5	6	0
J2	15	10	16	10	15	0
J3	6	12	14	10	13	0
J4	13	11	15	11	15	0
J5	15	6	16	10	6	0
J6	6	15	14	5	12	0

Let us bring zeros columnwise by subtracting the respective minima elements from each column respectively and the cost matrix, thus obtained, is as follows:

7	0	6	0	0	0
9	5	2	5	9	0
0	7	0	5	7	0
7	6	1	6	9	0
9	1	2	5	0	0
0	10	0	0	6	0

NOTES

Let us cover all the zeros by minimum number of horizontal and vertical lines and is given below :

NOTES

7	0	6	0	0	0
9	5	2	5	9	0
0	7	0	5	7	0
7	6	1	6	9	0
9	1	2	5	0	0
0	10	0	0	6	0

Now, the number of lines  $\neq$  order of the matrix. The minimum uncovered element by the lines is 1. Using Step-d of the Hungarian algorithm and covering all the zeros by minimum no. of lines we obtain as follows :

7	0	6	0	1	1
8	4	1	4	9	0
0	7	0	5	8	1
6	5	0	5	9	0
8	0	1	4	0	0
0	10	0	0	7	1

Now, the number of lines = order of the matrix and we have to select 6 zeros such that they cover all the rows and columns. This is done in the following :

7	0	6	0	1	1
8	4	1	4	9	0
0	7	0	5	8	1
6	5	0	5	9	0
8	0	1	4	0	0
0	10	0	0	7	1

Therefore, the optimal assignment is

J1→M2, J2→M6, J3→M1, J4→M3, J5→M5, J6→M4 and the minimum cost = Rs. (5 + 0 + 6 + 15 + 6 + 5) = Rs. 37.

In the above, the job J2 will not get processed since the machine M6 is dummy.

## MAX-TYPE ASSIGNMENT PROBLEMS

When the objective of the assignment is to maximize, the problem is called 'Max-type assignment problem'. This is solved by converting the profit matrix to an opportunity loss matrix by subtracting each element from the highest element of the profit matrix. Then the minimization of the loss matrix is the same as the maximization of the profit matrix.

**Example 5.** A company is faced with the problem of assigning 4 jobs to 5 persons. The expected profit in rupees for each person on each job are as follows :

Persons	Job			
	J1	J2	J3	J4
I	86	78	62	81
II	55	79	65	60
III	72	65	63	80
IV	86	70	65	71
V	72	70	71	60

Find the assignment of persons to jobs that will result in a maximum profit.

**Solution.** The above problem is unbalanced max.-type assignment problem. The maximum element is 86. By subtracting all the elements from it obtain the following opportunity loss matrix.

0	8	24	5
31	7	21	26
14	21	23	6
0	16	21	15
14	16	15	26

Now, a dummy job J5 is added with zero losses. Then bring zeros in each column by subtracting the respective minimum element from each column we obtain the following matrix.

0	1	9	0	0
31	0	6	21	0
14	14	8	1	0
0	9	6	10	0
14	9	0	21	0

Let us cover all the zeros by minimum number of lines and is as follows:

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0	1	9	0	0
31	0	6	21	0
14	14	8	1	0
0	9	6	10	0
14	9	0	21	0

Since, the no. of lines = order of the matrix, we have to select 5 zeros such that they cover all the rows and columns. This is done in the following :

0	1	9	0	0
31	0	6	21	0
14	14	8	1	0
0	9	6	10	0
14	9	0	21	0

The optimal assignment is

I→J4, II→J2, III→J5, IV→J1, V→J3 and maximum profit = Rs. (81 + 79 + 86 + 71) = Rs. 317. Here person III is idle.

Note. The max-type assignment problem can also be converted to a minimization problem by multiplying all the elements of the profit matrix by - 1. Then the Hungarian method can be applied directly.

**PROBLEMS**

1. Solve the following assignment problems :

(a)

	A	B	C	D	E
I	12	20	20	18	17
II	20	12	5	11	8
III	20	5	12	5	9
IV	18	11	5	12	10
V	17	8	9	10	12

(b)

	Jobs					
	J1	J2	J3	J4	J5	J6
A	18	10	25	10	11	22
B	20	15	21	-	20	18
C	11	17	19	15	18	17

Persons	D	18	16	20	16	20	21
	E	20	-	21	15	11	17
	F	11	15	19	12	15	20

2. A machine tool decides to make six sub-assemblies through six contractors A, B, C, D, E and F. Each contractor is to receive only one sub-assembly from A1, A2, A3, A4, A5 and A6. But the contractors C and E are not competent for the A4 and A2 assembly respectively. The cost of each subassembly by the bids submitted by each contractor is shown below (in hundred rupees) :

	A1	A2	A3	A4	A5	A6
A	15	10	11	18	13	22
B	9	12	18	10	14	11
C	9	15	11	-	22	11
D	14	13	9	12	15	10
E	10	-	11	22	13	18
F	10	14	15	12	13	14

Find the optimal assignments of the assemblies to contractors so as to minimize the total cost.

3. Five programmers, in a computer centre, write five programmes which run successfully but with different times. Assign the programmers to the programmes in a such a way that the total time taken by them is minimum taking the following time matrix:

		Programmes				
		P1	P2	P3	P4	P5
Programmers	A	80	66	65	65	73
	B	76	75	70	70	75
	C	74	73	72	70	66
	D	75	75	71	71	73
	E	76	66	66	70	75

4. Consider the problem of assigning seven jobs to seven persons. The assignment costs are given as follows :

		Jobs						
		I	II	III	IV	V	VI	VII
Persons	A	9	6	12	11	13	15	11
	B	14	13	14	14	10	20	15
	C	18	6	17	11	15	13	11
	D	10	11	12	15	15	14	13
	E	15	6	18	15	10	14	12
	F	9	18	15	20	14	13	11
	G	14	15	12	13	11	17	20

Determine the optimal assignment schedule.

5. Solve the following unbalanced assignment problems :

(a)

		Machines				
		M1	M2	M3	M4	M5
Jobs	J1	12	15	15	13	14
	J2	8	14	11	15	8

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J3	15	13	11	11	13
J4	8	12	8	11	10

(b)

		Machines			
		M1	M2	M3	M4
Jobs	J1	16	17	14	19
	J2	15	16	14	9
	J3	10	15	9	9
	J4	10	13	10	13
	J5	15	9	14	18

6. There are five operators and six machines in a machine shop. The assignment costs are given in the table below :

		Machine					
		M1	M2	M3	M4	M5	M6
Operator	A	5	-	22	6	8	6
	B	14	9	15	9	14	15
	C	8	12	12	10	8	5
	D	11	13	11	6	9	14
	E	8	9	11	13	-	12

Operator A cannot operate machine M2 and operator E cannot operate machine M5. Find the optimal assignment schedule.

7. A batch of 4 jobs can be assigned to 5 different machines. The setup time for each job on various machines is given below :

		Machine				
		1	2	3	4	5
Jobs	J1	3	9	6	5	6
	J2	4	5	5	7	4
	J3	5	5	3	4	4
	J4	6	8	4	5	5

Find an optimal assignment of jobs to machines which will minimize the total setup time.

8. A construction company has to move six large cranes from old construction sites to new construction sites. The distances (in miles) between the old and the new sites are given below :

		New sites				
		A	B	C	D	E
Old sites	I	12	9	8	11	7
	II	11	10	8	12	7
	III	9	12	7	6	9
	IV	9	8	11	10	10
	V	10	9	9	6	11
	VI	11	11	7	8	9

Determine a plan for moving the cranes such that the total distance involved in the move will be minimum.

9. A company wants to assign five salesperson to five different regions to promote a product. The expected sales (in thousand) are given below :

	Regions				
	I	II	III	IV	V
S1	27	54	37	100	85
S2	55	66	45	80	32
Salesperson S3	72	58	74	80	85
S4	39	88	74	59	72
S5	72	66	45	69	85

Solve the above assignment problem to find the maximum total expected sale.

10. A company makes profit (Rs.) while processing different jobs on different machines (one machine to one job only). Now, the company is facing problem of assigning 4 machines to 5 jobs. The profits are estimated as given below :

	Job				
	J1	J2	J3	J4	J5
A	21	16	35	42	16
B	15	20	30	35	15
Machine C	20	16	30	27	18
D	15	18	32	27	15

Determine the optimal assignment for maximum total profits.

## ANSWERS

- (a) I→A, II→E, III→D, IV→C, V→B. Min. cost = 38.

(b) A→J6, B→J2, C→J1, D→J3, E→J5, F→J4. Min. cost = 91.
- A→A2, B→A4, C→A1, D→A6, E→A3, F→A5

A→A2, B→A4, C→A6, D→A3, E→A1, F→A5

A→A2, B→A4, C→A6, D→A3, F→A5, E→A1

For each assignment, min. cost = Rs. 6300.
- A→P3, B→P4, C→P5, D→P1, E→P2

A→P4, B→P3, C→P5, D→P1, E→P2

Min. total time = 342 units.
- A→VII, B→V, C→IV, D→I, E→II, F→VI, G→III

A→VII, B→V, C→IV, D→VI, E→II, F→I, G→III

A→I, B→V, C→IV, D→VI, E→II, F→VII, G→III

Min. total cost = 73.
- (a) J1→M1, J2→M5, J3→M4, J4→M3, M2 is idle.

Min. total cost = 39.

(b) J2→M4, J3→M3, J4→M1, J5→M2, J1 is not processed.

Min. total cost = 37.

NOTES

**NOTES**

- 6. A→M5, B→M2, C→M6, D→M4, E→M1. M3 is idle.  
Min. total cost = 36.
- 7. J1→1, J2→5, J3→3, J4→4  
J1→1, J2→5, J3→4, J4→3  
Min. total time = 15, Machine 2 is idle.
- 8. I→E, III→A, IV→B, V→D, VI→C  
Min. total distance = 37 miles  
Crane II is not moved.
- 9. S1→IV, S2→I, S3→III, S4→II, S5→5, Max. total profit = Rs. 40200.
- 10. A→J4, B→J2, C→J1, D→J3, Job J5 is idle.  
Max. total profit = Rs. 114.

**ROUTING PROBLEMS**

There are various types of routing problems which occurs in a network. The most widely discussed problem is the 'Travelling Salesman Problem (TSP)': Suppose there is a network of  $n$  cities and a salesman wants to make a tour *i.e.*, starting from a city 1 he will visit each of the other  $(n - 1)$  cities once and will return to city 1. In this tour the objective is to minimize either the total distance travelled or the cost of travelling by the salesman.

**(a) Mathematical Formulation**

Let the cities be numbered as 1, 2, ...,  $n$  and the distance matrix as follows:

$$D = \begin{matrix} & \begin{matrix} \text{To} \\ \text{From} \end{matrix} & \begin{matrix} 1 & 2 & \dots & n \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ \vdots \\ n \end{matrix} & \begin{bmatrix} d_{11} & d_{12} & \dots & d_{1n} \\ d_{21} & d_{22} & \dots & d_{2n} \\ \dots & \dots & \dots & \dots \\ d_{n1} & d_{n2} & \dots & d_{nn} \end{bmatrix} \end{matrix}$$

Generally an infinity symbol is placed in the principal diagonal elements where there is no travelling. So  $d_{ij}$  represents the distance from city  $i$  to city  $j$  ( $i \neq j$ ). If the cost of travelling is considered then D is referred as cost matrix. It is also to be noted that D may be symmetric in which case the problem is called 'Symmetric TSP' or asymmetric in which case the problem is called 'Asymmetric TSP'.

Let us define the decision variables as follows :

$$x_{ij} = \begin{cases} 1, & \text{if he travels from city } i \text{ to city } j \\ 0, & \text{otherwise} \end{cases}$$

where  $i, j = 1, 2, \dots, n$

Then the linear programming formulation can be stated as follows :

Minimize 
$$z = \sum_{i=1}^n \sum_{j=1}^n d_{ij} x_{ij}$$

Subject to,

$$\sum_{i=1}^n x_{ij} = 1, \quad j = 1, 2, \dots, n$$

$$\sum_{j=1}^n x_{ij} = 1, \quad i = 1, 2, \dots, n$$

$$x_{ij} = 0 \text{ or } 1 \text{ for all } i \text{ and } j = 1, 2, \dots, n$$

and

$$x = (x_{ij}) \text{ is a tour.}$$

The above problem has been solved with various approaches e.g., Graph Theoretic Approach, Dynamic Programming, Genetic algorithm etc.

The above problem looks like a special type of Assignment problem. Consider a  $4 \times 4$  assignment problem and a solution as 1 - 4, 2 - 3, 3 - 1, 4 - 2 which can also be viewed as a tour i.e., 1 - 4 - 2 - 3 - 1. If the solution is 1 - 4, 2 - 3, 3 - 2, 4 - 1 then this consists of two sub-tours 1 - 4 - 1, 2 - 3 - 2.

Here one algorithm known as 'Branch and Bound' algorithm is described below :

### (b) Branch and Bound Algorithm for TSP

- (i) Ignoring tour, solve [D] using Hungarian Algorithm. The transformed matrix is denoted as  $[D_0]$ . If there is a tour, stop, else goto next step while storing the solution in a node denoted by TSP.
- (ii) Calculate the **evaluation** for the variables in  $[D_0]$  whose values are zero i.e.,  $x_{ij} = 0$  where evaluation means the sum of smallest elements of the  $i$ -th row and the  $j$ -th column excluding the  $(i, j)$ th entry.
- (iii) Select the variable with highest evaluation, say  $x_{ij}$ . If there is a tie, break it arbitrarily. The variable  $x_{ij}$  is called the branching variable.
- (iv) Create a left branch (TSP1) with  $x_{ij} = 0$ . To implement this put  $d_{ij} = \infty$  in  $[D_0]$  i.e., travelling from city  $i$  to city  $j$  is restricted.  
Set  $[D] =$  transformed  $[D_0]$  and goto step (i).
- (v) Create a right branch (TSP2) with  $x_{ij} = 1$ . This means the salesman must visit city  $j$  from city  $i$ . To implement this take  $[D_0]$  of the parent node. Delete the  $i$ -th row and  $j$ -th column and put  $d_{ji} = \infty$  (to prevent a subtour).  
Set  $[D] =$  transformed  $[D_0]$  and goto step (i).

Note.

- (a) There may be a situation arises in step (i) where further solution is not possible then we shall stop that branch.
- (b) There may be multiple tours. We shall select the tour with minimum distance or travelling cost.
- (c) Calculate total distance (TD) from the given [D] which increases with the level of the tree.

**Example 6.** Solve the following travelling salesman problem using branch and bound algorithm.

NOTES

NOTES

$$D = \begin{matrix} & \begin{matrix} \text{To} \\ 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} \text{From} \\ 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} \infty & 3 & 6 & 5 \\ 3 & \infty & 5 & 8 \\ 6 & 5 & \infty & 2 \\ 5 & 8 & 2 & \infty \end{bmatrix} \end{matrix}$$

**Solution.** Let us apply the Hungarian Algorithm on [D] and obtain the following matrix :

$$D_0 = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} \infty & \textcircled{0} & 3 & 2 \\ \textcircled{0} & \infty & 2 & 5 \\ 4 & 3 & \infty & \textcircled{0} \\ 3 & 6 & \textcircled{0} & \infty \end{bmatrix} \end{matrix}$$

The solution is 1 - 2, 2 - 1, 3 - 4, 4 - 3. i.e., there exists two subtours 1 - 2 - 1, 3 - 4 - 3. The total distance (TD) = 3 + 3 + 2 + 2 = 10 units.

Then we have to calculate the evaluations for the variables having the value zero in [D<sub>0</sub>].

Variable	Evaluation
$x_{12}$	$2 + 3 = 5$
$x_{21}$	$2 + 3 = 5$
$x_{34}$	$3 + 2 = 5$
$x_{43}$	$3 + 2 = 5$

Since there are ties in the values, let us select  $x_{12}$  as branching variable.

**Subproblem TSP1**

Let  $x_{12} = 0 \Rightarrow$  Put  $d_{12} = \infty$  in [D<sub>0</sub>] and obtain

$$D_0 = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} \infty & \infty & 3 & 2 \\ 0 & \infty & 2 & 5 \\ 4 & 3 & \infty & 0 \\ 3 & 6 & 0 & \infty \end{bmatrix} \end{matrix}$$

$$\Downarrow$$

$$\begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} \infty & \infty & 1 & \textcircled{0} \\ \textcircled{0} & \infty & 2 & 5 \\ 4 & \textcircled{0} & \infty & 0 \\ 3 & 1 & \textcircled{0} & \infty \end{bmatrix} \end{matrix}$$

(Apply Hungarian Algorithm)

The solution is 1 - 4, 2 - 1, 3 - 2, 4 - 3 i.e., 1 - 4 - 3 - 2 - 1 which is a tour and TD = 5 + 3 + 5 + 2 = 15 units from [D].

### Subproblem TSP2

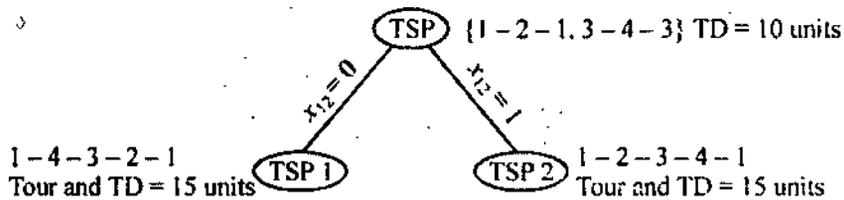
Let  $x_{12} = 1 \Rightarrow$  Delete row 1 and column 2, from  $[D_0]$  and put  $d_{21} = \infty$  to prevent subtour. The resultant transformed matrix is obtained as follows :

	1	3	4	
2	$\infty$	2	5	
3	4	$\infty$	0	
4	3	0	$\infty$	

↓

	1	3	4	
2	$\infty$	①	3	
3	1	$\infty$	①	(Applying Hungarian Algorithm)
4	①	0	$\infty$	

The solution is 1 - 2, 2 - 3, 3 - 4, 4 - 1 i.e., 1 - 2 - 3 - 4 - 1 which is a tour and TD = 3 + 5 + 2 + 5 = 15 units from [D]. The above calculations is presented in the following tree diagram :



Since, there are two tours with same TD, the given problem has two solutions.

### SUMMARY

- This is an efficient algorithm for solving the assignment problem developed by the Hungarian mathematician König. Here the optimal assignment is not affected if a constant is added or subtracted from any row or column of the balanced assignment cost matrix.
- When the objective of the assignment is to maximize, the problem is called 'Max-type assignment problem'. This is solved by converting the profit matrix to an opportunity loss matrix by subtracting each element from the highest element of the profit matrix.
- There are various types of routing problems which occurs in a network. The most widely discussed problem is the 'Travelling Salesman Problem (TSP)'.

NOTES

**PROBLEMS**

Solve the following travelling salesman problems using branch and bound algorithm :

NOTES

1.

		To			
		1	2	3	4
From	1	$\infty$	10	6	4
	2	8	$\infty$	5	8
	3	7	5	$\infty$	2
	4	4	10	2	$\infty$

2.

		To				
		1	2	3	4	5
From	1	$\infty$	4	9	5	10
	2	4	$\infty$	7	6	8
	3	10	6	$\infty$	5	4
	4	5	6	5	$\infty$	3
	5	8	7	4	3	$\infty$

		To				
		1	2	3	4	5
From	1	$\infty$	5	2	7	4
	2	5	$\infty$	3	5	6
	3	2	3	$\infty$	4	1
	4	7	5	4	$\infty$	3
	5	4	6	1	3	$\infty$

**ANSWERS**

1. 1 - 4 - 3 - 2 - 1, TD = 19 units.
2. 1 - 4 - 5 - 3 - 2 - 1, TD = 22 units.
3. 1 - 3 - 5 - 4 - 2 - 1 and 1 - 2 - 4 - 5 - 3 - 1, TD = 16 units.

## CHAPTER 4 QUEUEING THEORY

### ★ STRUCTURE ★

- Introduction
- Basic Elements of Queueing Model
- Notations
- Kendall's Notation
- Queueing Models Based on Birth-and-Death Processes
- M/M/1 : Fifo/ $\infty$  Model
- M/M/1 : Fifo/N Model
- M/M/S : Fifo/ $\infty$  Model
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- Non-Poisson Queueing Models
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## INTRODUCTION

Queueing systems are prevalent throughout society. The formation of waiting lines is a common phenomenon which occurs whenever the current demand for a service exceeds the current capacity to provide that service. Commuters waiting to board a bus, cars waiting at signals, machines waiting to be serviced by a repairman, letters waiting to be typed by a typist, depositors waiting to deposit the money to a counter in a bank provide some examples of queues. There are applications of queueing theory in several disciplines. A schematic diagram of a queueing system is given below :

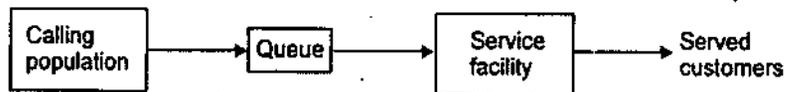


Fig. 4.1 The basic queueing system

## BASIC ELEMENTS OF QUEUEING MODEL

The basic elements of a queueing model depend on the following factors :

(a) *Arrival's distribution.* Customers arrive and join in the queue according to a probability distribution. The arrival may be single or bulk.

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(b) *Service-time distribution.* The service offered by the server also follows a probability distribution. The server(s) may offer single or bulk services e.g., one man barber shop, a computer with parallel processing.

(c) *Design of service facility.* The services can be offered by the servers in a series, parallel or network stations. A facility comprise a number of series stations through which the customer may pass for service is called 'Tandem queues'. Waiting lines may or may not be allowed between the stations. Similarly parallel queue and network queue are defined.

(d) *Service discipline/Queue discipline.* There are three types of discipline e.g.,

FIFO – First In First Out

LIFO – Last In First Out

SIRO – Service in Random Order

'Stack' is an example of LIFO and selling tickets in a bus is an example of SIRO. Sometimes FIFO is referred as GD (*i.e.*, General Discipline).

Also there is priority service which is two types.

Preemptive. The customers of high priority are given service over the low priority customer.

Non-preemptive. A customer of low priority is served before a high priority customer.

(e) *Queue Size.* Generally it is referred as length of the queue or line length. Queue size may be finite or infinite (*i.e.*, a very large queue). Queue size along with the server(s) form the capacity of the system.

(f) *Calling Population.* It is also called calling source. Customers join in the queue from a source is known as calling population which may be finite or infinite (*i.e.*, a very large number). To reserve a ticket in a railway reservation counter, customers may come from anywhere of a city. Then the population of the city forms the calling population which can be considered as infinite.

(g) *Human behaviour.* In a queueing system three types of human behaviours are observed.

Jockeying – If one queue is shorter then one join from a larger queue to it.

Balking – If the length of the queue is large, one decides not to enter into it.

Reneging – When a person becomes tired *i.e.*, loses patience on standing on a queue, the person leaves the queue.

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## NOTATIONS

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$P_n$  = Probability of  $n$  customers in a system (steady state)

$P_n(t)$  = Probability of  $n$  customers at time  $t$  in a system (transient state)

$L_s$  = Expected number of customers in a system

$L_q$  = Expected number of customers in queue

$W_s$  = Expected waiting time in a system (in queue + in service)

$W_q$  = Expected waiting time in queue

$\lambda_n$  = Mean arrival rate when  $n$  customers are in the system.

(If  $\lambda_n$  = Constant for all  $n$ , this constant is denoted by  $\lambda$ )

$\mu_n$  = Mean service rate for overall system when  $n$  customers are in the system.

(If  $\mu_n$  = Constant for all  $n$ , this constant is denoted by  $\mu$ )

$\rho$  =  $\frac{\lambda}{\mu}$  = Traffic intensity/utilization factor

$s$  = Number of servers

$N(t)$  = Number of customers in queuing system at time  $t$ .

It has been known that  $L_s = \lambda W_s$ ,  $L_q = \lambda W_q$ . These relations are called 'Little's formula'. Also we have,

$$W_s = W_q + \frac{1}{\mu}, \text{ for } n \geq 1$$

$$\text{and } L_s = L_q + \frac{\lambda}{\mu}$$

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## KENDALL'S NOTATION

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A convenient notation to denote queuing system is as follows :

*a/b/c : d/e/f*

where

$a$  = Arrivals' distribution

$b$  = Service time distribution

$c$  = Number of servers or channels

$d$  = Service discipline

$e$  = System capacity

$f$  = Calling population.

In the above,  $a$  and  $b$  usually take one of the following distribution with its symbol:

M = Exponential distribution for inter-arrival or service time and Poisson arrival.

$E_k$  = Erlangian or gamma distribution with parameter  $k$ .

D = Constant or deterministic inter-arrival or service time.

G = General distribution (of service time)

Generally  $f$  is taken as  $\infty$  (infinity) and it is omitted while representing a queue.

## NOTES

## QUEUEING MODELS BASED ON BIRTH-AND-DEATH PROCESSES

### NOTES

The assumptions of the birth-and-death processes are the following :

- (a) Given  $N(t) = n$ , ( $n = 0, 1, 2, \dots$ ), the current probability distribution of the remaining time until the next arrival is exponential with parameter  $\lambda_n$ .
- (b) Given  $N(t) = n$ , ( $n = 1, 2, \dots$ ), the current probability distribution of the remaining time until the next service completion is exponential with parameter  $\mu_n$ .
- (c) Only one birth or death can occur at a time.

In the queueing models, arrival of a customer implies a birth and departed customer implies a death. In queueing system both arrivals and departures take place simultaneously. This makes difference from birth-and-death process. In the following queueing models mean arrival rate and mean service rate are constant.

### M/M/1 : FIFO/ $\infty$ MODEL

In this queueing model, arrivals and departures are Poisson with rates  $\lambda$  and  $\mu$  respectively. There is one server and the capacity of the system is infinity i.e., very large. We shall derive the steady-state probabilities and other characteristics.

Let  $P_n(t) =$  Probability of  $n$  arrivals during time interval  $t$

If  $h > 0$  and small then

$$P_n(t+h) = P(n \text{ arrivals during } t \text{ and none during } h)$$

or

$$P(n-1 \text{ arrivals during } t \text{ and one during } h)$$

or

$$P(n+1 \text{ arrivals during } t \text{ and one departure during } h)$$

$$\left\{ \text{For Poisson process, } P\{X=n\} = \frac{(\alpha t)^n \cdot e^{-\alpha t}}{n!} \right\}$$

Now,

$$P(\text{zero arrival in } h) = e^{-\lambda h} \approx 1 - \lambda h$$

$$P(\text{one arrival in } h) = 1 - e^{-\lambda h} \approx \lambda h$$

Similarly,

$$P(\text{zero departure in } h) = h e^{-\mu h} \approx 1 - \mu h$$

$$P(\text{one departure in } h) = 1 - e^{-\mu h} \approx \mu h$$

Then for  $h > 0$ , we can write

$$P_n(t+h) = P_n(t) \cdot (1 - \lambda h)(1 - \mu h) + P_{n-1}(t)(\lambda h)(1 - \mu h) + P_{n+1}(t)(1 - \lambda h)(\mu h)$$

$$\Rightarrow \frac{P_n(t+h) - P_n(t)}{h} = \lambda P_{n-1}(t) + \mu P_{n+1}(t) - (\lambda + \mu) P_n(t)$$

Taking  $h \rightarrow 0$ , we obtain

$$P'_n(t) = \lambda P_{n-1}(t) + \mu P_{n+1}(t) - (\lambda + \mu) P_n(t) \quad \dots(1)$$

For  $n = 0$ , we can write

$$P_0(t+h) = P_0(t)(1 - \lambda h) + P_1(t)(1 - \lambda h)(\mu h)$$

$$\Rightarrow \frac{P_0(t+h) - P_0(t)}{h} = -\lambda P_0(t) + \mu P_1(t)$$

Taking  $h \rightarrow 0$ , we obtain

$$P'_0(t) = -\lambda P_0(t) + \mu P_1(t) \quad \dots(2)$$

These are called difference-differential equations.

The solution of (1) and (2) will give the transient-state probabilities,  $P_n(t)$ . But the solution procedure is complex. So with certain assumptions we shall obtain the steady state solution.

For steady state, let us consider

$$t \rightarrow \infty, \lambda < \mu,$$

$$P'_n(t) \rightarrow 0, P_n(t) \rightarrow P_n \text{ for } n = 0, 1, 2, \dots$$

(Here  $\lambda = \mu \Rightarrow$  No queue and  $\lambda > \mu \Rightarrow$  explosive state).

From (1) and (2) we obtain,

$$\lambda P_{n-1} + \mu P_{n+1} - (\lambda + \mu) P_n = 0, \quad n > 0 \quad \dots(3)$$

$$-\lambda P_0 + \mu P_1 = 0, \quad n = 0 \quad \dots(4)$$

From (4),  $P_1 = \frac{\lambda}{\mu} P_0$

From (3), for  $n = 1$ ,

$$\begin{aligned} P_2 &= \left(\frac{\lambda + \mu}{\mu}\right) P_1 - \frac{\lambda}{\mu} P_0 \\ &= \left(\frac{\lambda + \mu}{\mu}\right) \left(\frac{\lambda}{\mu}\right) P_0 - \left(\frac{\lambda}{\mu}\right) P_0 \\ &= \left(\frac{\lambda}{\mu}\right)^2 P_0 \end{aligned}$$

Similarly, for  $n = 2, 3, \dots$

$$P_3 = \left(\frac{\lambda}{\mu}\right)^3 P_0$$

$$P_n = \left(\frac{\lambda}{\mu}\right)^n P_0$$

Also, we have

$$\sum_{n=0}^{\infty} P_n = 1$$

$$\Rightarrow P_0 \sum_{n=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^n = 1$$

NOTES

## NOTES

$$= P_0 \left[ 1 - \frac{\lambda}{\mu} \right]^{-1} = 1, \frac{\lambda}{\mu} < 1$$

$$= P_0 = \frac{1}{1 - \frac{\lambda}{\mu}} = \frac{1}{1 - \rho}, \rho < 1.$$

Therefore, 
$$P_n = P[X=n] = \left( \frac{\lambda}{\mu} \right)^n \left( 1 - \frac{\lambda}{\mu} \right)$$

$$= P^n (1 - \rho), \rho < 1, n \geq 0.$$

Now  $L_s$  = Expected number of customers in the system.

$$= \sum_{n=0}^{\infty} n \cdot P_n = \sum_{n=0}^{\infty} n \cdot \rho^n (1 - \rho)$$

$$= (1 - \rho) \rho \sum_{n=0}^{\infty} n \cdot \rho^{n-1} = (1 - \rho) \rho \frac{d}{d\rho} \left( \sum_{n=0}^{\infty} \rho^n \right)$$

$$= (1 - \rho) \rho \cdot \frac{d}{d\rho} \left( \frac{1}{1 - \rho} \right) (\because \rho < 1)$$

$$= (1 - \rho) \rho \cdot \frac{1}{(1 - \rho)^2} = \frac{\rho}{1 - \rho} = \frac{\lambda}{\mu - \lambda}$$

$W_s$  = Expected waiting time in the system

$$= \frac{L_s}{\lambda} \text{ (By Little's formula)} = \frac{1}{\mu - \lambda}$$

$L_q$  = Average queue length

$$= L_s - \frac{\lambda}{\mu} = \frac{\rho^2}{1 - \rho}$$

$W_q$  = Expected waiting time in queue

$$= \frac{L_q}{\lambda} \text{ (By Little's formula)} = \frac{\rho}{\mu(1 - \rho)} = \frac{\lambda}{\mu(\mu - \lambda)}$$

**P (at least  $n$  customers in the system)**

$$= P(\text{queue size} \geq n)$$

$$= \sum_{j=n}^{\infty} P_j = \sum_{j=n}^{\infty} \rho^j (1 - \rho)$$

$$= (1 - \rho) \rho^n \sum_{j=n}^{\infty} \rho^{j-n} = (1 - \rho) \rho^n \sum_{k=0}^{\infty} \rho^k \text{ (let } k = j - n)$$

$$= (1 - \rho) \rho^n \cdot \frac{1}{1 - \rho} = \rho^n.$$

Let  $m$  = No. of customers in the queue

$$\begin{aligned} P[m > 0] &= P[n > 1] = 1 - P[n \leq 1] \\ &= 1 - \{P[n=0] + P[n=1]\} \\ &= 1 - \{(1-\rho) + \rho(1-\rho)\} = \rho^2. \end{aligned}$$

NOTES

Therefore, Average length of non-empty queue

$$\begin{aligned} &= E[m|m > 0] \\ &= \frac{E(m)}{P[m > 0]} = \frac{L_q}{P[m > 0]} = \frac{\rho^2}{1-\rho} \cdot \frac{1}{\rho^2} = \frac{1}{1-\rho}. \end{aligned}$$

Variance of system length/Fluctuation of queue

$$\begin{aligned} &= \sum_{n=0}^{\infty} [n - L_s]^2 P_n = \sum_{n=0}^{\infty} n^2 \cdot P_n - [L_s]^2 \\ &= \frac{\rho}{(1-\rho)^2}, \text{ after simplification.} \end{aligned}$$

### (a) Waiting Time Distributions

Let the time spent by a customer in the system be given as follows

$$T_s = t_1 + t_2 + \dots + t_n + t_{n+1}$$

where  $t_1$  is the additional time taken by the customer in service,  $t_2, \dots, t_n$  are the service times of other customers ahead of him and  $t_{n+1}$  is the service time of arriving customer. Here  $T_s$  is the sum of  $(n+1)$  independently identically exponentially distributed random variables and follows a gamma distribution with parameters  $\mu$  and  $n+1$ . The conditional pdf  $w(t | n+1)$  of  $T_s$  is given by

$$w(t | n+1) = \frac{\mu}{n!} (\mu t)^n \cdot e^{-\mu t}, \quad t > 0.$$

Then the pdf of  $T_s$  is obtained by first multiplying the expression  $w(t | n+1)$  with the probability that there are  $n$  customers in the system and then summing over all values of  $n$  from 0 to  $\infty$  and is given below :

$$\text{pdf of } T_s = (\mu - \lambda) e^{-(\mu - \lambda)t}, \quad t > 0$$

which is an exponential distribution with parameter  $(\mu - \lambda)$ . We can also compute the pdf  $T_q$  of waiting time of an incoming customer before he receives the service following a similar line of argument. Thus

$$\text{pdf of } T_q = \begin{cases} \rho(\mu - \lambda) e^{-(\mu - \lambda)t}, & t > 0 \\ 1 - \rho & t = 0 \end{cases}$$

The second component means the customer starts receiving service immediately after the arrival if there is no customer in the system.

Also we can obtain

$$W_s = \int_0^{\infty} t \cdot T_s dt = \frac{1}{\mu - \lambda}, \quad W_q = \int_0^{\infty} t \cdot T_q dt = \frac{\lambda}{\mu(\mu - \lambda)}.$$

## NOTES

**Example 1.** At a public telephone booth arrivals are considered to be Poisson with an average inter-arrival time of 10 minutes. The length of a phone call may be treated as service, assumed to be distributed exponentially with mean = 2.5 minutes. Calculate the following :

- Average number of customers in the booth
- Probability that a fresh arrival will have to wait for a phone call.
- Probability that a customer completes the phone call in less than 10 minutes and leave.
- Probability that queue size exceeds at least 5.

**Solution.** Here  $\lambda = \frac{1}{10}$  customers per minute.

$$\mu = \frac{1}{2.5} \text{ customers per minute.}$$

and 
$$\rho = \frac{\lambda}{\mu} = \frac{2.5}{10} = 0.25 < 1$$

- Average number of customers in the booth

$$= L_s = \frac{\rho}{1-\rho} = \frac{0.25}{1-0.25} = 0.33$$

- P (a fresh arrival will have to wait)

$$\begin{aligned} &= 1 - P(\text{a fresh arrival will not have to wait}) \\ &= 1 - P(\text{no customers in the booth}) \\ &= 1 - P[X=0] \\ &= 1 - (1 - \rho) = \rho = 0.25 \end{aligned}$$

- P (phone call completes in less than 10 min.)

$$\begin{aligned} &= P\{T_s < 10\} \\ &= \int_0^{10} (\mu - \lambda) e^{-(\mu - \lambda)t} dt = 1 - e^{-(\mu - \lambda)10} = 0.95 \end{aligned}$$

**Example 2.** At a one-man barber shop, customers arrive according to the Poisson distribution with a mean arrival rate of 4 per hour and his hair cutting time was exponentially distributed with an average hair cut taking 12 minutes. There is no restriction in queue length. Calculate the following :

- Expected time in minutes that a customer has to spend in the queue.
- Fluctuations of the queue length.
- Probability that there is at least 5 customers in the system.
- Percentage of time the barber is idle in 8-hr. day.

**Solution.**  $\lambda = 4$  per hour =  $\frac{1}{15}$  per minute.

$$\mu = \frac{1}{12} \text{ per minute.}$$

$$\rho = \frac{\lambda}{\mu} = \frac{12}{15} = 0.8 < 1$$

$$(a) \quad W_q = \frac{\lambda}{\mu(\mu - \lambda)} = \frac{1/15}{\frac{1}{12} \left( \frac{1}{12} - \frac{1}{15} \right)} = 48 \text{ minutes}$$

$$(b) \text{ Fluctuations of queue length} = \frac{\rho}{(1 - \rho)^2} = \frac{0.8}{(0.2)^2} = 20$$

$$(c) P(\text{at least 5 customers in the system}) = \rho^5 = (0.8)^5 = 0.33$$

$$(d) P(\text{barber is idle}) = P(\text{no customers in the shop}) \\ = 1 - \rho = 1 - 0.8 = 0.2$$

$$\text{Percentage of time barber is idle} = 8 \times 0.2 = 1.6.$$

**Example 3.** In the Central Railway station 15 computerised reservation counters are available. A customer can book the ticket in any train on any day in any one of these counters. The average time spent per customer by each clerk is 5 minutes. Average arrivals per hour during three types of activity periods have been calculated and customers have been surveyed to determine how long they are willing to wait during each type of period.

Type of period	Arrivals per hr.	Customer's acceptable waiting time
Peak	110	15 Minutes
Normal	60	10 Minutes
Low	30	5 Minutes

Making suitable assumptions on this queuing process, determine how many counters should be kept open during each type of period.

**Solution.** Assumptions are as follows :

- (i) Arrivals follow Poisson distribution with average arrival rate  $\lambda$ .
- (ii) Service time follows exponential distribution with average service rate  $\mu$ .
- (iii) The given system can be considered as  $m$  different single server queuing system ( $m \leq 15$ ) and there is no jockeying and balking.

$$\begin{aligned} \text{Then the arrival rate } \lambda \text{ (per hour)} &= 100/m \text{ (peak period)} \\ &= 60/m \quad \text{(normal period)} \\ &= 30/m \quad \text{(low period)} \end{aligned}$$

$$\text{The service rate at all the periods} = \frac{60}{5} = 12 \text{ per hour}$$

NOTES

Peak period : 
$$W_q = \frac{15}{60} = \frac{100/m}{12(12 - 100/m)} \Rightarrow m = 12.22$$

Hence, 13 counters must be kept open to ensure that the average waiting time does not exceed 15 minutes.

**NOTES**

Normal period : 
$$W_q = \frac{10}{60} = \frac{60/m}{12(12 - 60/m)} \Rightarrow m = 7.5$$

Hence 8 counters must be kept open during the normal period so that the waiting time does not exceed 10 minutes.

Low period : 
$$W_q = \frac{5}{60} = \frac{30/m}{12(12 - 30/m)} \Rightarrow m = 5$$

Hence, 5 counters must be kept open during the low period so that the waiting time does not exceed 5 minutes.

**M/M/1 : FIFO/N MODEL**

In this model the system capacity is restricted to N. Therefore, (N + 1)th customer will not join and the difference-differential equations of the previous model are valid if  $n < N$ . Then for  $n = N$ , we have

$$P_N(t + h) = P_N(t)(1 - \mu h) + P_{N-1}(t)(\lambda h)(1 - \mu h)$$

On simplification, the additional difference-differential equation is obtained as

$$P'_N(t) = -\mu P_N(t) + \lambda P_{N-1}(t).$$

Under steady-state this equation reduces to

$$0 = -\mu P_N + \lambda P_{N-1}.$$

Hence, we have three difference equations

$$\lambda P_{n+1} = (\lambda + \mu)P_n - \lambda P_{n-1}, \quad 1 \leq n \leq N-1$$

$$\mu P_1 = \lambda P_0, \quad n = 0$$

and

$$\mu P_N = \lambda P_{N-1}, \quad n = N.$$

As before, the first two equations give

$$P_n = \left(\frac{\lambda}{\mu}\right)^n P_0, \quad n \leq N-1$$

$\Rightarrow$

$$P_n = \rho^n P_0.$$

This equation satisfies the third difference equation for  $n = N$ .

To determine  $P_0$ , we use,  $\sum_{n=0}^N P_n = 1$ .

$$\Rightarrow 1 = P_0 \sum_{n=0}^N \rho^n = \begin{cases} P_0 \left( \frac{1 - \rho^{N+1}}{1 - \rho} \right), & \rho \neq 1 \\ P_0 (N + 1), & \rho = 1 \end{cases}$$

## NOTES

$$\Rightarrow P_0 = \begin{cases} \frac{1-\rho}{1-\rho^{N+1}}, & \rho \neq 1 \\ \frac{1}{N+1}, & \rho = 1 \end{cases}$$

$$\text{Hence } P_n = \begin{cases} \frac{(1-\rho)\rho^n}{1-\rho^{N+1}}, & \rho \neq 1 \\ \frac{1}{N+1}, & \rho = 1 \end{cases}$$

So in this model,  $\rho$  can be  $> 1$  or  $< 1$ .

$$L_s = \sum_{n=0}^N n \cdot P_n$$

For  $\rho \neq 1$ ,

$$\begin{aligned} L_s &= P_0 \cdot \sum_{n=0}^N n \cdot \rho^n \\ &= P_0 \cdot \rho \sum_{n=0}^N n \cdot \rho^{n-1} = P_0 \cdot \rho \frac{d}{d\rho} \left( \sum_{n=0}^N \rho^n \right) \\ &= P_0 \cdot \rho \frac{d}{d\rho} \left( \frac{1-\rho^{N+1}}{1-\rho} \right) \\ &= P_0 \cdot \rho \cdot \frac{(1-\rho)(-(N+1)\rho^N) - (1-\rho^{N+1})(-1)}{(1-\rho)^2} \\ &= P_0 \cdot \rho \cdot \frac{1-(N+1)\rho^N + N\rho^{N+1}}{(1-\rho)^2} \\ &= \left( \frac{1-\rho}{1-\rho^{N+1}} \right) \cdot \rho \cdot \frac{1-(N+1)\rho^N + N\rho^{N+1}}{(1-\rho)^2} \\ &= \frac{\rho[1-(N+1)\rho^N + N\rho^{N+1}]}{(1-\rho)(1-\rho^{N+1})} \end{aligned}$$

For  $\rho = 1$ ,

$$\begin{aligned} L_s &= \sum_{n=0}^N n P_n = \sum_{n=0}^N n \cdot \frac{1}{(N+1)} = \frac{1}{N+1} \sum_{n=0}^N n \\ &= \frac{1}{N+1} \cdot \frac{N(N+1)}{2} = \frac{N}{2} \end{aligned}$$

Thus

$$L_s = \begin{cases} \frac{\rho[1-(N+1)\rho^N + N\rho^{N+1}]}{(1-\rho)(1-\rho^{N+1})}, & \rho \neq 1 \\ \frac{N}{2}, & \rho = 1 \end{cases}$$

Let  $\bar{\lambda} =$  Effective arrival rate.

Here  $\lambda_n = 0$  for  $n \geq N$

**NOTES**

and 
$$\sum_{n=0}^N P_n = 1$$

$\Rightarrow P_N + \sum_{n=0}^{N-1} P_n = 1$

$\Rightarrow \sum_{n=0}^{N-1} P_n = 1 - P_N$

Therefore, 
$$\begin{aligned} \bar{\lambda} &= \sum_{n=0}^{N-1} \lambda_n P_n = \lambda \cdot \sum_{n=0}^{N-1} P_n \quad (\because \lambda_n = \lambda) \\ &= \lambda(1 - P_N) \end{aligned}$$

The other measures are obtained as follows :

$$L_s = L_q + \frac{\bar{\lambda}}{\mu} \Rightarrow L_q = L_s - \frac{\bar{\lambda}}{\mu}$$

$$W_q = \frac{L_q}{\lambda}$$

$$W_s = W_q + \frac{1}{\mu}$$

All will give two values i.e., one for  $\rho \neq 1$  and the other for  $\rho = 1$ ; e.g.,

$$L_q = \begin{cases} \frac{\rho^2 [1 - N\rho^{N-1} + (N-1)\rho^N]}{(1-\rho)(1-\rho^{N+1})}, & \rho \neq 1 \\ \frac{N(N-1)}{2(N+1)}, & \rho = 1 \end{cases}$$

**Example 4.** Assume that the trucks with goods are coming in a market yard at the rate of 30 trucks per day and suppose that the inter-arrival times follows an exponential distribution. The time to unload the trucks is assumed to be exponential with an average of 42 minutes. If the market yard can admit 10 trucks at a time, calculate P (the yard is empty) and find the average queue length.

If the unload time increases to 48 minutes, then again calculate the above two questions.

**Solution.** Here 
$$\lambda = \frac{30}{60 \times 24} = \frac{1}{48} \text{ trucks per minute.}$$

and 
$$\mu = \frac{1}{42} \text{ trucks per minute.}$$

$$N = 10, \rho = \frac{\lambda}{\mu} = \frac{42}{48} = 0.875$$

$$P(\text{yard is empty}) = P(\text{no trucks in the yard})$$

$$= P_0 = \frac{1-\rho}{1-\rho^{N+1}} = \frac{1-0.875}{1-(0.875)^{11}}$$

$$= \frac{0.125}{0.7698} = 0.16$$

$$\text{Average queue length} = \frac{\rho^2 [1 - N\rho^{N-1} + (N-1)\rho^N]}{(1-\rho)(1-\rho^{N+1})}$$

$$= \frac{(0.875)^2 [1 - 10 \cdot (0.875)^9 + 9 \cdot (0.875)^{10}]}{(1-0.875)(1-(0.875)^{11})}$$

$$= \frac{0.2765}{0.0962} = 2.87.$$

Next part :  $\mu = \frac{1}{48}$  trucks per minute

and  $\rho = \frac{\lambda}{\mu} = 1$

$$P(\text{yard is empty}) = P_0 = \frac{1}{N+1} = \frac{1}{11} = 0.09.$$

$$\text{Average queue length} = \frac{N(N-1)}{2(N+1)} = \frac{10 \times 9}{2 \times 11} = 4.09.$$

**Example 5.** Cars arrive in a pollution testing centre according to poisson distribution at an average rate of 15 cars per hour. The testing centre can accommodate at maximum 15 cars. The service time (i.e., testing time) per car is an exponential distribution with mean rate 10 per hour.

- (a) Find the effective arrival rate at the pollution testing centre.
- (b) What is the probability that an arriving car has not to wait for testing.
- (c) What is the probability that an arriving car will find a vacant place in the testing centre.
- (d) What is the expected waiting time until a car is left from the testing centre.

**Solution.**  $\lambda = 15$  cars/hour

$$\mu = 10 \text{ cars/hour}$$

$$\rho = \frac{\lambda}{\mu} = 1.5, N = 15$$

NOTES

(a)  $\bar{\lambda} = \lambda (1 - P_N) = 15(1 - P_{15}) = 15(1 - 0.333) = 10$  cars/hour.

(b) P (arriving car has not to wait for testing)

$$= P_0 = \frac{1 - 1.5}{1 - (1.5)^{16}} = 0.00076.$$

NOTES

(c)  $P_0 + P_1 + \dots + P_{14} = 1 - P_{15} = 1 - 0.333 = 0.667.$

(d)  $L_s = \frac{\rho [1 - (N+1)\rho^N + N\rho^{N+1}]}{(1-\rho)(1-\rho^{N-1})} = 13.016$

$\therefore W_s = \frac{L_s}{\lambda} = 1.301$  hours.

**M/M/S : FIFO/∞ MODEL**

In this model, the arrival rate of the customers is  $\lambda$ , but maximum of  $s$  customers can be served simultaneously.

If  $\mu$  be the average number of services per unit time per server, then we have

$$\lambda_n = \lambda, n = 0, 1, 2, \dots$$

$$\mu_n = \begin{cases} n\mu, & 0 \leq n \leq s \\ s\mu, & n \geq s \end{cases}$$

When  $n < s$ , there is no queue  $\rho = \frac{\lambda}{\mu}$ .

In this model the condition of existence of steady state solution is  $\frac{\rho}{s} < 1$ .

The steady-state probabilities are obtained as

$$P_n = \begin{cases} \frac{\rho^n}{n!} \cdot P_0, & 0 \leq n \leq s \\ \frac{\rho^n}{s^{n-s} \cdot s!} \cdot P_0, & n \geq s \end{cases}$$

where  $P_0 = \left[ \sum_{n=0}^{s-1} \frac{\rho^n}{n!} + \frac{\rho^s}{s!(1-\rho/s)} \right]^{-1}$

$$L_q = \sum_{n=s}^{\infty} (n-s)P_n = \sum_{n=s}^{\infty} (n-s) \frac{\rho^n}{s^{n-s} \cdot s!} \cdot P_0 = \frac{\rho^s}{s!} \cdot P_0 \cdot \sum_{i=0}^{\infty} i \cdot \left(\frac{\rho}{s}\right)^i \quad (\text{let } i = n - s)$$

$$= \frac{\rho^{s+1}}{s \cdot s!} \cdot P_0 \cdot \frac{d}{du} \left( \sum_{i=0}^{\infty} u^i \right) \quad (\text{let } u = \rho/s)$$

$$= \frac{\rho^{s+1}}{s \cdot s!} \cdot P_0 \cdot \frac{d}{du} \left( \frac{1}{1-u} \right) = \frac{\rho^{s+1}}{s \cdot s!} P_0 \cdot \frac{1}{(1-u)^2}$$

$$= \frac{\rho^{s+1}}{s \cdot s!} \cdot \frac{1}{\left(1 - \frac{\rho}{s}\right)^2} P_0 = \frac{\rho^{s+1}}{(s-1)! \cdot (s-\rho)^2} \cdot P_0$$

NOTES

The other measures are obtained as follows :

$$L_s = L_q + \rho, \quad W_q = L_q / \lambda, \quad W_s = W_q + \frac{1}{\mu}$$

Expected number of customers in the service =  $\rho$ .

Expected time for which a server is busy =  $\frac{\rho}{s}$

Expected time for which a server is idle =  $1 - \frac{\rho}{s}$

$$P(\text{all servers are busy}) = P_s + P_{s+1} + \dots$$

$$= \sum_{n=s}^{\infty} \frac{\rho^n}{s! \cdot s^{n-s}} \cdot P_0 = \frac{\rho^s}{s!} \left(1 - \frac{\rho}{s}\right)^{-1} \cdot P_0$$

P (an arrival has to wait) = P (all servers are busy).

**Example 6.** A post office has two counters, which handles the business of money orders, registration letters etc. It has been found that the service time distributions for both the counters are exponential with mean service time of 4 minutes per customer. The customers are found to come in each counter in a Poisson fashion with mean arrival rate of 11 per hour. Calculate

- (a) Probability of having to wait for service of a customer.
- (b) Average waiting time in the queue.
- (c) Expected number of idle counters.

**Solution.** Here

$$\lambda = 11 \text{ customers/hour.}$$

$$\mu = \frac{60}{4} = 15 \text{ customers/hour.}$$

$$s = 2$$

$$\rho = \frac{\lambda}{\mu} = \frac{11}{15}, \quad \frac{\rho}{s} = \frac{11}{30} < 1.$$

$$P_0 = \left[ \sum_{n=0}^1 \frac{\rho^n}{n!} + \frac{\rho^2}{2! \cdot (1 - \rho/2)} \right]^{-1}, \quad P_1 = \rho \cdot P_0 = 0.34$$

$$= \left[ 1 + \frac{11}{15} + \frac{(11/15)^2}{2 \cdot (1 - 11/30)} \right]^{-1} = 0.463$$

(a) P (an arrival has to wait) = P (all servers are busy)

$$= \left( \frac{11}{15} \right)^2 \cdot \frac{1}{2!} \left( 1 - \frac{11}{30} \right)^{-1} \cdot (0.463) = 0.197$$

$$(b) \quad L_q = \frac{\rho^{s+1}}{(s-1)!(s-\rho)^2} \cdot P_0$$

$$= \frac{(11/15)^3}{(2-11/15)^2} \cdot (0.463) = 0.114.$$

NOTES

∴ Average waiting time in queue ( $W_q$ ) =  $\frac{L_q}{\lambda} = \frac{0.114}{11} = 0.01 \text{ hour} = 0.62 \text{ min.}$

(c) Expected number of idle counters

$$= 2 \cdot P_0 + 1 \cdot P_1 = 1.266.$$

**M/M/S : FIFO/N, S ≤ N MODEL**

This is called *s*-server model with finite system capacity.

$$\text{Arrival rate } \lambda_n = \begin{cases} \lambda, & 0 \leq n < N \\ 0, & n \geq N \end{cases}$$

$$\text{Service rate } \mu_n = \begin{cases} n\mu, & 0 \leq n \leq s \\ s\mu, & s \leq n \leq N \end{cases}$$

$$\rho = \frac{\lambda}{\mu}$$

The steady-state probability are given as

$$P_n = \begin{cases} \frac{\rho^n}{n!} \cdot P_0 & 0 \leq n \leq s \\ \frac{\rho^n}{s! \cdot s^{n-s}} \cdot P_0 & s \leq n \leq N \end{cases}$$

where

$$P_0 = \left[ \sum_{j=0}^{s-1} \frac{\rho^j}{j!} + \sum_{j=s}^N \frac{\rho^j}{s! \cdot s^{j-s}} \right]^{-1}$$

The other characteristics of this model are given below :

$$L_q = \sum_{n=s}^N (n-s) \cdot P_n = P_0 \cdot \frac{\rho^{s+1}}{(s-1)!(s-\rho)^2} \{1 - x^{N-s} - (N-s)x^{N-s}(1-x)\}$$

where  $x = \frac{\rho}{s}$

Expected number of idle servers  $\bar{s} = \sum_{k=0}^s (s-k) \cdot P_k$ , *k* = arrival

$$\bar{\lambda} = \lambda(1 - P_N) = \mu(s - \bar{s})$$

$$W_q = \frac{L_q}{\lambda}$$

$$W_s = W_q + \frac{1}{\mu}$$

$$L_s = L_q + (s - \bar{s}).$$

Expected number of busy servers = Expected number of customers in service

$$= s - \bar{s} = \frac{\bar{\lambda}}{\mu}$$

Proportion of busy time for a server =  $\frac{s - \bar{s}}{s}$ .

### Special Cases : (I) M/M/s : FIFO/s

In this model  $s = N$ , the steady state probabilities are given by

$$P_n = \frac{\rho^n}{n!} P_0, \quad 0 \leq n \leq s$$

$$= 0, \quad n > s$$

where

$$P_0 = \left[ \sum_{n=0}^s \left( \frac{\rho^n}{n!} \right) \right]^{-1}$$

$$L_s = \sum_{n=1}^s n \cdot P_n = P_0 \sum_{n=1}^s \frac{\rho^n}{n! - 1}, \quad L_q = W_q = 0.$$

### (II) M/M/ $\infty$ : FIFO/ $\infty$ (Self-service queueing model)

In this model a customer joining the system becomes a server. So this is called self-service system. The steady state probabilities are given by

$$P_n = \frac{e^{-\rho} \cdot \rho^n}{n!}, \quad n = 0, 1, 2, \dots \text{ (Poisson distribution)}$$

$$\bar{\lambda} = \lambda, \quad L_s = \rho, \quad W_s = \frac{L_s}{\lambda} = \frac{1}{\mu}$$

$$L_q = 0, \quad W_q = 0.$$

**Example 7.** A barber shop has two barbers and four chairs for customers. Assume that customers arrive in a Poisson fashion at a rate of 4 per hour and that each barber services customers according to an exponential distribution with mean of 18 minutes. Further, if a customer arrives and there are no empty chairs in the shop he will leave. Calculate the following :

- Probability that the shop is empty.
- Effective arrival rate.

NOTES

(c) Expected number of busy servers.

(d) Expected number of customers in queue.

## NOTES

Solution. Here,  $s = 2$ ,  $N = 4$ ,  $\lambda = \frac{4}{60} = \frac{1}{15}$  customers per minute

$$\mu = \frac{1}{18} \text{ customers per minute.}$$

$$\rho = \frac{\lambda}{\mu} = \frac{18}{15} = 1.2, \quad \frac{\rho}{s} = 0.6$$

$$P_0 = \left[ 1 + \rho + \sum_{i=2}^4 \frac{\rho^i}{2i(2)^{i-2}} \right]^{-1} = [3.6112]^{-1} = 0.28$$

$$P_1 = \rho \cdot P_0 = 0.34, \quad P_4 = \frac{\rho^4 \cdot P_0}{2! \cdot 2^2} = 0.07, \quad P_3 = 0.12$$

(a) P (shop is empty) =  $P_0 = 0.28$

(b)  $\bar{\lambda} = \lambda(1 - P_4) = \frac{1}{15}(1 - 0.07) = 0.062$

(c) Expected no. of busy servers =  $\frac{\bar{\lambda}}{\mu} = 18 \times 0.062 = 1.116$

(d)  $L_q = \sum_{n=2}^4 (n-2) \cdot P_n = 1 \cdot P_3 + 2 \cdot P_4$

$$= 0.12 + 2 \cdot (0.07) = 0.26.$$

## NON-POISSON QUEUEING MODELS

In these queueing models, either arriving time distribution or service time distribution or both does not follow Poisson distribution. The results of two such models are summarize below :

### (a) $M/E_k/1$ : FIFO/ $\infty$ Model

This queueing model has single server and Poisson input process with mean arrival rate  $\lambda$ . However the service time distribution is Erlang distribution with  $k$ -phases. The density function of Erlang distribution is given as

$$f(t) = \frac{(\mu k)^k}{(k-1)!} t^{k-1} e^{-k\mu t}, \quad t \geq 0$$

where  $\mu$  and  $k$  are positive parameters and  $k$  is integer.

In this  $k$  phases of service, a new customer enters the service channel if the previous customer finishes the  $k$ -phases of the service. The characteristics of this model are listed below :

If  $\lambda$  = mean arrival rate,  $\frac{1}{k\mu}$  = Expected service time of each phase, then

$$(i) \quad P_0 = 1 - k\rho$$

$$(ii) \quad W_q = \frac{k+1}{2k} \cdot \frac{\lambda}{\mu(\mu-\lambda)}$$

$$(iii) \quad W_s = W_q + \frac{1}{\mu}$$

$$(iv) \quad L_s = \lambda \cdot W_s$$

$$(v) \quad L_q = \lambda \cdot W_q$$

**(b) M/G/1 : FIFO/ $\infty$  Model**

This queueing model has single server and Poisson input process with mean arrival rate  $\lambda$ . However there is a general distribution for the service time whose mean and variance are  $1/\mu$  and  $\sigma^2$  respectively. For this system the steady state condition is  $\rho = \frac{\lambda}{\mu} < 1$ . The characteristics of this model are given below :

$$(i) \quad P_0 = 1 - \rho$$

$$(ii) \quad L_q = \frac{\lambda^2 \sigma^2 + \rho^2}{2(1-\rho)}$$

$$(iii) \quad L_s = L_q + \rho$$

$$(iv) \quad W_q = \frac{L_q}{\lambda}$$

$$(v) \quad W_s = W_q + \frac{1}{\mu}$$

**Example 8.** A barber shop with a one-man takes exactly 20 minutes to complete one haircut. If customers arrive in a Poisson fashion at an average rate of 2 customers per hour calculate the average waiting time in the queue and expected number of customers in the shop.

**Solution.**  $\lambda = 2$  customers/hour =  $\frac{1}{30}$  customers/min.

$$\mu = \frac{1}{20} \text{ customers/min.}$$

$$\rho = \frac{\lambda}{\mu} = \frac{2}{3} < 1.$$

Since service time is constant, we can take  $k = \infty$ .

$$\therefore W_q = \lim_{k \rightarrow \infty} \frac{k+1}{2k} \cdot \frac{\lambda}{\mu(\mu-\lambda)} = \frac{1}{2} \cdot \frac{\lambda}{\mu(\mu-\lambda)} = 20 \text{ min.}$$

and 
$$L_s = \lambda \cdot W_s = \lambda \left( W_q + \frac{1}{\mu} \right) = \frac{1}{30} (20 + 20) = 1.33.$$

## SUMMARY

### NOTES

- Queue Size is referred as length of the queue or line length. Queue size may be finite or infinite (*i.e.*, a very large queue). Queue size alongwith the server(s) form the capacity of the system.
- Calling Population is also called calling source. Customers join in the queue from a source is known as calling population which may be finite or infinite (*i.e.*, a very large number).
- Human Behaviour is a Queuing System. Three types of human behaviour are observed, Jockeying, Balking and Reneging.

## PROBLEMS

1. Customers at a box office window, being manned by a single individual arrive according to a Poisson process with a rate of 30 per hr. The time taken to serve a customer has an exponential distribution with a mean of 80 sec. Find the average waiting time of a customer.
2. At a certain Petrol Pump, customers arrive according to a Poisson process with an average time of 6 min. between arrivals. The service time is exponentially distributed with mean time as 3 min. Calculate the following :
  - (a) What would be the average number of customers in the petrol pump ?
  - (b) What is the average waiting time of a car before receiving petrol ?
  - (c) The per cent of time that the petrol pump is idle.
3. A xerox machine in an office is operated by a person who does other jobs also. The average service time for a job is 6 minutes per customer. On an average, every 12 minutes one customer arrives for xeroxing. Find :
  - (a) the xerox machine utilisation.
  - (b) percentage of times that an arrival has not to wait.
  - (c) average time spent by a customer.
  - (d) average queue length.
  - (e) the arrival rate if the management is willing to deploy the person exclusively for xeroxing when the average time spent by a customer exceeds 15 minutes.
4. A repairman is to be hired to repair machines, which breakdown at an average rate of 3 per hour. Breakdowns are distributed in time in a manner that may be regarded as Poisson. Non-productive time on any one machine is considered to cost the company Rs. 5 per hour. The company has narrowed the choice to 2 repairmen—one slow but cheap, the other fast but expensive. The slow-cheap repairman asks Rs. 2 per hour, in return he will service breakdown machine exponentially at an average rate of 4 per hour. The second fast expensive repairman demands Rs. 8 per hour and will repair machines exponentially at an average rate of 5 per hour. Which repairman should be hired ? (Assume a day is 8 hours).
5. A computer manufacturing firm has a troubleshooting station that can replace a computer component in an average time of 3 minutes. Service is provided on a FIFO basis and the service rate is Poisson distributed. Arrival rates are also Poisson distributed with a mean of 18 per hour. Assuming that this is a single channel, single phase system.

## NOTES

- (a) What is the average waiting time before a component is replaced?
- (b) What is the probability that a component is replaced in less than 10 minutes?
6. A bank has two tellers working on savings accounts. The first teller handles withdrawals only while the second teller on deposits only. For both tellers the service time is exponential with mean service time 3 minutes per customer. Arrivals follow Poisson distribution with mean arrival rate of 16 per hour for depositors and 14 per hour for withdrawers respectively. What would be the effect on the average waiting time for depositors and withdrawers if each teller handles both withdrawals and deposits ?
7. Customers arrive at a bank counter manned by a single person according to a Poisson input process with a mean rate of 10 per hour. The time required to serve a customer has an exponential distribution with a mean of 4 minutes. Find :
- (a) the average number in the system.
- (b) the probability that there would be 3 customer in the queue.
- (c) the probability that the time taken by a customer in the queue is more than 3 minutes.
8. There are two booking clerks at a railway ticket counter. Passengers arrive according to Poisson process with an average rate of 176 per 8 hour day. The mean service time is 5 minutes. Find the idle time of a booking clerk in a day and the average number of customers in the queue.
9. In a cycle repair shop, the inter-arrival times of the customers are exponential with an average time of 10 minutes. The length of service time is assumed to be exponentially distributed with mean 5 minutes. Services are offered by a mechanic. Find the following :
- (a) the probability that an arrival will have to wait for more than 10 minutes before getting a service.
- (b) the probability that an arrival will have to spent at most 15 minutes in the shop.
- (c) the probability that there will be two or more customers in the shop.
10. In a petrol pump vehicles are arriving to buy petrol or diesel according to Poisson distribution with an inter-arrival time of 6 minutes for petrol and 4 minutes for diesel and two different queues are maintained. The service time for both the queues are assumed to be distributed exponentially with an average of 3 minutes. Compare the (a) average waiting time in the two queues, (b) average length of queues from time to time.
11. In a self service queuing system the customers are coming in a Poisson process with an average inter-arrival time of 8 minutes. The service time is exponential with a mean of 4 minutes. Calculate
- (a) probability that there are more than one customer in the system.
- (b) average number of customers in the system.
12. Five components of a certain machine needs to be testing by a mechanic for efficiency in production. For machines the testing is done in a Poisson fashion at an average rate of 2 per hour. Mechanic tests the components in a prescribed order. The testing times of all the five components are identical exponential distributions with mean 5 minutes.
- (a) Find the average time spent before taking service.
- (b) Find the average time the machine remains with the mechanic.

ANSWERS

## NOTES

1. 1.36 mins. per customer in queue and 2.03 mins. per customer in system.
2. (a) 1, (b) 3 min., (c) 50%
3. (a) 50% of time, (b)  $P_0 = 0.5 \Rightarrow 50\%$  of time, (c)  $W_q = 12$  minutes  
(d)  $L_q = 0.5$   
(e) If  $\lambda > 6$  customers per hr.,  $W_q$  exceeds 15 minutes.
4. Fast mechanic (Total cost = charge of mechanic + downtime cost)
5. (a)  $W_q = 27$  min., (b) 0.99
6. Before combining,  $W_q = 12$  min. for depositors and 7 min. for withdrawers. After combining,  $W_q = 3.86$  min. for both.
7. (a)  $L_s = 2$ , (b) 0.099, (c) 0.52
8. Idle time =  $\frac{2}{3}$  hrs.,  $L_q = 9.54$
9. (a) 0.18, (b) 0.78, (c) 0.25
10. (a)  $W_q$  (petrol) = 0.05,  $W_q$  (diesel) = 0.15, (b) (petrol) 2, (diesel)
11. (a) 0.09, (b) 0.5
12. M/E<sub>s</sub>/1 (i)  $W_q = 75$  min., (ii) 100 min.

## CHAPTER 5 INVENTORY CONTROL

### ★ STRUCTURE ★

- Introduction
- Notations
- Model I : Purchasing Model Without Shortage
- Model II: Purchasing Model With Shortage
- Model III: Manufacturing Model Without Shortage
- Model IV: Manufacturing Model with Shortage
- Probabilistic Models
- Summary
- Problems

### NOTES

## INTRODUCTION

The term inventory may be defined as stock in hand on a given time. Inventories of physical goods are maintained to meet up demands from commercial, governmental and military sectors. As the inventory of an item gets depleted to fulfil demands, it needs to be replenished through procurement actions. Different organizations have different inventory problems. Let us discuss the *basic terms* related to inventories.

**Demand.** It can be categorized according to their size, rate and pattern. Demand size refers to the magnitude of demand and has the dimension of quantity. Demand may be constant from period to period or may be variable. When the demand size is known, then it is called deterministic demand. When the demand size is not known, it is possible in some cases to ascertain its probability distribution and the demand is called probabilistic demand. The demand rate is simply the demand size per unit of time.

**Replenishments.** The replenishments are usually instantaneous, uniform or batch. Its size refers to the quantity or size of the order to be received into inventory which may be constant or variable.

**Lead Time.** This is the period between the time an order is placed (administrative lead time) and the time when it is received (delivery lead time). When the lead time is known, it is called deterministic. When it is not known, it can govern by a random variable.

The level of inventory of an item depends upon the length of its lead time.

NOTES

**Costs.** The main costs involved in inventory problems are described below :

(a) *Manufacturing or Purchase Cost.* The purchase cost of an item is the unit purchase price if it is obtained from an external source. The price breaks (quantity discounts) wherever allowed due to bulk purchasing must be taken into account. Manufacturing cost is the unit production cost when the item is produced internally.

(b) *Setup Cost.* This cost is incurred due to setting up of machinery before production.

(c) *Ordering Cost.* This cost originates from the expense of issuing a purchase order to the outside supplier.

(d) *Holding Cost/Inventory Carrying Cost/Storage Cost.* This cost is associated with investing in inventory and in general, it is directly proportional to the level of inventory and to the time the inventory is maintained.

(e) *Shortage Cost/Stockout Cost.* The shortage cost is the economic consequence of an external or internal shortage. An external shortage occurs when a consumer's order is not filled. An internal shortage occurs when an order of a group or department within the organization is not filled. Actually shortage cost is the penalty costs that are incurred as a result of running out of stock. It costs money, less business and goodwill loss.

(f) *Selling Price/Revenue Cost/Salvage Cost.* Actually the price and demand are not control of a company. When the company unable to meet the demand and the sale is lost then the revenue cost is included in the company's inventory policies.

**Time Horizon.** This is the period over which the inventory is controlled. It can be finite or infinite.

**Constraints.** These are the limitations placed on the inventory system such as space constraint, capital constraint etc.

**Economic Order Quantity.** This is also known as 'economic lot size' or EOQ which is the optimum quantity to be purchased or produced such that the total cost of the inventory is minimized.

**Re-order Level.** The level between the maximum and minimum stock at which purchasing (or manufacturing) activities must start for replenishment is known as re-order level.

**Order Cycle.** The time period between placement of two successive orders is referred to as order cycle.

The above basic terms highlight the various activities in inventory. Now the basic needs of an inventory are given below :

(a) It helps to run the business-smooth and efficiently.

(b) It gives the advantage of price discounts by bulk purchasing.

(c) It takes the advantage of batching and longer runs.

(d) It economizes the transportation, clearing and forwarding charges.

In inventory control attempts are made to answer the following two basic problems:

- (i) When should an order to be placed or the production to be run?
- (ii) How much quantity to be produced or to be ordered for each time interval?

## NOTES

**NOTATIONS**

- $D$  = total number of units produced or supplied per time period to meet the demand
- $Q$  = lot size in each production run/order size
- $C_s$  = setup cost per production run/ordering cost
- $c_1$  = holding cost per unit per unit time
- $c_2$  = shortage cost per unit per unit time
- $r$  = demand rate
- $k$  = production rate
- $c_A$  = Average total cost per unit time.

**MODEL I : PURCHASING MODEL WITHOUT SHORTAGE**

*Assumptions.* Demand is known and uniform, purchasing at equal interval, zero lead time, no shortages and instantaneous replenishment.

The corresponding model is shown in Fig. 5.1.

Let  $D$  = annual demand and  $c_1$  = holding cost/unit/year and time horizon = 1 year.

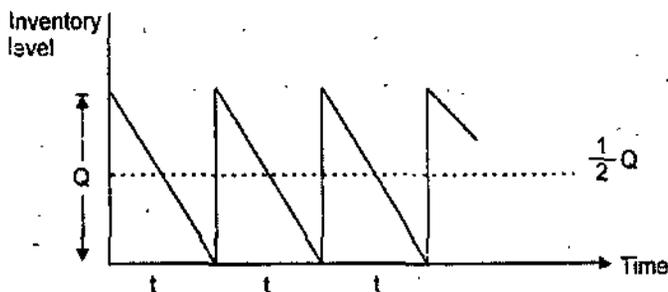


Fig. 5.1 Purchasing model without shortage

Total inventory over the time period  $t$  = Area of the first triangle

$$= \frac{1}{2} Q \cdot t$$

$$\therefore \text{Average inventory at any time} = \left( \frac{1}{2} Q \cdot t \right) / t = \frac{1}{2} Q$$

Total cost = Ordering cost + Holding cost + Purchase cost (constant)

$$\text{Minimize } TC(Q) = c_r \frac{D}{Q} + c_1 \frac{1}{2} Q + \text{Constant}$$

$$\Rightarrow \text{Minimize } TC(Q) = c_r \frac{D}{Q} + c_1 \frac{1}{2} Q$$

NOTES

We apply calculus method i.e.,  $\frac{dTC(Q)}{dQ} = 0$  and  $\frac{d^2TC(Q)}{dQ^2} > 0$ .

$$\therefore \frac{dTC(Q)}{dQ} = 0$$

$$\Rightarrow -c_s \cdot \frac{D}{Q^2} + \frac{1}{2}c_1 = 0$$

$$\Rightarrow Q^2 = \frac{2c_s D}{c_1}$$

$$\Rightarrow Q = \sqrt{\frac{2c_s D}{c_1}}$$

Also,  $\frac{d^2TC(Q)}{dQ^2} = \frac{2c_s D}{Q^3} > 0$  at  $Q = \sqrt{\frac{2c_s D}{c_1}}$

Hence optimum EOQ is  $Q^* = \sqrt{\frac{2c_s D}{c_1}}$

$$TC^*(Q) = c_s \cdot D \cdot \sqrt{\frac{c_1}{2c_s D}} + \frac{1}{2} \cdot c_1 \cdot \sqrt{\frac{2c_s D}{c_1}}$$

$$\Rightarrow c_A = \sqrt{2c_1 c_s D}$$

Time between orders  $t^* = \frac{Q^*}{D}$

$n^*$  = optimum number of orders placed per year

$$= \frac{D}{Q^*}$$

Note. If the holding cost is given as a percentage of average value of inventory held, then total annual holding cost,

$$c_1 = c \times I, \text{ where } c = \text{unit cost}$$

$I$  = % of the value of the average inventory.

The cost functions are shown in the Fig. 5.2

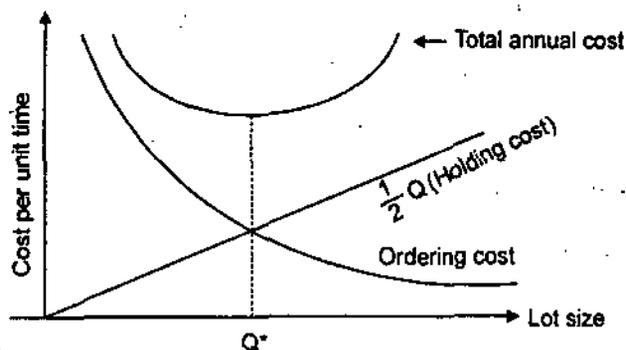


Fig. 5.2 Cost functions of Model I.

**Example 1.** A medical wholesaler supplies 30 bottles cough syrup each week to various shops. Cough syrups are purchased from the manufacturer in lots of 120 each for Rs. 1200 per lot. Ordering cost is Rs. 210 per order. All orders are filled the next day. The incremental cost is Rs. 0.60 per year to store a bottle in inventory. The wholesaler finances inventory investments by paying its holding company 2% monthly for borrowed funds. Suppose multiple and fractional lots also can be ordered. How many bottles should be ordered and how frequently he should order?

**Solution.** Consider 1 year = 52 weeks as working time.

$$\text{Annual demand, } D = 30 \times 52 = 1560.$$

$$\text{Unit cost of purchases} = \frac{1200}{120} = \text{Rs. } 10$$

$$\text{Ordering cost, } c_o = \text{Rs. } 210.$$

$$\text{Inventory carrying cost} = 0.6 + 10 \times 24/100 = \text{Rs. } 3 \text{ per unit/year.}$$

$$\therefore Q^* = \sqrt{\frac{2Dc_o}{c_1}} = \sqrt{\frac{2 \times 1560 \times 210}{3}} = 467.33$$

$$t^* = \frac{Q^*}{D} = \frac{467.33}{1560} = 0.3 \text{ year.}$$

**Example 2.** A company purchases in lots of 500 items which is a 3 month supply. The cost per item is Rs. 50 and the ordering cost is Rs. 100. The inventory carrying cost is estimated at 20% of unit value. What is the total cost of the existing inventory policy? How much money could be saved by employing the economic order quantity?

**Solution.** Given  $c_o = \text{Rs. } 100$

$$\text{Number of items per order} = 500$$

$$\therefore \text{Annual demand, } D = 500 \times 4 = 2000.$$

$$\begin{aligned} c_1 &= \text{Procurement price} \times \text{inventory carrying cost} \\ &\quad \text{per year} \\ &= 50 \times 0.20 = \text{Rs. } 10 \end{aligned}$$

Total annual cost of the existing inventory policy

$$= \frac{D}{Q} c_o + \frac{Q}{2} c_1 = \frac{2000}{500} \times 100 + \frac{500}{2} \times 10 = \text{Rs. } 2900$$

Now,

$$Q^* = \sqrt{\frac{2Dc_o}{c_1}} = \sqrt{\frac{2 \times 2000 \times 100}{10}} = 200$$

Then the corresponding annual cost

$$= \frac{2000}{200} \times 100 + \frac{200}{2} \times 10 = \text{Rs. } 2000.$$

Hence, by employing the economic order quantity, the company may save Rs.(2900 - 2000) = Rs. 900.

NOTES

## MODEL II : PURCHASING MODEL WITH SHORTAGE

**Assumptions.** All the assumptions of model I except shortage occurs here. Backlogs due to shortage to be met with penalty.

**NOTES**

One inventory cycle is given in the Fig. 5.3.

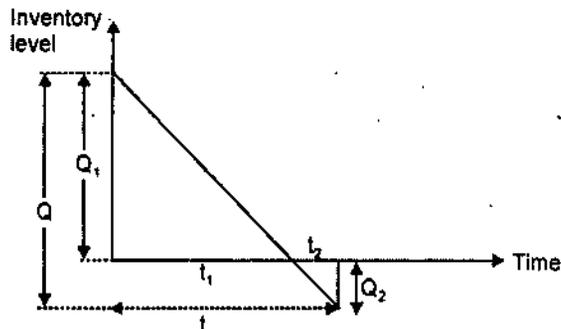


Fig. 5.3 Purchasing model with shortage

During time period  $t_1$  inventory exhaust and during time period  $t_2$  shortages developed.

Here

$Q_1$  = Actual inventory in hand.

$Q_2$  = Shortage/Stock out

$Q = Q_1 + Q_2, t = t_1 + t_2 =$  cycle time.

Total cost = Holding cost + Ordering cost + Shortage cost

The optimum values are given as

$$Q^* = \sqrt{\frac{2c_s D}{c_1} \cdot \frac{c_1 + c_2}{c_2}}$$

$$Q_1^* = \sqrt{\frac{2c_s D}{c_1} \cdot \frac{c_2}{c_1 + c_2}}$$

$$Q_2^* = Q^* - Q_1^*$$

$$i^* = \frac{Q^*}{D}, n^* = \frac{D}{Q^*}, t_1^* = \frac{Q_1^*}{D}, t_2^* = \frac{Q_2^*}{D}$$

$$\text{Total optimum cost} = \sqrt{2Dc_s c_1 \cdot \frac{c_2}{c_1 + c_2}}$$

**Example 3.** The demand for a certain item is 50 units per year. Unsatisfied demand causes a shortage cost of Rs. 0.45 per unit per short period. The ordering cost for purchase is Rs. 20 per order and the holding cost is 15% of average inventory valuation per year. Item cost is Rs. 5 per unit. Find the EOQ, the shortage inventory and the minimum cost.

**Solution.**

$D = 50$  units/year

$c_2 = \text{Rs. } 0.45/\text{unit/shortage period.}$

$c_s = \text{Rs. } 20$

$c_1 = 5 \times 0.15 = \text{Rs. } 0.75/\text{unit/year.}$

$$Q^* = \sqrt{\frac{2c_1 D}{c_1} \cdot \frac{c_1 + c_2}{c_2}} = \sqrt{\frac{2 \times 20 \times 50}{0.75} \cdot \frac{0.75 + 0.45}{0.45}}$$

$$= 84.33 \text{ units.}$$

$$Q_2^* = Q^* - \sqrt{\frac{2c_1 D}{c_1} \cdot \frac{c_2}{c_1 + c_2}}$$

$$= 84.33 - 31.62 = 52.71$$

$$\text{Total minimum cost} = \sqrt{2Dc_1c_2 \cdot \frac{c_1 + c_2}{c_1}}$$

$$= \sqrt{2 \times 50 \times 20 \times 0.75 \times \frac{0.45}{1.2}} = \text{Rs. } 23.72.$$

NOTES

### MODEL III : MANUFACTURING MODEL WITHOUT SHORTAGE

**Assumptions.** Items are manufactured, shortages are not allowed, demand is uniform, lead time is zero, items are produced and used to meet demand simultaneously for a portion of an inventory cycle.

One inventory cycle is illustrated in Fig. 5.4.

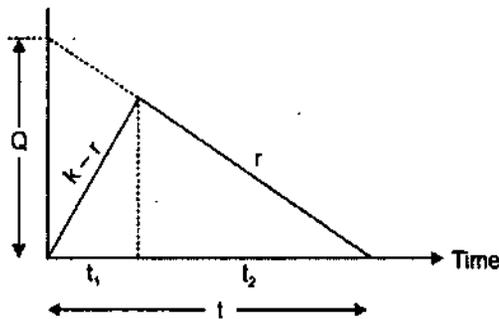


Fig. 5.4 Manufacturing model without shortage

In this model,

- (i) Inventory is building up at a constant rate of  $(k - r)$  units per unit time during  $t_1$ .
- (ii) No production during  $t_2$  and the demand is met at the rate of  $r$  per unit of time.

The optimal quantities are obtained as given below :

$$Q^* = \text{EOQ/Economic batch quantity}$$

$$= \sqrt{\frac{2c_1 r}{c_1(1-r/k)}}$$

$$t_1^* = \text{period of production as well as consumption}$$

$$= \frac{Q^*}{k}$$

$$t_2^* = \text{period of consumption only} = \frac{(k-r)t_1^*}{r}$$

NOTES

$n^*$  = optimum number of production runs per year

$$= \frac{r}{Q^*}$$

$$t^* = t_1^* + t_2^*$$

$$\text{Total minimum cost} = \sqrt{2rc_s c_1 (1-r/k)}$$

**Example 4.** A contractor has to supply 10000 bolts per day to a customer. He finds that during a production run he can produce 20000 bolts per day. The cost of holding bolt in stock for one year is 3 paise and set up cost of a production run is Rs. 20. How frequently should production run be made?

**Solution.**

$$r = 10000 \text{ bolts/day}$$

$$k = 20000 \text{ bolts/day}$$

$$c_1 = \text{Rs. } 0.03/\text{bolt/year}$$

$$= \text{Rs. } 0.000082/\text{bolt/day}$$

$$c_s = \text{Rs. } 20/\text{production run.}$$

$$Q^* = \sqrt{\frac{2rc_s}{c_1(1-r/k)}} = \sqrt{\frac{2 \times 10000 \times 20}{0.000082(1-1/2)}}$$

$$= 98772.96 \approx 98773 \text{ bolts.}$$

$$t^* = \frac{Q^*}{r} = 9.88 \text{ days.}$$

$$\text{Length of production cycle} = \frac{Q^*}{k} = 4.94 \text{ days.}$$

⇒ production cycle starts at an interval of 9.88 days and production continues for 4.94 days, so that in each cycle a batch of 98773 bolts is produced.

---

**MODEL IV : MANUFACTURING MODEL WITH SHORTAGE**

---

one inventory cycle of this model is given in Fig. 5.5.

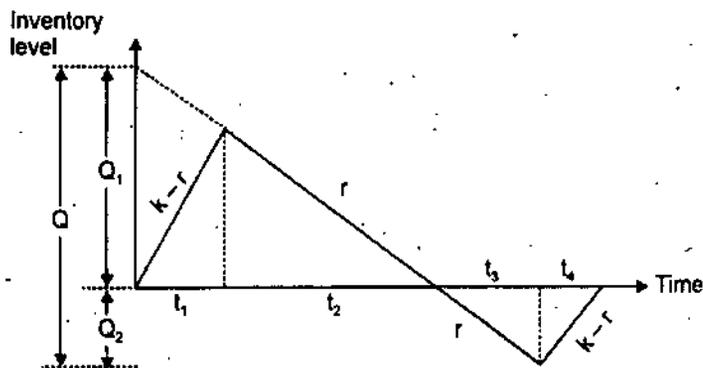


Fig. 5.5 Manufacturing model with shortage

Here during  $t_1$ , inventory is built up at the rate of  $(k - r)$ ,

during  $t_2$ , inventory is consumed at the rate of  $r$ ,

during  $t_3$ , shortage is building at the rate of  $r$ ,

during  $t_4$ , shortage is being filled at the rate of  $(k - r)$ .

The optimum quantities are obtained as given below :

$$Q^* = \sqrt{\frac{2c_s(c_1 + c_2)}{c_1c_2} \cdot \frac{kr}{k-r}}$$

$$Q_1 + Q_2^* = \left(1 - \frac{r}{k}\right)Q^*$$

$$Q_2^* = \text{no. of shortages} = \sqrt{\frac{2c_s c_1}{(c_1 + c_2)c_2} \cdot r \left(1 - \frac{r}{k}\right)}$$

$$Q_1^* = \text{maximum inventory level} = \left(1 - \frac{r}{k}\right)Q^* - Q_2^*$$

$$t^* = \text{production cycle time} = \frac{Q^*}{r}$$

$$\text{Manufacturing time} = \frac{Q^*}{k}$$

$$t_1^* = \frac{Q_1^*}{k-r}, t_2^* = \frac{Q_1^*}{r}, t_3^* = \frac{Q_2^*}{r}, t_4^* = \frac{Q_2^*}{k-r}$$

$$\text{Total minimum production inventory cost} = \sqrt{\frac{2c_s c_2 c r}{c_1 + c_2} \left(1 - \frac{r}{k}\right)}$$

**Example 5.** The demand for an item is 10000 units per year. Its production rate is 1500 units per month. The holding cost is Rs. 20/unit/year and the set up cost is Rs. 800 per set up. The shortage cost is Rs. 1000 per unit per year. Find the EOQ, maximum shortage and total minimum production inventory cost.

**Solution.**

$$r = 10000 \text{ units/year}$$

$$k = 1500 \text{ units/month} = 18000 \text{ units/year}$$

$$c_1 = \text{Rs. } 20/\text{unit/year}$$

$$c_s = \text{Rs. } 800$$

$$c_2 = \text{Rs. } 1000/\text{unit/year.}$$

$$Q^* = \sqrt{\frac{2c_s(c_1 + c_2)}{c_1c_2} \cdot \frac{kr}{k-r}}$$

$$= \sqrt{\frac{2 \times 800 (20 + 1000)}{20 \times 1000} \cdot \frac{18000 \times 10000}{(18000 - 10000)}} = 1355 \text{ units}$$

NOTES

NOTES

$$Q_2^* = \sqrt{\frac{2c_2c_1}{(c_1 + c_2)c_2} \cdot r \left(1 - \frac{r}{k}\right)}$$

$$= \sqrt{\frac{2 \times 800 \times 20}{(20 + 1000) 1000} \cdot 10000 \left(1 - \frac{10000}{18000}\right)} = 11.81 \text{ units.}$$

Total minimum production inventory cost

$$= \sqrt{\frac{2c_1c_2c_1r}{c_1 + c_2} \left(1 - \frac{r}{k}\right)}$$

$$= \sqrt{\frac{2 \times 20 \times 1000 \times 800 \times 10000}{(20 + 1000)} \left(1 - \frac{10000}{18000}\right)}$$

$$= \text{Rs. } 11808.20.$$

### PROBABILISTIC MODELS

It seldom happens that future demand is not known exactly *i.e.*, uncertain. We assume that the probability distribution of future demand is known which can be determined by some forecasting analysis. Probability distributions can be discrete or continuous. We will discuss some probabilistic models where only demand follows some probability distribution.

#### (a) Model I (Discrete Case)

*Assumptions.* Demand is instantaneous, total demand is filled at the beginning of the period, no setup cost, lead time is zero.

- Let
- $t$  = constant interval between orders.
  - $Q$  = stock (in discrete units) for time  $t$
  - $d$  = estimated (random) demand with probability  $p(d)$ .

Since, the total demand is filled at the beginning of the period, the inventory position just after the demand occurs may be either positive (surplus) or negative (shortage).

When the demand does not exceed the stock  $Q$  *i.e.*,  $d \leq Q$ , the holding cost per unit time is

$$= (Q - d) \cdot c_1, \quad d \leq Q$$

$$= 0 \cdot c_1, \quad d > Q.$$

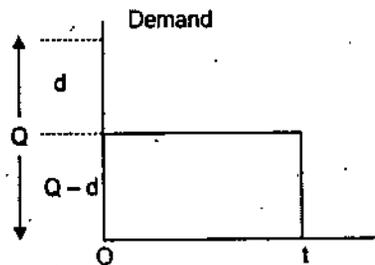


Fig. 5.6 Case  $d \leq Q$

When the demand  $d$  exceeds the stock  $Q$  i.e.,  $d > Q$ , the shortage cost per unit time becomes

$$= 0 \cdot c_2, \quad d \leq Q$$

$$= (d - Q) \cdot c_2, \quad d > Q.$$

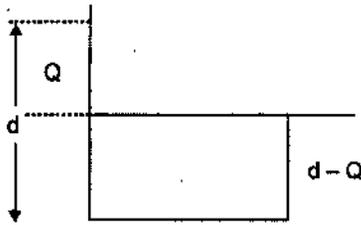


Fig. 5.7 Case  $d > Q$

The total expected cost per unit time is

$$c(Q) = \sum_{d=0}^Q (Q-d) \cdot c_1 p(d) + \sum_{d=Q+1}^{\infty} (d-Q) \cdot c_2 p(d)$$

For minimum  $c(Q)$ , the condition

$$\Delta c(Q-1) < c(Q) < \Delta c(Q).$$

must be satisfied.

Now,

$$\Delta c(Q) = (c_1 + c_2) \cdot \sum_{d=0}^Q p(d) - c_2$$

Using the condition for minimum, we must have

$$\Delta c(Q) > 0 \quad \text{i.e.,} \quad \sum_{d=0}^Q p(d) > \frac{c_2}{c_1 + c_2}$$

Also,

$$\Delta c(Q-1) < 0 \quad \text{i.e.,} \quad \sum_{d=0}^{Q-1} p(d) < \frac{c_2}{c_1 + c_2}$$

Thus combining the optimum value of stock level  $Q$  can be obtained from the relationship.

$$\sum_{d=0}^{Q-1} p(d) < \frac{c_2}{c_1 + c_2} < \sum_{d=0}^Q p(d).$$

**Example 6.** A newspaper boy buys papers for Rs. 2 and sells them for Rs. 3 each. He cannot return the unsold newspapers. Daily demand has the following distribution

No. of customers	22	23	24	25	26	27	28	29	30	31
Probability	0.02	0.02	0.06	0.15	0.15	0.25	0.15	0.1	0.05	0.05

If each day's demand is independent of the previous day's, how many papers should be ordered each day?

NOTES

Solution. Here

$$c_1 = \text{Rs. } 2$$

$$c_2 = \text{Rs. } (3 - 2) = \text{Rs. } 1$$

$$\frac{c_2}{c_1 + c_2} = \frac{1}{2 + 1} = \frac{1}{3} = 0.33.$$

## NOTES

Now, we have to calculate the cumulative distribution *i.e.*,

No. of customer	22	23	24	25	26	27	28	29	30	31
Cumulative probability	0.02	0.04	0.1	0.25	0.4	0.65	0.8	0.9	0.95	1

Since, the value 0.33 lies between 0.25 and 0.4, the optimal  $Q^*$  is taken as 26. *i.e.*, the newspaper boy should order 26 newspaper per day to minimize the cost.

**(b) Model II (Continuous Case)**

*Assumptions.* The assumptions are same as in the discrete case expect that  $f(x)$  will be used as probability density function of demand  $x$ .

Proceeding in the same manner as in Model I (discrete case), the cost equation can be formulated as follows :

$$c(Q) = c_1 \int_0^Q (Q-x)f(x) dx + c_2 \int_Q^{\infty} (x-Q)f(x) dx$$

The optimal value of  $Q$  can be obtained from  $\frac{dc(Q)}{dQ} = 0$  which gives

$$(c_1 + c_2) \int_0^Q f(x) dx - c_2 = 0$$

$$\Rightarrow \int_0^Q f(x) dx = \frac{c_2}{c_1 + c_2}$$

Thus, we can find out the optimum value of  $Q$ , satisfying the above equation.

**Example 7.** A baking company makes a profit of Rs. 5 per kg. on each kg. cakes sold on the day it is baked. It disposes all cakes not sold on the date, it is baked at a loss of Rs. 1.20 per kg. If demand is known to be rectangular between 2000 and 3000 kg., determine the optimal daily amount to be baked if the demand is instantaneous.

**Solution.** Let

$$c_1 = \text{Rs. } 5, c_2 = \text{Rs. } 1.20.$$

$$f(x) = \frac{1}{1000}, 2000 \leq x \leq 3000.$$

Also 
$$\frac{c_2}{c_1 + c_2} = \frac{1.2}{6.2} = 0.1935$$

Then 
$$\int_{2000}^Q f(x) dx = \frac{c_2}{c_1 + c_2}$$

$$\frac{1}{1000} \int_{2000}^Q dx = 0.1935$$

$$Q - 2000 = 193.5$$

$$Q = 2193.5$$

∴ The company should bake 2193.5 kg. daily.

### (c) Model III (Discrete Case with Uniform Demand)

Compare to Model I, the demand is uniform here rather than instantaneous and the other assumptions are the same.

The optimum stock (i.e., EOQ)  $Q$  is obtained by the same relationship i.e.,

$$\sum_{d=0}^{Q-1} p(d) < \frac{c_2}{c_1 + c_2} < \sum_{d=0}^Q p(d)$$

### (d) Model IV (Continuous Case with Uniform Demand)

Compare to Model III, the demand is uniform here rather than instantaneous and the other assumptions are the same.

Here the optimum stock  $Q$  is obtained by the following relationship:

$$\int_0^Q f(x) dx + \int_0^Q \frac{Q}{x} f(x) dx = \frac{c_2}{c_1 + c_2}$$

## SUMMARY

- The replenishments are usually instantaneous, uniform or batch. Its size refers to the quantity or size of the order to be received into inventory which may be constant or variable.
- Lead Time is the period between the time an order is placed (administrative lead time) and the time when it is received (delivery lead time). When the lead-time is known, it is called deterministic. When it is not known, it can govern by a random variable.
- Economic Order Quantity is also known as 'economic lot size' or EOQ which is the optimum quantity to be purchased or produced such that the total cost of the inventory is minimized.
- The level between the maximum and minimum stock at which purchasing (or manufacturing) activities must start for replenishment is known as re-order level.

## PROBLEMS

### (Model I and III)

1. A purchase manager places order each time for a lot of 500 units of a particular item. From the available data the following results are obtained :

NOTES

Ordering cost per order = Rs. 600

Cost per unit = Rs. 50

Annual demand = Rs. 1000

Inventory carrying cost = 40%

NOTES

Find out the loss to the organization due to his ordering policy.

2. An aircraft company uses rivets at an approximately constant rate of 5000 kg per year. The rivets cost Rs. 20 per kg. and the company personnel estimate that it costs Rs. 200 to place an order. and the carrying cost of inventory is 10% per year. How frequently should orders for rivets be placed and what quantities should be ordered for?
3. The Inventory company, after an analysis of its accounting and production records, has determined that it uses Rs. 36000 per year of a component part purchased at Rs. 18 per part. The purchasing cost is Rs. 40 per order, and its annual inventory carrying charges are  $16\frac{2}{3}\%$  of the average inventory. Determine
  - (a) the most EOQ at one time.
  - (b) the most economic number of times to order per year.
  - (c) the average days' supply for ordering the most EOQ.  
(year = 365 days)
4. A company uses 50,000 widgets per annum which costs Rs. 10 per piece to purchase. The ordering and handling costs are Rs. 150 per order and carrying costs are 15% per annum. Find the EOQ.  
Suppose the company decides to make the widgets in its own factory and installed a machine which has capacity of 2,50,000 widgets per annum. What is the EOQ?

**(Model III)**

5. A contractor has to supply 10000 bearing per day to an automobile manufacturer. He finds that when he starts production run, he can produce 25000 bearings per day. The cost of holding a bearing in stock for a year is Rs. 2 and the setup cost of a production run is Rs. 1800. How frequently should production run be made? (Assume 1 year = 300 working days)
6. In a paints manufacturing unit, the changeover from one type of paint to another is estimated to cost Rs. 100 per batch. The annual sales of a particular grade of paint are 20,000 litres and the inventory carrying cost is Rs. 1 per litre. Given that the rate of production is 3 times the sales rate, determine the economic batch size and number of batches per year and total optimum yearly cost.
7. A product is sold at the rate of 30 pieces per day and is manufactured at a rate of 200 pieces per day. The set up costs of the machines are Rs. 300 and the holding cost is found to be Rs. 0.05 per piece day. Find optimum batch size, period of production and the optimum number of production run. (Assume 1 year = 365 days)

**(Model II and I)**

8. An item is to be supplied at a constant rate of 100 unit per day. The ordering cost for each supply is Rs. 20, cost of holding the item in inventory is Rs. 1 per unit per day while delay in the supply of the item induces a penalty of Rs. 3 per unit per day. Find the optimal policy (Q, t) and optimum shortage.

9. The demand for a product is 150 units per month and the items are withdrawn uniformly. The setup cost each time a production run is Rs. 12. The holding cost is Rs. 0.25 per item per month.
- (a) Determine how often to make production run, if shortages are not allowed.
- (b) Determine how often to make production run, if shortages cost Rs. 1 per item per month.
10. The demand for a certain item is uniform at a rate of 30 units per month. The fixed cost is Rs. 10 each time a production run is made. The production cost is Rs. 2 per item and the holding cost is Rs. 0.3 per item per month. If the shortage cost is Rs. 1.5 per item per month, determine how often to make a production run and of what size should be?

## NOTES

**(Model IV)**

11. The demand for an item in a company is 24000 units per year, and the company can produce 2500 units per month. The one setup cost is Rs. 300 and the holding cost per unit per month is Rs. 0.3 and the shortage cost of one unit is Rs. 20 per month. Determine the optimum manufacturing quantity, the number of shortages and manufacturing time.

**(Probabilistic Models)**

12. Rework the Example 7, taking demand is uniform.
13. If the demand for a certain product has a rectangular distribution between 4000 and 500, find the optimum expected total cost if storage cost is Rs. 1 per unit and shortage cost is Rs. 7 per unit and the purchasing cost is Rs. 10 per unit, and the demand is instantaneous.
14. A certain children product is stocked by a company. The demand distribution is given below :

Demand	0	10	20	30	40	50
Probability	0.1	0.20	0.35	0.2	0.1	0.05

Inventory carrying cost is Rs. 5, the storage cost is Rs. 20, find the economic order quantity.

**ANSWERS**

1. Loss is about Rs. 1301.
2.  $Q^* = 1000$  unit,  $n^* = 5$  times in a year.
3. (a)  $Q^* = 231$ , (b)  $n^* = 9$ , (c) 40.6 days.
4.  $Q^* = 10,000,000$  and  $Q_2^* = 3535.53$ .
5.  $Q^* = 1,04,446$ ,  $t^* = 10.44$  days.
6.  $Q^* = 2449.49$ ,  $n^* = 2.25$ , total cost = Rs. 2000.
7.  $Q^* = 650.79$ .
8.  $Q^* = 73.03$ ,  $t^* = 0.73$  day.  
 $Q_2^* = 18.26$ .
9. (a) 126, (b) 134.16.
10.  $Q^* = 49$ ,  $t^* = 1.63$ .
11.  $Q^* = 4505.55$ ,  $Q_2^* = 13.32$ , Manufacturing time = 1.8 month.
13. Rs. 45187.
14.  $Q^* = 30$ .

# CHAPTER 6 SIMULATION

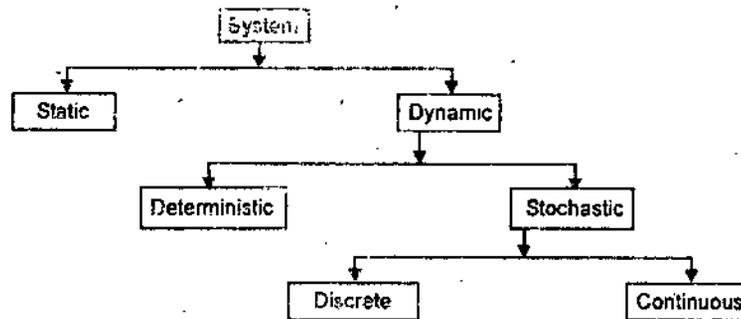
## NOTES

### ★ STRUCTURE ★

- Introduction
- Basic Concepts of Simulation
- Random Numbers
- Simulation Methods
- Applications
- Simulation Softwares
- Summary
- Problems

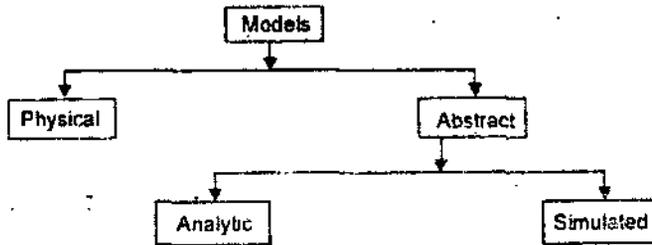
## INTRODUCTION

A system is made up of elements or components that are related functionally to one another. An element can be treated as a part of the system or as an independent system. A *classification* of systems is given below :



If the state of the system does not change then it is called static *e.g.*, breeze, else it is called dynamic. A dynamic system is called deterministic if for each system state the subsequent system state is uniquely determined. If the subsequent system state is random then it is called stochastic system. Again the randomness may be two types—discrete and continuous. A discrete system may be made up of elements some of which are mobile and some are stationary. When the state of a system changes due to the transactions, the system is called transaction oriented. When the state of a system changes due to stationary elements, the system is called an event oriented system.

A system is often represented by a model that describes the system sufficiently in detail so that the behaviour of the system can be predicted from the behaviour of the model. A *classification* of models is given below :



## NOTES

A simulated model is built up by simulation which is the activity of artificially creating probabilistic processes to represent system components with the sole purpose of studying the behaviour of the system.

Modelling and simulation have found applications in the most varied fields e.g., Business, Defence, Politics, Physics, Mathematics, Computer Science, Management etc.

## BASIC CONCEPTS OF SIMULATION

For developing a model the system under study is closely analyzed and each system component is identified along with its parameters. We summarize the ABC of simulation steps in below :

- A Analyze the system bottom-up  
(elements → relations → components → system)
- B Build the model
- C Consider strategy and setup cases for studying the behaviour of the system
- D Do simulation
- E Evaluate and criticise the results
- F If acceptable produce before, else repeat all steps
- G Goto C after modifying.

Also we write down the basic differences of two types of simulation *i.e.*, fixed time step simulation (FTSS) and event to event model simulation (EES).

FTSS	EES
1. Here the system clock is updated at fixed regular intervals of time.	1. Here the system clock is updated only when there is an occurrence of a potential event in the system.
2. The interval width is pre-determined	2. The interval width cannot be pre-determined.
3. In an interval same changes may or may not take place in the system.	3. Between one occurrence to next occurrence no change takes place in the system.
4. Machine efficiency or utilization is less.	4. Utilization of the system is very effective.
5. Less preferred.	5. Most preferred.

## RANDOM NUMBERS

### NOTES

The basic requirement of random numbers is that the order in which they occur should be independent. In simulation models, the use of  $[0, 1]$  random numbers is most preferred. In computer simulation truly random numbers cannot be generated because all computer activities are guided by algorithms which are deterministic in nature. However sequences of numbers, called *pseudo-random* numbers, are used as random numbers. These numbers satisfy some of the properties of random numbers. So the generation of random numbers (i.e., the pseudo-random numbers) is a fundamental step in simulation. We describe two random numbers generation technique below :

### (a) Mid-Square Technique

This is the oldest technique given by Neumann which is described below :

- (i) Take a number of  $2d$  digits (called seed).
- (ii) Square it.
- (iii) Take the middle  $2d$  digit (non-zero) and it will be the first pseudo-random number.

[This technique has a drawback. We may obtain zero digits in the middle resulting a stop of this technique.]

e.g., Let  $u_0 = 0.0400$  (seed)

Then  $u_0^2 = 0.00160000$

$\Rightarrow$  the first pseudo-random number is  $u_1 = 0.1600$ .

Then  $u_1^2 = 0.02560000$

$\Rightarrow$  the second pseudo-random number is  $u_2 = 0.5600$  and so on.

### (b) Linear Congruential Technique

This is also known as power residual technique. This recursion formula is

$$x_k \equiv (a x_{k-1} + c) \pmod{m}, \quad k = 1, 2, \dots$$

where  $a$ ,  $c$  and the seed  $x_0$  are all positive integers and less than  $m$ .  $c$  is relatively prime to  $m$ . If  $c = 0$  then this is called multiplicative congruential technique. Again  $m$  is chosen to be  $2^b$  or  $10^d$  ( $b = \text{binary}$ ,  $d = \text{decimals}$ ).  $\pmod{m}$  gives the remainder on division.

e.g., Let  $a = 31$ ,  $c = 7$ ,  $m = 100$  and  $x_0 = 21$ .

Then,  $x_1 = (31 \times 21 + 7) \pmod{100}$   
 $= 658 \pmod{100} = 58$

$\Rightarrow$  the first  $(0, 1)$  random number is  $u_1 = \frac{x_1}{m} = 0.58$

Again  $x_2 = (31 \times 58 + 7) \pmod{100}$   
 $= 1805 \pmod{100}$   
 $= 5$

$\Rightarrow$  the second (0, 1) random number is  $u_2 = \frac{x_2}{m} = 0.05$

Again

$$\begin{aligned}x_3 &= (31 \times 5 + 7) \pmod{100} \\ &= 162 \pmod{100} \\ &= 62.\end{aligned}$$

$\Rightarrow$  the third (0, 1) random number is  $u_3 = \frac{x_3}{m} = 0.62$

and so on.

If the seed is changed then another sequence of random numbers can be generated.

### (c) Random Numbers in [A, B]

Let  $u_i$  be the uniformly distributed random numbers in [0, 1]. Then the uniformly distributed random numbers in [A, B] can be generated using the following relation  $v_i = A + (B - A) u_i, \forall i$ .

**Example 1.** (Voting problem). Suppose 10 people wants to vote in a certain issue. Vote consists of yes or no. People give the vote by tossing a coin. Let head (H) corresponds to yes and tail (T) corresponds to no. Simulate a result using the following random numbers :

.38, .31, .78, .83, .42, .79, .53, .68, .77, .92.

**Solution.** We know  $P(H) = 0.5, P(T) = 0.5$ .

Let us construct the hypothesis

(i) If  $0 \leq u \leq 0.5$ , the outcome of toss is H.

(ii) If  $0 < u \leq 1$ , the outcome of toss is T.

The simulation is carried out in the following table :

Random numbers	Toss	Yes vote	No vote
0.38	H	1	-
0.31	H	1	-
0.78	T	-	1
0.83	T	-	1
0.42	H	1	-
0.79	T	-	1
0.53	T	-	1
0.68	T	-	1
0.77	T	-	1
0.92	T	-	1
	Total	3	7

**Conclusion.** The issue has got more no votes than yes votes.

NOTES

## SIMULATION METHODS

### NOTES

#### (a) Monte Carlo Technique

- (i) Generate random numbers  $u_i$  from (0, 1)
- (ii) Generate random observations from any desired probability distribution.

#### (b) Method of Inversion

- (i) Generate a random number  $u_i$  from (0, 1)
- (ii) Find the distribution function  $F(x)$  of the random variable  $X$ .

(iii) Set  $F(x_i) = u_i$

$$\Rightarrow x_i = F^{-1}(u_i)$$

which gives the random observation from the probability distribution of  $X$ .

#### (c) Rejection Method

Let the pdf  $f(x)$  over a finite interval  $[A, B]$  and  $C$  be the mode of the distribution.

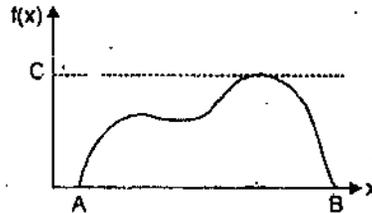


Fig. 7.1

- (i) Generate two random numbers  $u_1$  and  $u_2$  from (0, 1).

- (ii) Take a point on  $x$ -axis as

$$p = A + (B - A) u_1$$

- (iii) Take a point on the vertical axis as

$$q = C u_2$$

- (iv) If  $q \leq f(p)$  accept the pair  $(p, q)$  else reject the pair. All the rejected pairs lie outside the boundary of the curve  $f(x)$ .

## APPLICATIONS

#### (a) Sampling of Exponential Distribution

Let the pdf be  $f(x) = \lambda e^{-\lambda x}, x > 0$

$$\therefore F(x) = \int_0^x f(x) dx = 1 - e^{-\lambda x}$$

Let  $u$  be random number in (0, 1).

$$\therefore F(x) = u$$

$$\begin{aligned} \Rightarrow 1 - e^{-\lambda x} &= u \\ \Rightarrow e^{-\lambda x} &= 1 - u \\ \Rightarrow -\lambda x &= \ln(1 - u) \\ \Rightarrow x &= -\frac{1}{\lambda} \ln(1 - u) = -\frac{1}{\lambda} \ln u \\ &(\because u \text{ is random, replace } 1-u \text{ by } u). \\ \Rightarrow x_i &= -\frac{1}{\lambda} \ln u_i, \quad i = 1, 2, \dots \end{aligned}$$

### (b) Sampling of Gamma Distribution

Let  $x_i$  ( $i = 1, 2, \dots, n$ ) be the exponential random variable with parameter  $\lambda$ . They are independent and identically distributed. By a statistical property,

$T = x_1 + x_2 + \dots + x_n$  be a gamma distributed with the parameter  $n$  and  $\lambda$  i.e., gamma ( $n, \lambda$ ).

$$\text{Then } T_i = -\frac{1}{\lambda} \ln u_1 - \frac{1}{\lambda} \ln u_2 - \dots - \frac{1}{\lambda} \ln u_n,$$

where  $u_1, u_2, \dots, u_n$  are  $n$  random numbers from  $(0, 1)$

$$\begin{aligned} \Rightarrow T_i &= \frac{1}{\lambda} [\ln u_1 + \ln u_2 + \dots + \ln u_n] \\ &= -\frac{1}{\lambda} \ln(u_1 u_2 \dots u_n) \end{aligned}$$

$\Rightarrow$  to generate one gamma sample we need  $n$  random numbers from  $[0, 1]$ .

### (c) Sampling the Normal Distribution

We know that the mean and variance of  $(0, 1)$  uniform distribution as  $\frac{1}{2}$  and  $\frac{1}{12}$  respectively.

Let  $u_1, u_2, \dots, u_n$  be  $n$  random numbers from  $(0, 1)$ .

$$\begin{aligned} \text{Let } T &= u_1 + u_2 + \dots + u_n \\ \text{Then, } E(T) &= \frac{1}{2} + \frac{1}{2} + \dots + \frac{1}{2} = \frac{n}{2} \end{aligned}$$

$$V(T) = \frac{1}{12} + \frac{1}{12} + \dots + \frac{1}{12} = \frac{n}{12}$$

$$\text{Consider } z = \frac{T - n/2}{\sqrt{n/12}} \sim N(0, 1)$$

To generate a sample from  $N(\mu, \sigma^2)$ .

$$\frac{y - \mu}{\sigma} = z = \frac{T - n/2}{\sqrt{n/12}}$$

$$\Rightarrow y = \mu + \frac{\sigma}{\sqrt{n/12}} \left( T - \frac{n}{2} \right)$$

$$= \mu + \frac{\sigma}{\sqrt{n/12}} \left( \sum_{i=1}^n u_i - \frac{n}{2} \right)$$

In practice  $n = 12$  is taken, so we get

## NOTES

$$y = \mu + \sigma \left[ \sum_{i=1}^{12} u_i - 6 \right]$$

$\Rightarrow$  to generate one random sample we need 12 random numbers from (0, 1).

**(d) Evaluation of an Integral  $I = \int_a^b f(x) dx$  (Using Rejection Method)**

Here  $f(x)$  is a continuous curve in  $[a, b]$  and also assume that  $0 \leq f(x) \leq f_{\max}$ .

- (i) Generate  $u_1 \in [a, b]$ , uniformly distributed random number.
- (ii) Generate  $u_2 \in [0, f_{\max}]$  uniformly distributed random number.
- (iii) If  $u_2 \leq f(u_1)$ , then the pair will fall on or under the given curve  $f(x)$  and accept the pair else reject them.
- (iv) Repeat the steps (i) to (iii) N times and determine

$$FP = \frac{\text{Total no. of accepted pairs}}{N}$$

- (v) The value of the integral is calculated as

$$I = FP \times (b-a) \times f_{\max}$$

**(e) Evaluation of  $\pi$**

Consider a quadrant of a unit circle in the positive octant i.e.,

$$x^2 + y^2 \leq 1, x, y \geq 0$$

$$\Rightarrow y^2 \leq 1 - x^2$$

$$\Rightarrow y \leq \sqrt{1 - x^2}$$

Here

$$c = 1 \text{ and } f(x) = \sqrt{1 - x^2}$$

- (i) Generate two random numbers  $u_1$  and  $u_2$  from (0, 1).

$$\text{Then } p = 0 + u_1 = u_1 \text{ and } q = u_2$$

- (ii) If  $q \leq f(p)$  i.e.,  $u_2 \leq \sqrt{1 - u_1^2}$ , accept the pair.

- (iii) Repeat the steps (i)-(ii), say N times.

- (iv) Compute the ratio,  $\frac{\text{No. of accepted pairs}}{\text{Total no. of generated pairs}}$  which approaches to  $\frac{\pi}{4}$ .

Hence we can calculate the approximate value of  $\pi$ .

**Example 2.** Find a scheme for generating a random sample from the following distribution :

$$f(x) = \frac{2}{5}(x+1), 1 \leq x \leq 2.$$

**Solution.** Here

$$F(x) = \int_1^x \frac{2}{5}(x+1) dx = \frac{2}{5} \left( \frac{x^2}{2} + x - \frac{3}{2} \right)$$

Let  $u$  be a random number from  $(0, 1)$ .

By method of inversion,

$$\frac{2}{5} \left( \frac{x^2}{2} + x - \frac{3}{2} \right) = u.$$

$$\Rightarrow x^2 + 2x - 3 = 5u$$

$$\Rightarrow x^2 + 2x - (3 + 5u) = 0$$

$$\Rightarrow x = -1 + \sqrt{1 + (3 + 5u)^2}.$$

**Example 3.** Assume the inter-arrival time  $X$  and service time  $Y$  are exponentially distributed with mean 3 and 2 min. respectively. Simulate the model for 10 minutes by using the following random numbers:

RN for $X$	0.82	0.23	0.37	0.75	0.15	0.27
RN for $Y$	0.66	0.31	0.48	0.92	0.38	0.72

Find the following characteristics for one server model.

- Average no. of customers in the system.
- Average no. of customers in the queue.
- Average waiting time in the system.
- Average waiting time in the queue.
- Proportion of idle time of the server.

**Solution.** Consider the short time = 0. Simulation table is given below (Here  $\lambda = 1/3$ ,  $\mu = 1/2$ ).

$u_i(X)$	$X_i = \frac{1}{\lambda} \ln u_i$	Arrival	Service start time	Queue	Waiting time	$u_i(Y)$	$X_i = \frac{1}{\lambda} \ln u_i$	Departure
0.82	0.60	0.60	0.60	-	-	0.66	0.83	1.43
0.23	4.41	5.01	5.01	-	-	0.31	2.34	7.35
0.37	2.98	7.99	7.99	-	-	0.48	1.47	9.46
0.75	0.86	8.85	9.46	1	0.61	0.92	0.17	9.63
0.15	5.69	-	-	-	-	-	-	-

(a) Total time spent (by the four customers) in the system

$$= 0.83 + 2.34 + 1.47 + 0.78 = 5.42.$$

$$\text{Average no. of customers in the system} = \frac{5.42}{10} = 0.542.$$

NOTES

(b) Total time spent in the queue = 0.61.

$$\text{Average no. of customers in the queue} = \frac{0.61}{10} = 0.061.$$

**NOTES**

(c) Average waiting time in the system =  $\frac{5.42}{4} = 1.36$ .

(d) Average waiting time in the queue =  $\frac{0.61}{1} = 0.61$ .

(e) Total idle time of the server =  $0.60 + 3.58 + 0.64 + 0.37 = 5.19$ .

$$\therefore \text{Proportion of idle time of the server} = \frac{5.19}{10} = 0.52.$$

**Example 4.** A newspaper boy estimates the daily demand with a probability as given below :

Daily demand	0	10	20	30	40	50
Probability	0.02	0.18	0.15	0.50	0.10	0.04

Use the following sequence of random numbers to simulate the demand for next 10 days :

RN : 25, 65, 39, 76, 05, 70, 12, 81, 32, 43.

Also estimate the daily average, demand for the newspaper on the basis of simulated data.

**Solution.** Random number interval is formulated as follows :

Demand	Probability	Cumulative probability	Random number interval
0	0.02	0.02	00-01
10	0.18	0.20	02-19
20	0.15	0.35	20-34
30	0.50	0.85	35-84
40	0.10	0.95	85-94
50	0.05	1.00	95-99

Next demand is simulated as follows :

Days	Random number	Demand
1	25	20
2	65	30
3	39	30
4	76	30
5	05	10
6	70	30

7	12	10
8	81	30
9	32	20
10	43	30

$$\text{Average demand} = \frac{\text{Total demand}}{\text{No. of days}} = \frac{240}{10} = 24.$$

**Example 5.** The demand for a certain product has the following distribution:

Demand/week (x) ('000 units)	0	1	2	3	4	5
P(x)	0.05	0.05	0.1	0.4	0.35	0.05

The distribution of the lead time is:

Lead time (y) (weeks)	2	3	4
P(y)	0.3	0.5	0.2

Consider : Beginning inventory = 3000 units

An order is placed whenever the inventory level  $\leq 2000$

Order size = Difference between the current inventory level and the maximum replenishment level of 4000 units.

No back orders are permitted. Simulate the first ten weeks of the inventory.

Demand RN : 32, 26, 64, 45, 12, 99, 52, 43, 84, 38

Lead RN : 73, 19, 41, 87.

**Solution.**

Demand/week (thousand) (x)	P(x)	Cumulative probability	Random numbers
0	0.05	0.05	00-04
1	0.05	0.1	05-09
2	0.1	0.2	10-19
3	0.4	0.6	20-59
4	0.35	0.95	60-94
5	0.05	1	95-99

Lead time (Weeks) (y)	P(y)	Cumulative probability	Random numbers
2	0.3	0.3	00-29
3	0.5	0.8	30-79
4	0.2	1	80-99

NOTES

The simulation is carried out in the following table :

NOTES

Week	Beginning Inventory	Demand		Ending inventory	Lead time		Quantity Ordered	Shortage
		RN	Unit		RN	Week		
0	3000	-	-	3000	-	-	-	-
1	3000	32	3000	0	73	3	1000	-
2	0	26	3000	0	-	-	-	3000
3	0	64	4000	0	-	-	-	4000
4	0	45	3000	0	-	-	-	3000
5	4000	12	2000	2000	19	2	2000	-
6	2000	99	5000	0	-	-	-	3000
7	0	52	3000	0	-	-	-	3000
8	2000	43	3000	0	41	3	4000	1000
9	0	84	4000	0	-	-	-	4000
10	0	38	3000	0	-	-	-	3000

## SIMULATION SOFTWARES

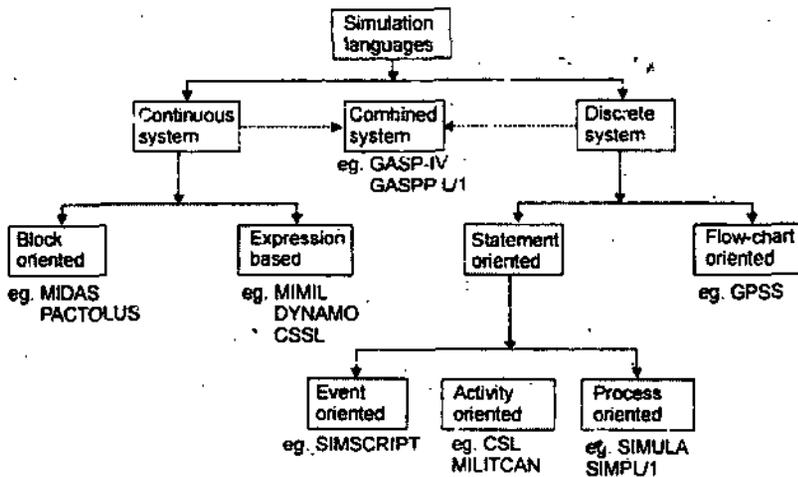
Computer implementation of simulation is costly because of the following reasons : (i) a large number of trials are required in order to arrive at a reliable result, (ii) the requirements of storage is large and (iii) complexity of program development is considerable.

The program development cost is greatly reduced if special purpose simulation languages are used for program development. These languages are simple to use.

Simulation languages are application oriented. Continuous systems are usually described by systems of differential equations. Languages like DYNAMO and CSMP are equipped to describe continuous state transition and have tools to solve differential equations. For discrete systems where the state changes occur in single discontinuous steps, languages like GPSS, GASP, SIMSCRIPT and SIMULA are appropriately used.

GPSS is a popular simulation language well suited for queueing system. Since its inception in 1962 by G. Gordon, GPSS has gone through many changes to increase its power and simplicity.

A classification of simulation languages is given below :



## NOTES

## SUMMARY

- A system is made up of elements or components that are related functionally to one another. An element can be treated as a part of the system or as an independent system.
- A system is often represented by a model that describes the system sufficiently in detail so that the behaviour of the system can be predicted from the behaviour of the model.
- A simulated model is built up by simulation which is the activity of artificially creating creating probabilistic processes to represent system components with the sole purpose of studying the behaviour of the system.
- Computer implementation of simulation is costly because of the following reasons: (i) a large number of trials are required in order to arrive at a reliable result, (ii) the requirements of storage is large and (iii) complexity of program development is considerable.
- Simulation languages are application oriented. Continuous systems are usually described by systems of differential equations. Languages like DYNAMO and CSMP are equipped to describe continuous state transition and have tools to solve differential equations.

## PROBLEMS

1. Give a scheme to generate random observations from (a) binomial distribution, (b) Poisson distribution.
2. Simulate the value of the integral  $\int_1^2 \frac{dx}{1+\sqrt{x}}$  by taking 20 random numbers from (0, 1).
3. Consider gamma (3, 2) and the following random numbers :  
0.35, 0.36, 0.77, 0.89, 0.39, 0.79, 0.52, 0.64, 0.75.  
Generate the first three random observations.

NOTES

4. Rework the Example 3 with the following set of random numbers :

<i>RN for X</i>	0.91	0.78	0.35	0.16	0.23	0.81	0.66
<i>RN for Y</i>	0.21	0.39	0.71	0.61	0.82	0.59	0.22

5. A company is evaluating an investment proposal which has uncertainty associated with the three important aspects : the original cost, the useful life and the annual net cash flows. The three probability distributions for these variables are shown below :

<i>Original</i>		<i>Useful life</i>		<i>Annual net cash inflows</i>	
<i>Value</i>	<i>Probability</i>	<i>Period</i>	<i>Probability</i>	<i>Value</i>	<i>Probability</i>
Rs. 60000	0.6	6 yr.	0.3	Rs. 10000	0.1
Rs. 70000	0.3	7 yr.	0.4	Rs. 20000	0.2
Rs. 80000	0.1	8 yr.	0.3	Rs. 30000	0.3
				Rs. 40000	0.4

The firm wants to perform five simulation runs of this projects life. To simulate, use the following sets of random numbers : 09, 84, 41, 92, 65; 24, 38, 74, 17, 45; 06, 48, 67, 54, 73 respectively.

6. People arrive at a bus stop with inter-arrival times  $3 \pm 1$  minutes. A bus arrives with inter-arrival times of  $\pm 5$  minutes. The bus has a capacity of 30 people and number of seats occupied when the bus arrives is equally likely to be any number from 0 to 30. The bus takes on board as many passengers as it can seat and passengers that cannot be seated walk away. Write a flow chart to simulate the arrival of 100 buses and count how many people do not get on board.

## CHAPTER 7 NETWORK SCHEDULING BY CPM/PERT

NOTES

### ★ STRUCTURE ★

- Introduction
- Time Calculations in Network
- Critical Path Method (CPM)
- Program Evaluations and Review Technique (Pert)
- Elements of Crashing A Network
- Summary
- Problems

### INTRODUCTION

Let us define 'Project'.

A project can be considered to be any series of activities and tasks that

- (i) have a specific objective to be completed within certain specifications.
- (ii) have defined start and end dates.
- (iii) have funding limits and consume resources.

A number of techniques have been developed to assist in planning, scheduling and control of projects. The most popular methods are the Critical Path Method (CPM) and the Program Evaluation and Review Technique (PERT). These techniques decompose the project into a number of activities, represent the precedence relationships among activities through a network and then determine a critical path through the network.

The basic concepts are described below :

#### (a) Activity

An activity is an item of work to be done that consumes time, effort, money or other resources. It is represented by an arrow. Tail represents start and head represents end of that activity.

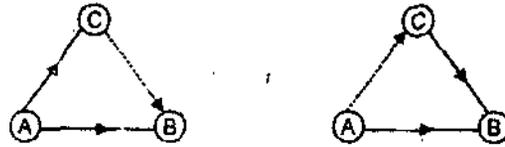
#### (b) Event/Node

It represents a point time signifying the completion of an activity and the beginning of another new activity. Here beginning of an activity represents tail event and end of an activity represents head event.

NOTES

**(c) Dummy Activity**

This shows only precedence relationship and they do not represent any real activity and is represented by a dashed line arrow or dotted line arrow and does not consume any time. e.g.,

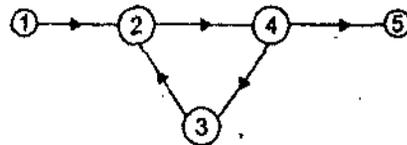


**(d) Rules for Construction of a Network**

- (1) Each activity is shown by one and only one arrow.
- (2) There will be only one beginning node/event and only one end node/event.
- (3) No two activities can be identified by the same head and tail events.
- (4) All events/node should be numbered distinctly.
- (5) Time flows from left to right.

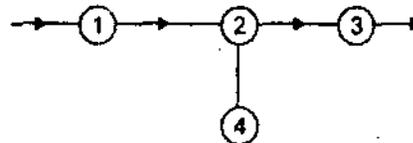
**(e) Common Errors in Network**

(1) Loops :



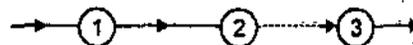
This situation can be avoided by checking the precedence relationship of the activities and by numbering them in a logical order.

(2) Dangling :



This situation can be avoided by keeping in mind that all events except the starting and ending event of the whole project must have at least one entering and one leaving activity. A dummy activity can be introduced to avoid this dangling.

(3) Redundancy :



The dummy activity is redundant and can be eliminated.

**(f) Critical Path**

It is the longest path in the project network. Any activity on this path is said to be critical in the sense that any delay of that activity will delay the completion time of the project.

# TIME CALCULATIONS IN NETWORK

Let  $t_{ij}$  be the duration of an activity  $(i, j)$ .

(a) *Earliest Start Time (ES)*. This is the earliest occurrence time of the event from which the activity emanates.

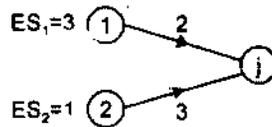
For the beginning event,  $ES_1 = 0$  and let  $ES_j = ES$  of all the activities emanating from node  $i$ . Then

$$ES_j = \text{Max}\{ES_i + t_{ij}\}$$

(b) *Earliest Finish/Completion Time (EF)*. This is the ES plus the activity duration

$$EF_j = ES_i + t_{ij}$$

For example, Consider a part of the network.



$$ES_j = \text{Max.}\{ES_1 + t_{1j}, ES_2 + t_{2j}\}$$

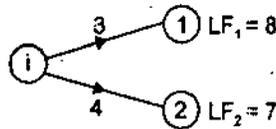
$$= \text{Max.}\{3 + 2, 1 + 3\} = 5$$

$$EF_1 = 3 + 2 = 5, EF_2 = 1 + 3 = 4.$$

(c) *Latest Finish/Completion Time (LF)*. This is the latest occurrence time of the event at which the activity terminates.

$$LF_i = \text{Min}_j\{LF_j - t_{ij}\}$$

For example, consider a part of the network



Then

$$LF_i = \text{Min.}\{LF_1 - t_{i1}, LF_2 - t_{i2}\}$$

$$= \text{Min.}\{8 - 3, 7 - 4\} = 3.$$

(d) *Latest Start Time (LS<sub>i</sub>)*. This is the last time at which the event can occur without delaying the completing of the project.

(e) *Total Floats (TF)*. It is a time duration in which an activity can be delayed without affecting the project completion time.

$$\therefore TF_{ij} = LF_j - ES_i - t_{ij}$$

$$= LF_j - (ES_i + t_{ij})$$

$$= LF_j - EF_{ij}$$

Also

$$TF_{ij} = LS_{ij} - ES_i$$

$$= (LF_j - t_{ij}) - ES_i$$

## NOTES

(f) *Free Floats (FF)*. It is a time duration in which the activity completion time can be delayed without affecting the earliest start time of immediate successor activities in the network.

## NOTES

$$\begin{aligned} FF_{ij} &= ES_j - ES_i - t_{ij} \\ &= ES_j - (ES_i + t_{ij}) \\ &= ES_j - EF_{ij} \end{aligned}$$

An activity ( $i, j$ ) is said to be critical if all the following conditions are satisfied:

$$ES_i = LF_i, ES_j = LF_j, ES_j - ES_i = LF_j - LF_i = t_{ij}$$

Thus any critical activity will have zero total float and zero free float.

(g) *Independent Floats*. It is defined as the difference between the free float and the tail slack.

Note. Slack is with reference to an event and float is with respect to an activity. Slack is generally used with PERT and float with CPM, but they may be used interchangeably used.

### CRITICAL PATH METHOD (CPM)

CPM was developed by E.I. duPont in 1957 and was first applied to construction and maintenance of chemical plants. Since then, the use of CPM has grown at a rapid rate. There are computer programs to perform the calculations.

Let the project network be drawn. Then this method consists of two phases calculations. In Phase 1, which is also called *forward pass*, Earliest start times (ES) of all the nodes are calculated.

In Phase 2, which is also called *backward pass*, Latest finish time (LF) of all the nodes are calculated.

These two calculations are displayed in the network diagram in a two chamber boxes. Upper chamber represents LF and the lower one as ES.

The critical activities (*i.e.*,  $ES = LF$ ) are identified. The critical path is obtained by joining them using double arrow.

**Example 1.** A project schedule has the following characteristics :

Activity	Time	Activity	Time
1-2	3	5-6	5
1-3	1	5-7	8
2-4	1	6-8	1
2-5	1	7-9	2
3-5	5	8-10	4
4-9	6	9-10	6

Draw the project network and find the critical path. Also calculate the total floats and free floats.

**Solution.**

NOTES

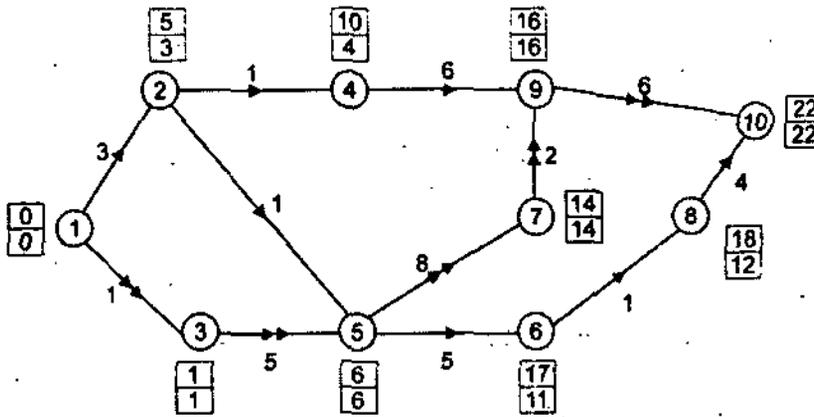


Fig.

- Set
- $ES_1 = 0$
  - $ES_2 = ES_1 + t_{12} = 0 + 3 = 3$
  - $ES_3 = ES_1 + t_{13} = 0 + 1 = 1$
  - $ES_4 = ES_2 + t_{24} = 4$
  - $ES_5 = \text{Max. } \{ES_3 + t_{35}, ES_2 + t_{25}\} = \text{Max. } \{6, 4\} = 6$
  - $ES_6 = ES_5 + t_{56} = 11$
  - $ES_7 = ES_5 + t_{57} = 14$
  - $ES_8 = ES_6 + t_{68} = 12$
  - $ES_9 = \text{Max. } \{ES_4 + t_{49}, ES_7 + t_{79}\} = \text{Max. } \{10, 16\} = 16$
  - $ES_{10} = \text{Max. } \{ES_9 + t_{910}, ES_8 + t_{810}\} = \text{Max. } \{22, 16\} = 22$
- Set
- $LF_{10} = ES_{10} = 22$
  - $LF_9 = LF_{10} - t_{910} = 22 - 6 = 16$
  - $LF_8 = LF_{10} - t_{810} = 22 - 4 = 18$
  - $LF_7 = LF_9 - t_{79} = 16 - 2 = 14$
  - $LF_6 = LF_8 - t_{68} = 17$
  - $LF_5 = \text{Min. } \{LF_7 - t_{57}, LF_6 - t_{56}\} = \text{Min. } \{6, 12\} = 6$
  - $LF_4 = LF_9 - t_{49} = 10$
  - $LF_3 = LF_5 - t_{35} = 1$
  - $LF_2 = \text{Min. } \{LF_4 - t_{24}, LF_5 - t_{25}\} = \text{Min. } \{9, 5\} = 5$
  - $LF_1 = \text{Min. } \{LF_3 - t_{13}, LF_2 - t_{12}\} = \text{Min. } \{0, 2\} = 0$

Activity (i, j)	Duration $t_{ij}$	Total Float $TF_{ij}$	Free float $FF_{ij}$
1 - 2	3	2	0
1 - 3	1	0	0
2 - 4	1	6	0

NOTES

2 - 5	1	2	2
3 - 5	5	0	0
4 - 9	6	6	6
5 - 6	5	6	0
5 - 7	8	0	0
6 - 8	1	6	0
7 - 9	2	0	0
8 - 10	4	6	6
9 - 10	6	0	0

The critical path is 1—3—5—7—9—10.

**Example 2.** Consider the following informations :

Activity	Immediate predecessors	Duration
A	None	2
B	None	3
C	A	1
D	B	4
E	C, D	3
F	D	1
G	E	2
H	F	3

Draw the project network and find the critical path.

**Solution.** The network is drawn below :

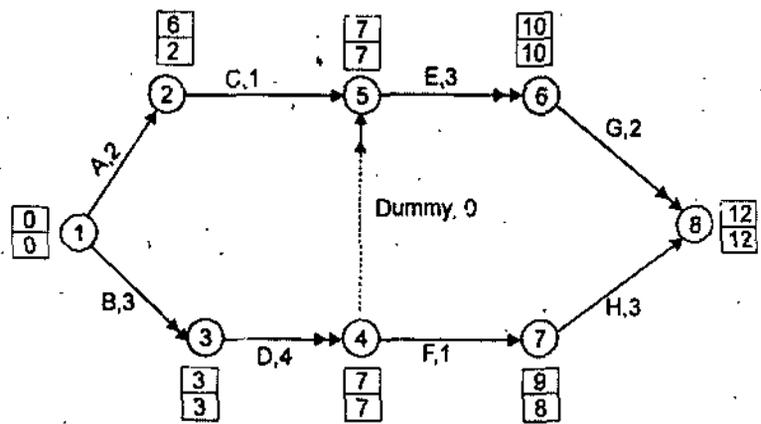


Fig.

Set

$$ES_1 = 0$$

$$ES_2 = ES_1 + t_{12} = 0 + 2 = 2$$

$$ES_3 = ES_1 + t_{13} = 0 + 3 = 3$$

$$ES_4 = ES_3 + t_{34} = 3 + 4 = 7$$

$$ES_5 = \text{Max. } \{ES_2 + t_{25}, ES_4 + t_{45}\}$$

$$= \text{Max } \{2 + 1, 7 + 0\} = 7$$

$$ES_6 = ES_5 + t_{56} = 10$$

$$ES_7 = ES_4 + t_{47} = 8$$

$$ES_8 = \text{Max. } \{ES_6 + t_{68}, ES_7 + t_{78}\}$$

$$= \text{Max } \{10 + 2, 8 + 3\} = 12$$

Set

$$LF_8 = ES_8 = 12$$

$$LF_7 = LF_8 - t_{78} = 12 - 3 = 9$$

$$LF_6 = LF_8 - t_{68} = 12 - 2 = 10$$

$$LF_5 = LF_6 - t_{56} = 10 - 3 = 7$$

$$LF_4 = \text{Min. } \{LF_5 - t_{54}, LF_7 - t_{47}\}$$

$$= \text{Min. } \{7, 8\} = 7$$

$$LF_3 = LF_4 - t_{34} = 7 - 4 = 3$$

$$LF_2 = LF_5 - t_{25} = 7 - 1 = 6$$

$$LF_1 = \text{Min. } \{LF_2 - t_{12}, LF_3 - t_{13}\}$$

$$= \text{Min. } \{4, 0\} = 0.$$

Thus the critical path is B—D—(dummy)—E—G.

NOTES

## PROGRAM EVALUATION AND REVIEW TECHNIQUE (PERT)

PERT was originally developed in 1958 to 1959 as part of the Polaris Fleet Ballistic Missile Program of the United States' Navy.

The primary difference between PERT and CPM is that PERT takes explicit account of the uncertainty in the activity duration estimates. CPM is activity oriented whereas PERT is event oriented. CPM gives emphasis on time and cost whereas PERT is primarily concerned with time.

In PERT, the probability distribution is specified by three estimates of the activity duration—a most likely duration ( $t_m$ ), an optimistic duration ( $t_o$ ) and a pessimistic duration ( $t_p$ ). This type of activity duration is assumed to follow the beta distribution with

$$\text{Mean} = \frac{t_o + 4t_m + t_p}{6}$$

and

$$\text{Variance} = \left( \frac{t_p - t_o}{6} \right)^2$$

The network construction phase of PERT is identical to that of CPM. Furthermore, once mean and variance are computed for each activity, the critical path determination is identical to CPM. The earliest and latest event times for the network are

random variables. Once the critical path is determined, probability statements may be made about the total project duration and about the slack at any event.

**Example 3.** A project consists of the following activities and different time estimates :

NOTES

Activity	$t_0$	$t_m$	$t_p$
1-2	3	5	8
1-3	2	4	8
1-4	6	8	12
2-5	5	9	12
3-5	3	5	9
4-6	3	6	10
5-6	2	4	8

- (a) Draw the network.
- (b) Determine the expected time and variance for each activity.
- (c) Find the critical path and the project variance.
- (d) What is the probability that the project will be completed by 22 days?

**Solution.** (a) Using the given information the resulting network is drawn as follows:

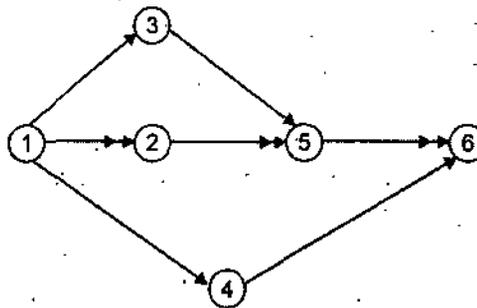


Fig.

(b) Expected time =  $\frac{t_0 + 4t_m + t_p}{6} = \bar{t}_{ij}$

$\bar{t}_{12} = 5.17$     $\bar{t}_{25} = 8.83$     $\bar{t}_{36} = 4.33$   
 $\bar{t}_{13} = 4.33$     $\bar{t}_{35} = 5.33$   
 $\bar{t}_{14} = 8.33$     $\bar{t}_{46} = 6.17$

Variance =  $\left(\frac{t_p - t_0}{6}\right)^2$

$\sigma_{12}^2 = 0.694$     $\sigma_{25}^2 = 1.361$     $\sigma_{36}^2 = 1$   
 $\sigma_{13}^2 = 1$     $\sigma_{35}^2 = 1$   
 $\sigma_{14}^2 = 1$     $\sigma_{46}^2 = 1.361$

(c) Set  $ES_1 = 0$

Then  $ES_2 = ES_1 + \bar{t}_{12} = 5.17$   
 $ES_3 = ES_1 + \bar{t}_{13} = 4.33$   
 $ES_4 = ES_1 + \bar{t}_{14} = 8.33$   
 $ES_5 = \text{Max} \{ES_3 + \bar{t}_{35}, ES_2 + \bar{t}_{25}\}$   
 $= \text{Max} \{9.66, 14\} = 14$   
 $ES_6 = \text{Max} \{ES_5 + \bar{t}_{56}, ES_4 + \bar{t}_{46}\}$   
 $= \text{Max} \{18.33, 14.5\} = 18.33$

Set  $LF_6 = ES_6 = 18.33$

Then  $LF_5 = LF_6 - \bar{t}_{56} = 14$   
 $LF_4 = LF_6 - \bar{t}_{46} = 12.16$   
 $LF_3 = LF_5 - \bar{t}_{35} = 8.67$   
 $LF_2 = LF_5 - \bar{t}_{25} = 5.17$   
 $LF_1 = \text{Min.} \{LF_3 - \bar{t}_{13}, LF_2 - \bar{t}_{12}, LF_4 - \bar{t}_{14}\}$   
 $= \text{Min.} \{4.34, 0, 3.83\} = 0.$

NOTES

Hence the critical path is (1)—(2)—(5)—(6)

$$\text{Project variance} = \sigma_{12}^2 + \sigma_{25}^2 + \sigma_{56}^2$$

$$= 0.694 + 1.361 + 1 = 3.055.$$

(d) Here mean project length is 18.33.

Set  $z = \frac{x - 18.33}{\sqrt{3.055}} \sim N(0, 1)$

For  $x = 22, z = 2.1$

Then the required probability  $= P(X \leq 22)$   
 $= P(z \leq 2.1)$   
 $= 0.5 + 0.4821$   
 $= 0.9821$

⇒ there is 98.21% chance that the project will be completed by 22 days.

**Example 4:** A PERT network consists of 10 activities. The precedence relationships and expected time and variance of activity times, in days, are given below:

Activity	a	b	c	d	e	f	g	h	i	j
Immediate predecessor (s)	-	a	a	-	b	c	d	d	e, f, g	h
Expected activity time	4	2	6	2	3	9	5	7	1	10
Variance of activity time	1	1	2	1	1	5	1	8	1	16

Construct an arrow diagram. Find the critical path based on expected times. Based on this critical path find the probability of completing the project in 25 days.

NOTES

**Solution.** The resulting network is given in Fig. 8.4.

Set

$$\begin{aligned}
 ES_1 &= 0 \\
 ES_2 &= ES_1 + \bar{t}_{12} = 4 \\
 ES_3 &= ES_2 + \bar{t}_{23} = 6 \\
 ES_4 &= ES_2 + \bar{t}_{24} = 10 \\
 ES_5 &= ES_1 + \bar{t}_{15} = 2 \\
 ES_6 &= \text{Max. } \{ES_3 + \bar{t}_{36}, ES_4 + \bar{t}_{46}, ES_5 + \bar{t}_{56}\} = 19 \\
 ES_7 &= ES_5 + \bar{t}_{57} = 9 \\
 ES_8 &= \text{Max. } \{ES_6 + \bar{t}_{68}, ES_7 + \bar{t}_{78}\} = 20
 \end{aligned}$$

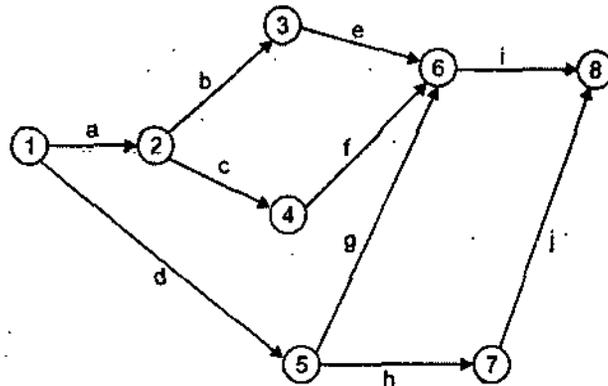


Fig.

Set

$$\begin{aligned}
 LF_8 &= 20 \\
 LF_7 &= LF_8 - \bar{t}_{78} = 10 \\
 LF_6 &= LF_8 - \bar{t}_{68} = 19 \\
 LF_5 &= \text{Min. } \{LF_7 - \bar{t}_{57}, LF_6 - \bar{t}_{56}\} = 3 \\
 LF_4 &= LF_6 - \bar{t}_{46} = 10 \\
 LF_3 &= LF_6 - \bar{t}_{36} = 16 \\
 LF_2 &= \text{Min. } \{LF_3 - \bar{t}_{23}, LF_4 - \bar{t}_{24}\} = 4 \\
 LF_1 &= \text{Min. } \{LF_2 - \bar{t}_{12}, LF_3 - \bar{t}_{13}\} = 0
 \end{aligned}$$

Hence the critical path is  $a \rightarrow c \rightarrow f \rightarrow i$  on which  $ES = LF$

$$\text{Total expected time} = 4 + 6 + 9 + 1 = 20$$

$$\text{Project variance} = 1 + 2 + 5 + 1 = 9$$

Set

$$z = \frac{x - 20}{3} \sim N(0, 1)$$

For  $x = 25, z = 1.67$

$$\begin{aligned}
 \text{Then the required probability} &= P(X \leq 25) \\
 &= P(z \leq 1.67) \\
 &= 0.5 + \Phi(1.67) \\
 &= 0.5 + 0.4525 \\
 &= 0.9525
 \end{aligned}$$

⇒ There is 95.25% chance that the project will be completed by 25 days.

NOTES

**PROBLEMS**

1. For a small project of 12 activities, the details are given below :

Activity	Dependence	Duration (days)
A	-	9
B	-	4
C	-	7
D	BC	8
E	A	7
F	C	5
G	E	10
H	E	8
I	D,F,H	6
J	L	9
K	I,J	10
L	G	2

- (a) Draw the network.
- (b) Find the critical path.

2. (a) Draw a network for the following project:

Activities	1-2	1-3	1-4	2-5	2-6	3-6	5-7	6-7	4-7
Time (days)	8	12	4	9	3	6	5	10	5

- (b) Determine total slack time for all activities and identify the critical path.
- (c) Calculate total float and free floats of each activities

3. Consider the following informations :

Job	1-2	2-3	2-4	3-4	3-5	3-6	4-5	5-6
Time (days)	10	9	7	6	9	10	6	7

- (a) Draw the network.
  - (b) Find the critical path.
  - (c) Calculate total floats and free floats of each activities.
4. Draw the network using the given precedence conditions. Calculate the critical path and floats (total and free).

NOTES

Activity	A	B	C	D	E	F	G	H	I	J
Immediate Predecessor(s)	-	A	A	A	D	D	E	F, G	C, H	B
Duration (months)	1	4	2	2	3	3	2	1	3	2

5. A project consists of eight activities with the following time estimates:

Activity	Immediate predecessor	Time (days)		
		$t_o$	$t_m$	$t_p$
A	-	1	1	7
B	-	1	4	7
C	-	2	2	8
D	A	1	1	1
E	B	2	5	14
F	C	2	5	8
G	D, E	3	6	15
H	F, G	1	2	3

- Draw PERT network.
  - Find the expected time for each activity.
  - Determine the critical path.
  - What is the probability that the project will be completed in (i) 22 days, (ii) 18 days ?
  - What project duration will have 95% chance of completion ?
6. Consider the following project :

Activity	Time estimates (in weeks)			Predecessor
	$t_o$	$t_m$	$t_p$	
A	3	6	9	None
B	2	5	8	None
C	2	4	6	A
D	2	3	10	B
E	1	3	11	B
F	4	6	8	C, D
G	1	5	15	E

Find the critical path and its standard deviation. What is the probability that the project will be completed by 18 weeks ?

7. A project has the following activities and other characteristics :

Activity	Preceding activity	Time estimates (in weeks)		
		$t_o$	$t_m$	$t_p$
A	-	4	7	16
B	-	1	5	15
C	A	6	12	30
D	A	2	5	8
E	C	5	11	17
F	D	3	6	15

G	B	3	9	27
H	E, F	1	4	7
I	G	4	19	28

NOTES

- (a) Draw the PERT diagram.  
 (b) Identify the critical path.  
 (c) Find the probability that the project is completed in 36 weeks.
8. A small project is composed of eight activities whose time estimates are given below :

Activity	Time estimates		Pessimistic
	Optimistic	Most likely	
0 - 1	2	3	10
0 - 2	4	5	6
1 - 2	0	3	0
1 - 3	9	7	8
1 - 4	1	5	9
2 - 5	3	5	19
3 - 4	0	0	0
4 - 5	1	3	5

- (a) Draw the project network.  
 (b) Compute the expected duration of each activity.  
 (c) Compute the variance of each activity.
9. Suppose the computer centre of your institute is planning to organize a national seminar. Consider the possible activities and prepare a PERT network for this seminar.
10. Draw the PERT network diagram using the given precedence conditions. Calculate the expected time, variance of each activity and the critical path.

Activity	Predecessor(s)	Duration (weeks)		
		$t_o$	$t_m$	$t_p$
A	-	1	2	3
B	-	1	2	8
C	A	6	7	8
D	B	1	2	3
E	A	1	4	7
F	C, D	1	5	9
G	C, D, E	1	2	3
H	F	1	2	9

**NOTES**

1. CP : A—E—H—I—K, Project length = 40.

2. (b) CP : (1) — (3) — (6) — (7)

(c)  $\mu$  : 6, 0, 19, 6, 7, 0, 6, 0, 19

FF : 0, 0, 0, 0, 7, 0, 6, 0, 19

3. (b) CP : (1) — (2) — (3) — (4) — (5) — (6)

TF : 0, 0, 8, 0, 3, 9, 0, 0

FF : 0, 0, 8, 0, 3, 9, 0, 0

4. A—D—E—G—H—I, Project length = 12 months.

5. (c) B—E—G—H (or, 1—3—5—6—7)

Variance =  $\frac{82}{9}$ , Mean project length = 19

(d)  $P(X \leq 22) = 0.8389$  or 83.89%

$P(X \leq 18) = 0.3707$  or 37.07%

(e) 23.97 or 24 days.

6. CP : A—C—F, expected duration = 16 weeks

Standard deviation = 1.374.

$P(X \leq 18) = 0.928$ .

7. (b) A—C—E—H (i.e., 1—2—4—6—7—8)

(c)  $P(X \leq 36) = 0.4207$ .

10. CP : A—C—F—H, expected duration = 17 weeks

Project variance = 1.568.

**ELEMENTS OF CRASHING A NETWORK**

Every activity may have two types of completion times—normal time and crash time. Accordingly costs are also two types i.e., normal cost and crash cost. Obviously, the crash cost is higher than the normal cost and the normal time is higher than the crash time.

Crashing of a network implies that crashing of activities. During crashing direct cost increases and there is a trade-off between direct cost and indirect cost. So the project can be crashed till the total cost is economical. The following procedures are carried out :

(a) Calculate the critical path (CP) with normal times of the activities.

(b) Calculate the slope as given below of each activity.

$$\text{Slope} = \frac{\text{Crashing cost} - \text{Normal cost}}{\text{Normal time} - \text{Crash time}}$$

- (c) Identify the critical activity with lowest slope.  
 (d) Compress that activity within crash limit. Compression time can also be calculated by taking min. (crash limit, free float limit).

If there are more than one critical path then select a common critical activity with least slope. If there is no such activity then select the critical activity with least slope from each critical path and compress them simultaneously within the crash limit.

- (e) Continue crashing until it is not possible to crash any more.  
 (f) Calculate the total cost (TC) after each crashing as follows:

$$TC = \text{Previous TC} + \text{Increase in direct cost} - \text{Decrease in indirect cost.}$$

If the current TC is greater than the previous TC then the crashing is uneconomical and stop. Suggest the previous solution as optimal crashing solution.

**Example 5.** A project consists of six activities with the following times and costs estimates :

Activity	Normal time (weeks)	Normal cost (Rs.)	Crash time (weeks)	Crash cost (Rs.)
1-2	9	400	7	900
1-3	5	500	3	800
1-4	10	450	6	1000
2-5	8	600	6	1000
3-5	7	1000	5	1300
4-5	9	900	6	1200

If the indirect cost per week is Rs. 120, find the optimal crashed project completion time.

**Solution.** The slope calculations and the crash limit are given in the following table :

Activity	Slope	Crash limit (weeks)
1-2	250	2
1-3	150	2
1-4	137.5	4
2-5	200	2
3-5	150	2
4-5	100	3

NOTES

### Iteration 1

The CP calculations are shown in Fig.

NOTES

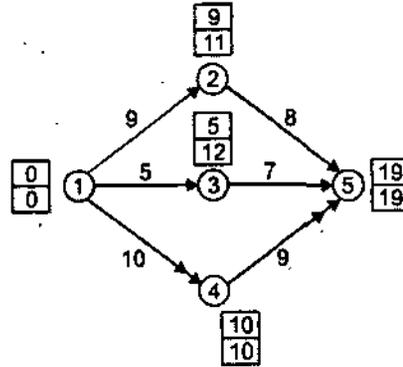


Fig.

CP : 1-4-5

Normal project duration = 19 weeks

Total direct (i.e., normal) cost = Rs. 3850

Indirect cost = Rs. (19 × 120) = Rs. 2280

Total Cost (TC) = Rs. 3850 + Rs. 2280 = Rs. 6130

The slopes and crash limits of critical activities are summarised below:

Critical activity	Slope	Crash limit (weeks)
1-4	137.5	4
4-5	100*	3

Since 100 is the minimum slope, crash the activity 4-5 by 1 week i.e., from 9 weeks to 8 weeks.

### Iteration 2

The CP calculations are shown in Fig.

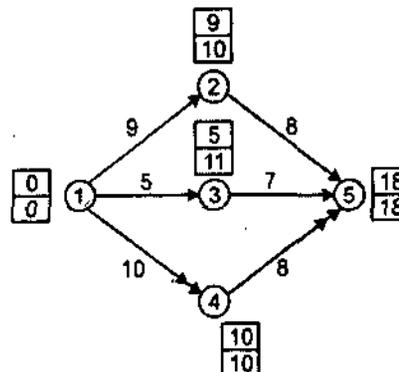


Fig.

CP : 1-4-5

Under the crashing, the project duration reduces to 18 weeks.

New TC = Rs. (6130 + 100 - 120) = Rs. 6110

Since the new TC is less than the previous TC, the present crashing is economical and proceed for further crashing.

The slopes and crash limits of critical activities are summarised below :

Critical activity	Slope	Crash limit (weeks)
1-4	137.5	4
4-5	100*	2

NOTES

Crash the activity 4-5 by 1 week i.e., from 8 weeks to 7 weeks.

### Iteration 3

The CF calculations are shown in Fig.

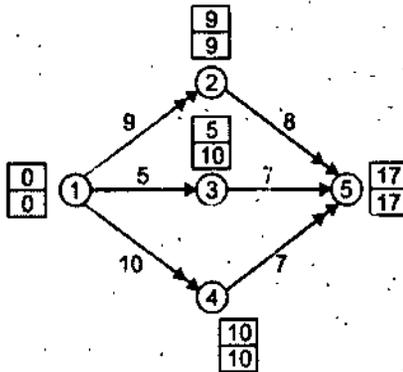


Fig.

We obtain two CPS : 1-4-5 and 1-2-5.

$$\text{New TC} = \text{Rs. } (6110 + 100 - 120) = \text{Rs. } 6090.$$

Since the new TC is less than the previous TC, the present crashing is economical and proceed for further crashing. The slopes and crash limits of critical activities are summarised below :

Critical activity	Slope	Crash limit (weeks)
1-4	137.5	4
4-5	100	1
1-2	250	2
2-5	200	2

Since there is no common critical activity, let us crash 4-5 by 1 week and 2-5 by 1 week.

### Iteration 4

The CP calculations are shown in Fig.

NOTES

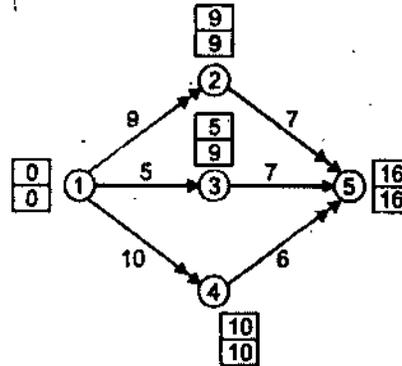


Fig.

New TC = Rs. 6090 + Rs. 100 + Rs. 200 - Rs. 120 = Rs. 6270.

Since the new TC is greater than previous TC, stop the iteration.

The previous iteration solution is the best for implementation.

Therefore, the final crashed project completion time is 17 weeks and the CPS are 1-2-5 and 1-4-5.

### SUMMARY

- An activity is an item of work to be done that consumes time, effort, money or other resources. It is represented by an arrow.
- An event represents a point time signifying the completion of an activity and the beginning of another new activity.
- Dummy Activity shows only precedence relationship and they do not represent any real activity and is represented by a dashed line arrow or dotted line arrow and does not consume any time.
- The primary difference between PERT and CPM is that PERT takes explicit account of the uncertainty in the activity duration estimates. CPM is activity oriented whereas PERT is event oriented. CPM gives emphasis on time and cost whereas PERT is primarily concerned with time.

### PROBLEMS

1. A project consists of seven activities with the following times and costs estimates:

Activity	Normal time (weeks)	Normal cost (Rs.)	Crash time (weeks)	Crash cost (Rs.)
1-2	12	500	8	900
1-3	6	600	5	700
1-4	8	700	5	850
2-5	11	500	10	820
3-5	7	1000	5	1200
4-6	6	900	4	1000
5-6	10	1200	8	1450

If the indirect cost per week is Rs. 150, find the optimal crashed project completion time.

2. Consider the data of a project as shown in the following table :

Activity	Normal time (weeks)	Normal cost (Rs.)	Crash time (weeks)	Crash cost (Rs.)
1-2	9	500	8	600
1-3	7	800	6	1100
1-4	8	900	6	1200
2-5	6	850	5	950
3-4	10	1200	8	1400
4-5	4	700	3	870
5-6	5	1000	4	1200

If the indirect cost per week is Rs. 160, find the optimal crashed project completion time.

NOTES

3. The table below provides the costs and times for a seven activity project :

Activity (i, j)	Time estimates (weeks)		Direct cost estimates (Rs. '000)	
	Normal	Crash	Normal	Crash
(1, 2)	2	1	10	15
(1, 3)	8	5	15	21
(2, 4)	4	3	20	24
(3, 4)	1	1	7	7
(3, 5)	2	1	8	15
(4, 6)	5	3	10	16
(5, 6)	6	2	12	36

- (i) Draw the project network corresponding to normal time.
- (ii) Determine the critical path and the normal duration and cost of the project.
- (iii) Crash the activities so that the project completion time reduces to 11 weeks irrespective of the costs.

## ANSWERS

1. Total crashed cost = Rs. 10100 with crashed project completion time = 27 weeks.
2. Total crashed cost = Rs. 9990 with crashed project completion time = 24 weeks.
3. (ii) CP : 1-3-5-6. Normal project duration 15 weeks and normal cost = Rs. 82000.  
(iii) All paths becomes CP.

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## CHAPTER 8    GAME THEORY

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## NOTES

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### INTRODUCTION

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The mathematical theory of games was invented by John Von Neumann and Oskar Morgenstern (1944). Game theory is the study of the ways in which strategic interactions among rational players produce **outcomes** with respect to the **preferences** (or utilities) of those players, none of which might have been intended by any of them.

Game theory has found its applications in various fields such as Economics, Social Science, Political Science, Biology, Computer Science etc.

The famous example of a game is the **Prisoner's Dilemma** game. Suppose that the police have arrested two people whom they know have committed an armed robbery together. Unfortunately they lack enough admissible evidence to get a jury to convict. They do, however, have enough evidence to send each prisoner away for two years for theft. The chief inspector now makes the following offer to each prisoner. If you will confess the robbery implicating your partner and he does not also confess, then you shall go free and he will get ten years. If you both confess, you shall each get 5 years. If neither of you confess, then you shall each get two years for the theft.

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### BASIC DEFINITIONS

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We assume that players are economically rational *i.e.*, a player can (i) assess outcomes, (ii) choose actions that yield their most preferred outcomes, given the actions of the other players.

(i) **Game** : All situations in which at least one player can only act to maximize his utility through anticipating the responses to his actions by one or more other players is called a game.

(ii) **Strategy** : A strategy is a possible course of action open to the player.

(iii) **Pure strategy** : A pure strategy is defined by a situation in which a course of action is played with probability one.

(iv) **Mixed strategy** : A mixed strategy is defined by a situation in which no course of action is taken with probability one.

(v) **Payoff matrix (or Reward matrix)** : A payoff matrix is an array in which any  $(i, j)$ th entry shows the outcome. Positive entry is the gain and negative entry is the loss for the row-player.

Matrix games are referred to as 'normal form' or 'strategic form' games, and games as trees are referred to as 'extensive form' games. The two sorts of games are not equivalent.

(vi) **Maximin criterion** : This is a criterion in which a player will choose the strategies with the largest possible payoff given an opponent's set of minimising countermoves.

(vii) **Minimax criterion** : This is a criterion in which a player will choose the strategies with the smallest possible payoff given an opponent's set of maximising countermoves.

(viii) **Saddle point** : If a payoff matrix has an entry that is simultaneously a maximum of row minima and a minimum of column maxima, then this entry is called a saddle point of the game and the game is said to be strictly determined.

(ix) **Value of the game** : If the game has a saddle point then the value at that entry is called the value of the game. If this value is zero then the game is said to be fair.

(x) **Zero-sum game** : A zero-sum game is a game in which the interests of the players are diametrically opposed *i.e.*, what one player wins the other loses. When two person play such game then it is called two person zero-sum game.

In this chapter we shall consider only matrix games.

**Note.** If in a game the total payoff to be divided among players is invariant *i.e.*, it does not depend upon the mix of strategies selected, then the game is called constant-sum game.

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## TWO-PERSON ZERO-SUM GAME WITH PURE STRATEGIES

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To identify the saddle point and value of game the following procedure to be adopted on the payoff matrix :

- (i) Identify the minimum from each row and place a symbol \* in that cell/entry.

Take the maximum of these minima.

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(ii) Identify the maximum from each column and place a symbol  $\times$  in that cell/entry.

Take the minimum of these maxima.

(iii) If both the symbols  $*$  and  $\times$  occurs in a/an cell/entry, then that cell/entry is called saddle point and the value in that cell/entry is called value of the game ( $v$ ).

Also  $v = \text{Maximum (row minima)} = \text{Minimum (column maxima)}$ . There may be more than one saddle point but the value of the game is unique.

**Example 1.** Solve the following game :

		Player B			
		B1	B2	B3	B4
Player A	A1	1	5	4	2
	A2	2	3	5	3
	A3	3	4	5	3

**Solution.** The calculations are displayed in the following table :

		Player B				Min.
		B1	B2	B3	B4	
Player A	A1	1*	5 $\times$	4	2	1
	A2	2*	3	5 $\times$	3 $\times$	2
	A3	3**	4	5 $\times$	3**	3
Max.		3	5	5	3	

$$\text{Max. (Row Min.)} = 3,$$

$$\text{Min. (Column Max.)} = 3$$

In the above game, there are two saddle points at (A3, B1) and (A3, B4).

The value of the game is 3. Here the optimal strategy for player A is A3 and the optimal strategy for player B is B1 and B4.

**Example 2.** Determine the solution of the following game :

		Player B				
		B1	B2	B3	B4	B5
Player A	A1	0	1	7	8	2
	A2	6	4	5	5	4
	A3	7	3	2	1	2
	A4	1	4	1	4	5

**Solution.** In the given game, player A has 4 strategies and player B has 5 strategies. The calculations are displayed in the following table :

	B1	B2	B3	B4	B5	Row Min.
A1	0*	1	7×	8×	2	0
A2	6	4*×	5	5	4*	4
A3	7×	3	2	1*	2	1
A3	1*	4×	1*	4	5×	1
Col. Max.	7	4	7	8	5	

Max. (Row Min.) = 4, Min. (Col Max.) = 4

In this game there is one saddle point at (A2, B2)

The value of the game is 4.

The optimal strategy for player A is A2

and the optimal strategy for player B is B2.

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## TWO-PERSON ZERO-SUM GAME WITH MIXED STRATEGIES

Consider the following game :

		Player B	
		I	II
Player A	I	$a_{11}$	$a_{12}$
	II	$a_{21}$	$a_{22}$

If this game does not have saddle point, then we assume that both players use mixed strategies.

Let player A select strategy I with probability  $p$  and strategy II with probability  $1 - p$ . Suppose player B select strategy I, then the expected gain to player A is given by  $a_{11} p + a_{21} (1 - p)$ .

If player B select strategy II, then the expected gain to player A is given by  $a_{12} p + a_{22} (1 - p)$ .

The optimal plan for player A requires that its expected gain to be equal for each strategies of player B. Thus we obtain

$$a_{11} p + a_{21} (1 - p) = a_{12} p + a_{22} (1 - p)$$

$$\Rightarrow p = \frac{a_{22} - a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}$$

Similarly, let player B selects strategy I with probability  $q$  and strategy II with probability  $1 - q$ . The expected loss to player B with respect to the strategies of player A are

$$a_{11} q + a_{12} (1 - q) \text{ and } a_{21} q + a_{22} (1 - q).$$

By equating the expected losses of player B we obtain

$$q = \frac{a_{22} - a_{12}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}$$

The value of game  $v$  is found by substituting the value of  $p$  in one of the equations for the expected gain of A and on simplification, we obtain

$$v = \frac{a_{11} a_{22} - a_{12} a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}$$

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**Example 3.** Determine the solution of the following game :

		Player B	
		B1	B2
Player A	A1	3	1
	A2	2	4

**Solution.** Clearly the given game has no saddle point. So the players have to use mixed strategies.

Let the mixed strategies for A as  $S_A = \begin{pmatrix} A1 & A2 \\ p_1 & p_2 \end{pmatrix}$

where  $p_2 = 1 - p_1$

and the mixed strategies for B as  $S_B = \begin{pmatrix} B1 & B2 \\ q_1 & q_2 \end{pmatrix}$

where  $q_2 = 1 - q_1$

$$\therefore p_1 = \frac{4 - 2}{(3 + 4) - (1 + 2)} = \frac{2}{4} = \frac{1}{2}, p_2 = 1 - p_1 = \frac{1}{2}$$

$$q_1 = \frac{4 - 1}{(3 + 4) - (1 + 2)} = \frac{3}{4}, q_2 = 1 - q_1 = \frac{1}{4}$$

$$v = \frac{12 - 2}{(3 + 4) - (1 + 2)} = \frac{10}{4} = \frac{5}{2}$$

Thus the optimal strategy for A is  $S_A = \begin{pmatrix} A1 & A2 \\ 1/2 & 1/2 \end{pmatrix}$

and for B is  $S_B = \begin{pmatrix} B1 & B2 \\ 3/4 & 1/4 \end{pmatrix}$

and the value of the game is  $5/2$ .

### DOMINANCE RULES

**(a) For rows :** (i) In the payoff matrix if all the entries in a row  $i_1$  are greater than or equal to the corresponding entries of another row  $i_2$ , then row  $i_2$  is said to be dominated by row  $i_1$ . In this situation row  $i_2$  of the payoff matrix can be deleted.

e.g.,  $i_2 = (1, 2, -1)$  is dominated by  $i_1 = (2, 2, 1)$ , hence  $(1, 2, -1)$  can be deleted.

(ii) If sum of the entries of any two rows is greater than or equal to the corresponding entry of a third row, then that third row is said to be dominated by the above two rows and hence third row can be deleted.

(b) For columns : (i) In the payoff matrix if all the entries in a column  $j_1$  are less than or equal to the corresponding entries of another column  $j_2$ , then column  $j_1$  is said to be dominated by column  $j_2$ . In this situation column  $j_1$  of the payoff matrix can be deleted.

e.g.,  $j_2 = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$  is dominated by  $j_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ . Hence  $\begin{pmatrix} 2 \\ 4 \end{pmatrix}$  can be deleted.

(ii) If sum of the entries of any two columns is less than or equal to the corresponding entry of a third column, then that third column is said to be dominated by the above two columns and hence third column can be deleted.

**Example 4.** Using the rules for dominance solve the following game :

		Player B		
		I	II	III
Player A	I	5	2	-2
	II	2	3	-1
	III	3	-2	3

**Solution .** The given game has no saddle point. Let us apply the rules for dominance. It is observed that column 1 is dominated by column 3. Hence delete column 1 and the payoff matrix is reduced as follows :

		II	III
I	-	2	-2
II	-	3	-1
III	-	-2	3

Again, row 1 is dominated by row 2. Hence delete row 1 and the payoff matrix is reduced to a  $2 \times 2$  matrix.

		II	III
II	-	3	-1
III	-	-2	3

Let the mixed strategy for player A be  $S_A = \begin{pmatrix} 1 & 0 \\ 0 & p_1 & p_2 \end{pmatrix}$  with  $p_2 = 1 - p_1$  and

the mixed strategy for player B be

$$S_B = \begin{pmatrix} 1 & 0 \\ 0 & q_1 & q_2 \end{pmatrix} \text{ with } q_2 = 1 - q_1$$

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$$p_1 = \frac{3 - (-2)}{(3 + 3) - (-1 - 2)} = \frac{5}{9}, p_2 = \frac{4}{9}$$

$$q_1 = \frac{3 - (-1)}{(3 + 3) - (-1 - 2)} = \frac{4}{9}, q_2 = \frac{5}{9}$$

$$v = \frac{9 - 2}{(3 + 3) - (-1 - 2)} = \frac{7}{9}$$

Hence the optimal mixed strategies are

$$S_A = \begin{pmatrix} I & II & III \\ 0 & 5/9 & 4/9 \end{pmatrix}$$

$$S_B = \begin{pmatrix} I & II & III \\ 0 & 4/9 & 5/9 \end{pmatrix}$$

$$v = 7/9$$

**Example 5.** Solve the following game:

		Player B			
		I	II	III	IV
Player A	I	4	3	0	3
	II	3	4	3	0
	III	4	3	4	3
	IV	0	5	4	4

**Solution.** The given game has no saddle point. Let us apply the rules for dominance to reduce the size of the payoff matrix. It is observed that row II is dominated by row III, hence row I can be deleted and the payoff matrix reduces as follows :

		I	II	III	IV
II	3	4	3	0	
III	4	3	4	3	
IV	0	5	4	4	

It is observed that column III is dominated by column I. Hence column III can be deleted. Also it is observed that column II is dominated by column IV. Hence column IV can also be deleted. Hence the payoff matrix reduces as follows :

		I	IV
II	3	0	
III	4	3	
IV	0	4	

Here row II is dominated by row III. Hence row II can be deleted and the payoff matrix reduces to  $2 \times 2$  matrix.

$$\begin{array}{c} \text{III} \\ \text{IV} \end{array} \begin{array}{cc} \text{I} & \text{IV} \\ \left[ \begin{array}{cc} 4 & 3 \\ 0 & 4 \end{array} \right] \end{array}$$

Let the mixed strategies for A be  $S_A = \begin{pmatrix} \text{I} & \text{II} & \text{III} & \text{IV} \\ 0 & 0 & p_1 & p_2 \end{pmatrix}$

with  $p_2 = 1 - p_1$

and the mixed strategies for B be  $S_B = \begin{pmatrix} \text{I} & \text{II} & \text{III} & \text{IV} \\ q_1 & 0 & 0 & q_2 \end{pmatrix}$

with  $q_2 = 1 - q_1$

$$p_1 = \frac{4 - 0}{(4 + 4) - (0 + 3)} = \frac{4}{5}, p_2 = 1 - p_1 = \frac{1}{5}$$

$$q_1 = \frac{4 - 3}{(4 + 4) - (0 + 3)} = \frac{1}{5}, q_2 = 1 - q_1 = \frac{4}{5}$$

$$v = \frac{16 - 0}{(4 + 4) - (0 + 3)} = \frac{16}{5}$$

Hence the optimal mixed strategies are

$$S_A = \begin{pmatrix} \text{I} & \text{II} & \text{III} & \text{IV} \\ 0 & 0 & 4/5 & 1/5 \end{pmatrix}$$

$$S_B = \begin{pmatrix} \text{I} & \text{II} & \text{III} & \text{IV} \\ 1/5 & 0 & 0 & 4/5 \end{pmatrix}$$

and  $v = \frac{16}{5}$ .

**Note.** If we add a fixed number  $x$  to each element of the payoff matrix, then the strategies remain unchanged while the value of the game is increased by  $x$ .

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## GRAPHICAL METHOD FOR GAMES

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(a) Let us consider a  $2 \times n$  game i.e., the payoff matrix will consist of 2 rows and  $n$  columns. So player A (or, row-player) will have two strategies. Also assume that there is no saddle point. Then the problem can be solved by using the following procedure :

- (i) Reduce the size of the payoff matrix using the rules of dominance, if it is applicable.
- (ii) Let  $p$  be the probability of selection of strategy I and  $1 - p$  be the probability of selection of strategy II by player A.

Write down the expected gain function of player A with respect to each of the strategies of player B.

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- (iii) Plot the gain functions on a graph. Keep the gain function on y-axis and  $p$  on x-axis. Here  $p$  will take the value 0 and 1.
- (iv) Find the highest intersection point in the lower boundary (i.e., lower envelope) of the graph. Since player A is a maximin player, then this point will be a maximin point.
- (v) If the number of lines passing through the maximin point is only two, then obtain a  $2 \times 2$  payoff matrix by retaining the columns corresponding to these two lines. Go to step (vii) else go to step (vi).
- (vi) If more than two lines passing through the maximin point then identify two lines with opposite slopes and form the  $2 \times 2$  payoff matrix as described in step (v).
- (vii) Solve the  $2 \times 2$  game.

**Example 6.** Consider the following game and solve it using graphical method.

		Player B				
		I	II	III	IV	V
Player A	I	3	1	6	-1	5
	II	-2	4	-1	2	1

**Solution.** It is observed that there is no saddle point. Column V is dominated by column I and column II is dominated by column IV. Therefore delete column V and column II and the payoff matrix is reduced as follows :

		Player B		
		I	III	IV
Player A	I	3	6	-1
	II	-2	-1	2

Let  $p$  be the probability of selection of strategy I and  $(1-p)$  be the probability of selection of strategy II by player A. Therefore, the expected gain (or payoff) function to player A with respect to different strategies of player B is given below :

B's strategy	A's expected gain function	A's expected gain	
		$p = 0$	$p = 1$
I	$3p - 2(1 - p) = 5p - 2$	-2	3
III	$6p - (1 - p) = 7p - 1$	-1	6
IV	$-p + 2(1 - p) = -3p + 2$	2	-1

Now the A's expected gain function is plotted in Fig. It is observed that line I and IV passes through the highest point of the lower boundary. Hence we can form  $2 \times 2$  payoff matrix by taking the columns due to I and IV for player A and it is displayed below :

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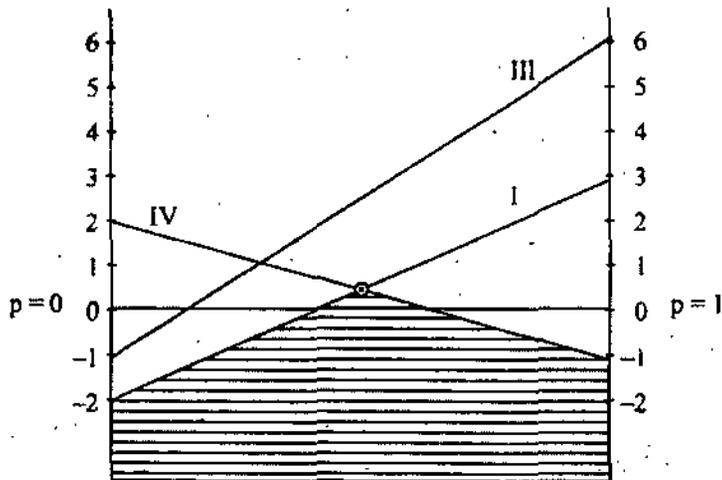


Fig.

		Player B	
		I	IV
Player A	I	3	-1
	II	-2	2

Let the mixed strategies for A be  $S_A = \begin{pmatrix} I & II \\ p_1 & p_2 \end{pmatrix}$

with  $p_2 = 1 - p_1$

and the mixed strategies for B be  $S_B = \begin{pmatrix} I & II & III & IV & V \\ q_1 & 0 & 0 & q_2 & 0 \end{pmatrix}$

with  $q_2 = 1 - q_1$

Therefore,

$$p_1 = \frac{2 - (-2)}{(3 + 2) - (-1 - 2)} = \frac{1}{2}, p_2 = 1 - p_1 = \frac{1}{2}$$

$$q_1 = \frac{2 - (-1)}{(3 + 2) - (-1 - 2)} = \frac{3}{8}, q_2 = 1 - q_1 = \frac{5}{8}$$

$$v = \frac{6 - 2}{(3 + 2) - (-1 - 2)} = \frac{1}{2}$$

The optimal mixed strategies for A is

$$S_A = \begin{pmatrix} I & II \\ 1/2 & 1/2 \end{pmatrix}$$

the optimal mixed strategies for B is

$$S_B = \begin{pmatrix} I & II & III & IV & V \\ 3/8 & 0 & 0 & 5/8 & 0 \end{pmatrix}$$

**NOTES**

and value of game. =  $\frac{1}{2}$ .

(b) Let us consider a  $m \times 2$  game i.e., the payoff matrix will consist of  $m$  rows and 2 columns. Also assume that there is no saddle point. Then the problem can be solved by using the following procedure :

- (i) Reduce the size of the payoff matrix using the rules of dominance, if it is applicable.
- (ii) Let  $q$  be the probability of selection of strategy I and  $1-q$  be the probability of selection of strategy II by the player B.  
Write down the expected gain function of player B with respect to each of the strategies of player A.
- (iii) Plot the gain functions on a graph. Keep the gain function on  $y$ -axis and  $q$  on  $x$ -axis. Here  $q$  will take the value 0 and 1.
- (iv) Find the lowest intersection point in the upper boundary (i.e., upper envelope) of the graph. Since player B is a minimax player, then this point will be a minimax point.
- (v) If the number of lines passing through the minimax point is only two, then obtain a  $2 \times 2$  payoff matrix by retaining the rows corresponding to these two lines. Go to step (vii) else goto step (vi).
- (vi) If more than two lines passing through the minimax point then identify two lines with opposite slopes and form a  $2 \times 2$  payoff matrix as described in step (v).
- (vii) Solve the  $2 \times 2$  game.

**Example 7.** Consider the following game and solve it using graphical method.

		Player B	
		I	II
Player A	I	2	1
	II	1	3
	III	4	-1
	IV	5	-2

**Solution.** The given game dose not have saddle point. Also it is observed that none of the rows can be deleted using the rules of dominance.

Let  $q$  be the probability of selection of strategy I and  $1 - q$  be the probability of selection of strategy II by player B. Therefore, the expected gain (or payoff) function to player B with respect to different strategies of player A is given below:

A's strategy	B's expected gain function	B's expected gain	
		$q = 0$	$q = 1$
I	$2q + (1 - q) = q + 1$	1	2
II	$q + 3(1 - q) = -2q + 3$	3	1
III	$4q - (1 - q) = 5q - 1$	-1	4
IV	$5q - 2(1 - q) = 7q - 2$	-2	5

Now the B's expected gain function is plotted in Fig.

It is observed that the line II and IV passes through the lowest point of the upper boundary. Hence we can form  $2 \times 2$  payoff matrix by taking the rows due to II and IV for player B and it is displayed below :

		Player B	
		I	II
Player A	II	1	3
	IV	5	-2

Let the mixed strategies for A be  $S_A = \begin{pmatrix} 1 & 0 \\ p_1 & p_2 \end{pmatrix}$  (I II III IV)

with  $p_2 = 1 - p_1$

and the mixed strategies for B be  $S_B = \begin{pmatrix} q_1 & q_2 \end{pmatrix}$  (I II)

with  $q_2 = 1 - q_1$

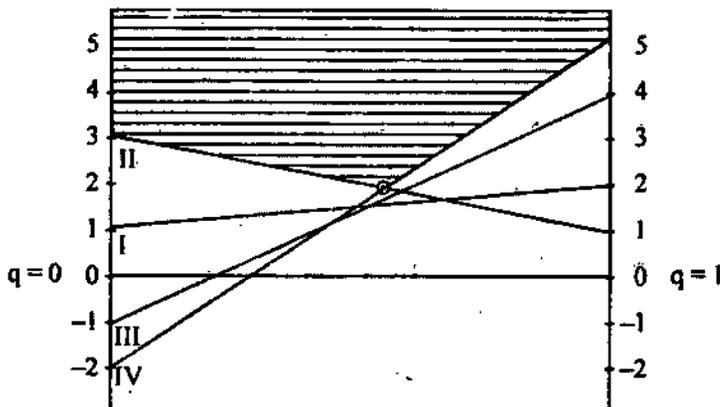


Fig.

Therefore,

$$p_1 = \frac{-2 - 5}{(1 - 2) - (5 + 3)} = \frac{7}{9}, p_2 = 1 - p_1 = \frac{2}{9}$$

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$$q_1 = \frac{-2-3}{(1-2)-(5+3)} = \frac{5}{9}, q_2 = 1 - q_1 = \frac{4}{9}$$

$$v = \frac{-2-15}{(1-2)-(5+3)} = \frac{17}{9}$$

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∴ The optimal mixed strategies for A is

$$S_A = \begin{pmatrix} I & II & III & IV \\ 0 & 7/9 & 0 & 2/9 \end{pmatrix}$$

the optimal mixed strategies for B is

$$S_B = \begin{pmatrix} I & II \\ 5/9 & 4/9 \end{pmatrix}$$

and value of game =  $\frac{1}{2}$

### LINEAR PROGRAMMING METHOD FOR GAMES

The linear programming method is used in solving mixed strategies games of dimensions greater than  $(2 \times 2)$  size. Consider an  $m \times n$  payoff matrix in which player A (i.e., the row player) has  $m$  strategies and player B (i.e., the column player) has  $n$  strategies.

The elements of payoff matrix be  $\{(a_{ij}); i = 1, 2, \dots, m \text{ and } j = 1, 2, \dots, n\}$ .

Let  $p_i$  be the probability of selection of strategy  $i$  by player A and  $q_j$  be the probability of selection of strategy  $j$  by player B.

#### LPP FOR PLAYER A

B's strategy	Expected gain function for A
1	$\sum_{i=1}^m a_{i1} p_i$
2	$\sum_{i=1}^m a_{i2} p_i$
...	...
$n$	$\sum_{i=1}^m a_{in} p_i$

Let 
$$v = \text{Min.} \left\{ \sum_i a_{i1} p_i, \sum_i a_{i2} p_i, \dots, \sum_i a_{in} p_i \right\}$$

Since the player A is maximin type, the LPP can be written as follows :

Maximize  $v$

Subject to, 
$$\sum_i a_{i1} p_i \geq v$$

$$\sum_i a_{i2} p_i \geq v$$

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$$\sum_i a_{im} p_i \geq v$$

$$p_1 + p_2 + \dots + p_m = 1$$

$$\text{all } p_i \geq 0$$

⇒ Maximize  $v$

Subject to, 
$$\sum_i a_{i1} (p_i / v) \geq 1$$

$$\sum_i a_{i2} (p_i / v) \geq 1$$

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$$\sum_i a_{im} (p_i / v) \geq 1$$

$$\frac{p_1}{v} + \frac{p_2}{v} + \dots + \frac{p_m}{v} = 1$$

$$\text{all } p_i \geq 0$$

Set  $p_i / v = x_i, i = 1, 2, \dots, m$ . Therefore

$$\text{Maximize } v = \text{Minimize } \left(\frac{1}{v}\right)$$

$$= \text{Minimize } \left(\frac{p_1}{v} + \frac{p_2}{v} + \dots + \frac{p_m}{v}\right)$$

$$= \text{Minimize } (x_1 + x_2 + \dots + x_m)$$

Subject to, 
$$\sum_i a_{i1} x_i \geq 1$$

$$\sum_i a_{i2} x_i \geq 1$$

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$$\sum_i a_{im} x_i \geq 1$$

and  $x_i \geq 0, i = 1, 2, \dots, m$ .

**I.P.P FOR PLAYER B**

A's strategy	Expected loss/gain function to B
1	$\sum_j a_{1j} q_j$
2	$\sum_j a_{2j} q_j$
...	...
m	$\sum_j a_{mj} q_j$

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Let 
$$u = \text{Max.} \left\{ \sum_j a_{1j} q_j, \sum_j a_{2j} q_j, \dots, \sum_j a_{mj} q_j \right\}$$

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Since the player B is minimax type, the LPP can be written as follows :

Minimize  $u$

Subject to, 
$$\sum_j a_{1j} q_j \leq u$$

$$\sum_j a_{2j} q_j \leq u$$

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$$\sum_j a_{mj} q_j \leq u$$

$$q_1 + q_2 + \dots + q_n = 1$$

all  $q_j \geq 0$

$\Rightarrow$  Minimize  $u$

Subject to, 
$$\sum_j a_{1j} (q_j / u) \leq 1$$

$$\sum_j a_{2j} (q_j / u) \leq 1$$

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$$\sum_j a_{mj} (q_j / u) \leq 1$$

$$\frac{q_1}{u} + \frac{q_2}{u} + \dots + \frac{q_n}{u} = 1$$

all  $q_j \geq 0$

Set  $q_j / u = y_j, j = 1, 2, \dots, n$ . Therefore

$$\text{Minimize } u = \text{Maximize} \left( \frac{1}{u} \right)$$

$$= \text{Maximize} \left( \frac{q_1}{u} + \frac{q_2}{u} + \dots + \frac{q_n}{u} \right)$$

$$= \text{Maximize} (y_1 + y_2 + \dots + y_n)$$

Subject to, 
$$\sum_j a_{1j} y_j \leq 1$$

$$\sum_j a_{2j} y_j \leq 1$$

---


$$\sum_j a_{mj} y_j \leq 1$$

and  $y_j \geq 0, j = 1, 2, \dots, n$ .

**Note.** 1. In the above approach we may face two problems. Value of the game may be zero or less than zero. First case constraints will become infinite and in the second case, the type of each constraint will get changed. Therefore to obtain a non-negative value of the game, a constant  $c = \text{Max. (abs. (negative values))} + 1$  is to be added to each elements in the payoff matrix. The optimal strategy will not change. However the value of the original game will be the value of the new game minus constant.

2. The above LPP formulations for player A and B are primal-dual pair. So solving one problem, we can read the solution of the other problem from the optimal table.

**Example 8.** Solve the following game by linear programming technique :

$$\begin{array}{c} \text{Player B} \\ \text{Player A} \end{array} \begin{bmatrix} -1 & -1 & 1 \\ -1 & 1 & 2 \\ 1 & 1 & -1 \end{bmatrix}$$

**Solution.** The game has no saddle point. Since the payoff matrix has negative values, let us add a constant  $c = 2$  to each element. The revised payoff matrix is given below :

$$\begin{array}{c} \text{Player B} \\ \text{Player A} \end{array} \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & 4 \\ 3 & 3 & 1 \end{bmatrix}$$

Let the strategies of the two players be

$$S_A = \begin{pmatrix} \text{I} & \text{II} & \text{III} \\ p_1 & p_2 & p_3 \end{pmatrix}, \quad S_B = \begin{pmatrix} \text{I} & \text{II} & \text{III} \\ q_1 & q_2 & q_3 \end{pmatrix}$$

where  $p_1 + p_2 + p_3 = 1$  and  $q_1 + q_2 + q_3 = 1$ .

The LPP for player A:

$$\text{Maximize } v = \text{Minimize } \frac{1}{v} = x_1 + x_2 + x_3$$

Subject to,  $x_1 + x_2 + 3x_3 \geq 1$

$$x_1 + 3x_2 + 3x_3 \geq 1$$

$$3x_1 + 4x_2 + x_3 \geq 1$$

$$x_1, x_2, x_3 \geq 0$$

where  $x_j = p_j/v, j = 1, 2, 3$ .

The LPP for player B:

$$\text{Minimize } u = \text{Maximize } \frac{1}{u} = y_1 + y_2 + y_3$$

Subject to,  $y_1 + y_2 + 3y_3 \leq 1$

$$y_1 + 3y_2 + 4y_3 \leq 1$$

$$3y_1 + 3y_2 + y_3 \leq 1$$

$$y_1, y_2, y_3 \geq 0$$

**NOTES**

where  $y_j = q/u, j = 1, 2, 3.$

Let us now solve the problem for player B.

The standard form can be written as follows :

NOTES

$$\text{Maximize } \frac{1}{u} = y_1 + y_2 + y_3 + 0.s_1 + 0.s_2 + 0.s_3$$

$$\text{Subject to, } y_1 + y_2 + 3y_3 + s_1 = 1$$

$$y_1 + 3y_2 + 4y_3 + s_2 = 1$$

$$3y_1 + 3y_2 + y_3 + s_3 = 1$$

$$y_1, y_2, y_3 \geq 0, s_1, s_2, s_3 \text{ slacks } \geq 0.$$

Iteration 1

			$c_j$	1	1	1	0	0	0	Min.
$c_B$	$x_B$	Soln.	$y_1$	$y_2$	$y_3$	$s_1$	$s_2$	$s_3$	Ratio	
0	$s_1$	1	1	1	3	1	0	0	1	
0	$s_2$	1	1	3	4	0	1	0	1	
0	$s_3$	1	3	3	1	0	0	1	1/3	→
$z_j - c_j$			-1	-1	-1	0	0	0		

↑

Iteration 2

			$c_j$	1	1	1	0	0	0	Min.
$c_B$	$x_B$	Soln.	$y_1$	$y_2$	$y_3$	$s_1$	$s_2$	$s_3$	Ratio	
0	$s_1$	2/3	0	0	8/3	1	0	-1/3	$\frac{2}{8} = \frac{1}{4}$	
0	$s_2$	2/3	0	2	11/3	0	1	-1/3	2/11	→
1	$y_1$	1/3	1	1	1/3	0	0	1/3	1	
$z_j - c_j$			0	0	-2/3	0	0	1/3		

↑

Iteration 3

			$c_j$	1	1	1	0	0	0	Min.
$c_B$	$x_B$	Soln.	$y_1$	$y_2$	$y_3$	$s_1$	$s_2$	$s_3$	Ratio	
0	$s_1$	2/11	0	-16/11	0	1	-8/11	-1/11		
1	$y_3$	2/11	0	6/11	1	0	3/11	-1/11		
1	$y_1$	3/11	1	9/11	0	0	-1/11	4/11		
$z_j - c_j$			0	4/11	0	0	2/11	3/11		

$$\therefore y_1^* = \frac{3}{11}, y_2^* = 0, y_3^* = \frac{2}{11}, \text{Max. } \frac{1}{u} = \frac{5}{11} \Rightarrow u^* = \frac{11}{5} = v^*$$

$$\text{original } u^* = \frac{11}{5} - 2 = \frac{1}{5} = \text{original } v^*$$

Using duality,  $x_1^* = 0, x_2^* = \frac{2}{11}, x_3^* = \frac{3}{11}$

Now,  $q_1 = y_1 \cdot u = \frac{3}{11} \cdot \frac{11}{5} = \frac{3}{5}, p_1 = x_1 \cdot v = 0$

$$q_2 = y_2 \cdot u = 0, p_2 = x_2 \cdot v = \frac{2}{11} \cdot \frac{11}{5} = \frac{2}{5}$$

$$q_3 = y_3 \cdot u = \frac{2}{11} \cdot \frac{11}{5} = \frac{2}{5}, p_3 = x_3 \cdot v = \frac{3}{11} \cdot \frac{11}{5} = \frac{3}{5}$$

$$S_A = \begin{pmatrix} I & II & III \\ 0 & 2/5 & 3/5 \end{pmatrix}$$

$$S_B = \begin{pmatrix} I & II & III \\ 3/5 & 0 & 2/5 \end{pmatrix} \text{ and } v^* = \frac{1}{5}$$

### SUMMARY

- All situations in which at least one player can only act to maximize his utility through anticipating the responses to his actions by one or more other players is called a game.
- A mixed strategy is defined by a situation in which no course of action is taken with probability one.
- If a payoff matrix has an entry that is simultaneously a maximum of row minima and a minimum of column maxima, then this entry is called a **saddle point** of the game and the game is said to be **strictly determined**.
- If the game has a saddle point then the value at that entry is called the value of the game.
- A zero-sum game is a game in which the interests of the players are diametrically opposed *i.e.*, what one player wins the other loses. When two person play such game then it is called **two person zero-sum game**.
- The linear programming method is used in solving mixed strategies games of dimensions greater than  $(2 \times 2)$  size.

### PROBLEMS

1. Solve the following games :

(a)

		Player B			
		B1	B2	B3	B4
Player A	A1	0	1	-4	6
	A2	3	4	4	5
	A3	2	0	3	-2

NOTES

(b)

	B1	B2	B3	B4	B5
A1	2	0	-2	0	5
A2	1	1	4	1	2
A3	5	0	-2	-3	-2
A4	-4	-2	4	4	2

(c)

	B1	B2	B3	B4	B5
A1	2	2	1	3	5
A2	3	4	7	5	2
A3	5	5	8	5	6
A4	2	8	0	2	9

(d)

	B1	B2	B3	B4	B5	B6
A1	3	2	-1	1	5	-2
A2	2	4	0	-2	3	1
A3	5	1	2	1	2	3

2. Determine the values of  $a$  and  $b$  such that the following game is determinable :

	B1	B2	B3
A1	2	$a$	3
A2	-3	2	$b$
A3	1	5	4

3. Determine the value of  $a$  such that the following game is determinable :

	B1	B2	B3
A1	$a$	1	2
A2	0	-2	$a$
A3	3	$a$	1

4. Solve the following two person zero-sum games :

(a)

		Player B	
		B1	B2
Player A	A1	2	4
	A2	5	3

(b)

		Player B	
		B1	B2
Player A	A1	-1	2
	A2	3	-1

5. Using the rules for dominance solve the following games :

(a)

		Player B			
		I	II	III	IV
Player A	I	3	0	-3	1
	II	11	2	0	5
	III	-1	5	1	-4
	IV	6	6	-4	1

(b)

		Player B		
		I	II	III
Player A	I	3	8	2
	II	8	3	7
	III	7	2	6

NOTES

(c)

	B1	B2	B3	B4
A1	-1	6	2	1
A2	3	2	4	5
A3	2	0	3	6

(d)

	I	II	III	IV	V
I	-3	3	2	-5	4
II	-3	4	7	0	7
III	0	1	4	8	7

(e)

	B1	B2	B3	B4	B5
A1	0	0	0	0	0
A2	4	2	0	2	1
A3	4	3	1	3	2
A4	4	3	4	-1	2

(f)

	B1	B2	B3	B4	B5
A1	8	7	6	-1	2
A2	12	10	12	0	4
A3	14	6	8	14	16

6. Use dominance property to reduce the game in  $2 \times 4$  game and then solve graphically

		Player B			
		I	II	III	IV
Player A	I	4	0	5	-1
	II	0	2	-1	3
	III	-2	0	-3	1

7. Solve the following games graphically :

(a)

		Player B		
		I	II	III
Player A	I	3	8	5
	II	6	2	7
	III	4	5	6

(b)

	B1	B2	B3	B4
A1	2	2	3	-1
A2	4	3	2	6

(c)

	B1	B2
A1	2	7
A2	3	5
A3	11	2

(d)

	B1	B2
A1	1	-3
A2	3	5
A3	-1	6
A4	4	1
A5	2	2
A6	-5	0

NOTES

8. Solve the following games by linear programming :

(a) 
$$\begin{array}{c} \text{Player B} \\ \begin{bmatrix} 1 & 6 \\ 4 & -5 \\ -5 & 3 \end{bmatrix} \\ \text{Player A} \end{array}$$

(b) 
$$\begin{array}{c} \text{Player B} \\ \begin{array}{ccc} \text{B1} & \text{B2} & \text{B3} \\ \text{Player A} & \begin{bmatrix} 1 & -1 & 2 \\ -2 & 3 & -1 \end{bmatrix} \end{array} \end{array}$$

(c) 
$$\begin{array}{c} \text{Player B} \\ \begin{bmatrix} 2 & -4 \\ 3 & 1 \\ 0 & 2 \end{bmatrix} \\ \text{Player A} \end{array}$$

(d) 
$$\begin{array}{c} \text{Player B} \\ \begin{bmatrix} -2 & 3 & 2 \\ 1 & -2 & 5 \end{bmatrix} \\ \text{Player A} \end{array}$$

(e) 
$$\begin{array}{c} \text{Player B} \\ \begin{bmatrix} -1 & 2 & 1 \\ 1 & -2 & 1 \\ 2 & 2 & -3 \end{bmatrix} \\ \text{Player A} \end{array}$$

(f) 
$$\begin{array}{c} \text{Player B} \\ \begin{bmatrix} 1 & -1 & 2 \\ -1 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix} \\ \text{Player A} \end{array}$$

(g) 
$$\begin{array}{c} \text{Player B} \\ \begin{bmatrix} 4 & 2 & 6 \\ 6 & 8 & 0 \\ 9 & 5 & 1 \end{bmatrix} \\ \text{Player A} \end{array}$$

9. Two competitive brands rely on advertising for securing a greater shares of the market. They select three media: TV, Newspaper and Mobile Phone. The expected change in their market shares depends on the type of media chosen. Consider the following payoff matrix of brand A:

		Brand B		
		TV	Newspaper	Mobile Phone
Brand A	TV	3	-2	4
	Newspaper	-1	4	2
	Mobile Phone	2	2	-6

Find the optimal solution for both brands.

10. An MNC has decided to establish a plant in either Singapore, Denmark or India. The degree of competition in the next five years is not certain. The company's expected return will depend on whether this competition is weak, mild or strong as shown in the following matrix:

	Weak	Mild	Strong
Singapore	16	13	4
Denmark	13	11	6
India	11	9	8

If the company's managing board is conservative, where should they decide to establish the plant?

## ANSWERS

1. (a) Saddle Point (A2, B1),  $v = 3$   
 (b) Saddle Point (A2, B2),  $v = 1$   
 (c) Saddle Point (A3, B1) and (A3, B4),  $v = 5$   
 (d) Saddle Point (A3, B4),  $v = 11$
2.  $2 < a < 5, -3 < b < 4$  [Hint: ignore  $a$  and  $b$  to find saddle point]
3.  $a = 1$  [Saddle point at (A1, B2)]
4. (a)  $S_A = \begin{pmatrix} A1 & A2 \\ 1/2 & 1/2 \end{pmatrix}, S_B = \begin{pmatrix} B1 & B2 \\ 1/4 & 3/4 \end{pmatrix}, v = 3.5$   
 (b)  $S_A = \begin{pmatrix} A1 & A2 \\ 4/7 & 3/7 \end{pmatrix}, S_B = \begin{pmatrix} B1 & B2 \\ 3/7 & 4/7 \end{pmatrix}, v = \frac{5}{7}$
5. (a)  $S_A = \begin{pmatrix} I & II & III & IV \\ 0 & 1/2 & 1/2 & 0 \end{pmatrix}, S_B = \begin{pmatrix} I & II & III & IV \\ 0 & 0 & \frac{9}{10} & \frac{1}{10} \end{pmatrix}, v = \frac{1}{2}$   
 (b)  $S_A = \begin{pmatrix} I & II & III \\ 2/5 & 3/5 & 0 \end{pmatrix}, S_B = \begin{pmatrix} I & II & III \\ 0 & 1/2 & 1/2 \end{pmatrix}, v = 5$   
 (c)  $S_A = \begin{pmatrix} A1 & A2 & A3 \\ 1/8 & 7/8 & 0 \end{pmatrix}, S_B = \begin{pmatrix} B1 & B2 & B3 & B4 \\ 1/2 & 1/2 & 0 & 0 \end{pmatrix}, v = 2.5$   
 (d)  $S_A = \begin{pmatrix} I & II & III \\ 0 & 0 & 1 \end{pmatrix}, S_B = \begin{pmatrix} I & II & III & IV & V \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}, v = 0$   
 (e)  $S_A = \begin{pmatrix} A1 & A2 & A3 & A4 \\ 0 & 0 & \frac{5}{7} & \frac{2}{7} \end{pmatrix}, S_B = \begin{pmatrix} B1 & B2 & B3 & B4 & B5 \\ 0 & 0 & \frac{4}{7} & \frac{3}{7} & 0 \end{pmatrix}, v = \frac{13}{7}$   
 (f)  $S_A = \begin{pmatrix} A1 & A2 & A3 \\ 0 & \frac{4}{9} & \frac{5}{9} \end{pmatrix}, S_B = \begin{pmatrix} B1 & B2 & B3 & B4 & B5 \\ 0 & \frac{7}{9} & 0 & \frac{2}{9} & 0 \end{pmatrix}, v = \frac{70}{9}$
6.  $S_A = \begin{pmatrix} I & II & III \\ 3/8 & 5/8 & 0 \end{pmatrix}, S_B = \begin{pmatrix} I & II & III & IV \\ 0 & 3/4 & 1/4 & 0 \end{pmatrix}, v = \frac{5}{4}$
7. (a)  $S_A = \begin{pmatrix} I & II & III \\ 4/9 & 5/9 & 0 \end{pmatrix}, S_B = \begin{pmatrix} I & II & III \\ 2/3 & 1/3 & 0 \end{pmatrix}, v = \frac{14}{3}$   
 (b)  $S_A = \begin{pmatrix} A1 & A2 \\ 1/2 & 1/2 \end{pmatrix}, S_B = \begin{pmatrix} B1 & B2 & B3 & B4 \\ 0 & 1/2 & 1/2 & 0 \end{pmatrix}, v = \frac{5}{2}$   
 (c)  $S_A = \begin{pmatrix} A1 & A2 & A3 \\ 9/14 & 0 & 5/14 \end{pmatrix}, S_B = \begin{pmatrix} B1 & B2 \\ 5/14 & 9/14 \end{pmatrix}, v = \frac{73}{14}$   
 (d)  $S_A = \begin{pmatrix} A1 & A2 & A3 & A4 & A5 & A6 \\ 0 & 3/5 & 0 & 2/5 & 0 & 0 \end{pmatrix}, S_B = \begin{pmatrix} B1 & B2 \\ 4/5 & 1/5 \end{pmatrix}, v = \frac{17}{5}$

### NOTES

NOTES

$$8. (a) S_A = \begin{pmatrix} I & II & III \\ 9/14 & 5/14 & 0 \end{pmatrix}, S_B = \begin{pmatrix} I & II \\ 11/14 & 3/14 \end{pmatrix}, v = \frac{29}{14} \text{ (3lt)}$$

$$(b) S_A = \begin{pmatrix} A1 & A2 \\ 5/7 & 2/7 \end{pmatrix}, S_B = \begin{pmatrix} B1 & B2 & B3 \\ 4/7 & 3/7 & 0 \end{pmatrix}, v = \frac{1}{7} \text{ (4lt)}$$

$$(c) S_A = \begin{pmatrix} A1 & A2 & A3 \\ 1/2 & 1/2 & 0 \end{pmatrix}, S_B = \begin{pmatrix} B1 & B2 \\ 3/4 & 1/4 \end{pmatrix}, v = \frac{5}{2} \text{ (3lt)}$$

$$(d) S_A = \begin{pmatrix} A1 & A2 \\ 3/8 & 5/8 \end{pmatrix}, S_B = \begin{pmatrix} B1 & B2 & B3 \\ 5/8 & 3/8 & 0 \end{pmatrix}, v = -\frac{1}{8} \text{ (3lt)}$$

$$(e) S_A = \begin{pmatrix} A1 & A2 & A3 \\ 5/12 & 5/12 & 1/6 \end{pmatrix}, S_B = \begin{pmatrix} B1 & B2 & B3 \\ 4/9 & 2/9 & 1/3 \end{pmatrix}, v = \frac{1}{3} \text{ (4lt)}$$

$$(f) S_A = \begin{pmatrix} A1 & A2 & A3 \\ 2/9 & 5/9 & 2/9 \end{pmatrix}, S_B = \begin{pmatrix} B1 & B2 & B3 \\ 2/9 & 5/9 & 2/9 \end{pmatrix}, v = \frac{1}{9} \text{ (4lt)}$$

$$(g) S_A = \begin{pmatrix} A1 & A2 & A3 \\ 2/3 & 1/3 & 0 \end{pmatrix}, S_B = \begin{pmatrix} B1 & B2 & B3 \\ 0 & 1/2 & 1/2 \end{pmatrix}, v = 1 \text{ (3lt)}$$

$$9. S_A = \begin{pmatrix} T & N & M \\ 0 & 0 & 1 \end{pmatrix}, S_B = \begin{pmatrix} T & N & M \\ 2/5 & 3/5 & 0 \end{pmatrix}, v = 2$$

10. In India with an expected return of 8.

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## APPENDIX-1

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## NOTES

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### A BRIEF INTRODUCTION TO INTEGER PROGRAMMING

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I. In linear programming the nature of the decision variables are taken as non-negative, non-positive and unrestricted in sign, the values may be real or integer obtained from solution procedure. For some real world problems the real number solution is impractical. For example, the optimal number of a certain computer system to be produced is 10.23, the number of floppies to be supplied as 20.14 etc. In these cases we require integer solution. So in integer programming problems, all or some decision variables must take integer values. There are three classifications :

(a) *All Integer Programming Problem (All IPP)*. Here all the decision variables must take integer values e.g.,

$$\begin{aligned} \text{Max } z &= cx \\ \text{S/t } Ax &= b, \\ x &\geq 0 \text{ and are integers.} \end{aligned}$$

(b) *Mixed Integer Programming Problem*. Here some decision variables take integer values and some are real numbers e.g.,

$$\begin{aligned} \text{Max } z &= cx \\ \text{S/t } Ax &= b \\ x_i &\geq 0 \text{ and are integers, } x_j \geq 0 \text{ (} i \neq j \text{)} \end{aligned}$$

To solve (a) and (b) we have two widely used methods e.g., Gomory's cutting plane method and Branch and Bound method. Both the methods have different computational complexity.

(c) *0-1 Linear Problem*. Here the decision variables will take either 0 or 1 value. Widely used method is Balas Algorithm. Most of time, 0-1 problem arises in project selection, portfolio management, assignment problems etc.

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### II. BRANCH AND BOUND METHOD

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Ignoring the integer restrictions, solve the given problem by a suitable method. Then for non integer variables branches will be created one by one. Every branch will have one subproblem. Solve the subproblem. If the integer solution is obtained stop branching else go for branching. Also the branching will be stopped if the solution is infeasible or unbounded.

For mixed integer problems, the branches will be created only for integer restricted variables. Display the method in binary tree.

**Example 1.** (All integer) *Solve the following by branch and bound method :*

$$\text{Maximize } z = 2x_1 + x_2$$

$$\text{S/t } 2x_1 + 3x_2 \leq 12, 5x_1 + 2x_2 \leq 10,$$

$$x_1, x_2 \geq 0 \text{ and are integers.}$$

NOTES

**Solution.** Ignoring the integer restrictions, let us solve the given problem by graphical method and the optimal solution is  $x_1 = \frac{6}{11}, x_2 = \frac{40}{11}, z = \frac{52}{11}$ . Let the two branches be created by  $x_1$  variable. The nearest left side integer of  $x_1$  is 0 and the nearest right side integer is 1. So introduce two constraints  $x_1 \leq 0$  and  $x_1 \geq 1$  and the following two subproblems are obtained :

S1 : 
$$\text{Max } z = 2x_1 + x_2$$
  

$$\text{S/t } 2x_1 + 3x_2 \leq 12, 5x_1 + 2x_2 \leq 10, x_1 \leq 0, x_1, x_2 \geq 0.$$

S2 : 
$$\text{Max } z = 2x_1 + x_2$$
  

$$\text{S/t } 2x_1 + 3x_2 \leq 12, 5x_1 + 2x_2 \leq 10, x_1 \geq 1, x_1, x_2 \geq 0.$$

Using graphical method.

The solution of S1 : 
$$x_1 = 0, x_2 = 4, z^* = 4.$$

The solution of S2 : 
$$x_1 = 1, x_2 = 2.5, z^* = 4.5.$$

All integer solution has obtained in subproblem S1. So there will be no branching on S1. But the solution of S2 is not all integer. The value of  $x_2$  is real which requires branching.

The nearest left side integer is 2 and the nearest right side integer is 3. So introduce two constraints  $x_2 \leq 2$  and  $x_2 \geq 3$  and the following two subproblems are obtained :

S21 : 
$$\text{Max } z = 2x_1 + x_2$$
  

$$\text{S/t } 2x_1 + 3x_2 \leq 12, 5x_1 + 2x_2 \leq 10, x_1 \geq 1, x_2 \leq 2, x_1, x_2 \geq 0.$$

S22 : 
$$\text{Max } z = 2x_1 + x_2$$
  

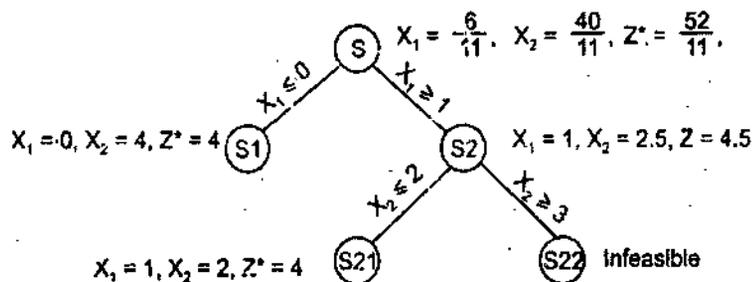
$$\text{S/t } 2x_1 + 3x_2 \leq 12, 5x_1 + 2x_2 \leq 10, x_1 \geq 1, x_2 \geq 3, x_1, x_2 \geq 0.$$

Using graphical method,

The solution of S21 : 
$$x_1 = 1, x_2 = 2, z^* = 4.$$

The solution of S22 : Infeasible.

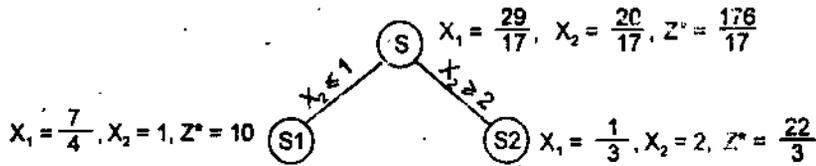
The subproblems S1 and S21 give same objective function value for two integer solutions. Hence these solutions are called 'multiple optima'. The binary tree presentation is given below :



**Example 2. (Mixed Integer)** Solve the following by branch and bound method:

$$\begin{aligned} &\text{Maximize } z = 4x_1 + 3x_2 \\ &\text{S/t } 3x_1 + 5x_2 \leq 11, 4x_1 + x_2 \leq 8, \\ &x_1, x_2 \geq 0 \text{ and } x_2 \text{ is an integer.} \end{aligned}$$

**Solution.** The binary tree presentation of the given problem is as follows:



Both the subproblems have obtained mixed integer feasible solutions. But  $z^*$  at S1 is greater than  $z^*$  at S2. So the solution at S1 is considered to be optimal solution.

## PROBLEMS

**Solve the following:**

1. Maximize  $z = 3x_1 + 2x_2$   
S/t  $3x_1 + 5x_2 \leq 5, 5x_1 + 3x_2 \leq 15, x_1, x_2 \geq 0$  and all integers.
2. Minimize  $z = x_1 + 8x_2$   
S/t  $x_1 \leq 3, x_2 \leq 7, 3x_1 + 4x_2 \geq 24, x_1 \geq 0, x_2 \geq 0$  is an integer.

## ANSWERS

1.  $x_1 = 1, x_2 = 0, z^* = 3.$
2.  $x_1 = \frac{8}{3}, x_2 = 4, z^* = 34.6.$

## NOTES

## AN INTRODUCTION TO NON-LINEAR PROGRAMMING

### I. Some basics

Let  $X$  be a non-empty convex set. A function  $f(x)$  on  $X$  is said to be convex if for any two vectors  $x_1$  and  $x_2$  in  $X$ ,

$$f[\lambda x_1 + (1 - \lambda)x_2] \leq \lambda f(x_1) + (1 - \lambda)f(x_2), \quad 0 \leq \lambda \leq 1.$$

For ' $\geq$ ' type relation, the  $f(x)$  is said to be concave *some basic properties* : Sum of convex functions is convex. If  $f(x)$  is convex then  $-f(x)$  and  $\frac{1}{f(x)}$

are concave functions.

Consider the non-linear programming problem (NLPP) as follows :

$$\text{Max/Min } f(x_1, x_2, \dots, x_n)$$

$$\text{S/t, } g_i(x_1, x_2, \dots, x_n) \{ \leq, =, \geq \} c_i$$

$$i = 1, 2, \dots, m. \text{ and } x_i \geq 0 \quad \forall i.$$

Here either  $f$  or  $g_i$  or both are non-linear functions. Let us consider only equality linear constraints in NLPP.

### II. Lagrange Multiplier's method

Let us construct the Lagrange function as given below :

$$L = f - \sum_{i=1}^m \lambda_i (g_i - a_i), \text{ where } \lambda_i \text{ is called Lagrange multiplier.}$$

The necessary conditions for optimum, set up the following :

$$\frac{\partial L}{\partial x_i} = 0, \quad i = 1, 2, \dots, n.$$

$$\frac{\partial L}{\partial \lambda_i} = 0, \quad i = 1, 2, \dots, m.$$

Solve these system of  $(n + m)$  variables and the solution is called stationary point.

These necessary conditions become sufficient conditions for a max. (min.) if the given objective function is concave (convex) and the constraints are the equalities.

If the type of the objective function is not known then bordered Hessian matrix test is to be performed.

**Example 1.** Solve Minimize  $z = x_1^2 + x_2^2 + x_3^2$

$$\text{S/t, } x_1 + 2x_2 + x_3 = 4, \quad 2x_1 + x_2 + x_3 = 6, \quad x_1, x_2, x_3 \geq 0.$$

**Solution.** The Lagrange function is

$$L = x_1^2 + x_2^2 + x_3^2 - \lambda_1(x_1 + 2x_2 + x_3 - 4) - \lambda_2(2x_1 + x_2 + x_3 - 6)$$

For necessary conditions,

$$\frac{\partial L}{\partial x_1} = 0 \Rightarrow 2x_1 - \lambda_1 - 2\lambda_2 = 0$$

$$\frac{\partial L}{\partial x_2} = 0 \Rightarrow 2x_2 - 2\lambda_1 - \lambda_2 = 0$$

$$\frac{\partial L}{\partial x_3} = 0 \Rightarrow 2x_3 - \lambda_1 - \lambda_2 = 0$$

$$\frac{\partial L}{\partial \lambda_1} = 0 \Rightarrow x_1 + 2x_2 + x_3 = 4$$

$$\frac{\partial L}{\partial \lambda_2} = 0 \Rightarrow 2x_1 + x_2 + x_3 = 6$$

Solving we obtain the stationary points as

$$x_1 = \frac{26}{11}, x_2 = \frac{4}{11}, x_3 = \frac{10}{11}, \lambda_1 = -\frac{12}{11}, \lambda_2 = \frac{32}{11}.$$

Since  $x_i^2$  is convex and the sum of convex function is convex, then the objective function is convex. Hence the optimal solution is

$$x_1^* = \frac{26}{11}, x_2^* = \frac{4}{11}, x_3^* = \frac{10}{11} \text{ and } z^* = \frac{792}{121}.$$

## PROBLEMS

Solve the following using Lagrange multiplier method :

1. Maximize  $z = x_1^2 + 2x_2^2 + x_3^2$

S/t  $2x_1 + x_2 + 2x_3 = 30, x_1, x_2 \geq 0.$

2. Maximize  $z = x_1^2 + x_2^2 + x_3^2$

S/t  $x_1 + x_2 + 2x_3 = 20, 2x_1 + 3x_2 + x_3 = 20, x_1, x_2, x_3 \geq 0.$

3. Maximize  $z = 2x_1^2 + x_2^2 + 2x_3^2$

S/t  $x_1 + x_2 + 2x_3 = 30, x_1 + 2x_2 + 3x_3 = 40; x_1, x_2, x_3 \geq 0.$

4. If the Lagrange multiplier takes the value 1 (other values neglected) for the following NLPP then find the optimal solution. Maximize  $z = (x_1 - 4)^2 + (x_2 - 3)^2$ ,  
S/t,  $36(x_1 - 2)^2 + (x_2 - 3)^2 = 9.$

5. Minimize

$$z = x_1^2 + x_2^2 + x_3^2$$

S/t,  $x_1 + 2x_2 + 3x_3 = 5$

$$2x_1 + 5x_2 + 4x_3 = 10$$

$$x_1, x_2, x_3 \geq 0.$$

**NOTES**

**ANSWERS****NOTES**

1.  $x_1 = \frac{120}{17}$ ,  $x_2 = \frac{30}{17}$ ,  $x_3 = \frac{120}{17}$ ,  $\lambda = \frac{120}{17}$ ,  $z^* = 105.88$
2.  $x_1 = \frac{20}{7}$ ,  $x_2 = \frac{16}{7}$ ,  $x_3 = \frac{52}{7}$ ,  $\lambda_1 = \frac{56}{7}$ ,  $\lambda_2 = -\frac{8}{7}$ ,  $z^* = 68.57$
3.  $x_1 = 10$ ,  $x_2 = 0$ ,  $x_3 = 10$ ,  $\lambda_1 = 80$ ,  $\lambda_2 = -40$ ,  $z^* = 400$
4.  $\left(\frac{68}{35}, 5.99\right)$  and  $\left(\frac{68}{35}, 0.02\right)$
5.  $x_1 = \frac{25}{54}$ ,  $x_2 = \frac{35}{27}$ ,  $x_3 = \frac{35}{54}$ ,  $\lambda_1 = -\frac{5}{9}$ ,  $\lambda_2 = \frac{20}{27}$

**III. Kuhn-Tucker Conditions**

Consider the NLPP : 
$$\text{Max. } z = f(x_1, x_2, \dots, x_n)$$

$$\text{S/t, } g_i(x_1, x_2, \dots, x_n) \leq 0, i = 1, 2, \dots, m.$$

$$x_j \geq 0, j = 1, 2, \dots, n.$$

Where 
$$g_i = G_i(x_1, x_2, \dots, x_n) - b_i.$$

Let us add the slack variables which is always positive as follows :

$$g_i(x_1, x_2, \dots, x_n) + S_i^2 = 0, i = 1, 2, \dots, m.$$

Then the Lagrange function is

$$L = f(x_1, x_2, \dots, x_n) - \sum_{i=1}^m \lambda_i [g_i(x_1, x_2, \dots, x_n) + S_i^2].$$

For the above maximization problem with concave objective function and with all less than or equal to type constraints (convex type constraints), the value of  $\lambda_i$  should be  $\geq 0$ .

Kuhn-Tucker has established the following necessary conditions :

(a) 
$$\lambda_i \geq 0, i = 1, 2, \dots, m.$$

(b) 
$$\frac{\partial L}{\partial x_j} = 0, j = 1, 2, \dots, n.$$

(c) 
$$\lambda_i g_i(x_1, x_2, \dots, x_n) = 0, i = 1, 2, \dots, m.$$

(d) 
$$g_i(x_1, x_2, \dots, x_n) \leq 0, i = 1, 2, \dots, m.$$

For minimization problem with convex objective function and with all greater than or equal to type constraints (concave type constraints), the value of  $\lambda_i$  should be  $\geq 0$ .

**Example 1. Solve :** 
$$\text{Maximize } z = 2x_1^2 + 3x_2^2$$

$$\text{S/t, } x_1 + 2x_2 \leq 4, x_1, x_2 \geq 0.$$

**Solution.** The Lagrange function is

$$L = (2x_1^2 + 3x_2^2) - \lambda_1 [x_1 + 2x_2 + s_1^2 - 4]$$

Then the Kuhn-Tucker conditions are constructed as follows :

$$(i) \quad \lambda_1 \geq 0$$

$$(ii) \quad \frac{\partial L}{\partial x_1} = 0 \Rightarrow 4x_1 - \lambda_1 = 0$$

$$(iii) \quad \frac{\partial L}{\partial x_2} = 0 \Rightarrow 3x_2 - \lambda_1 = 0$$

$$(iv) \quad \lambda_1(x_1 + 2x_2 - 4) = 0$$

$$(v) \quad x_1 + 2x_2 \leq 4$$

Solving, (i)-(v), we obtain  $x_1^* = \frac{12}{11}$ ,  $x_2^* = \frac{16}{11}$  and  $z^* = \frac{1056}{121}$ .

### PROBLEMS

Using Kuhn-Tucker conditions solve the following :

1. Maximize  $z = 3x_1^2 + 14x_1x_2 - 8x_2^2$

S/t,  $3x_1 + 6x_2 \leq 72$ ;  $x_1, x_2 \geq 0$ .

2. Maximize  $z = 2x_1 + 3x_2$

S/t,  $x_1^2 + x_2^2 \leq 4$ ,  $x_1, x_2 \geq 0$ .

### ANSWERS

1.  $x_1^* = 22$ ,  $x_2^* = 1$ ,  $z^* = 1752$

2.  $x_1^* = \frac{4}{\sqrt{13}}$ ,  $x_2^* = \frac{6}{\sqrt{13}}$ ,  $z^* = \frac{26}{\sqrt{13}} = 2\sqrt{13}$ .

NOTES

