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SYLLABUS

C-133 COMPUTER BASED OPTIMIZATION TECHNIQUES

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UNIT I: PRELIMINARIES: INVENTORY MODELS AND REPLACEMENT PROBLEMS

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★ STRUCTURE ★

1.0 LEARNING OBJECTIVES

After going through this unit, you should be able to:

- describe inventory models and its applications.
- explain inventory cost, EOQ and Stochastic model.
- state that how to develop an inventory management model.

1.1 INTRODUCTION

Inventory in general and in wider sense is defined as an idle resource, which has some economic value. The word inventory is loosely used as listing of materials of interest. But related with financial aspects, it is the total of raw materials, spare parts, maintenance materials, fuels and lubricants, paints and acids, tools, gadgets, semi-processed materials, semi finished goods and finished goods etc. Though an idle inventory is a resource, which is idle when kept in stores and this costs the enterprise money, some amount of inventory has to be maintained for smooth functioning of the enterprise. If no inventory is maintained, the enterprise may be forced to buy the raw material or other goods necessary for meeting the production targets, at very high price. Inventories have to be maintained for achieving a planned operational

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smoothness. Also, the monetary value of inventory will indicate the investments required to be made for achieving desired production/service requirement. So, the managers need to be aware of the nature of the production distribution system/service system in a particular industry and different functions that inventories perform in the system. Inventory provides the management with flexibility or choice of using it selectivity to support and implement the corporate strategies, whether it is buying the raw material in bulk to take advantage of discounts *etc*, or to increase production to enlarge market share at a particular time.

Inventory control is a subject of study under the broad discipline of materials management. Materials need to be handled and managed by any enterprise be it a production unit, a workshop engaged in overhauling of vehicles, any engineering department engaged in any of the operations on machines or a service industry like hospitals, hotels, educational institutions or an army unit. The main objective of handling the materials are :

1. Operate the plant and machinery installed or services designed to its optimal capacity so that maximum production is achieved or best service is provided in the interest of the user/customer.
2. Minimum investment on material is made. As funds are always in scarce quantity, their optimum use is one of the vital areas of management.

Inventory management has become extremely important as scientific management of inventory can increase the profits of the enterprise or reduce its costs. Organisations have to make substantial investments in materials, many enterprises have 20-25 percent of these total funds committed to inventory. Some industries like pharmaceuticals and chemical/paints industry may have even 70% cost contribution to the total funds of the organization. Managerial tools are being extensively used to control inventories as this is an area which can immediately provide concrete results in terms of reduced expenditure and increased profits.

Just the right amount of inventory being made available at the right place and right time is very complex and difficult to achieve. It is common knowledge that government departments overstock inventories without any consideration for cost it involves. On the other hand Japan has brought in the concept of Just in Time (JIT) so that no stocks are kept and the raw material directly moves from the manufacturer to the machine where it is to be used.

Different terms like Inventory Management, Inventory Control, Inventory System, Inventory Theory, Inventory process *etc*, are used while explaining the same objective as explained above.

1.2 INVENTORY COSTS

In an inventory system the following costs are significant :

Set-up Costs

These costs are incurred in setting-up of a plant *i.e.*, the costs of land, construction of buildings, purchase of machinery etc. These are not directly related with the numbers produced or ordered. But plants or factories are set up for a particular capacity to be manufactured.

Purchase Costs

It is the price that is paid for purchasing/processing of any item. Purchase management is a special subject of study. Purchase costs become important when large quantities purchased attract discounts. Also, economies of scale recommend manufacture of definite numbers to reduce the cost per unit.

Ordering Costs (Costs of Replenishing Inventory)

This is the money and effort spent on processing the materials required by any organization. The cost is expressed as Rs per order. These are the costs marked each time an order is placed with the suppliers and include everything from the purchase requisition placed on supplier to the cost of calls made, visits of the purchase officer etc. This cost has the following components :

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1. **Purchasing** – The administrative cost and cost of the clerks and the material they use is included in it, cost of advertisements, stationery and postage, telephone charges *etc,* are included in the cost.
2. **Accountings** – This is the cost incurred in checking whether what was ordered has been received, sending payments etc.
3. **Inspection and Storage** – After the material has been received it needs to be checked whether it meets the specifications ordered. Also it has to be stocked in a suitable store.
4. **Transport Cost** – These costs could be borne by the enterprise ordering the material or by the supplier. In any case, the supplier may include it in the price of the material or hide it to suit him.
5. **Inventory Carrying Costs (Holding Costs)** – This is the cost of holding an item in inventory. This cost depends on two factors, one the amount of inventory and second the period for which it is to be held and includes the following :
   (*i*) Storage cost – Cost of storage space, bins, shelves, etc.
   (*ii*) Salaries of staff engaged in store, security etc.
   (*iii*) Interest on capital blocked in purchase of inventory.
   (*iv*) Insurance against fire, theft *etc,* and any tax charged like octorai etc.
   (*v*) Reduced value due to deterioration of material, spillage, storage evaporation and other types of inventories.
   (*vi*) Obsolescence – The material lying in stock may become unusable due to technical advancements/changes.

   It is obvious from the above that different enterprises will have different inventory carrying costs. It is of the order of approximately 30% for a typical Indian industry. The inventory carrying costs may be classified as fixed and variables. The fixed carrying costs do not change irrespective of the number of orders placed. A store whether empty of full is a fixed cost unless it can be put to some other use.

   Carrying costs can be worked out as follows :

   (*a*) (Costs of carrying one unit of an item for a given time) × (Average number of units carried in the inventory for same lengths of time).
   (*b*) (Cost of carrying – one rupee worth of the inventory item for 'unit time period) × (Rupee value of total units carried).
6. **Shortage Costs or Stock-out Costs** – If an organization is not able to meet the demand of the customers altogether or can do so but at a later date, this will cost the organization some amounts in terms of money, this is the storage cost of stock-out cost. This is very important component of costs. As the penalty the enterprise has to pay is not only in terms of money but also the loss of good will, lost sales and even the business may have to be wound up.

   Shortage cost = (Cost of one unit short) × (Average number of short units)

   If the shortage can be met at a later stage, these costs will vary proportional to the short quantity and delay time. But if there is a stock-out and no demand is materialized, shortage cost is proportional only to the short quantity as there is no question of delay in meeting the demand, it is just not being met when it was due.

7. **Over-stocking Costs** – This is the inventory carrying cost for a period for which the material is stocked more than the requirement.

1.3 DEVELOPING AN INVENTORY MANAGEMENT MODEL

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The history of inventory models goes back to 1915 when FN Harris developed a simple model. The approach of inventory models has attracted a large number of work based on mathematical analysis because the pay-off in this area are substantial. However, no single model can take into account all possible situations of real life and suggest how much be ordered and when. In fact, even if such a model can be developed, it may not be possible to solve it. Some of the inventory models are discussed in this unit. Certain terms, which will be used in these models, are explained below :

(*a*) **Demand Patterns** – The very purpose of holding inventories is to meet the demands of the market. These demands are not within the control of the organization. However, the following factors related to demands need to be studied while developing an inventory model. The demands could be classified as :

(i) *Deterministic Demand* – It means that demand is known with certainty either at a particular point of time or over a period of time. This is not a very common situation because the demand may keep changing due to a large number of factors beyond the control of organization, consumer's tastes, technological developments, government policy and host of other factors related with the business environment.

(*ii*) *Probabilistic Demand* – Where the demand of product can be determined based on a probability of occurrence. If the demand is not known, it may be possible to determine its probability distribution.

(*b*) **Lead Time** – This is the time between ordering a replenishment and the time when it is actually received and is ready for use. When an enterprise places order for a particular item, it may be altered immediately or it may be received over a period of time. Lead time can either be determined *i.e.*, it is exactly known when after placing an order the item will be delivered or probabilistic or the probability of receiving an order in a particular time is know. Incase the lead time is zero, the orders need not be placed in advance as the moment it is placed, the demand is met, if it is known to be a finite time, say 6 weeks then the demand must be placed 6 weeks ahead. How much should be ordered will depend upon the consumption, which will last for the lead time period.

(*c*) **Stock Replenishment Policy** – One of the important factors while designing an inventory model is that at what rate the inventory is being added.

(*d*) **Time Horizon** – Inventory control models cannot be developed for an infinite period Time for which this model is applicable is called the time period or time horizon.

(*e*) **Items Required or Demanded** – The inventory model may consider only one item or a number of items demanded. The total inventory cost depends upon the number of items demanded.

(*f*) **Safety Stock ; Minimum Stock, or Buffer Stock** – This is stock an organization must cater for the delay in delivery or higher demand than average expected demand for which the organization has planned.

(*g*) **Reorder Level** – This is the level between the maximum and minimum level of stock at which purchasing action must be initiated or manufacturing actions taken for fresh supply of the material in question.

(*h*) **Reorder Quantity** – This is the quantity for which the order must be placed to replace and replenish the stock. In some inventory control systems this is the economic order quantity *i.e.*, the quantity which according to the scientific basis must be ordered and is the most economic order considering many factors.

(i) **Number of Supply Points** – The order may be placed on one or more supply points for replenishing the material.

Steps Involved in Developing an Inventory Model

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The purpose of all inventory decisions is only one and that is to meet the production requirements at minimum cost. This involves two issues ; how much to order and when to order? For this, researchers have developed many models of inventory management, though the basic consideration for development of inventory models are essentially the same. These are described in succeeding paragraphs :-

1. Carry out an audit of all the inventory of the enterprise *i.e.*, taking a physical stock of all the items at a particular period of time.
2. Classify the inventory as determined in step I above. Inventory items may be classified as raw material, work in process, semi-finished goods, purchased or bought out components, maintenance spares, tools and gadgets, lubricants, coolants and oils, finished goods etc.
3. The classification of inventory done in step 2 is further classified into different groups for example, maintenance inventory may be grouped into spare parts for plant and machinery, spare parts for special maintenance tools (SMTs), oils, coolants, lubricants etc.
4. Each item of inventory is allotted a suitable code. A suitable coding system to include all existing items of inventory as also with capacity to include new items, is selected.
5. Deciding which inventory model is suitable for what category of items. each organization may have thousands of items, it is not desirable that management pays some attention to all kinds of items of inventory whether they are vital, fast moving, slow moving, very expensive or very cheap.
6. Work out the annual value of each item of inventory. These are listed in descending order of annual usage value. This classification is called ABC analysis. What set of items (say A) need to be managed by top management or middle or junior levels has to be decided. This will decide what kind of inventory management has to be done for what classification of items.
7. Another classification of inventory may be based on V-Vital, E-essential or D-desirable. Also, FNS may be another classification F – Fast, N – Normal and S – Slow.
8. Decide the type of model suitable for each classification of materials, for example, the safety stock for vital items like gear box or Fuel Injection Pump (FIP) which are also 'A' items being very costly cannot be the same as for nuts and bolts. Certain items may be overstocked without locking huge funds, some other need to be stocked selectively as they are very expensive.
9. At this stage data relevant to ordering cost, inventory carrying cost or other important costs needs to be collected.
10. Work out the requirement of each inventory item and its associated market price.
11. Decide the quality of service being planned to be provided to the customers and estimate lead time, reorder level and safety stock to ensure the level of satisfaction of the customer.
12. Develop a suitable inventory model based on the above data.
13. Modify the model to include any changes.

1.4 THE ECONOMIC ORDER QUANTITY (EOQ) OR WILSONS LOT SIZE FORMULA

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EOQ was first developed by Ford W Harris in 1915. The idea was to balance the cost of holding or stocking too much against the cost of ordering too small quantity of materials. This is one of the oldest and most commonly used inventory control models. It is still used by many organizations, as it is relatively easy to use. But it gives an approximate solution as it makes the following assumptions :

- Demand is known (certain) and does not vary (constant). It is continuous at a constant rate.
- The lead time is known and is constant and is equal to or greater than 0.
- Once the order is placed, the items of inventory ordered are received instantly *i.e.*, the inventory arrives in one go and at the same time.
- Inventory ordering and inventory carrying costs are the only variables.
- If the orders are placed at the right time, there will be no shortage or stock-outs *i.e.*, there are no stock-out costs.
- The process continues infinitely.
- No constraints are imposed on the quantity or an item ordered, budget and storage capacity.
- No quantity discounts *i.e.*, bulk purchase discounts are available.

Graphical Method

From the definition of the term Economic Order Quantity, it may be seen that it would be that quantity of material for which the ordering costs and the carrying costs are minimum. This has been explained in the diagram below.

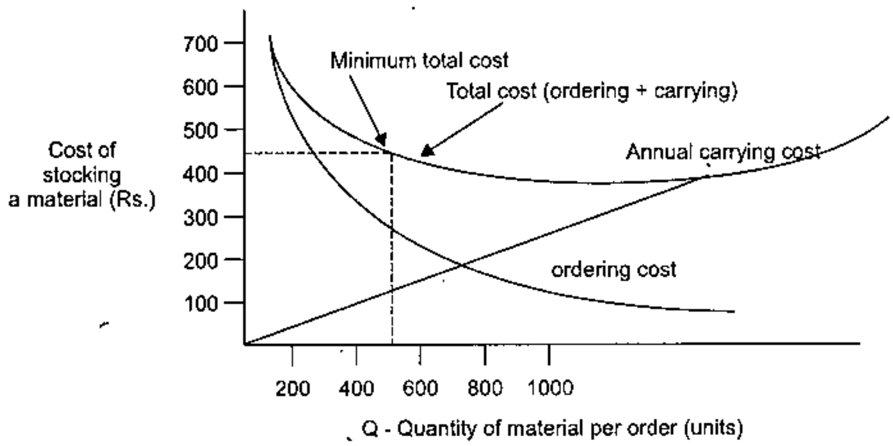

Fig. 1.1

The above diagram shows a typical inventory model. On X-axis the quantity of material per order is shown as units and on the Y-axis the costs of stocking of materials are shown in Rs. The curve for ordering costs keeps decreasing as more and more material is ordered as holding large inventory means a smaller number of orders. The carrying cost curve increases as shown as the order quantity increases, as the capital cost and other related costs for holding large quantity will increase. The total cost curve is obtained by adding the two curves for ordering cost and carrying cost. Minimum total cost is shown in the figure. Also, the point at which the carrying cost curve and ordering cost curve intersect is the optimal order quantity point. It can be seen from the diagram that the point at which the ordering and carrying costs are equal, at that point the total cost curve dips and is minimum at that point. The diagram also shows that the two costs plotted behave in opposite manner to each other. If order quantity increases and becomes more than the optimal or economic order quantity, the ordering cost will decrease, however for the same quantity, the carrying costs will increase.

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Algebraic Method

Here a relationship between the demand, the costs and optimal order quantity is established and the equation can be used to solve the problem directly. Let us use the following notations.

Q = Optimal number of units per order

D = Annual demand (units) of the inventory item

C_0 = Ordering cost/order

C_c = Carrying or holding cost/unit/year.

(*i*) Annual ordering cost = number of orders placed/year × ordering cost per order
= Annual demand × Ordering cost/order

Number of units in one order $= \frac{D}{Q} \times C_0$

(*ii*) Annual carrying cost = Average level of inventory × carrying cost/unit/year
= Ordered quantity/2 × carrying cost/unit/year
$= Q/2 \times C_c$

At optimal order or economic order quantity, these two costs are equal

$$\frac{D}{Q} \times C_0 = Q/2 \times C_c$$

Hence $$Q = \sqrt{\frac{2DC_0}{C_c}}$$

Average level of inventory has been worked out as the average of maximum inventory and minimum inventory *i.e.*, Q + 0/2 = Q/2. This can be seen in the figure below.

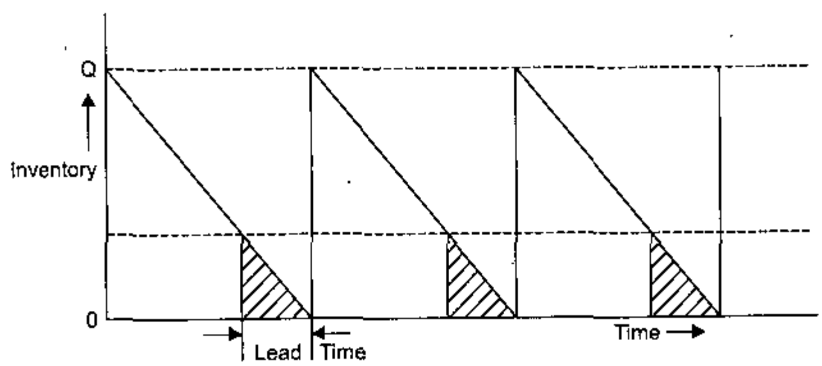

Fig. 1.2

This figure has been drawn assuming that there is no safety stock, maximum quantity Q *is received in one go and consumed uniformally over a* period of time when again quantity Q is received at once. Though these conditions rarely are available in practical life, yet Q/2 as average inventory is a reasonable assumption.

Example 1.1. *Annual demand for a particular product is known to be 40,000 units, ordering cost have been estimated to be Rs 30 per order. Whereas the annual carrying or holding costs are 15% of* *the inventory value. Cost of material per unit is Rs 100. Making the usual assumptions, use EOQ formula and find out the economic order quantity.*

Solution. D = 40000

C_0 = Rs 30 per order

C_c = 15% of inventory costs = $\frac{15}{100} \times 100$ = Rs 15 per unit

$$EOQ = \sqrt{\frac{2DC_0}{C_c}} = \sqrt{\frac{2 \times 40,000 \times 30}{15}} = \sqrt{1,60,000} = 400 \text{ units.}$$

EOQ in rupees = D = Annual consumption in rupees = 40000 × 100 = Rs 4000000

Using the formula for EOQ

$$EOQ = \sqrt{\frac{2 \times 40,00000 \times 30}{15\%}} = \sqrt{\frac{2 \times 40,00000 \times 30 \times 1000}{15}} = \sqrt{16,00000000} = \text{Rs } 4,0000.$$

It should be noted that for EOQ in units we have used Rs 15/unit as the carrying cost but for EOQ in rupees we have used 15% and not Rs 15.

Example 1.2. *Annual consumption of steel in a utensil manufacturing company is 36,00 ton. The ordering cost is Rs 400 per ton carrying cost is 40% of the stock value and price per ton of material is Rs 8000. Determine the EOQ in units and rupees.*

Solution. D = 3600

C_0 = Rs 400

C_c = 40% 8000 = Rs 3200

$$EOQ = \sqrt{\frac{(2 \times 3600 \times 400)}{3200}} = \sqrt{(900)} = 30 \text{ tons}$$

$$\text{EOQ in rupees} = \sqrt{\frac{(2 \times 3600 \times 400)}{40\%}} = \sqrt{\frac{(2 \times 3600 \times 400 \times 100)}{40}} = \text{Rs } 2683$$

Example 1.3. *If the annual demand of a particular item is 20,000 units, the estimated ordering cost is Rs 100, estimated inventory carrying cost is 20 percent and unit price of the item is Rs 20. Find out the optimal order quantity. What will be the EOQ if ordering cost is doubled and carrying cost is reduced to 10% ?*

Solution. $$EOQ = \sqrt{\frac{2DC_0}{C_c}}$$

Here D = 20,000

C_c = Rs 100

C_c = 20% of Rs 20 = Rs 4

$$EOQ = \sqrt{\frac{(2 \times 2000 \times 100)}{4}} = \sqrt{(1000000)} = 1000 \text{ units}$$

II case

$$EOQ = \sqrt{\frac{\frac{(2 \times 20000 \times 200)}{10}}{100 \times 20}} = 2000 \text{ units}$$

If the order quantity was 1000 units, number of orders placed in a year is 20000/1000 = 20 orders.

Example 1.4. *ABC Ltd manufacturs auto spare parts, his sale is 8000 parts per year if he does not supply this quantity to his customer, he will suffer irreparable loss of good will and future orders. Inventory carrying cost is Rs 5 per unit per year. The setup cost per run is 200. Determine:*

(a) *EOQ*

(b) *Increase in total cost associated with ordering*

  (i) *40% more than EOQ*

  (ii) *30% less than EOQ.*

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Solution. $D = 8000$ units

C_0 = Rs 200 setup cost per run

C_c = Rs 5 per unit per year

(a) $$\text{EOQ} = \sqrt{\frac{2DC_0}{C_c}} = \sqrt{\left(2 \times 8000 \times \frac{200}{5}\right)} = 800 \text{ units}$$

(b) $$\text{Ordering 40\% more than EOQ} = \frac{140}{100} \times 800 = 1120 \text{ units}$$

(c) $$\text{Ordering 30\% less than EOQ } i.e., \frac{70}{100} \times 800 = 560 \text{ units}$$

Using these two figure the difference in total cost can be worked out.

Example 1.5. *Yearly demand for a particular item is 6400 units. The cost of item is Rs 10 per unit. Cost of one procurement is Rs 200 and the inventory carrying cost is 20% per annum. Determine:*

(a) *EOQ*

(b) *No of orders per year*

(c) *Time between two orders*

Solution. $D = 6400$ units per year

Cost of item = Rs 8 per unit

C_0 = Rs 200

$C_c = 10 \times 1{\cdot}2$ = Rs 12

(a) $$\text{EOQ} = \sqrt{\frac{2DC_0}{C_c}} = \sqrt{\frac{(2 \times 6400 \times 200)}{12}} = 462$$

(b) $$\text{No of order per year} = \frac{6400}{462} = 14$$

(c) $$\text{Time between two orders} = 462 \times \frac{12}{6400} = 0{\cdot}87 \text{ months} = 26 \text{ days}$$

Example 1.6. *A repair and maintenance company is engaged in providing service to a particular brand of popular vehicle. It uses 8000 units of a moving part per year as replacement of the old parts. each part cost Rs 250. The set up costs are estimated at Rs 100 and the inventory carrying cost is the average of such industry at 20% of the inventory cost. Supply of the part is at the rate of 80 per day. Calculate the following.*

(a) *Optimal order quantity*

(b) *Optimal number of set ups.*

(c) *Total variable cost based on optimal policy.*

Assume 310 working days in a year.

Solution. $D = 8000$ units

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Cost = Rs 250

C_0 = Rs 100 per set up

C_c = 20% of 250 = Rs 50 per year

p = 80 parts per day

$$d = \frac{8000}{310} = 26$$

(*a*) Optimal order quantity = $EOQ = \sqrt{\frac{2DC_0}{C_c}\left(\frac{p}{p-d}\right)} = \sqrt{\frac{2 \times 8000 \times 100}{50}\left(\frac{80}{80-26}\right)}$

$$= \sqrt{\frac{32000 \times 80}{54}}$$

= 218 units

(*b*) Optimal number of set ups = $\frac{D}{EOQ} = \frac{8000}{218}$ = 37 setups

(*c*) Total variable cost = $\left(\frac{D}{EOQ} \times C_0\right) + \frac{EOQ}{2}\left(\frac{p-d}{p}\right)C_c$

$$= \frac{8000}{218} \times 100 + \frac{218}{2}\left(\frac{80-26}{80}\right) \times 50$$

= 3669·71 + 3678·75 = Rs 7348·5 per year

1.5 DETERMINSTIC INVENTORY MODEL WITH SHORTAGE (BACK ORDER MODEL)

We made the assumptions in the above discussed problems so far that no shortage are permitted. For working out EOQ, we have equated the ordering costs with the inventory carrying cost. Planned shortages however can be economical in certain cases where the ordering costs can be spread over a period of time. Also if the price of the items is high ('A' items) or if the inventory carrying cost is high, planned shortages may be economical in the long run.

Let us assume that stock outs and back ordering is permitted. Back ordering a situation where the user awaits the arrival of orders already placed but not materialised even after the stock outs have occurred. The materialisedevil effects of stock out (loss of orders due to inability to meet delivery schedule, loss of reputation and good will) are assumed to be negligible. It is assumed that back orders will materialise before new demand for the product arises. The following notations, in addition to the usual ones, which we are already familiar with, will be used.

S = units of inventory after the back order materializes

C_b = Back order (Stock out) cost per back order per unit of time

Q – S = Back order quantity or number of shortages per order

t_1 = Time during which inventory is available.

t_2 = Time of shortage or stock out

T = Time between orders received *i.e.*, $T = t_1 + t_2$

The situation where stock outs are permitted is depicted in the following figure.

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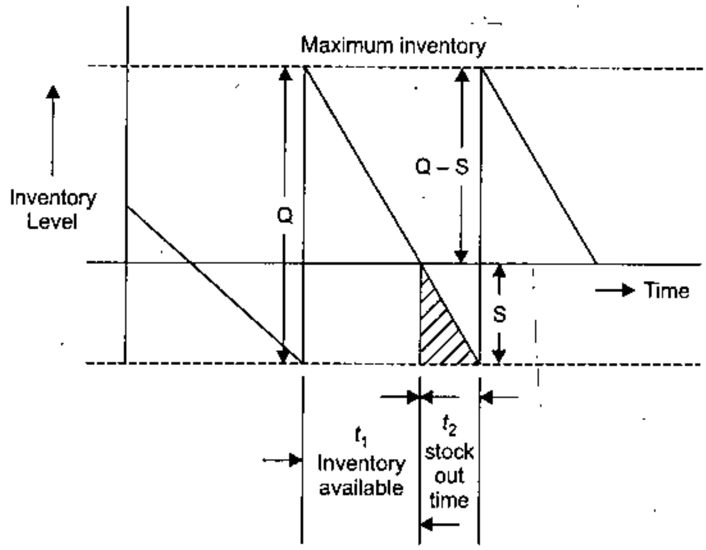

Fig. 1.3

It can be seen from the figure that quantity Q is ordered but received after S has been consumed for a time period of t_2. So stock out occurs for t_2 time and thus cycle of planned shortages continues. Every time after stock out period when quantity Q is received the inventory level reaches its maximum level as shown.

Formulae to be Used in Back Order Model

1. $\text{EOQ} = \sqrt{\frac{2DC_0}{C_c}\left(\frac{C_B + C_C}{C_B}\right)}$

2. Maximum Number of back orders = $\text{EOQ}\left(\frac{C_C}{C_B + C_C}\right)$

3. Number of order = $\frac{D}{\text{EOQ}}$ per year

4. Time between order = $\frac{\text{EOQ}}{D} = \sqrt{\frac{2C_0}{DC_C}\left(\frac{C_B + C_C}{C_B}\right)}$

5. Maximum Inventory Level = EOQ – Maximum number of back orders

$$= \text{EOQ}\left(1 - \frac{C_C}{C_B + C_C}\right)$$

$$= \text{EOQ}\left(\frac{C_B}{C_B + C_C}\right)$$

Example 1.7. *A scooter manufacturer company XYZ has a contract with a foundry from where it buys its engine castings. The foundry is able to provide the delivery of the engine costing on 10th days from the day on which order has been placed by the company. The demands of scooters varies between 500-800 scooters per day but on an average 650 scooters are sold. The company wishes to find out the safety stock so that there are no stock outs. Also suggest the reordering point.*

Solution. Average number of engine castings required = 650 × lead time (10) = 6500 units

Maximum number used during lead time = 800 × 100 = 8000 units

The company should establish ROP at level of 8000 units and his safety stock at (8000 – 6500) = 1500 units. This is shown in figure below.

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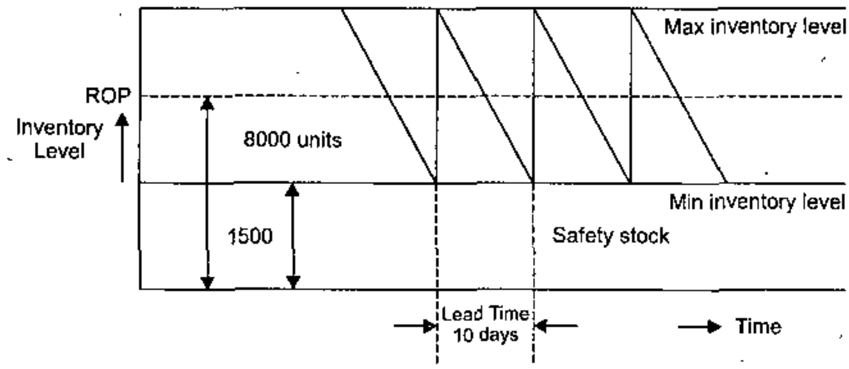

Fig. 1.4

Suppose the lead time is variable and varies between 8-12 days.

Average engine casting requires = 650 × 10 = 6500 units

Maximum required = 800 × 12 = 9600 units

Minimum required = 500 × 8 = 4000 units

ROP = maximum lead time usage = 9600 units

Safety stock = 9600 – 6500 = 3100 units.

Example 1.8. *In a system which has a uniform consumption rate of 1200 items per year, from previous experience the lead time were estimated as 20, 12, 16, 24, 30 days. Determine the safety stock and the reorder level.*

Solution. Average lead time = $\frac{(20+12+16+24+30)}{5} = 21$ days

Consumption rate per day = $\frac{1200}{320} = 4$ units per day

Assuming that there are 320 working days per year

Maximum lead time = 30 days

Minimum lead time = 12 days

Reorder point = Maximum lead time × usage per day = 30 × 4 = 120 units.

Safety stock = Maximum lead time consumption – average lead time consumption

$$= 120 - 21 \times 4 = 120 - 84 = 36 \text{ units}$$

Example 1.9. *A Company uses fixed order quantity inventory system. The annual demand of the item is 20,000 units, cost per unit is Rs 15, set up or ordering cost is Rs 200 per production run, carrying or holding cost as per industry average, which is 22%. Lead time in the past has been 10, 12, 20, 25, 28 days. Calculate safety stock, Reorder level and average level of inventory.*

Solution.

$$EOQ = \sqrt{\frac{2DC_O}{C_C}} = \sqrt{\left(2 \times 20{,}000 \times \frac{200}{3{\cdot}3}\right)} = 1557 \text{ units}$$

$$C_C = \frac{22}{100} \times 15 = \text{Rs } 3{\cdot}3$$

Daily Demand = Annual demand/No. of working days in a year = $\frac{20{,}000}{315} = 63{\cdot}5$ units

(Assuming 315 working days in a year)

Average lead time = $\frac{10+12+20+25+28}{5} = 19$ days

Safety Stock = (Maximum lead time-average lead time) × demand per day

$$= (28 - 19) \times 63{\cdot}5 = 9 \times 63{\cdot}5 = 571{\cdot}5 = 572 \text{ units}$$

Reorder level

Safety stock + average lead time demand

$= 572 + 19 + 635 = 572 + 1206{\cdot}5 = 1779$ units (approx)

Also, maximum lead time × usage per day $= 28 \times 63{\cdot}5 = 1778$ units

Which is the same as above

Maximum inventory level = safety stock + reorder level

$= 572 + 1778 = 2350$ units

Minimum level = Safety stock = 572 units

Average inventory level $= \frac{EOQ}{2}$ + safety stock $= \frac{1557}{2} + 572 = 1350$ units (approx)

NOTES

1.6 PROBABILISTIC INVENTORY MODELS

When we assumes that there is no uncertainty associated with demand and replenishment, the models are relatively simple but unrealistic as in real life situations these two assumptions are not valid. There is always some uncertainty related to demand pattern and lead time of material. As these uncertainties keep increasing, this increases the inventory as the manager has to keep extra stock (safety stock) to account for these uncertainties and avoid stock-out situations and costs which may be far higher and larger than the amount blocked in extra inventory. Reorder can be easily planned in a deterministic model where the demand is at uniform rate and the lead time is know. For example, if the demand is uniform at 10 units of an item per day and the lead time is 4 days, then in the deterministic system, Reorder point is 40 units. However due to uncertainties, an extra safety stock must be added to the expected demand during the lead time period to obtain the reorder point. Still there may be a stock out.

Let us assume that X_L is the average demand during the lead time and σ_L is the standard distribution of lead time demand ; then the ROP is $X_L + K\sigma_L$. This is shown in the diagram.

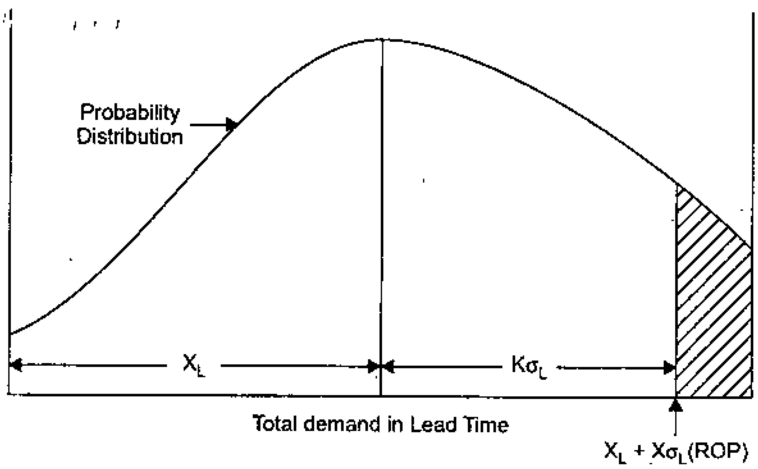

Fig. 1.5

The shaded area in the graph gives the probability of stock out during the lead time. This value can be found out from normal distribution tables for different value of K.

ROP can be determined by using the simple formula as shown below.

ROP = Expected demand during lead time + safety stock $= X_L + K\sigma_L$

Different value of X_L and σ_L, a lead time demand distribution have to be obtained.

NOTES

Single Period Model

There are certain typical cases where the inventory problems are such that number ordered decisions have to be taken only once during the complete demand cycle. In situations of uncertainties, the order should be such that over stocking and under – stocking must be optimized to minimize losses. a typical example could be flowerist who must sell only fresh flowers everyday and these cannot be sold – the following day. He must find out what quantities of orders for fresh flowers must be placed so that he maximizes his profits in-spite of the fact that he does not know how many customers will order the flowers. Similar decision problem is faced by the newspaper vendor, baker *etc*, who sell fresh items every day. In such cases, a critical ratio based on potential loss is prepared.

Critical ratio = $\frac{C_2}{C_1 + C_2}$ when C_1 = potential loss per item unsold, C_2 = potential profit per unit sold.

Probability of demand is calculated as $\sum^{\infty} p(d)$ this probability of demand of different number of units is calculated. It could be EOQ number or any other number, the critical ratio $\frac{C_2}{C_1 + C_2}$ and probability of demand $\sum_{EOQ}^{\infty} p(d)$ are compared and the nearest integer value is taken. This can be explained with the help of an example.

Example 1.10. *A flower vendor buys gladiola sticks at Rs 3 each and sells it at Rs 5 each. The unsold sticks cannot be sold the next day. daily demand of his flowers has the following distribution.*

No. of customers	30	10	15	20	10	22	25	10	5	20	28	30
Probability P (d)	0·02	0·03	0·10	0·10	0·10	0·20	0·15	0·05	0·10	0·10	0·03	0·02

If demand of each day is independent of that of the previous day, how many flower sticks should he order every day to maximize his profit ?

Sol. Potential for loss per stick unsold C_1 = Rs 3

Potential for profit per item sold C_2 = Rs (5 – 3) = Rs 2

$$\text{Critical ratio or } p(d) = \frac{C_2}{C_1 + C_2} = \frac{2}{3} = 0{\cdot}67$$

No of customers	*Probability*	*probability of demand not exceeding d < p (d)*
30	0·02	0·02
10	0·03	0·05
15	0·10	0·15
20	0·10	0·25
10	0·10	0·35
22	0·20	0·55
25	0·15	0·70
10	0·05	0·75
5	0·10	0·85
20	0·10	0·95
28	0·03	0·98
30	0·02	1·00

NOTES

$$\text{Since } p(22) = 0.55 < \frac{C_2}{C_1 + C_2} = 0.67 < p(23) = 0.70$$

Therefore the flower vendor should place order for 22 sticks per day

Example 1.11. *Calculate EOQ and total cost from the following information*

Annual Demand = 2500 units
Carrying cost = 25%
Unit cost = Rs 100
Ordering cost = 100 per order

Sol.

Annual Demand = (D) = 2500 units
Carrying Cost = (C_C) = 25% of 100 = 25 per unit
Ordering Cost (C_0) = 100/-
Unit cost (Cpu) = Rs 100/-

$$\text{EOQ} = \sqrt{\left(\frac{2DC_O}{C_C}\right)} = \sqrt{\frac{2 \times 2500 \times 100}{25}} = \sqrt{\frac{5{,}00{,}000}{25}} = \sqrt{20{,}000}$$

= 141·42 units

Total cost = Material cost + Ordering cost + carrying cost

$$= \text{Material Cost} + \text{TVC} = D \times \text{Cpu} + \sqrt{2DC_O C_C}$$

$$= 2500 \times 100 + \sqrt{2 \times 2500 \times 100 \times 25}$$

= 250000 + 3535×53 = 253535×53.

Example 1.12. *The annual requirement for a particular raw material is 2000. Unit costing Rs 1/- each. The ordering cost is Rs 10/- per order and carrying cost is 16% per annum of the average inventory value. Find EOQ and total inventory cost per annum.*

Solution.

Demand (D) = 2000 units per annum
Unit cost (Cpu) = Rs 1/-
Ordering cost (C_O) = Rs 10/-
Carrying cost (C_C) = 16% of 1 = 0·16 per unit

$$\text{EOQ} = \sqrt{\left(\frac{2DC_O}{C_C}\right)} = \sqrt{\frac{2 \times 2000 \times 10}{0.16}} = 500 \text{ units}$$

Total cost = material cost + Ordering cost + carrying cost

$$TC = (2000 \times 1) + \left(\frac{2000}{500} \times 10\right) + \left(\frac{500}{2} \times 0.16\right) = 2000 + 40 + 40 = 2080$$

Example 1.13. *A newspaper boy buys paper at Rs 1·30 each & he sells at Rs 1·70 each. He can't return the unsold newspaper. Daily demand has the following distribution.*

No of customers	23	24	25	26	27	28	29	30	31	32
Prob	0·01	0·03	0·06	0·10	0·20	0·25	0·15	0·10	0·05	0·05

If each day demand is independent of the previous days demand how many papers shall he order each day ?

Solution. Potential Profit per item sold (1·70 – 1·30) = 0·40

Potential Loss per item unsold (1·30 – 00) = 1·30

$$\text{Critical prob} = \frac{\cdot 40}{1{\cdot}30 + \cdot 40} = 0{\cdot}235$$

NOTES

Determination of optimum no of customers.

No of Customers	*Prob*	*Cumulative Prob*
23	0·01	1·00 > 0·2353
24	0·03	0·99 > 0·2353
25	0·06	0·96 > 0·2353
26	·10	0·90 > 0·2353
27	0·20	0·80 > 0·2353
28	0·25	0·60 > 0·2353
29	0·15	0·35 > 0·2353
30	0·10	0·20 < 0·2353
31	0·05	0·10 < 0·2353
32	0·05	0·10 < 0·2353

Optimum no of customer is 29.

Example 1.14. *Amit manufactures 50,000 bottles of tomato ketchup in a year. The factory cost per bottle is Rs 5, the set up cost per production run is estimated to be Rs 90 & carrying cost on finished goods inventory amounts to 20% of the cost per annum. The production is 600 bottles per day and sales amount to 150 bottles per day. What is optimal lot size and the no of production run? If the factory costs increase to Rs 7·50 per bottle what will be the optimum production lot size ?*

Solution.

Production Rate (P) = 600

Consumption Rate (D) = 150

Carrying Cost (C_C) = 20% of Rs 5 = Rs 1

Ordering Cost (C_O) = Rs 90

$$\text{Production Lot Size} = \sqrt{\frac{2DC_O}{C_C\left(1-\frac{d}{p}\right)}} = \sqrt{\frac{2\times 50000\times 90}{1\left(1-\frac{150}{600}\right)}} = \sqrt{\frac{90{,}00000}{1\,(0{\cdot}75)}} = \sqrt{120{,}00000}$$

= 3464 units

$$\text{No of production Runs} = \frac{\text{Annual Demand}}{\text{EOQ}} = \frac{50000}{3464} = 14.$$

If factory cost increases to Rs 7·50 per bottle carrying cost = 20% of 7·50 = 1·50.

$$\text{Production Lot Size} = \sqrt{\frac{2\times 50{,}000\times 90}{1{\cdot}5\left(1-\frac{150}{600}\right)}} = \sqrt{\frac{90{,}00000}{1{\cdot}5\,(0{\cdot}75)}} = \sqrt{\frac{90{,}00000}{1{\cdot}125}} = \sqrt{80{,}00000}$$

= 2828 units.

SUMMARY

- Inventory control is a subject of study under the broad discipline of materials management. Materials need to be handled and managed by any enterprise be it a production unit, a workshop engaged in overhauling of vehicles, any engineering department engaged in any of the operations on machines or a service industry like hospitals, hotels, educational institutions or an army unit. The main objective of handling the materials are :
- Operate the plant and machinery installed or services designed to its optimal capacity so that maximum production is achieved or best service is provided in the interest of the user/customer.

- Minimum investment on material is made. As funds are always in scarce quantity, their optimum use is one of the vital areas of management.
- Purchase management is a special subject of study. Purchase costs become important when large quantities purchased attract discounts.
- The history of inventory models goes back to 1915 when FN Harris developed a simple model. The approach of inventory models has attracted a large number of work based on mathematical analysis because the pay-off in this area are substantial.
- This is one of the oldest and most commonly used inventory control models. It is still used by many organizations, as it is relatively easy to use.

GLOSSARY

- **Inventory:** In general and in wide sense it is defind as an idle resource, which has some economic value.
- **Inventory Management:** Is a scientific management which can increase the profit of the enterprise or reduce its costs.
- **Just-in-time (JIT):** It is an inventory concept used in Japan. In which no stocks are kept by the government and the raw material directly moves form the manufacture to the machine where it is to be used.
- **Obsolescence:** The material lying in stock may become unusable due to technical advancement or charges.

REVIEW QUESTIONS

1. What are the economic parameters of inventory ?
2. Describe the basic characteristics of an inventory system.
3. (*a*) Enumerate the various types of inventory models.
   (*b*) Distinguish between deterministic and stochastic models in inventory theory.
4. Define the terms set-up cost, holding cost and shortage or penalty cost as applied to an inventory problem.
5. Explain the terms Lead time, Re-order point, Stock-out cost, and Set-up cost. Derive Wilson's formula.
6. Obtain an expression for the EOQ for any one inventory model, stating the assumptions made.
7. With usual notations derive an expression for the economic order quantity, for a production-inventory situation, with known demand.
8. Prove that in the inventory problem of Economic Lot Size with uniform demand and unequal times of production run, the optimal lot size Q_0 for each production run is given by $Q_0 = \sqrt{\frac{2DC_S}{C_1}}$, and the optimal total cost C_0 is given by $C_0 = \sqrt{2DC_1C_s}$ where D denotes the total number of units produced per unit time, C_s is the set-up cost per production run and C_1 is the holding cost per unit of inventory per unit time. Production is assumed to be instantaneous and shortage cost infinite.
9. In a certain manufacturing situation the production is instantaneous and the demand is R. Show that the optimal order quantity is

$$Q = \sqrt{\frac{2C_s(C_1 + C_2)}{C_1 C_2}}$$

where C_1, C_2 are the shortage and shortage costs per unit per year and C_s is the set-up cost per run.

NOTES

10. Derive an expression for "Economic Batch Size" in case of a single item deterministic model with uniform demand and finite rate of replenishment.
11. (*a*) Formulate and solve the purchase inventory problem with one price break.
    (*b*) Describe the single item static model with any number of price breaks.
12. Discuss the problem of inventory control when the stochastic demand is uniform, production of commodity is instantaneous and lead time is negligible (discrete case).
13. (*a*) Discuss any one stochastic model of inventory management. Derive the formula of optimum level of the inventory.
    (*b*) Show that for a probabilistic discrete inventory model with instantaneous demand and no set-up cost, the optimum stock level *z* can be obtained by

$$\sum_{d=0}^{z} p(d) \geq \frac{c_2}{c_1 + c_2} \geq \sum_{d=0}^{z-1} p(d)$$

14. Discuss the continuous case of a probabilistic inventory model with instantaneous demand and no set-up cost.
15. Ten items kept in inventory of school of management studies of a state university are listed below. Which items should be classified as A items, B items and C items ? What percentage of items is in each class ? What percentage of total annual value is in each class ?

Item	*Annual usage*	*Value per unit (Rs.)*
1	200	40.00
2	100	260.00
3	200	0.20
4	400	20.00
5	6000	0.04
6	1200	0.80
7	120	100.00
8	2000	0.70
9	1000	1.00
10	80	400.00

FURTHER READINGS

- *Operational Research,* by col. D.S. Cheema, University Science Press.
- *Statistics and Operational Research —A Unified Approach,* by Dr. Debashis Dutta, Laxmi Publications (P) Ltd.

NOTES

UNIT II: LINEAR PROGRAMMING PROBLEMS (LPP)

★ STRUCTURE ★

2.0 LEARNING OBJECTIVES

After going through this unit, you should be able to:

- enumerate linear programming problems through various methods, such as simplex method, two phase method etc.
- describe duality and sensitivity analysis.

2.1 INTRODUCTION

The roots of Operations Research (O.R.) can be traced many decades ago. First this term was coined by Mc Closky and Trefthen of United Kingdom in 1940 and it came in existence during world war II when the allocations of scarce resources were done to the various military operations. Since then the field has developed very rapidly. Some chronological events are listed below :

1952 – Operations Research Society of America (ORSA).
1957 – Operations Research Society of India (ORSI)
– International Federation of O.R. Societies
1959 – First Conference of ORSI
1963 – Opsearch (the journal of O.R. by ORSI).

NOTES

However the term 'Operations Research' has a number of different meaning. The Operational Research Society of Great Britain has adopted the following illaborate definition :

"Operational Research is the application of the methods of science to complex problems arising in the direction and management of large systems of men, machines, materials and money in industry, business, government and defence. The distinctive approach is to develop a scientific method of the system, incorporating measurements of factors such as chance and risk, with which to predict and compare the outcomes of alternative decisions, strategies and controls. The purpose is to help management to determine its policy and actions scientifically."

Whereas ORSA has offered the following shorter definition :

"Operations Research is concerned with scientifically deciding how to best design and operate man-machine systems, usually under conditions requiring the allocation of scarce resources".

Many individuals have described Operational Research according to their own view. Only three are quoted below :

"Operational Research is the art of giving bad answers to problems which otherwise have worse answers" —*T.L. Saaty*

"Operational Research is a scientific approach to problems solving for executive management"

—*H.M. Wagner*

"Operational Research is a scientific knowledge through interdisciplinary team effort for the purpose of determining the best utilization of limited resources". —*H.A. Taha*

An abbreviated *list of applications* of Operational Research techniques are given below :

1. Manufacturing	:	Production scheduling Inventory control Product mix Replacement policies
2. Marketing	:	Advertising budget allocation Supply chain management
3. Organizational behaviour	:	Personnel planning Scheduling of training programs Recruitment policies
4. Facility planning	:	Factory location Hospital planning Telecommunication network planning Warehouse location
5. Finance	:	Investment analysis Portfolio analysis
6. Construction	:	Allocation of resources to projects Project scheduling
7. Military		
8. Different fields of engineering.		

2.2 INTRODUCTION TO LINEAR PROGRAMMING PROBLEMS (LPP)

When a problem is identified then the attempt is to make an mathematical model. In decision making all the decisions are taken through some variables which are known as decision variables. In engineering design, these variables are known as design vectors. So in the formation of mathematical model the following **three phases** are carried out :

(*i*) Identify the decision variables.

(*ii*) Identify the objective using the decision variables and

(*iii*) Identify the constraints or restrictions using the decision variables.

Let there be n decision variable $x_1, x_2, ..., x_n$ and the general form of the mathematical model which is called as Mathematical programming problem under decision making can be stated as follows :

Maximize/Minimize $\quad z = f(x_1, x_2, ..., x_n)$

Subject to, $\quad g_i(x_1, x_2, ..., x_n) \{\leq, \geq \text{ or } =\} b_i$

$i = 1, 2, ..., m.$

and the type of the decisions *i.e.*, $x_j \geq 0$

or, $\quad x_j \leq 0$ or x_j's are unrestricted

or combination types decisions.

In the above, if the functions f and g_i $(i = 1, 2, ..., m)$ are all linear, then the model is called *"Linear Programming Problem (LPP)"*. If any one function is non-linear then the model is called *"Non-linear Programming Problem (NLPP)"*.

Basic Aspects of LPP

(*i*) We define some basic aspects of LPP in the following :

Convex set : A set X is said to be convex if

$x_1, x_2 \in X$, then for $0 \leq \lambda \leq 1$,

$$x_3 = \lambda x_1 + (1 - l)x_2 \in X$$

Some examples of convex sets are :

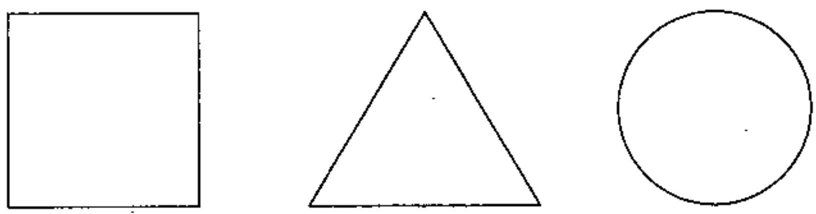

Fig. 2.1 Convex sets

Some examples of non-convex sets are :

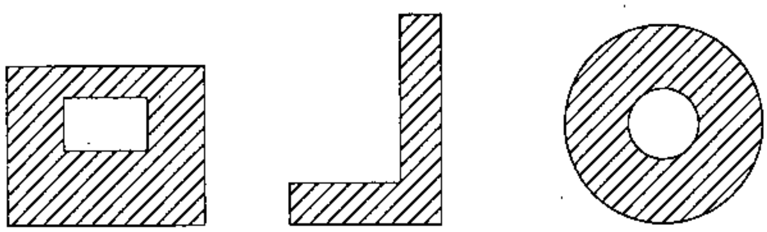

Fig. 2.2 Non-convex sets

Basically if all the points on a line segment forming by two points lies inside the set/geometric figure then it is called convex.

(*ii*) **Extreme point or vertex or corner point of a convex set :** It is a point in the convex set which can not be expressed as $\lambda x_1 + (1 - \lambda)x_2$ where x_1 and x_2 are any two points in the convex set.

For a triangle, there are three vertices, for a rectangle there are four vertices and for a circle there are infinite number of vertices.

(*iii*) Let $Ax = b$ be the constraints of an LPP. The set $X = \{x \mid Ax = b, x \geq 0\}$ is a convex set.

Feasible Solution : A solution which satisfies all the constraints in LPP is called feasible solution.

NOTES

Basic Solution : Let m = no. of constraints and n = no. of variables and $m < n$. Then the solution from the system $Ax = b$ is called basic solution. In this system there are ${}^{n}c_m$ number of basic solutions. By setting $(n - m)$ variables to zero at a time, the basic solutions are obtained. The variables which is set to zero are known as 'non-basic' variables. Other variables are called basic variables.

Basic Feasible Solution (BFS) : A solution which is basic as well as feasible is called basic feasible solution.

Degenerate BFS : If a basic variable takes the value zero in a BFS, then the solution is said to be degenerate.

Optimal BFS : The BFS which optimizes the objective function is called optimal BFS. Linear programming problems can be farmulated by various methods. Here, we will discuss as follows:

2.3 GRAPHICAL METHOD OF LINEAR PROGRAMMING PROBLEMS

Let us consider the constraint $x_1 + x_2 = 1$. The feasible region of this constraint comprises the set of points on the straight line $x_1 + x_2 = 1$.

If the constraint is $x_1 + x_2 \geq 1$, then the feasible region comprises not only the set of points on the straight line $x_1 + x_2 = 1$ but also the points above the line. Here above means away from origin.

If the constraint is $x_1 + x_2 \leq 1$, then the feasible region comprises not only the set of points on the straight line $x_1 + x_2 = 1$ but also the points below the line. Here below means towards the origin.

The above three cases depicted below :

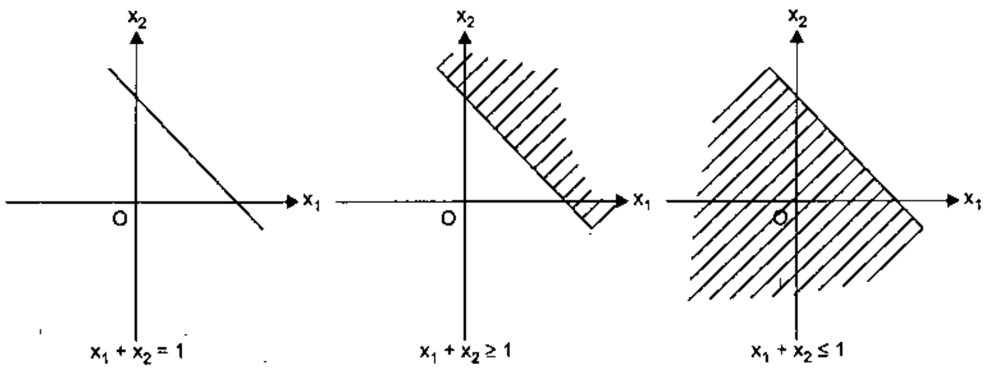

Fig. 2.3

For the constraints $x_1 \geq 1$, $x_1 \leq 1$, $x_2 \geq 1$, $x_2 \leq 1$ the feasible regions are depicted below :

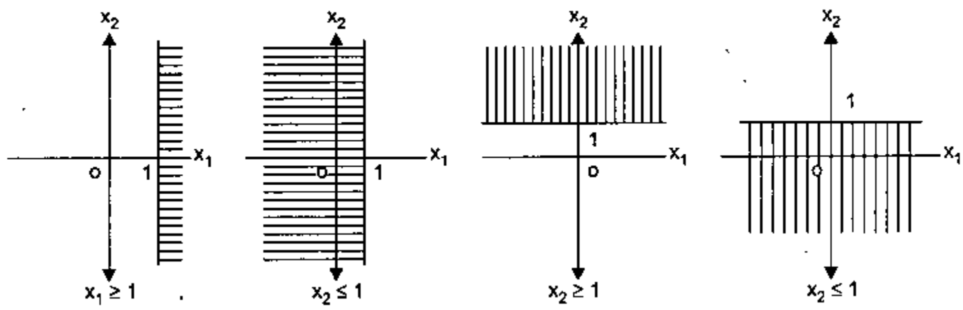

Fig. 2.4

For the constraints $x_1 - x_2 = 0$, $x_1 - x_2 \geq 0$ and $x_1 - x_2 \leq 0$ the feasible regions are depicted in Fig. 2.5.

The *steps of graphical method* can be stated as follows :

(*i*) Plot all the constraints and identify the individual feasible regions.

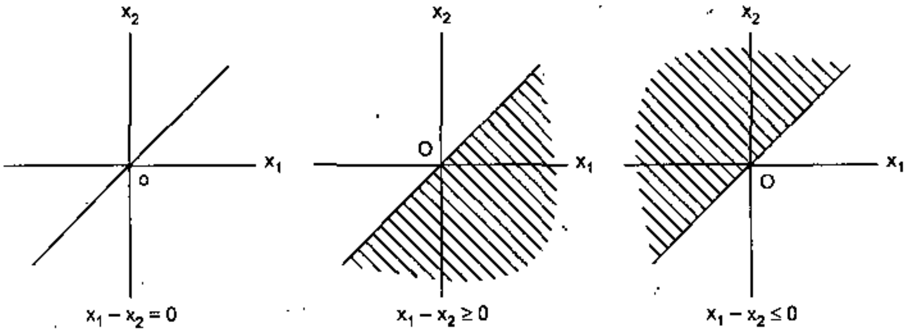

Fig. 2.5

(*ii*) Identify the common feasible region and identify the corner points *i.e.,* vertices of the common feasible region.

(*iii*) Identify the optimal solution at the corner points if exists.

Example 2.1. *Using graphical method solve the following LPP :*

Maximize $z = 5x_1 + 3x_2$

Subject to, $2x_1 + 5x_2 \le 10,$

$5x_1 + 2x_2 \le 10,$

$2x_1 + 3x_2 \ge 6,$

$x_1 \ge 0, x_2 \ge 0.$

Solution. Let us present all the constraints in intercept form *i.e.,*

$$\frac{x_1}{5} + \frac{x_2}{2} \le 1 \quad \text{...(I)}$$

$$\frac{x_1}{2} + \frac{x_2}{5} \le 1 \quad \text{...(II)}$$

$$\frac{x_1}{3} + \frac{x_2}{2} \ge 1 \quad \text{...(III)}$$

The common feasible region ABC is shown in Fig. 2.6 and the individual regions are indicated by arrows. (Due to non-negativity constraints *i.e.,* $x_1 \ge 0$, $x_2 \ge 0$, the common feasible region is obtained in the first quadrant).

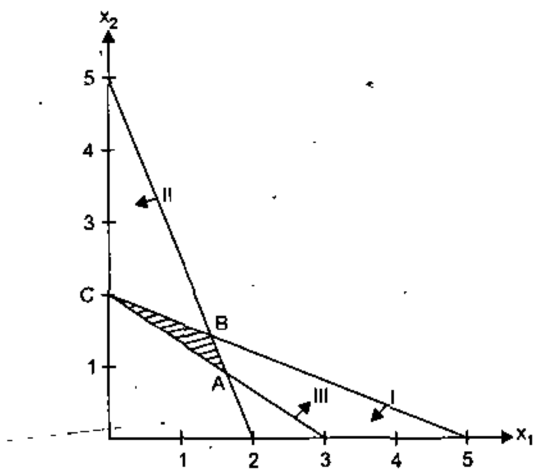

Fig. 2.6

The corner points are $A\left(\frac{18}{11}, \frac{10}{11}\right)$, $B\left(\frac{10}{7}, \frac{10}{7}\right)$ and C (0, 2). The value of the objective function at the corner points are $z_A = \frac{120}{11} = 10.91$, $z_B = \frac{80}{7} = 11.43$ and $z_C = 6$.

NOTES

Here the common feasible region is bounded and the maximum has occurred at the corner point B. Hence the optimal solution is

$$x_1^* = \frac{10}{7}, x_2^* = \frac{10}{7} \text{ and } z^* = \frac{80}{7} = 11.43.$$

Exceptional Cases in Graphical Method

NOTES

There are three cases may arise. When the value of the objective function is maximum/minimum at more than one corner points then *'multiple optima'* solutions are obtained.

Sometimes the optimum solution is obtained at infinity, then the solution is called *'unbounded solution'*. Generally, this type of solution is obtained when the common feasible region is unbounded and the type of the objective function leads to unbounded solution.

When there does not exist any common feasible region, then there does not exist any solution. Then the given LPP is called *infeasible i.e.,* having *no solution*. For example, consider the LPP which is infeasible

Maximize $z = 5x_1 + 10x_2$

Subject to, $x_1 + x_2 \le 2,$

$x_1 + x_2 \ge 3,$

$x_1, x_2 \ge 0.$

Example 2.2. *Solve the following LPP using graphical method :*

Maximize $z = x_1 + \frac{3}{5}x_2$

Subject to, $5x_1 + 3x_2 \le 15,$

$3x_1 + 4x_2 \le 12,$

$x_1, x_2 \ge 0.$

Solution. Let us present all the constraints in intercept forms *i.e.,*

$$\frac{x_1}{3} + \frac{x_2}{5} \le 1 \quad \text{...(I)}$$

$$\frac{x_1}{4} + \frac{x_2}{3} \le 1 \quad \text{...(II)}$$

Due to non-negativity constraints *i.e.,* $x_1 \ge 0$, $x_2 \ge 0$ the common feasible region is obtained in the first quadrant as shown in Fig. 2.7 and the individual feasible regions are shown by arrows.

The corner points are O(0, 0), A (3, 0), B$\left(\frac{24}{11}, \frac{15}{11}\right)$ and C(0, 3). The values of the objective function at the corner points are obtained as $z_O = 0$, $z_A = 3$, $z_B = 3$, $z_C = \frac{9}{4}$.

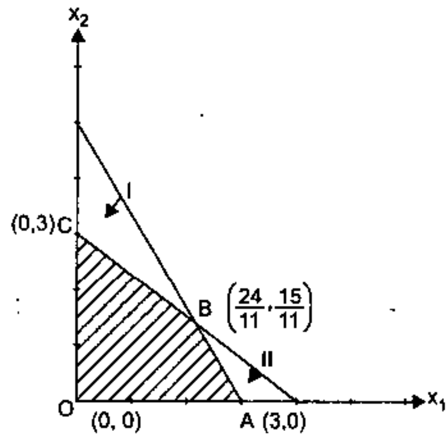

Fig. 2.7

Since the common feasible region is bounded and the maximum has occurred at two corner points *i.e.,* at A and B respectively, these solutions are called multiple optima. So the solutions are

$$x_1^* = 3, x_2^* = 0 \text{ and } x_1^* = \frac{15}{11}, x_2^* = \frac{24}{11} \text{ and } z^* = 3.$$

2.4 SIMPLEX METHOD

The algorithm is discussed below with the help of a numerical example *i.e.*, consider

Maximize $z = 4x_1 + 8x_2 + 5x_3$

Subject to, $x_1 + 2x_2 + 3x_3 \le 18,$

$2x_1 + 6x_2 + 4x_3 \le 15,$

$x_1 + 4x_2 + x_3 \le 6,$

$x_1, x_2, x_3 \ge 0.$

NOTES

Step 1. If the problem is in minimization, then convert it to maximization as Min $z = -$ Max $(-z)$.

Step 2. All the right side constants must be positive. Multiply by -1 both sides for negative constants. All the variables must be non-negative.

Step 3. Make standard form by adding slack variables for '$\le$' type constraints, surplus variables for '$\ge$' type constraints and incorporate these variables in the objective function with zero coefficients.

For example, Maximum $z = 4x_1 + 8x_2 + 5x_3 + 0, s_1 + 0.s_2 + 0.s_3$

Subject to, $x_1 + 2x_2 + 3x_3 + s_1 = 18,$

$2x_1 + 6x_2 + 4x_3 + s_2 = 15,$

$x_1 + 4x_2 + x_3 + s_3 = 6,$

$x_1, x_2, x_3 \ge 0, s_1, s_2, s_3 \ge 0$

Note that an unit matrix due to s_1, s_2 and s_3 variables is present in the coefficient matrix which is the key requirement for simplex method.

Step 4. Simplex method is an iterative method. Calculations are done in a table which is called simplex table. For each constraint there will be a row and for each variable there will be a column. Objective function coefficients c_j are kept on the top of the table. x_B stands for basis column in which the variables are called 'basic variables'. Solution column gives the solution, but in iteration 1, the right side constants are kept. At the bottom $z_j - c_j$ row is called 'net evaluation' row.

In each iteration one variable departs from the basis and is called departing variable and in that place one variable enter which is called entering variable to improve the value of the objective function.

Minimum ratio column determines the departing variable.

Iteration 1.

	c_j		4	8	5	0	0	0	Min. ratio
c_B	x_B	soln.	x_1	x_2	x_3	s_1	s_2	s_3	
0	s_1	18	1	2	3	1	0	0	
0	s_2	15	2	6	4	0	1	0	
0	s_3	6	1	4	1	0	0	1	
	$z_j - c_j$								

Note. Variables which are forming the columns of the unit matrix enter into the basis column. In this table the solution is $s_1 = 18, s_2 = 15, s_3 = 6, x_1 = 0, x_2 = 0, x_3 = 0$ and $z = 0$.

To test optimality we have to calculate $z_j - c_j$ for each column as follows :

$$z_j - c_j = c_B^T . [x_j] - c_j$$

For first column, $(0,0,0)\begin{pmatrix}1\\2\\1\end{pmatrix}-4=-4$

For second column, $(0,0,0)\begin{pmatrix}2\\6\\4\end{pmatrix}-8=-8$ and so on.

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These are displayed in the following table :

c_j			4	8	5	0	0	0	Min. ratio
c_B	x_B	soln.	x_1	x_2	x_3	s_1	s_2	s_3	
0	s_1	18	1	2	3	1	0	0	
0	s_2	15	2	6	4	0	1	0	
0	s_3	6	1	4	1	0	0	1	
$z_j - c_j$			– 4	– 8	– 5	0	0	0	
				↑					

Decisions : If all $z_j - c_j \geq 0$. then the current solution is optimal and stop. Else, Select the negative most value from $z_j - c_j$ and the variable corresponding to this value will be the entering variable and that column is called *'key column'*. Indicate this column with an upward arrow symbol.

In the given problem '– 8' is the most negative and variable x_2 is the entering variable. If there is a tie in the most negative, break it arbitrarily.

To determine the *departing variable*, we have to use minimum ratio. Each ratio is calculated as $\frac{[\text{soln.}]}{[\text{key column}]}$, componentwise division only for positive elements (*i.e.*, > 0) of the key column. In this example,

$$\min.\left\{\frac{18}{2},\frac{15}{6},\frac{6}{4}\right\} = \min.\{9, 2.5, 1.5\} = 1.5$$

The element corresponding to the min. ratio *i.e.*, here s_3 will be the departing variable and the corresponding row is called 'key row' and indicate this row by an outward arrow symbol. The intersection element of the key row and key column is called key element. In the present example, 4 is the key element which is highlighted. This is the end of this iteration, the final table is displayed below :

Iteration 1:

c_j			4	8	5	0	0	0	Min. ratio
c_B	x_B	soln.	x_1	x_2	x_3	s_1	s_2	s_3	
0	s_1	18	1	2	3	1	0	0	$\frac{18}{2}=9$
0	s_2	15	2	6	4	0	1	0	$\frac{15}{6}=2.5$
0	s_3	6	1	4	1	0	0	1	$\frac{6}{4}=1.5\rightarrow$
$z_j - c_j$			– 4	– 8	– 5	0	0	0	
				↑					

Step 5. For the construction of the next iteration (new) table the following calculations are to be made :

(a) Update the x_B column and the c_B column.

(b) Divide the key row by the key element.

(c) Other elements are obtained by the following formula :

$$\begin{pmatrix}\text{new}\\ \text{element}\end{pmatrix} = \begin{pmatrix}\text{old}\\ \text{element}\end{pmatrix} - \frac{\begin{pmatrix}\text{element}\\ \text{corresponding to}\\ \text{key row}\end{pmatrix}\cdot\begin{pmatrix}\text{element}\\ \text{corresponding to}\\ \text{key column}\end{pmatrix}}{\text{key element}}$$

(d) Then go to step 4.

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Iteration 2.

c_j			4	8	5	0	0	0	Min. ratio
c_B	x_B	soln.	x_1	x_2	x_3	s_1	s_2	s_3	
0	s_1	15	$\frac{1}{2}$	0	$\frac{5}{2}$	1	0	$-\frac{1}{2}$	$15\times\frac{3}{5}=6$
0	s_2	6	$\frac{1}{2}$	0	$\frac{5}{2}$	0	1	$-\frac{3}{2}$	$6\times\frac{2}{5}=2.4\rightarrow$
8	s_3	$\frac{3}{2}$	$\frac{1}{4}$	1	$\frac{1}{4}$	0	0	$\frac{1}{4}$	$\frac{3}{4}\times 4=3$
z_j-c_j			-2	0	-3 ↑	0	0	2	

Iteration 3.

c_j			4	8	5	0	0	0	Min. ratio
c_B	x_B	soln.	x_1	x_2	x_3	s_1	s_2	s_3	
0	s_1	9	0	0	0	1	-1	1	-
5	s_2	$\frac{12}{5}$	$\frac{1}{5}$	0	1	0	$\frac{2}{5}$	$-\frac{3}{5}$	$\frac{12}{5}\times\frac{2}{1}=12$
8	s_3	$\frac{9}{10}$	$\frac{1}{5}$	1	0	0	$-\frac{1}{10}$	$\frac{2}{5}$	$\frac{9}{10}\times\frac{5}{1}=4.5\rightarrow$
z_j-c_j			$-\frac{7}{5}$ ↑	0	0	0	$\frac{6}{5}$	$\frac{1}{5}$	

Iteration 4.

c_j			4	8	5	0	0	0	Min. ratio
c_B	x_B	soln.	x_1	x_2	x_3	s_1	s_2	s_3	
0	s_1	9	0	0	0	1	-1	1	
5	s_2	$\frac{3}{2}$	0	-1	1	0	$\frac{1}{2}$	-1	
4	s_3	$\frac{9}{2}$	1	5	0	0	$-\frac{1}{2}$	2	
z_j-c_j			0	7	0	0	$\frac{1}{2}$	3	

Since all $z_j - c_j \geq 0$, the current solution is optimal.

$$\therefore \quad x_1^* = \frac{9}{2}, x_2^* = 0, x_3^* = \frac{3}{2}, z^* = \frac{51}{2}.$$

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Note (exceptional cases).

(*a*) If in the key column, all the elements are non-positive *i.e.*, zero or negative, then min. ratio can- not be calculated and the problem is said to be unbounded.

(*b*) In the net evaluation of the optimal table all the basic variables will give the value zero. If any non-basic variable give zero net evaluation then it indicates that there is an alternative optimal solution. To obtain that solution, consider the corresponding column as key column and apply one simplex iteration.

(*c*) For negative variables, $x \leq 0$, set $x = -x'$, $x' \geq 0$.
For unrestricted variables set $x = x' - x''$ where $x', x'' \geq 0$.

Example 2.3. *Solve the following by simplex method :*

Maximize $z = x_1 + 3x_2$

Subject to, $-x_1 + 2x_2 \leq 2,\ x_1 - 2x_2 \leq 2,\ x_1, x_2 \geq 0.$

Solution. Standard form of the given LPP can be written as follows :

Maximum $z = x_1 + 3x_2 + 0.s_1 + 0.s_2$

Subject to, $-x_1 + 2x_2 + s_1 = 2,\ x_1 - 2x_2 + s_2 = 2,$

$x_1, x_2 \geq 0,\ s_1, s_2$ slacks $\geq 0.$

Iteration 1.

c_j			1	3	0	0	Min. ratio
c_B	x_B	soln.	x_1	x_2	s_1	s_2	
0	s_1	9	-1	2	1	0	$\frac{2}{2} = 1$
0	s_2	2	1	-2	0	1	–
$z_j - c_j$			-1	-3	0	0	
				↑			

Iteration 2.

c_j			1	3	0	0	Min. ratio
c_B	x_B	soln.	x_1	x_2	s_1	s_2	
0	x_1	1	$-\frac{1}{2}$	1	$\frac{1}{2}$	0	
0	s_2	4	0	0	1	1	
$z_j - c_j$			$-\frac{5}{2}$	0	$\frac{3}{2}$	0	
			↑				

Since all the elements in the key column are non-positive, we can not calculate min. ration. Hence the given LPP is said to be unbounded.

2.5 SENSITIVITY ANALYSIS

The solution to LPP is based on a number of deterministic assumptions like the prices are known exactly and are fixed, resources are known with certainty and time needed to manufacture/assemble/produce a product is fixed. In real life situations, which are dynamic and changing, the effect of variation of these variables must be studied and understood. This process of knowing the impact of variables on the outcome of optimal result is known as sensitivity analysis of linear programming problems. Let us say, for example, that if originally we had assumed the cost per unit to be Rs. 10 but it turns out be Rs. 11, how will the final profit and solution mix vary. Also, if we start with the

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assumption of certain fixed resources like man hours or machine hours and as we proceed we realise the availability can be improved, how will this change our optimal solution.

Sensitivity analysis can be used to study the impact of changes in

(a) Addition or deletion of variables initially selected.
(b) Change in the cost or price of the product under consideration.
(c) Increase or decrease in the resources.

Sensitivity Analysis uses the following two approaches :

(a) It involves solving the entire problem by trial and error approach and involves very cumbersome calculations.
Every time data of a variable is changed, it becomes another set of the problem and has to be solved independently.
(b) The last simplex table may be investigated. This reduces completion and computations considerably.

Limitations of Sensitivity Analysis

Sensitivity analysis does take into account the uncertainty element, yet, it suffers from the following limitations.

(a) Only one variable can be taken into account at one time. Hence, the impact of many variables changing cannot be considered simultaneously.
(b) It suffers from the linearity limitations as only linear relationship between the variable is considered.
(c) The extent of uncertainty cannot be studied.
(d) As the result can be judged by individual analysts depending upon their skills and experience, it is to that extent subjective in nature.

2.6 BIG M METHOD

The method is also known as 'penalty method' due to Charnes. If there is '≥' type constraint, we add surplus variable and if there is '=' type, then the constraint is in equilibrium. Generally, in these cases there may not be any unit matrix in the standard form of the coefficient matrix.

To bring unit matrix we take help of another type of variable, known as 'artificial variable'. The addition of artificial variable creates infeasibility in the system which was already in equilibrium. To overcome this, we give a very large number denoted as M to the coefficient of the artificial variable in the objective function. For maximization problem, we add "– M. (artificial variable)" in the objective function so that the profit comes down. For minimization problem we add "M.(artificial variable)" in the objective function so that the cost goes up. Therefore the simplex method tries to reduce the artificial variable to the zero level so that the feasibility is restored and the objective function is optimized.

The only **drawback** of the big M method is that the value of M is not known but it is a very large number. Therefore we cannot develop computer program for this method.

Note. (a) Once the artificial variable departs from the basis, it will never again enter in the subsequent iterations due to big M. Due to this we drop the artificial variable column in the subsequent iterations once the variable departs from the basis.

(*b*) If in the optimal table, the artificial variable remains with non-zero value, then the problem is said to be '*infeasible*'.

If the artificial variable remains in the optimal table with zero value, then the solution is said to be 'pseudo optimal'.

(*c*) The rule for 'multiple solution' and 'unbounded solution' are same as given by simplex method. The big-M method is a simple variation of simplex method.

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Example 2.4. *Using Big-M method solve the following LPP :*

Minimize $z = 10x_1 + 3x_2$

$S/t,\ x_1 + 2x_2 \geq 3,\ x_1 + 4x_2 \geq 4;\ x_1, x_2 \geq 0$

Solution. Standard form of the given LPP is

$$\text{Min. } z = -\text{Max. } (-z = -10x_1 - 3x_2 + 0.s_1 + 0.s_2 - Ma_1 - Ma_2)$$

$S/t,\ x_1 + 2x_2 - s_1 + a_1 = 3$

$x_1 + 4x_2 - s_2 + a_2 = 4$

$x_1, x_2 \geq 0,\ s_1, s_2$ surplus $\geq 0,\ a_1, a_2$ artificial ≥ 0

Iteration 1.

c_j			-10	-3	0	0	$-M$	$-M$	Min. ratio	
c_B	x_B	soln.	x_1	x_2	s_1	s_2	a_1	a_2		
$-M$	a_1	3	1	2	-1	0	1	0	$\frac{3}{2}=1.5$	
$-M$	a_2	4	1	4	0	-1	0	1	$\frac{4}{4}=1$	→
$z_j - c_j$			$-2M + 10$	$-6M + 3$	M	M	0	0		
				↑						

Iteration 2.

c_j			-10	-3	0	0	$-M$	Min. ratio	
c_B	x_B	soln.	x_1	x_2	s_1	s_2	a_1		
$-M$	a_1	1	$\frac{1}{2}$	0	-1	$\frac{1}{2}$	1	$\frac{1}{1/2}=2$	→
-3	x_2	1	$\frac{1}{4}$	1	0	$-\frac{1}{4}$	0	$\frac{1}{1/4}=4$	
$z_j - c_j$			$-\frac{M}{2}+\frac{37}{4}$	0	M	$-\frac{M}{2}+\frac{3}{4}$	0		
			↑						

Iteration 3.

c_j			-10	-3	0	0	Min. ratio	
c_B	x_B	soln.	x_1	x_2	s_1	s_2		
-10	x_1	2	1	0	-2	1	$\frac{2}{1}=2$	→
-3	x_2	$\frac{1}{2}$	0	1	$\frac{1}{2}$	$-\frac{1}{2}$		
$z_j - c_j$			0	0	$\frac{37}{2}$	$-\frac{17}{2}$		
				↑				

Iteration 4. (Optimal)

c_j			-10	-3	0	0	Min. ratio
c_B	x_B	soln.	x_1	x_2	s_1	s_2	
0	s_2	2	1	0	-2	1	
-3	x_2	$\frac{3}{2}$	$\frac{1}{2}$	1	$-\frac{1}{2}$	0	
$z_j - c_j$			$\frac{17}{2}$	0	$\frac{3}{2}$	0	

Since all $z_j - c_j \geq 0$, the current solution is optimal.

$$\therefore \qquad x_1^* = 0, x_2^* = \frac{3}{2}, z^* = \frac{9}{2}.$$

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2.7 TWO PHASE METHOD

To overcome the drawback of Big-M method, two phase method has been framed. In the first phase an auxiliary LP Problem is formulated as follows :

Minimize T = Sum of artificial variables

S/t, original constraints

which is solved by simplex method. Here artificial variables act as decision variables. So Big-M is not required in the objective function. If $T^* = 0$, then go to phase two calculations, else ($T^* \neq 0$) write the problem is infeasible. In phase two, the optimal table of phase one is considered with the following modifications :

Delete the artificial variable's columns and incorporate the original objective function and also update the c_B values. Calculate $z_j - c_j$ values. If all $z_j - c_j \geq 0$, the current solution is optimal else go to the next iteration.

Note. (*a*) Multiple solutions, if it exists, can be detected from the optimal table of phase two.

(*b*) In phase I, the problem is always minimization type irrespective of the type of the original given objective function.

Example 2.5. *Using two phase method solve the following LPP :*

Minimize $\qquad z = 10x_1 + 3x_2$

$\qquad$ S/t, $x_1 + 2x_2 \geq 3$, $x_1 + 4x_2 \geq 4$; $x_1, x_2 \geq 0$

Solution. Standard form of the given LPP is

$$\text{Min. } z = -\text{Max. } (-z = -10x_1 - 3x_2 + 0.s_1 + 0.s_2 - Ma_1 - Ma_2)$$

S/t, $x_1 + 2x_2 - s_1 + a_1 = 3,$

$x_1 + 4x_2 - s_2 + a_2 = 4,$

$x_1, x_2 > s_1, s_2$, surplus $> a_1, a_2$ artificial ≥ 0

Phase I $\qquad \text{Min } T = a_1 + a_2 = -\text{Max } (-T = 0.x_1 + 0.x_2 + 0.s_1 + 0.s_2 - a_1 - a_2)$

S/t., $x_1 + 2x_2 - s_1 + a_1 = 3$; $x_1 + 4x_2 - s_2 + a_2 = 4,$

$x_1, x_2, s_1, s_2, a_1, a_2 > 0$

Iteration 1.

C_j			0	0	0	0	−1	−1	Min. ratio	
c_B	x_B	soln.	x_1	x_2	s_1	s_2	a_1	a_2		
−1	a_1	3	1	2	−1	0	1	0	$\frac{3}{2}=1.5$	
−1	a_2	4	1	4	0	−1	0	1	$\frac{4}{4}=1$	→
z_j-c_j			−2	−6	1	1	0	0		
				↑						

Iteration 2.

C_j			0	0	0	0	−1	−1	Min. ratio	
c_B	x_B	soln.	x_1	x_2	s_1	s_2	a_1	a_2		
−1	a_1	1	$\frac{1}{2}$	0	−1	$\frac{1}{2}$	1	$-\frac{1}{2}$	$\frac{1}{1/2}=2$	
0	x_2	1	$\frac{1}{4}$	1	0	$-\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{1/4}=4$	→
z_j-c_j			$-\frac{1}{2}$	0	1	$-\frac{1}{2}$	0	$\frac{3}{2}$		
				↑						

Iteration 3.

C_j			0	0	0	0	−1	−1	Min. ratio
c_B	x_B	soln.	x_1	x_2	s_1	s_2	a_1	a_2	
0	x_1	2	1	0	−2	1	2	−1	
0	x_2	$\frac{1}{2}$	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	
z_j-c_j			0	0	0	0	1	1	

Since all $z_j - c_j \geq 0$, the solution is optimal $a_1^* = 0, a_2^* = 0$ and $T^* = 0$. Therefore we go to phase II calculations.

Phase II

Iteration 1.

C_j			−10	−3	0	0	Min. ratio	
c_B	x_B	soln.	x_1	x_2	s_1	s_2		
−10	x_1	2	1	0	−2	1	$\frac{2}{1}=2$	→
−3	x_2	$\frac{1}{2}$	0	1	$\frac{1}{2}$	$-\frac{1}{2}$		
z_j-c_j			0	0	$\frac{37}{2}$	$-\frac{17}{2}$		

Iteration 2.

C_j			−10	−3	0	0	Min. ratio
c_B	x_B	soln.	x_1	x_2	s_1	s_2	
0	s_2	2	1	0	−2	1	
−3	x_2	$\frac{3}{2}$	$\frac{1}{2}$	1	$-\frac{1}{2}$	0	
$z_j - c_j$			$\frac{17}{2}$	0	$\frac{3}{2}$	0	

Since all $z_j - c_j \geq 0$, the current solution is optimal.

$$\therefore \qquad x_1^* = 0, * = x_2^* = \frac{3}{2},\ z^* = \frac{9}{2}.$$

Solve the following LPP using *Big-M* method and *Two phase* method :

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2.8 REVISED SIMPLEX METHOD (RSM)

I. Algorithm

Step 1. Write the standard form of the given LPP and convert it into maximization type if it is in minimization type *i.e.,*

$$\text{Max. } z = cx$$
$$\text{S/}t,\ Ax = b,\ x \geq 0.$$

Use the following notations :

$c^T = [c_1, c_2, \ldots, c_n]$ Profit coefficients.

Columns of A as $A_1, A_2, \ldots, A_m$.

$\pi = (\pi_1, \pi_2, \ldots)$ Simplex multipliers

x_B = Basis vector

c_B^T = Profit coefficient in the basis

B = Basis matrix, B^{-1} = Basis inverse

$\overline{c}_j$ = Net evaluations, j = Index of non-basic variables

$\overline{b}$ = Current BFS

Step 2. For Iteration 1

$$B = I,\ B^{-1} = I$$

else for other iterations

Find $\qquad B = \left[x_{B_i}\right] = \left[A_{x_{B_i}}\right]$ and hence find B^{-1}.

Step 3. Calculate

$$\pi = c_B^T.B^{-1} \text{ and } \overline{b} = B^{-1}.b \text{ (current solution)}$$

$$\overline{c}_j = \pi A_j - c_j$$

Decisions : If all $\overline{c}_j \geq 0$ then the current BFS is optimal,

else

select the negative most of $\overline{c}_j$, say $\overline{c}_k$. Then x_k will be the '*Entering Variable*' and $\overline{A}_k$ = key column = $B^{-1}.A_k$.

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Step 4. Produce the following revised simplex table :

x_B	B^{-1}	$\bar{b}$	Entering Variable	Key Column

Encircle the key element obtained from the min. ratio $\{[\bar{b}]/[\text{Key column}]\}$.

Element corresponding to the key element will depart from $[x_B]$.

Step 5. Go to step 2.

Repeat the procedure until optimal BFS is obtained.

Note. (*a*) If, in step 4, all the elements in the key column are non-positive, then the given problem is unbounded.

(*b*) If, in the optimal BFS, artificial variables (if any) take zero value then the solution is degenerate else, for non-zero value, the given problem is said to be infeasible.

II. Advantages

In computational point of view, the Revised Simplex Method is superior than ordinary simplex method. Due to selected column calculations in revised simplex method, less memory is required in computer. Whereas the ordinary simplex method requires more memory space in computer.

Example 2.6. *Using revised simplex method solve the following LPP :*

Maximize $z = 5x_1 + 2x_2 + 3x_3$

$S/t,\ x_1 + 2x_2 + 2x_3 \le 8$

$3x_1 + 4x_2 + x_3 \le 7$

$x_1, x_2, x_3 \ge 0.$

Solution. Standard form of the given LPP is

Maximize $z = 5x_1 + 2x_2 + 3x_3 + 0.s_1 + 0.s_2$

$S/t,\ x_1 + 2x_2 + 2x_3 + s_1 = 8$

$3x_1 + 4x_2 + x_3 + s_2 = 7$

$x_1, x_2, x_3 \ge 0, s_1, s_2$ are slacks and ≥ 0

Then, $$A_1 = \begin{bmatrix}1\\3\end{bmatrix}, A_2 = \begin{bmatrix}2\\4\end{bmatrix}, A_3 = \begin{bmatrix}2\\1\end{bmatrix}, A_4 = \begin{bmatrix}1\\0\end{bmatrix}, A_5 = \begin{bmatrix}0\\1\end{bmatrix}, b = \begin{bmatrix}8\\7\end{bmatrix}$$

Let us consider the index of the variables x_1 be 1, x_2 be 2, x_3 be 3, s_1 be 4, s_2 be 5.

Iteration 1.

$$x_B = (s_1, s_2), B = [A_4, A_5] = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} = I, B^{-1} = I.$$

$$c_B^T = (0, 0), \bar{b} = B^{-1}.b = b, J = (1, 2, 3).$$

$$\pi = c_B^T.B^{-1} = (0, 0)\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} = (0, 0) = (\pi_1, \pi_2).$$

Net evaluations :

$$\bar{c}_1 = \pi A_1 - c_1 = -5 \leftarrow \text{negative most and entering variable is } x_1$$

$$\bar{c}_2 = \pi A_2 - c_2 = -2$$

$$\bar{c}_3 = \pi A_3 - c_3 = -3.$$

Key column: $B^{-1}.A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$

Table 1

x_B	B^{-1}	$\bar{b}$	Entering Variable	Key Column
s_1	1 0	8	x_1	1
s_2	0 1	7		③

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This indicates the departing variable as s_2.

Iteration 2.

$$x_B = (s_1, x_1), B = [A_4, A_1] = \begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix}, B^{-1} = \begin{bmatrix} 1 & -1/3 \\ 0 & 1/3 \end{bmatrix}.$$

$$\bar{b} = B^{-1}.b = \begin{bmatrix} 1 & -1/3 \\ 0 & 1/3 \end{bmatrix}\begin{bmatrix} 8 \\ 7 \end{bmatrix} = \begin{bmatrix} 17/3 \\ 7/3 \end{bmatrix}, J = (2, 3, 5).$$

$$\pi = c_B^T.B^{-1} = [0, 5]\begin{pmatrix} 1 & -1/3 \\ 0 & 1/3 \end{pmatrix} = \left(0, \frac{5}{3}\right).$$

Net evaluations :

$$\bar{c}_2 = \pi A_2 - c_2 = \left(0, \frac{5}{3}\right)\begin{pmatrix} 2 \\ 4 \end{pmatrix} - 2 = \frac{14}{3}.$$

$$\bar{c}_3 = \pi A_3 - c_3 = \left(0, \frac{5}{3}\right)\begin{pmatrix} 2 \\ 1 \end{pmatrix} - 3 = -\frac{4}{3} \leftarrow \text{Entering variable } x_3$$

$$\bar{c}_5 = \pi A_5 - c_5 = \left(0, \frac{5}{3}\right)\begin{pmatrix} 0 \\ 1 \end{pmatrix} - 0 = \frac{5}{3}$$

Key column: $B^{-1}.A_3 = \begin{pmatrix} 1 & -1/3 \\ 0 & 1/3 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 5/3 \\ 1/3 \end{pmatrix}.$

Table 2

x_B	B^{-1}	$\bar{b}$	Entering Variable	Key Column
s_1	1 −1/3	17/3	x_3	(5/3)
x_2	0 1/3	7/3		1/3

(This indicates the departing variable as s_1).

Iteration 3.

$$x_B = (x_3, x_1), B = [A_3, A_1] = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}, B^{-1} = \begin{pmatrix} 3/5 & -1/5 \\ -1/5 & 2/5 \end{pmatrix}$$

$$J = 2, 4, 5$$

$$\bar{b} = B^{-1}.b = \begin{pmatrix} 3/5 & -1/5 \\ -1/5 & 2/5 \end{pmatrix}\begin{pmatrix} 8 \\ 7 \end{pmatrix} = \begin{pmatrix} 17/5 \\ 6/5 \end{pmatrix}$$

$$\pi = c_B^T . B^{-1} = (3, 5)\begin{pmatrix} 3/5 & -1/5 \\ -1/5 & 2/5 \end{pmatrix} = \left(\frac{4}{5}, \frac{7}{5}\right).$$

Net evaluations :

NOTES

$$\bar{c}_2 = \pi A_2 - c_2 = \left(\frac{4}{5}, \frac{7}{5}\right)\begin{pmatrix} 2 \\ 4 \end{pmatrix} - 2 = \frac{26}{5}.$$

$$\bar{c}_4 = \pi A_4 - c_4 = \left(\frac{4}{5}, \frac{7}{5}\right)\begin{pmatrix} 1 \\ 0 \end{pmatrix} - 0 = \frac{4}{5}.$$

$$\bar{c}_5 = \pi A_5 - c_5 = \left(\frac{4}{5}, \frac{7}{5}\right)\begin{pmatrix} 0 \\ 1 \end{pmatrix} - 0 = \frac{7}{5}$$

As all $\bar{c}_j > 0 \Rightarrow$ the current $\bar{b}$ is optimal.

$\therefore$ $$x_1^* = \frac{6}{5}, x_2^* = 0, x_3^* = \frac{17}{5} \text{ and } z^* = \frac{81}{5}.$$

Using revised simplex method solve the following LPP :

2.9 INTRODUCTION AND FORMULATION OF DUALITY

For every LP Problem we can construct another LP problem using the same data. These two problems try to achieve two different objectives within the same data. The original problem is called *Primal* problem and the constructed problem is called *Dual*. This is illustrated through the following example :

A company makes three products X, Y, Z using three raw materials A, B and C. The raw material requirement is given below : (for 1 unit of product).

	X	Y	Z	*Availability*
A	1	2	1	36 units
B	2	1	4	60 units
C	2	5	1	45 units
Profit	Rs. 40	Rs. 25	Rs. 50	

Let the company decide to produce x_1, x_2 and x_3 units of the products X, Y and Z respectively in order to maximize the profit. We obtain the following LP problems :

Maximize profit $= 40x_1 + 25x_2 + 50x_3$

Subject to, $x_1 + 2x_2 + x_3 \le 36,$

$2x_1 + x_2 + 4x_3 \le 60,$

$2x_1 + 5x_2 + x_3 \le 45,$

$x_1, x_2, x_3 \ge 0.$

Adding slack variables s_1, s_2 and s_3 to the constraints, we solve the problem by simplex method. The optimal solution is

$$x_1 = 20, x_2 = 0, x_3 = 5 \text{ and optimal profit} = \text{Rs. } 1050.$$

Suppose the company wishes to sell the three raw materials A, B and C instead of using them for production of the products X, Y and Z. Let the selling prices be Rs. y_1, Rs. y_2 and Rs. y_3 per unit of raw material A, B and C respectively.

The cost of the purchaser due to all raw materials is

$$36y_1 + 60y_2 + 45y_3.$$

Then the purchaser forms the following LP problem :

Minimize $T = 36y_1 + 60y_2 + 45y_3$

Subject to, $y_1 + 2y_2 + 2y_3 \geq 40,$

$2y_1 + y_2 + 5y_3 \geq 25,$

$y_1 + 4y_2 + y_3 \geq 50,$

$y_1, y_2, y_3 \geq 0.$

The solution is obtained as :

$$y_1 = 0,\ y_2 = 10,\ y_3 = 10,\ \text{Optimal cost} = \text{Rs. } 1050.$$

In the above, the company's problem is called primal problem and purchaser's problem is called dual problem. Also we can use these two terms interchangeably. In the primal problem, the company achieve a profit of Rs. 1050 by producing 20 units of X and 5 units of Z. Instead, if the company sells the raw material B with Rs. 10 per unit and C with Rs. 10 per unit then also the company achieve a sale of Rs. 1050.

Formulation

In the above, both the problems are called **symmetric** problem since the objective function is maximization (minimization), all the constraints are '≤' type (≥ type) and non-negative decision variables.

The decision variables in the primal are called primal variables and the decision variables in the dual are called dual variables.

Let us consider the following table for formulation of the dual.

Primal (Maximization)	*Dual (Minimization)*
Right hand side constants	Cost vector
Cost vector	Right hand side constants.
Coefficient matrix	Transpose of coefficient matrix
'≤'	'≥'
Max. $z = cx$	Min. $T = b^T y$
S/t, $Ax \leq b$	S/t $A^T y \geq c^T$
$x \geq 0$	$y \geq 0$

Asymmetric Primal-Dual Problems

Primal (Maximization)	*Dual (Minimization)*
(*a*) Coefficient matrix	A^T
(*b*) Right hand side constants	Cost vector
(*c*) Cost vector	Right hand side constants
i-th constraint	*i*-th dual variable
≤ type	$y_i \geq 0$
≥ type	$y_i \leq 0$
= type	y_i unrestricted in sign
j-th primal variable	*j*-th dual constraint
x_j unrestricted in sign	= type
$x_j \leq 0$	≤ type
$x_j \geq 0$	≥ type

NOTES

Also in (*a*) and (*b*),

No. of primal constraints = No. of dual variables.

No. of primal variables = No. of dual constraints.

Note. The dual of the dual is the primal.

Example 2.7. *Obtain the dual of*

Minimize $z = 8x_1 + 3x_2 + 15x_3$

Subject to, $2x_1 + 4x_2 + 3x_3 \geq 28,$

$3x_1 + 5x_2 + 6x_3 \geq 30,$

$x_1, x_2, x_3 \geq 0.$

Solution. Let y_1 and y_2 be the variables corresponding to the first and second constraints respectively. Objective function, maximize $T = 28y_1 + 30y_2$. There will be three dual constraints due to three primal variables. In primal

$$A = \begin{bmatrix} 2 & 4 & 3 \\ 3 & 5 & 6 \end{bmatrix}, c = [8, 3, 15]$$

$\therefore$ In dual $A^T \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \leq c^T$

$$\Rightarrow \begin{pmatrix} 2 & 3 \\ 4 & 5 \\ 3 & 6 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \leq \begin{pmatrix} 8 \\ 3 \\ 15 \end{pmatrix}$$

$\Rightarrow$ $2y_1 + 3y_2 \leq 8$ (due to x_1)

$4y_1 + 5y_2 \leq 3$ (due to x_2)

$3y_1 + 6y_2 \leq 15$ (due to x_3)

Hence the dual problem is

Maximize $T = 28y_1 + 30y_2$

Subject to, $2y_1 + 3y_2 \leq 8$

$4y_1 + 5y_2 \leq 3$

$3y_1 + 6y_2 \leq 15$

$y_1, y_2 \geq 0.$

Example 2.8. *Find the dual of*

Maximize $z = 5x_1 + 4x_2 - 3x_3$

Subject to, $2x_1 + 4x_2 - x_3 \leq 14,$

$x_1 - 2x_2 + x_3 = 10,$

$x_1 \geq 0$, *x_2 unrestricted in sign,* $x_3 \leq 0.$

Solution. First we have introduce non-negative variables.

$\therefore$ Set $x_2 = x'_2 - x''_2;\ x'_2, x''_2 \geq 0$ and $x_3 = -x'_3,\ x'_3 \geq 0.$

The given problem reduces to

Maximize $z = 5x_1 + 4x'_2 - 4x''_2 + 3x'_3$

Subject to, $2x_1 + 4x'_2 - 4x''_2 + x'_3 \geq 14$

$x_1 - 2x'_2 + 2x''_2 - x'_3 \geq 10$

$x_1, x'_2, x''_2, x'_3 \geq 0$

The second constraint is expressed as

$x_1 - 2x'_2 + 2x''_2 - x'_3 \leq 10$

and $-x_1 + 2x'_2 - 2x''_2 + x'_3 \leq -10$

Let y_1, y_2, y_3 be the three dual variables corresponding to the three constraints respectively. Then the symmetric dual is

Minimize $T = 14y_1 + 10y_2 - 10y_3$

Subject to, $2y_1 + y_2 - y_3 \geq 5$ (due to x_1)

$4y_1 - 2y_2 + 2y_3 \geq 4$ (due to x'_1)

$-4y_1 + 2y_2 - 2y_3 \geq -4$ (due to x''_2)

$y_1 - y_2 + y_3 \geq -3$ (due to x'_3)

$y_1, y_2, y_3 \geq 0$

Set $w_1 = y_1, w_2 = y_2 - y_3 \Rightarrow w_1 \geq 0$ and w_2 unrestricted.

Also the second and third constraint reduces to

$4y_1 - 2y_2 + 2y_3 = 4$

Therefore the dual is

Minimize $T = 14w_1 + 10w_2$

Subject to, $2w_1 + w_2 \geq 5,$

$4w_1 - 2w_2 = 4,$

$-w_1 + w_2 \leq 3$

$w_1 \geq 0$ and w_2 unrestricted in sign.

2.10 DUALITY OF SIMPLEX METHOD

The fundamental theorem of duality helps to obtain the optimal solution of the dual from optimal table of the primal and vice-versa. Using C.S.C., the correspondence between the primal (dual) variables and slack and/or surplus variables of the dual (primal) to be identified. Then the optimal solution of the dual (primal) can be read off from the net evaluation row of the primal (dual) of the simplex table.

For example, if the primal variable corresponds to a slack variable of the dual, then the net evaluation of the slack variable in the optimal table will give the optimal solution of the primal variable.

Example 2.9. *Using the principle of duality solve the following problem :*

Minimize $z = 4x_1 + 14x_2 + 3x_3$

S/t, $-x_1 + 3x_2 + x_3 \geq 3,$

$2x_1 + 2x_2 - x_3 \geq 2,$

$x_1, x_2, x_3 \geq 0.$

Solution. The dual problem is

Maximize $T = 3y_1 + 2y_2$

S/t, $-y_1 + 2y_2 \leq 4$

$3y_1 + 2y_2 \leq 14$

$y_1 - y_2 \leq 3$

$y_1, y_2 \geq 0$

Standard form :

Maximize $T = 3y_1 + 2y_2 + 0.u_1 + 0.u_2 + 0.u_3$

S/t, $-y_1 + 2y_2 + u_1 = 4$

$3y_1 + 2y_2 + u_2 = 14$

$y_1 - y_2 + u_3 = 3$

$y_1, y_2 \geq 0$, u_1, u_2, u_3 are slacks and ≥ 0.

Let the surplus variables of the dual v_1 and v_2.

Then by C.S.C., $y_1v_1 = 0, y_2v_2 = 0,$

$x_1u_1 = 0, x_2u_2 = 0, x_3u_3 = 0.$

Let us solve the dual by simplex method and the optimal table is given below (Iteration 3) :

NOTES

	c_j		3	2	0	0	0
c_B	x_B	soln.	y_1	y_2	u_1	u_2	u_3
0	u_1	6	0	0	1	$-\frac{1}{5}$	$\frac{3}{5}$
2	y_2	1	0	1	0	$\frac{1}{5}$	$-\frac{3}{5}$
3	y_1	4	1	0	0	$\frac{1}{5}$	$\frac{2}{5}$
	$z_j - c_j$		0	0	0	1	0

The optimal solution of the dual is $y_1^* = 4, y_2^* = 1, T^* = 14$.

The optimal solution of the primal can be read off from the $(z_j - c_j)$-row. Since x_1, x_2, x_3 corresponds to u_1, u_2, u_3 respectively, then $x_1^* = 0, x_2^* = 1, x_3^* = 0$, and $z^* = 14$.

2.11 THE DUAL SIMPLEX METHOD

Step 1. Convert the minimization LP problem into an symmetric maximization LP problem (*i.e.*, all constraints are ≤ type) if it is in the minimization form.

Step 2. Introduce the slack variables and obtain the first iteration dual simplex table.

	c_j		
c_B	x_B	Soln.	(x)
	$[x_{B_1}]$		
	$z_j - c_j$		
	Max. ration		

Step 3. (*a*) If all $z_j - c_j$ and x_{B_i} are non-negative, then an optimal basic feasible solution has been attained.

(*b*) If all $z_j - c_j \geq 0$ and at least one of x_{B_i} is negative then go to step 4.

(*c*) If at least one $(z_j - c_j)$ is negative, the method is not applicable.

Step 4. Select the most negative of x_{B_i}'s and that basic variable will leave the basis and the corresponding row is called 'key-row'.

Step 5. (*a*) If all the elements of the key row is positive, then the problem is infeasible.

(*b*) If at least one element is negative then calculate the maximum ratios as follows :

$$\text{Max}\left\{\frac{(z_j - c_j)\text{ value}}{\text{Negative element of the key row}}\right\}$$

The maximum ratio column is called 'key column' and the intersection element of key row and key column is called 'key element'.

Step 6. Obtain the next table which is the same procedure as of simplex method.

Step 7. Go to step 3.

Note. 1. Difference between simplex method and dual-simplex method : In simplex method, we move from a feasible non-optimal solution to feasible optimal solution. Whereas in dual simplex method, we move from an infeasible optimal solution to feasible optimal solution.

2. The term 'dual' is used in dual simplex method because the rules for leaving and entering variables are derived from the dual problem but are used in the primal problem.

Example 2.10. *Using dual simplex method solve the following LP problem.*

Minimize $z = 4x_1 + 2x_2$

$S/t,\ x_1 + 2x_2 \geq 2,\ 3x_1 + x_2 \geq 3,$

$4x_1 + 3x_2 \geq 6,\ x_1, x_2 \geq 0.$

Solution. Min. $z = -$ Max. $(-z) = -$ Max. $(-z = -4x_1 - 2x_2)$.

Multiply -1 to all the $\geq$ constraints to make £ type.

Then the standard form is obtained as follows :

$$\text{Max} -z = -4x_1 - 2x_2 + 0.s_1 + 0.s_2 + 0.s_3$$

$S/t,\ -x_1 - 2x_2 + s_1 = -2$

$-3x_1 - x_2 + s_2 = -3$

$-4x_1 - 3x_2 + s_3 = -6$

$x_1, x_2 \geq 0,\ s_1, s_2, s_3$ are slacks and ≥ 0.

Iteration 1.

c_j			-4	-2	0	0	0	
c_B	x_B	soln.	x_1	x_2	s_1	s_2	s_3	
0	s_1	-2	-1	-2	1	0	0	
0	s_2	-3	-3	-1	0	1	0	
0	s_3	-6	-4	-3	0	0	1	→ Key row
$z_j - c_j$			4	2	0	0	0	
Max-ration			$\frac{4}{-4}$	$\frac{2}{-3}$				
				↑ Key column				

Iteration 2.

c_j			-4	-2	0	0	0	
c_B	x_B	soln.	x_1	x_2	s_1	s_2	s_3	
0	s_1	2	5/3	0	1	0	$-2/3$	
0	s_2	-1	$-5/3$	0	0	1	$-1/3$	→ Key row
-2	x_2	2	4/3	1	0	0	$-1/3$	
$z_j - c_j$			4/3	0	0	0	2/3	
Max-ration			$-\frac{4}{5}$	–	–	–	-2	
			↑ Key column					

Iteration 3.

c_j			-4	-2	0	0	0
c_B	x_B	soln.	x_1	x_2	s_1	s_2	s_3
0	s_1	.1	0	0	1	1	-1
-4	x_1	3/5	1	0	0	$-3/5$	1/5
-2	x_2	6/5	0	1	0	4/5	$-3/5$
$z_j - c_j$			0	0	0	4/5	2/5

Hence optimal feasible solution is $x_1^* = \frac{3}{5}, x_2^* = \frac{6}{5}$ and $z^* = \frac{24}{5}$.

NOTES

SUMMARY

- "Operations Research is concerned with scientifically deciding how to best design and operate man-machine systems, usually under conditions requiring the allocation of scarce resources".
- **Basic Solution :** Let m = no. of constraints and n = no. of variables and $m < n$. Then the solution from the system $Ax = b$ is called basic solution. In this system there are ${}^nc_m$ number of basic solutions. By setting $(n - m)$ variables to zero at a time, the basic solutions are obtained.
- Simplex method is an iterative method. Calculations are done in a table which is called simplex table. For each constraint there will be a row and for each variable there will be a column. Objective function coefficients c_j are kept on the top of the table. x_B stands for basis column in which the variables are called 'basic variables'.
- In real life situations, which are dynamic and changing, the effect of variation of these variables must be studied and understood. This process of knowing the impact of variables on the outcome of optimal result is known as sensitivity analysis of linear programming problems.
- For every LP Problem we can construct another LP problem using the same data. These two problems try to achieve two different objectives within the same data. The original problem is called *Primal* problem and the constructed problem is called *Dual*.

GLOSSARY

- **Feasible:** A solution which satisfied all the constraints solution in linear programming problems.
- **Decision:** In decision making, all the decisions are variables taken through some variables which are known as decision variable.
- **Basic feasible:** A solution which is basic as well as solution (BFS) feasible in called basic feasible solution.
- **Degenerate:** If a basic variable takes the value zero BFS in a BFS, then the solution is said to be BFS degenerate.
- **Optimal:** The BFS which optimizes the objective function BFS is called optimal BFS.

REVIEW QUESTIONS

Using graphical method solve the following LPP :

1. Maximize $z = 13x_1 + 117x_2$

   Subject to, $x_1 + x_2 \le 12,$

   $x_1 - x_2 \ge 0,$

   $4x_1 + 9x_2 \le 36,$

   $0 \le x_1 \le 2$ and $0 \le x_2 \le 10.$ (**Ans.** $x_1 = 2, x_2 = 2, z^* = 260$)

2. Maximize $z = 3x_1 + 15x_2$

   Subject to, $4x_1 + 5x_2 \le 20,$

   $x_2 - x_1 \le 1,$

   $0 \le x_1 \le 4$ and $0 \le x_2 \le 3.$ $\left(\textbf{Ans. } x_1 = \frac{5}{3}, x_2 = \frac{8}{3}, z^* = 45\right)$

3. Maximize $z = 5x_1 + 7x_2$

   Subject to, $3x_1 + 8x_2 \le 12,$

$x_1 + x_2 \le 2,$

$2x_1 \le 3,$

$x_1, x_2 \ge 0.$ $\left(\textbf{Ans. } x_1 = \frac{4}{5}, x_2 = \frac{6}{5}, z^* = \frac{62}{5}\right)$

4. Minimize $z = 2x_1 + 3x_2$

   Subject to, $x_2 - x_1 > 2,$

   $5x_1 + 3x_2 \le 15,$

   $2x_1 \ge 1,$

   $x_2 \le 4,$

   $x_1, x_2 \ge 0.$ $\left(\textbf{Ans. } x_1 = \frac{1}{2}, x_2 = \frac{5}{2}, z^* = \frac{17}{2}\right)$

5. Minimize $z = 10x_1 + 9x_2$

   Subject to, $x_1 + 2x_2 \le 10,$

   $x_1 - x_2 \le 0,$

   $x_1 \le 0, x_2 \ge 0.$ (**Ans.** $x_1 = 0, x_2 = 0, z^* = 0$)

NOTES

Solve the following LPP by simplex method:

6. Maximize $z = 3x_1 + 2x_2$

   S/t, $5x_1 + x_2 \le 10, 4x_1 + 5x_2 \le 60; x_1, x_2 \ge 0.$ (**Ans.** $x_1 = 0, x_2 = 10, z^* = 20$ (It 3))

7. Maximize $z = 5x_1 + 4x_2 + x_3$

   S/t, $6x_1 + x_2 + 2x_3 \le 12, 8x_1 + 2x_2 + x_3 \le 30,$

   $4x_1 + x_2 - 2x_3 \le 16, x_1, x_2, x_3 \ge 0.$ (**Ans.** $x_1 = 0, x_2 = 12, x_3 = 0, z^* = 48$ (It 3))

8. Maximize $z = 3x_1 + 2x_2$

   S/t, $3x_1 + 4x_2 \le 12, 2x_1 + 5x_2 \le 10, x_1, x_2 \ge 0.$ (**Ans.** $x_1 = 4, x_2 = 0, z^* = 12$ (It 2))

9. Maximize $z = x_1 + x_2$

   S/t, $x_1 - 2x_2 \le 2, -x_1 + 2x_2 \le 2, x_1, x_2 \ge 0.$ (**Ans.** Unbounded solution)

10. Find all the optimal BFS to the following :

    Maximize $z = x_1 + x_2 + x_3 + x_4$

    S/t, $x_1 + x_2 \le 2, x_3 + x_4 \le 5, x_1, x_2, x_3, x_4 \ge 0.$

    (**Ans.** (2, 0, 5, 0), (0, 2, 5, 0), (0, 2, 0, 5), (2, 0, 0, 5))

11. Explain the following terms.
    (*a*) Basic feasible solution.
    (*b*) Optimal solution.

12. Explain step by step the method used in solving LPP using simplex method.

13. Write a detailed note on the sensitivity analysis.

14. Minimize $z = 2x_1 + 3x_2$

    S/t, $2x_1 + x_2 \ge 1, x_1 + 2x_2 \ge 1; x_1, x_2 \ge 0.$ (**Ans.** $x_1 = \frac{1}{3}, x_2 = \frac{1}{3}, z^* = \frac{5}{3}$ (Big-M 3 It))

15. Maximize $z = 5x_1 + 3x_2$

    S/t, $2x_1 - 4x_2 \le 16, 3x_1 + 4x_2 \ge 12; x_1, x_2 \ge 0.$

    (**Ans.** Unbounded Solution (Big-M 4 It))

16. Maximize $z = 2x_1 + 3x_2 + 2x_3$

    S/t, $3x_1 + 2x_2 + 2x_3 = 16, 2x_1 + 4x_2 + x_3 = 20,$

    $x_1 \ge 0$, x_2 unrestricted in sign, $x_3 \ge 0.$

    (**Ans.** $x_1 = 0, x_2 = 4, x_3 = 4, z^* = 20$ (Big-M 4 It))

NOTES

17. Maximize $z = 2x_1 + 3x_2 + x_3$

$S/t,\ 3x_1 + 2x_2 + x_3 = 15,\ x_1 + 4x_2 = 10,$

x_1 unrestricted in sign, $x_2, x_3 \geq 0$. (**Ans.** Unbounded Solution (Big-M 2 It))

18. Find a BFS of the following system :

$$x_1 + x_2 \geq 1,\ -2x_1 + x_2 \geq 2,\ 2x_1 + 3x_2 \leq 6;\ x_1, x_2 \geq 0.$$

19. A manufacturer of furniture makes only chair and tables. A chair requires two hours on m/c A and six hours on m/c B. A table requires five hours on m/c A and two hours on m/c B. 16 hours are available on m/c A and 22 hours. on m/c B per day. Profits for a chair and table be Rs. 1 and Rs. 5 respectively. Formulate the LPP of finding daily production of these items for maximum profit and solve graphically.

(**Ans.** No chairs and 3.2 tables to be produced for max. profit of Rs. 16).

20. A tailor has 80 sq. m. of cotton material and 120 sq. m. of woolen material. A suit requires 1 sq. m. of cotton and 3 sq. m. of woolen material and a dress requires 2 sq. m. of each. A suit sells for Rs. 500 and a dress for Rs. 400. Pose a LPP in terms of maximizing the income. (**Ans.** Max. sells $= 500x_1 + 400x_2$ S/t, $x_1 + 2x_2 \leq 80,\ 3x_1 + 2x_2 \leq 120,$ $x_1 =$ no. of suits ≥ 0 and $x_2 =$ no. of dresses ≥ 0).

21. A company owns two mines : mine A produces 1 tonne of high grade ore, 3 tonnes of medium grade ore and 5 tonnes of low grade ore each day; and mine B produces 2 tonnes of each of the three grades of ore each day. The company needs 80 tonnes of high grade ore, 160 tonnes of medium grade ore and 200 tonnes of low grade ore. If it costs Rs. 200 per day to work each mine, find the number of days each mine has to be operated for producing the required output with minimum total cost.

(**Ans.** Mine A to be operated for 40 days and mine B to be operated for 20 days and min. cost = Rs. 12000).

22. A company manufactures two products A and B. The profit per unit sale of A and B is Rs. 10 and Rs. 15 respectively. The company can manufacture at most 40 units of A and 20 units of B in a month. The total sale must not be below Rs. 400 per month. If the market demand of the two items be 40 units in all, write the problem of finding the optimum number of items to be manufactured for maximum profit, as a problem of LP. Solve the problem graphically or otherwise.

(**Ans.** Max. profit $= 10x_1 + 15x_2$ S/t, $x_1 \leq 40,\ x_2 \leq 20,\ x_1 + x_2 \geq 40,\ 10x_1 + 15x_2 \geq 400,$ $x_1, x_2 \geq 0$ and $x_1^* = 40, x_2^* = 20,$ max. profit = Rs. 700).

23. Maximize $z = x_1 + x_2 + 3x_3$

$S/t,\ 3x_1 + 2x_2 + x_3 \leq 3,\ 2x_1 + x_2 + 2x_3 \leq 2;\ x_1, x_2, x_3 \geq 0.$

(**Ans.** $x_1 = 0, x_2 = 0, x_3 = 1, z^* = 3$)

24. Maximize $z = 3x_1 + 4x_2$

$S/t,\ x_1 - x_2 \geq 0,\ -x_1 + 3x_2 \leq 3;\ x_1, x_2 \geq 0.$ (**Ans.** Unbounded solution)

25. Minimize $z = x_1 + x_2$

$S/t,\ 2x_1 + x_2 \geq 4,\ x_1 + 7x_2 \geq 7;\ x_1, x_2 \geq 0.$ (**Ans.** $x_1 = \frac{21}{13}, x_2 = \frac{10}{13}, z^* = \frac{31}{13}$).

26. Minimize $z = 2x_1 - x_2 + 2x_3$

$S/t,\ -x_1 + x_2 + x_3 = 4,\ -x_1 + x_2 - x_3 \leq 6,$

$x_1 \leq 0,\ x_2 \geq 0,\ x_3$ unrestricted in sign.

(**Ans.** $x_1 = -5, x_2 = 0, x_3 = -1, z^* = -12$ (It 3)

27. Use principle of duality to solve the following LP problems :

(*a*) Minimize $z = 4x_1 + 3x_2$

$S/t,\ 2x_1 + x_2 \geq 40,\ x_1 + 2x_2 \geq 50,\ x_1 + x_2 \geq 35$

$x_1, x_2 \geq 0.$ (**Ans.** $x_1 = 5, x_2 = 30, z^* = 110$)

(*b*) Maximize $z = 2x_1 + x_2$

S/t, $x_1 + 2x_2 \le 10, x_1 + x_2 \le 6, x_1 - x_2 \le 2, x_1 - 2x_2 \le 1$

$x_1, x_2 \ge 0.$ (**Ans.** $x_1 = 4, x_2 = 2, z^* = 10$)

(*c*) Minimize $z = 6x_1 + x_2$

S/t, $2x_1 + x_2 \ge 3, x_1 - x_2 \ge 0, x_1, x_2 \ge 0.$ (**Ans.** $x_1 = 1, x_2 = 1, z^* = 7$)

(*d*) Minimize $z = 30x_1 + 30x_2 + 10x_3$

S/t, $2x_1 + x_2 + x_3 \ge 6, x_1 + x_2 + 2x_3 \le 8, x_1, x_2, x_3 \ge 0.$

(**Ans.** $x_1 = \frac{4}{3}, x_2 = 0, x_3 = \frac{10}{3}, z^* = \frac{220}{3}$)

(*e*) Maximize $z = 5x_1 + 2x_2$

S/t, $x_1 - x_2 \le 1, x_1 + x_2 \ge 4, x_1 - 3x_2 \le 3, x_1, x_2 \ge 0.$

(**Ans.** Unbounded solution)

28. Use dual-simplex method to solve the following LP problems :

(*a*) Minimize $z = x_1 + 3x_2$

S/t, $2x_1 + x_2 \ge 4, 3x_1 + 2x_2 \ge 5, x_1, x_2 \ge 0.$

(**Ans.** $x_1 = 2, x_2 = 0, z^* = 2$ (It – 3)

(*b*) Minimize $z = 2x_1 + x_2$

S/t, $x_1 + x_2 \ge 2, 3x_1 + 2x_2 \ge 4, x_1, x_2 \ge 0.$

(**Ans.** $x_1 = 0, x_2 = 2, z^* = 2$ (It – 2))

(*c*) Minimize $z = 2x_1 + 3x_2 + 10x_3$

S/t, $2x_1 - 5x_2 + 4x_3 > 30,$

$3x_1 + 2x_2 - 5x_3 > 25,$

$x_1 + 3x_2 + x_3 < 30,$

$x_1, x_2, x_3 > 0.$ (**Ans.** $x_1 = 15, x_2 = 0, x_3 = 0, z^* = 30$ (It – 2))

(*d*) Maximize $z = -2x_1 - x_2 - 3x_3$

S/t, $-3x_1 + x_2 - 2x_3 - x_4 = 1, x_1 - 2x_2 + x_3 - x_5 = 2, x_i > 0\ \forall i$

(**Ans.** Infeasible solution (It – 3))

(*e*) Minimize $z = 2x_1 + 3x_2 + 4x_3$

S/t, $3x_1 + 10x_2 + 5x_3 \ge 3, 3x_1 - 10x_2 + 9x_3 \le 30,$

$x_1 + 2x_2 + x_3 \ge 4, x_1, x_2, x_3 \ge 0.$

(**Ans.** $x_1 = 0, x_2 = 2, x_3 = 0, z^* = 6$ (It – 2))

(*f*) Minimize $z = 6x_1 + 2x_2 + 5x_3 + 3x_4$

S/t, $3x_1 + 2x_2 - 3x_3 + 5x_4 \ge 10, 4x_2 + 3x_3 - 5x_4 \ge 12,$

$5x_1 - 4x_2 + x_3 + x_4 \ge 10, x_1, x_2, x_3, x_4 \ge 0.$

(**Ans.** $x_1 = 0, x_2 = \frac{11}{3}, x_3 = 0, x_4 = \frac{8}{15}, z^* = \frac{134}{15}$ (It – 3))

(*g*) Maximize $z = -x_1 - 2x_2 - 3x_3$

S/t, $2x_1 - x_2 - x_3 \ge 4, x_1 - x_2 + 2x_3 \le 8, x_1, x_2, x_3 \ge 0.$

(**Ans.** $x_1 = 2, x_2 = 0, x_3 = 0, z^* = -2$ (It – 2))

FURTHER READINGS

- *Statistics and Operational Research—A Unified Approach,* by Dr. Debashis Dutta, Laxmi Publication (P) Ltd.
- *Operational Research,* by col. D.S. Cheema, University Science Press.

NOTES

UNIT III: INTEGER LINEAR PROGRAMMING, TRANSPORTATION AND ASSIGNMENT PROBLEMS

★ STRUCTURE ★

3.0 LEARNING OBJECTIVES

After going through this unit, you should be able to:

- explain integer linear programming.
- enumerate transportation problems.
- describe methods for obtaining basic feasible solution through matrix minima and vogel's approximation methods respectively.
- describe assignment problems.

PART I: INTEGER LINEAR PROGRAMMING PROBLEMS

NOTES

3.1 INTRODUCTION: INTEGER LINEAR PROGRAMMING PROBLEMS

In linear programming, each of the decision variable as well as slack and surplus variable is allowed to take any real or fractional value. However there are certain practical problems in which the fractional value of the decision variable has no meaning. For example, it does not make sense saying 1.3 men working in a project or 2.2 machines in a workshop. The integer solution to a problem can be obtained by rounding off the optimum value of the variable to the nearest integer value. The approach is easy in terms of efforts involved in deriving an integer solution, but this may not satisfy all the given constraints. Moreover the value of the objective function so obtained may not be optimum value. Such difficulties can be avoided, if the given problem, where integer solution is required is solved by integer programming techniques.

Types of Integer Programming Problems

1. **Pure (all) integer programming problems** in which all decision variables are required to be integer variables.
2. **Mixed integer programming problems** in which some variables should be integers and others need not be.
3. **Zero-one integer programming problems** in which all decision variables should be either 0 or 1.

Let us discuss two major methods namely:

1. Gomory's cutting plane method
2. Branch and bound method

3.2 GOMORY'S CUTTING PLANE METHOD

It was developed by **Gomory** in 1966, which uses Dual Simplex Method. First the optimum solution is obtained, relaxing the integer requirement. Special constraints (called cuts) are added to the solution space in a manner that renders an optimum **Integer Extreme Point**.

(Note: we know that one of the extreme points is the optimum solution).

By introducing special constraints (cuts) we created extreme points with integer values, which cuts out a part of the feasible region of the LP problem leaving behind the integer portion.

In the example shown below we first determine graphically, how cuts are used to produce an integer solution and then implement the idea algebraically.

Example 3.1.

Maximize $\quad Z = 7x_1 + 10x_2$

Subject to $\quad -x_1 + 3x_2 \le 6 \quad$ ...(1)

$7x_1 + x_2 \le 35 \quad$ ...(2)

$x_1, x_2 \ge 0$ *and integer.*

The cutting plane algorithm modifies the solution space by adding cuts that produces an optimum integer extreme point. Fig. 3.1 gives an example of two such cuts. Initially we start with continuous LPoptimum solution. Fig. 3.1(*a*).

NOTES

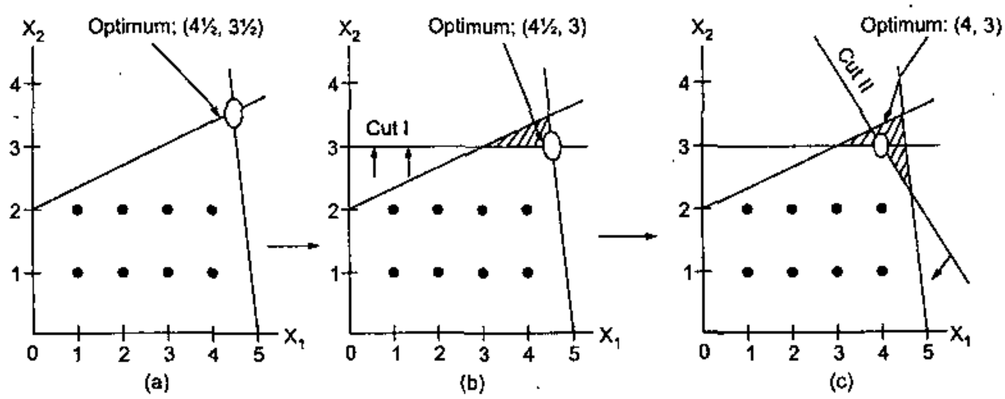

Fig. 3.1 Illustration of the use of cuts in LP

$$Z = 66\tfrac{1}{2}\ x_1 = 4\tfrac{1}{2}\ x_2 = 3\tfrac{1}{2}.$$

The dots in Fig. 3.1 referred to as lattice pts, represent all of the integer solutions that lie within the feasible space of LP Problem.

We add cut I which produces the LP optimum solution $Z = 62$, $x_1 = 4\tfrac{1}{2}$ and $x_2 = 3$. Fig. 3.1(*b*). Then we add cut II which together with cut I and the original constraints, produces the LP optimum solution $Z = 58$, $x_1 = 4$ and $x_2 = 3$. (Fig. 3.1(*c*)). The last solution is all integers as desired.

The added cuts do not eliminate any of the original feasible integer points, but must pass through at least one feasible or infeasible integer point. These are basic requirements of any cut.

Now we use the same example to show how the cuts are constructed and implemented algebraically.

Given the slacks as S_1 and S_2 for constraints 1 and 2 the optimum LP table is given below.

Basic		X_1	X_2	S_1	S_2	*Solution*
X_2	10	0	1	$\frac{7}{22}$	$\frac{1}{22}$	$3\frac{1}{2}$
X_1	7	1	0	$-\frac{1}{22}$	$\frac{3}{22}$	$4\frac{1}{2}$
C_j		7	10	0	0	
$C_j - Z_j$		0	0	$-\frac{63}{22}$	$-\frac{31}{22}$	

The optimum continuous solution is

$$Z = 66\tfrac{1}{2},\ x_1 = 4\tfrac{1}{2}\ x_2 = 3\tfrac{1}{2}\ s_1 = 0,\ s_2 = 0$$

In the current optimal solution, all basic variables in the basis are not integers and the solution is not acceptable.

The cut is developed under the assumption that all the variables (including the slacks S_1 and S_2) are integers.

3.3 METHOD FOR CONSTRUCTING ADDITIONAL CONSTRAINTS (CUTS)

In the optimum solution, if any basic variable is not integer, an additional linear constrainst called the Gomory constraint (or cut) is generated. After having generated a linear constraint it is added to the bottom of the optimal simplex table so that the solution no longer remains feasible. This is then solved by using dual simplex method until integer solution is obtained. The information in the optimum table can be written explicitly as

$$Z - \frac{63}{22}s_1 - \frac{31}{22}s_2 = 66\frac{1}{2}$$

$$x_2 + \frac{7}{22}s_1 + \frac{1}{22}s_2 = 3\frac{1}{2}$$

$$x_1 - \frac{1}{22}s_1 + \frac{3}{22}s_2 = 4\frac{1}{2}$$

A constraint equation can be used as a source row for generating a cut provided its right hand side is fractional. If we select x_1 row as source row then write the row as below.

$$x_1 - \frac{1}{22}s_1 + \frac{3}{22}s_2 = 4\frac{1}{2} \quad ...(1)$$

First we factor out all the non integer coefficients of the equation into an integer value and fractional components, such that the resulting in fractional component is strictly positive.

Factoring each coefficient of variable into integer and fraction, yields.

$$1x_1 + \left(-1 + \frac{21}{22}\right)s_1 + \frac{3}{22}s_2 = 4 + \frac{1}{2} \quad ...(2)$$

Moving all the integer component to the left hand side and all the fractional components to right hand side we get.

$$1x_1 - s_1 - 4 = \frac{1}{2} - \frac{21}{22}s_1 - \frac{3}{22}s_2 \quad ...(3)$$

Because s_1 and s_2 are non negative the right hand side must also satisfy the following inequality.

$$\frac{1}{2} - \frac{21}{22}s_1 - \frac{3}{22}s_2 \le \frac{1}{2} \quad ...(4)$$

Next since the left hand side in equation (3) $x_1 - s_1 - 4$ is an integer value by construction the right hand side must also be integer. It then follows that (4) can be replaced by the inequality.

$$\frac{1}{2} - \frac{21}{22}s_1 - \frac{3}{22}s_2 \le 0$$

This is the desired cut and it represents a necessary condition for obtaining an integer solution. It is also called as fractional cut because all its coefficients are fractions.

If we select x_2 row as the source row. The cut generated will be, as given below.

$$x_2 + \frac{7}{22}s_1 + \frac{1}{22}s_2 = 3\frac{1}{2}$$

$$(1+0)x_2 + \left(0 + \frac{7}{22}\right)s_1 + \frac{1}{22}s_2 = 3 + \frac{1}{2}$$

$$x_2 - 3 = \frac{1}{2} - \frac{7}{22}s_1 - \frac{1}{22}s_2$$

$$\frac{1}{2}-\frac{7}{22}s_1-\frac{1}{22}s_2\le\frac{1}{2}$$

or
$$\frac{1}{2}-\frac{7}{22}s_1-\frac{1}{22}s_2\le 0$$

NOTES

Any one of the two cuts given above can be used in the first iteration of the cutting plane algorithm. Usually the basic variable having higher fraction is selected as the *source row. There is no hard and fast rule.*

Arbitrarily selecting the cut generated from the x_2 row we can write it in the equation form as.

$$-\frac{7}{22}s_1-\frac{1}{22}s_2+sc_1=-\frac{1}{2}$$

$$sc_1\ge 0$$

Where sc_1 is the new non negative (integer) slack variable.

This constraint is added as a secondary constraint to the LP optimum table as follows:

Basic	X_1	X_2	S_1	S_2	S_4	*Solution*	
X_2 10	0	1	$\frac{7}{22}$	$\frac{1}{22}$	0	$3\frac{1}{2}$	
X_1 7	1	0	$-\frac{1}{22}$	$\frac{3}{22}$	0	$4\frac{4}{7}$	
Sc_1 0	0	0	$-\frac{7}{22}$	$-\frac{1}{22}$	1	$-\frac{1}{2}$	→ LV
C_j	7	10	0	0	0		
C_j-Z_j	0	0	$-\frac{63}{22}$	$-\frac{31}{22}$	0		
AR			9 ↑	31			

The tableus is optimum, but infeasible. We apply the dual simplex method to recover feasibility, which yields.

Basic	X_1	X_2	S_1	S_2	Sc_1	*Solution*
X_2 10	0	1	0	0	1	3
X_1 7	1	0	0	$\frac{1}{7}$	$-\frac{1}{7}$	$4\frac{4}{7}$
S_1 0	0	0	1	$\frac{1}{7}$	$-\frac{22}{7}$	$1\frac{4}{7}$
C_j	7	10	0	0	0	
C_j-Z_j	0	0	0	−1	−9	

The last solution is still non integer in x_1 and s_1. Let us arbitrarily select x_1 as the next source now, that is

$$x_1+\left(0+\frac{1}{7}\right)s_2+\left(-1+\frac{6}{7}\right)sc_1=4+\frac{4}{7}$$

$$x_1-4-sc_1=\frac{4}{7}-\frac{1}{7}s_2-\frac{6}{7}sc_1$$

Associated cut is $-\frac{1}{7}s_2-\frac{6}{7}sc_1+sc_2=-\frac{4}{7}$ (cut II)

Basic	X_1	X_2	S_1	S_2	Sc_1	Sc_2	*Solution*
X_2 10	0	1	0	0	1	0	3
X_1 7	1	0	0	$\frac{1}{7}$	$-\frac{1}{7}$	0	$1\frac{4}{7}$
S_1 0	0	0	1	$\frac{1}{7}$	$-\frac{22}{7}$	0	$1\frac{4}{7}$
S_2 0	0	0	0	$-\frac{1}{7}$	$-\frac{6}{7}$	0	$-\frac{4}{7}$
C_j	7	10	0	0	0	0	
$C_j - Z_j$	0	0	0	−1	−9	0	
AR				7↑	10.5		
X_2 10	0	1	0	0	1	0	3
X_1 7	1	0	0	0	−1	1	4
S_1 0	0	0	1	0	−4	1	1
S_2 0	0	0	0	1	6	−7	4
C_j	7	10	0	0	0	0	
$C_j - Z_j$	0	0	0	0	−3	−7	

It is optimum and all the basic variables are integers.

$$X_1 = 4 \quad x_3 = 3 \quad z = 7x_1 + 10x_2 = 28 + 30 = 58$$

It is not accidental that all the coefficients of the last tableu are integers. This is typical property of the implementation of the fractional cut.

Example 3.2. *Solve the following integer linear programming using the cutting plane algorithm.*

Maximize $Z = 2x_1 + 20x_2 - 10x_3$

Subject to $2x_1 + 20x_2 + 4x_3 \leq 5$

$6x_1 + 20x_2 + 4x_3 = 20$

$x_1, x_2, x_3 \geq 0$ *and integers.*

Solution.

Maximize $Z = 2x_1 + 20x_2 - 10x_3 + OS_1 - MA_2$

Subject to $2x_1 + 20x_2 + 4x_3 + S_1 = 15$

$6x_1 + 20x_2 + 4x_3 + A_2 = 20$

$x_1, x_2, x_3, S_1, A_2 \geq 0$ and integers.

The optimum solution for the problem relaxing integer constraint is

Basic	X_1	X_2	X_3	S_1	*Solution*
X_2 20	0	1	$\frac{1}{5}$	$\frac{3}{40}$	$\frac{5}{8}$
X_1 2	1	0	0	$-\frac{1}{4}$	$\frac{5}{4}$
C_j	2	20	−10	0	
$C_j - Z_j$	0	0	−14	−1	

A_2 column is deleted since it is artificial variable.

Optimum Table

$$x_1 = \frac{5}{4}, x_2 = \frac{5}{8}, x_3 = 0 \quad \text{and} \quad Z_{max} = 15$$

NOTES

To obtain the integer valued solution, additional constraint (fractional cut) is introduced.

Since fractional part of the value of $x_2\left(0+\frac{5}{8}\right)$ is more x_2 row is selected as the source row.

$$0x_1 + x_2 + \frac{1}{5}x_3 + \frac{3}{40}s_1 = \frac{5}{8}$$

Factorising the row

$$x_2 + \left(0+\frac{1}{5}\right)x_3 + \frac{3}{40}s_1 = 0+\frac{5}{8}$$

Rearranging, we obtain the Gomory's cut as

$$sc_1 - \frac{1}{5}x_3 - \frac{3}{40}s_1 = -\frac{5}{8}$$

Adding the additional constraint to the optimum table.

Basic	X_1	X_2	X_3	S_1	Sc_1	*Solution*
X_2 20	0	1	$\frac{1}{5}$	$\frac{3}{40}$	0	$\frac{5}{8}$
X_1 2	1	0	0	$-\frac{1}{4}$	0	$\frac{5}{4}$
Sc_1 0	0	0	$-\frac{1}{5}$	$-\frac{3}{40}$	1	$-\frac{5}{8}$
C_i	2	20	−10	0	0	
Δ_j	0	0	−14	−1	0	
AR			70	$-\frac{40}{3}$ ↑		
X_2 20	0	1	0	0	1	0
X_1 2	1	0	$\frac{2}{3}$	0	$-\frac{10}{3}$	$\frac{10}{3}$
S_1 0	0	0	$\frac{8}{3}$	1	$-\frac{40}{3}$	$\frac{25}{3}$
C_j	2	20	−10	0	0	
$C_j - Z_j$	0	0	$-\frac{34}{3}$	0	$-\frac{40}{3}$	

The solution is still non integer. Fractional cut II is introduced. Consider x_1 row as source row.

$$x_1 + 0x_2 + \frac{2}{3}x_3 + 0s_1 - \frac{10}{3}sc_1 = \frac{10}{3}$$

$$x_1 + \left(0+\frac{2}{3}\right)x_3 + \left(-4+\frac{2}{3}\right)sc_1 = 3+\frac{1}{3}$$

Fractional cut II $\quad sc_2 - \frac{2}{3}x_3 - \frac{2}{3}sc_1 = -\frac{1}{3}$

Add this cut to the optimum table.

Basic	X_1	X_2	X_3	S_2	Sc_1	Sc_2	Solution
X_2 20	0	1	0	0	1	0	0
X_1 2	1	0	$\frac{2}{3}$	0	$-\frac{10}{3}$	0	$\frac{10}{3}$
Sc_1 0	0	0	$\frac{8}{3}$	1	$-\frac{40}{3}$	0	$\frac{25}{3}$
Sc_2 0	0	0	$-\frac{2}{3}$	0	$-\frac{2}{3}$	1	$-\frac{1}{3}$
C_j	2	20	– 10	0	0	0	
Δ_j	0	0	$-\frac{34}{3}$	0	$-\frac{40}{3}$	0	
AR			17 ↑		20		
X_2 20	0	1	0	0	1	0	0
X_1 2	1	0	0	0	– 4	1	3
S_1 0	0	0	0	1	– 16	4	7
X_3 – 10	0	0	1	0	1	$-\frac{3}{2}$	$\frac{1}{2}$
C_j	2	20	– 10	0	0	0	
Δ_j	0	0	0	0	– 2	– 17	

Integer solution is not yet got. Now select x_3 row as the source row to get III cut.

$$x_3 + sc_1 - \frac{3}{2} sc_2 = \frac{1}{2}$$

$$x_3 + sc_1 + \left(-2 + \frac{1}{2}\right) sc_2 = \frac{1}{2}$$

Fractional cut III

$$sc_3 - \frac{1}{2}\, sc_2 = -\frac{1}{2}$$

Add this constraint to the optimum table.

Basic	X_1	X_2	X_3	S_2	Sc_1	Sc_2	Sc_3	Solution
X_2 20	0	1	0	0	1	0	0	0
X_1 2	1	0	0	0	– 4	1	0	3
S_1 0	0	0	0	1	– 16	4	0	7
x_3 – 10	0	0	1	0	1	$-\frac{2}{3}$	0	$\frac{1}{2}$
Sc_3 0	0	0	0	0	0	$-\frac{1}{2}$	1	$-\frac{1}{2}$
C_j	2	20	– 10	0	0	0	0	
Δ_j	0	0	0	0	– 2	– 17	0	
AR						34 ↑		
X_2 20	0	1	0	0	1	0	0	0
X_1 2	1	0	0	0	– 4	0	2	2
S_1 0	0	0	0	1	– 16	0	8	3
x_3 – 10	0	0	1	0	1	0	– 3	2
Sc_2 0	0	0	0	0	0	1	– 2	1
C_j	2	20	– 10	0	0	0	0	
Δ_j	0	0	0	0	– 2	0	– 34	

All basic variables are integers

Solution is $x_1 = 2\ x_2 = 0\ x_3 = 2$

$Z = 4 + 0 - 20 = -16$

3.4 MIXED INTEGER PROGRAMMING ALGORITHM

NOTES

If in the problem, only some basic variables have to be integers, the same procedure is followed and get the solution obtaining the required basic variable as integer.

Example 3.3. *Solve the following Mixed integer programming problem.*

Maximize $Z = -3x_1 + x_2 + 3x_3$

Subject to $-x_1 + 2x_2 + x_3 \le 4$

$$2x_2 - \frac{3}{2}x_3 \le 1$$

$$X_1 - 3x_2 + 2x_3 \le 3$$

$x_1, x_2 \ge 0$, x_3 non negative integer.

Solution. $Z = 3x_1 + x_2 + 3x_3 + OS_1 + OS_2 + OS_3$

Subject to $-x_1 + 2x_1 + x_3 + S_1 = 4$

$$2x_2 - \frac{3}{2}x_3 + S_2 = 1$$

$$x_1 - 3x_2 + 2x_3 + S_3 = 3$$

$x_1, x_2, S_1, S_2, S_3 \ge 0$ x_3 non negative integer

Relaxing the integer constraint, the optimal solution is obtained and is given in the table below.

Basic	X_1	X_2	X_3	S_1	S_2	S_3	*Solution*
X_2 1	$-\frac{3}{7}$	1	0	$\frac{2}{7}$	0	$-\frac{1}{7}$	$\frac{5}{7}$
S_2 0	$\frac{9}{7}$	0	0	$\frac{1}{7}$	1	$\frac{10}{7}$	$\frac{48}{7}$
X_3 3	$\frac{5}{14}$	0	1	$\frac{3}{7}$	0	$\frac{2}{7}$	$\frac{13}{7}$
C_j	-3	1	3	0	0	0	
$C_j - Z_j$	$-\frac{51}{14}$	0	0	$-\frac{11}{7}$	0	$-\frac{5}{7}$	

$x_1 = 0\ x_2 = 5/7\ x_3 = 13/7\ z = 44/7$

Since basic variable x_3 is required to be integer mixed integer cut is applied taking x_3 row as source row.

$$\frac{5}{14}x_1 + x_3 + \frac{3}{7}s_1 + \frac{2}{7}s_3 = \frac{13}{7}$$

$$\left(0 + \frac{5}{14}\right)x_1 + (1+0)x_3 + \left(0 + \frac{3}{7}\right)s_1 + \left(0 + \frac{2}{7}\right)s_3 = 1 + \frac{6}{7}$$

$$sc_1 - \frac{5}{14}x_1 - \frac{3}{7}s_1 - \frac{2}{7}s_3 = -\frac{6}{7}$$

Adding this constraint to the optimum solution table.

Basic	X_1	X_2	X_3	S_1	S_2	S_3	Sc_1	*Solution*
X_2 1	$-\frac{3}{7}$	1	0	$\frac{2}{7}$	0	$-\frac{1}{7}$	0	$\frac{5}{7}$
S_2 0	$\frac{9}{7}$	0	0	$\frac{1}{7}$	1	$\frac{10}{7}$	0	$\frac{48}{7}$
X_3 3	$\frac{5}{14}$	0	1	$\frac{3}{7}$	0	$\frac{2}{7}$	0	$\frac{13}{7}$
Sc_1 0	$-\frac{5}{14}$	0	0	$-\frac{3}{7}$	0	$-\frac{2}{7}$	1	$-\frac{6}{7}$ →
C_j	−3	1	3	0	0	0	0	0
$C_j - Z_j$	$-\frac{51}{14}$	0	0	$-\frac{11}{7}$	0	$-\frac{5}{7}$	0	
AR	$\frac{51}{5}$			$\frac{11}{3}$		$\frac{5}{2}$ ↑		
X_2 1	$-\frac{1}{4}$	1	0	$\frac{1}{2}$	0	0	$-\frac{1}{2}$	$\frac{8}{7}$
S_2 0	$\frac{1}{2}$	0	0	−2	1	0	5	$\frac{18}{7}$
X_3 3	0	0	1	0	0	0	1	1
S_3 0	$\frac{5}{4}$	0	0	$\frac{3}{2}$	0	1	$-\frac{7}{2}$	3
C_j	−3	1	3	0	0	0	0	
$C_j - Z_j$	$-\frac{11}{4}$	0	0	$-\frac{1}{2}$	0	0	$-\frac{5}{2}$	

Since x_3 has attained integer value, as required, the solution has been reached

Soln. $\quad x_1 = 0, x_2 = \frac{8}{7}, x_3 = 1$

and $\quad \text{Max } Z = 0 + \frac{8}{7} + 3 = 4\frac{1}{7}$.

NOTES

3.5 THE BRANCH AND BOUND ALGORITHM

The branch and bound algorithm is the most widely used method for solving both pure and mixed integer programming problems in practice. Basically the branch and bound algorithm is just an efficient enumeration procedure for examining all possible integer feasible solutions. A practical approach to solving an integer programme is to ignore the integer restrictions initially, and solve the problem as a linear programme. If the LP optimal solution contains fractional values for some integer variables, then by the use of truncation and rounding off procedures, one can attempt to get an approximate optimal integer solution. For example, if there are two integer variables x_1 and x_2 with fractional values 3.5 and 4.5, then one could examine four possible integer solutions (3, 4), (4, 4), (4, 5), (3, 5) obtained by truncation and rounding methods. We also observe that the true optimal integer solution may not correspond to any of these integer solutions, since it is possible for x_1 to have an optimal (integer) value less than 3 or greater than 4. Hence, to obtain the true optimal integer solution one has to consider all possible integer values of x_1 which are smaller and larger than 3.5. In other words the optimal integer solution must satisfy,

$$\text{either } x_1 \leq 3 \quad \text{or} \quad x_1 \geq 4$$

When a problem contains a large number of integer variables, it is essential to have a systematic method that will look into all possible combinations of integer solutions obtained from the LP optimal solution. The branch and bound algorithm does this in the most efficient manner.

Basic Principle

Consider the following example to illustrate the basic principle of branch and bound method.

Example. 3.4.

Maximize $Z = 3x_1 + 2x_2$

Subject to

$x_1 \le 2$ ...(1)

$x_2 \le 2$ ...(2)

$x_1 + x_2 \le 3.5$ ...(3)

$x_1, x_2 \ge 0$ *and integer.*

First solve the programme as a linear programme by ignoring the integer restrictions on x_1 and x_2. Call this linear programme as LP-1. Graphical solution is shown in Fig. 3.2 LP-1 optimal solution is $x_1 = 2$, $x_2 = 1.5$ $Z_{max} = 9$.

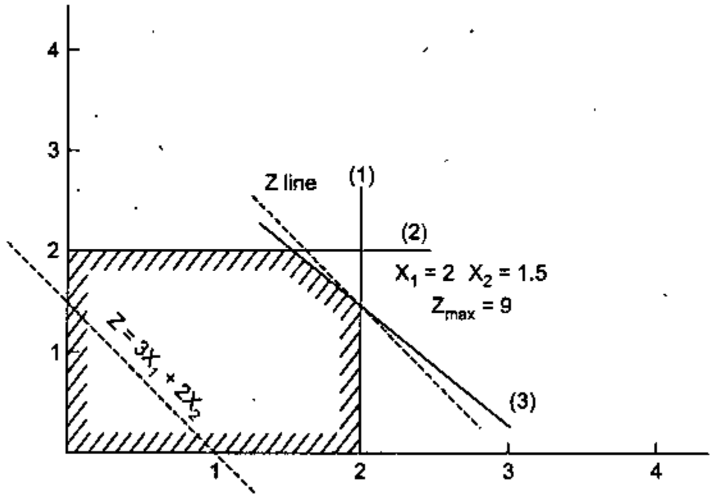

Fig. 3.2 Solution to LP-1

This is not the solution since x_2 is not integer. The optimal integer solution cannont be more than 9, since the imposition of integer restriction on x_2 can only make the LP solution worse.

Thus we have an upper bound on the maximum value of Z for the integer programme given by the optimal value of LP-1.

The next step is to examine other integer values of x_2, larger or smaller that 15. This is done by adding a new constraint either $x_1 \le 1$ or $x_2 \ge 2$. to the original problem (LP-1). This creates two new linear programmes (LP-2 and LP-3)

LP-2

Max $Z = 3x_1 + 2x_2$

Subject to $x_1 \le 2$ ...(1)

$x_2 \le 2$ (Redundant) ...(2)

$x_1 + x_2 \le 3.5$ (Redundant) ...(3)

(New Constraint) $x_2 \le 1$ ...(4)

$x_1, x_2 \ge 0$

LP-3

Max $Z = 3x_1 + 2x_2$

Subjec to $x_1 \le 2$ (Redundant) ...(1)

$x_1 \le 2$ ...(2)

$x_1 + x_2 \le 3.5$ ...(3)

(New Constraint) $x_2 \ge 2$ ...(5)

$x_1, x_2 \ge 0$

The feasible regions corresponding to LP-2 and LP-3 are shown graphically in Fig. 3.3 and 3. 4

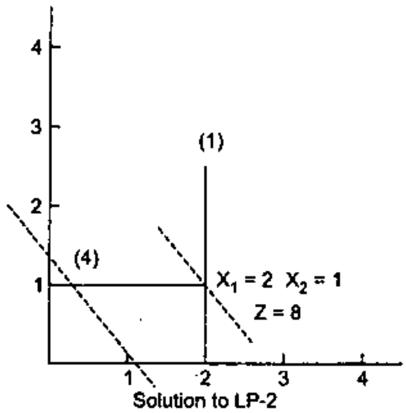

Fig. 3.3

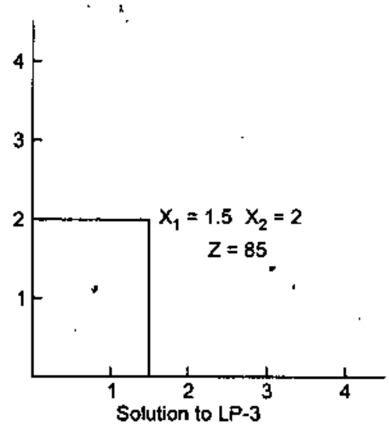

Fig. 3.4

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Observe that the feasible regions of LP-2 and LP-3 satisfy the following.

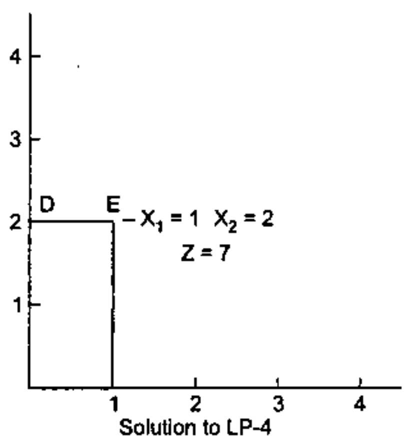
1. The optimal solution to LP-1 ($x_1 = 2$, $x_2 = 1.5$) is infeasible to both LP-2 and LP-3. Thus the old fractional optimal solution will not be repeated.
2. Every integer (feasible) solution to the original problem (lattice pts) is contained in either LP-2 or LP-3. Thus none of the feasible. (integer) solution to ILP (integer Linear problem) is lost due to the creation of two new linear program.

The optimal solution of LP-2 (Fig. 3.3) is $x_1 = 2$, $x_1 = 1$ and Z = 8. Hence Z = 8 is lower bound on the maximum value of Z for ILP. We cannot call the LP-2 solution as the optimal integer solution without examining L.P-3.

The optimal solution to LP-3 (Fig. 3.4) is $x_1 = 1.5$ and $x_2 = 2$ and $Z_{max} = 8.5$. This is not the solution since x_1 is taking a fractional values. But the max Z value(8.5) is larger than the lower bound (8). Hence it is necessary to examine whether there exists an integer solution in the feasible region of LP-3 whose value of Z is larger than 8. To determine this we add the constraints either $x_1 \leq 1$ or $x_1 \geq 2$ to LP-3. This gives two new linear programs LP-4 and LP-5.

LP-4			**LP-5**		
Max	$Z = 3x_1 + 2x_2$		Max	$Z = 3x_1 + 22x_2$	
Subject to	$x_1 + x_2 \leq 3.5$	...(3)	(Redundant) Subject to–	$x_1 + x_2 \leq 3.5$	...(3)
	$X_2 = 2$	...(2, 5)		$x_2 = 2$	...(2, 5)
	$x_1 \leq 1$	...(6)		$x_1 \geq 2$	...(7)
	$x_1; x_2 \geq 0$			$x_1, x_2 \geq 0$	
				Infeasible	

The feasible region of LP-4 is the straight line DE shown in Fig. 3.5. while LP-5 becomes infeasible.



Fg. 3.5

The optimal solution to LP-4 (Fig. 3.5) is given by $x_1 = 1$, $x_2 = 2$ and Z = 7. Which is less than 8. Hence the integer solution obtained while solving LP-2 namely $x_1 = 2$, $x_2 = 1$ and Z = 8 is the optimal integer solution to the problem. The sequence of linear programming problem solved under the branch and bound procedure for the Example 3.4 may be represented in the form of a *network* or *tree* diagram as shown in Fig. 3.6.

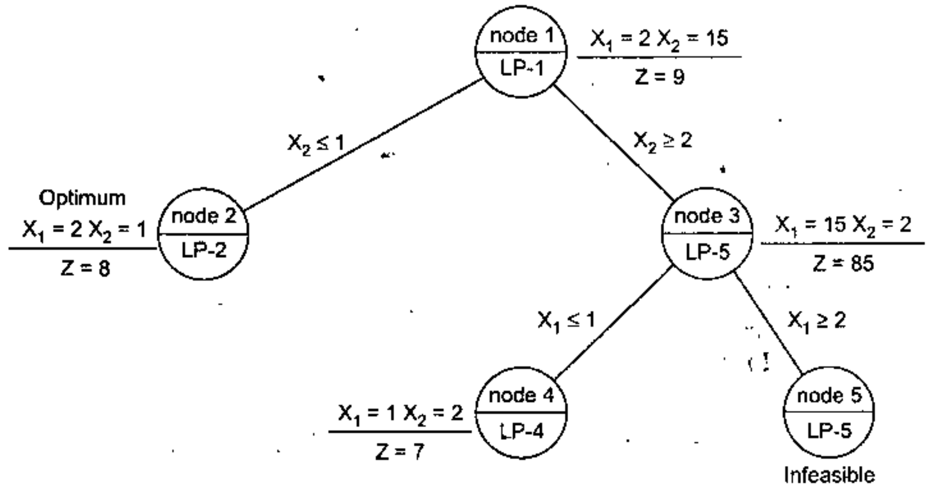


Fig. 3.6 Network representation of the branch and bound method for Example 3.4

Node 1 represents the equivalent linear programming problem (LP-1) of the integer programming ignoring the integers restriction. From node 1 we branch to node 2(LP-2) with the help of the integer variable x_2 by adding the constraint $x_2 \leq 1$ to LP-1. Since we have an integer optimal solution for node 2, no further branching from node 2 is necessary. Once this type of decision can be made, we say that node 2 has been *fathomed*. Branching on $x_2 \geq 2$ from node 1 results in LP-3 (node 3). Since the optimal solution to LP-3 is fractional, we branch further from node 3 using the integer variable x_1. This results in the creation of node 4 and 5. Both have been fathomed since LP-4 has an integer solution while LP-5 is infeasible. The best integer solution obtained at a fathomed node (in this case node 2) becomes the optimal solution to the integer programme.

Example 3.5. *Solve the following integer programming problem using the branch and bound method.*

Maximize $Z = 3x_1 + 5x_2$

Subject to $2x_1 + 4x_2 \leq 25$ ...(1)

$x_1 \leq 8$ ...(2)

$2x_2 \leq 10$ ...(3)

$x_1, x_2 \geq 0$ *and integers.*

First solve the programme as a linear programming by ignoring the integer restrictions on x_1 and x_2. Call this linear programme as LP-1. Graphical solution is shown in Fig. 3.7.

LPI optimal solution is $x_1 = 8$ $x_2 = 2.25$ and $Z_{max} = 35.25$

This is not the solution since x_2 is not integer. The optimal integer solution cannot be more than 35.25. The upper bound is 35.25.

The next step is to examine other integer values of x_2 larger or smaller than 2.25. This is done by adding a new constraint either $x_2 \leq 2$ or $x_2 \geq 3$ to the original problem (LP-1). This creates two new linear programmes LP-2 and LP-3.

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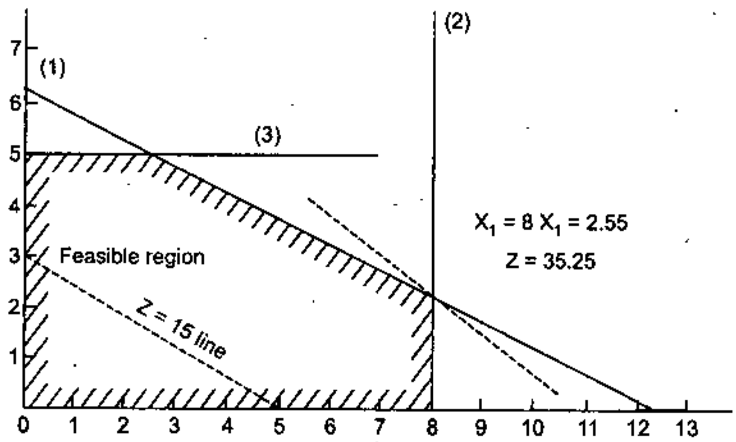

Fig. 3.7 Graphical solution of LP-1 dotted line is Z line

LP-2

Maximize $Z = 3x_1 + 5x_2$

Subject to $2x_1 + 4x_2 \leq 25$ (Redundant) ...(1)

$x_1 \leq 8$ ...(2)

$2x_2 \leq 10$ (Redundant) ...(3)

$x_2 \leq 2$ ...(4)

$x_1, x_2 \geq 0$

LP-3

Maximize $Z = 3x_1 + 5x_2$

Subject to $2x_1 + 4x_2 \leq 25$ ...(1)

$X_1 \leq 8$ (Redundant) ...(2)

$2x_2 \leq 10$ ...(3)

$x_2 \geq 3$ ...(5)

$X_1, x_2 \geq 0$

The feasible regions corresponding to LP-2 and LP-3 are shown graphically in Figs. 3.8 and 3.9.

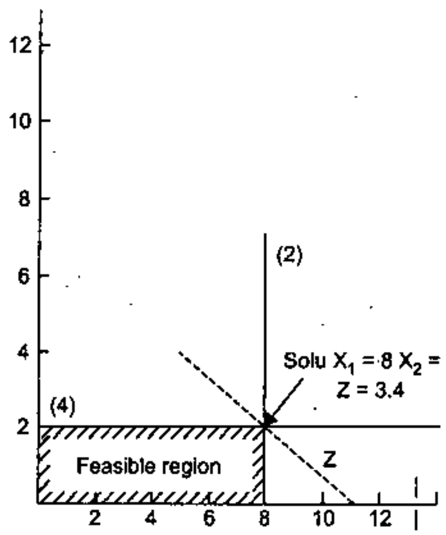

Fig. 3.8 Graphical solution to LP-2 dotted line is Z line

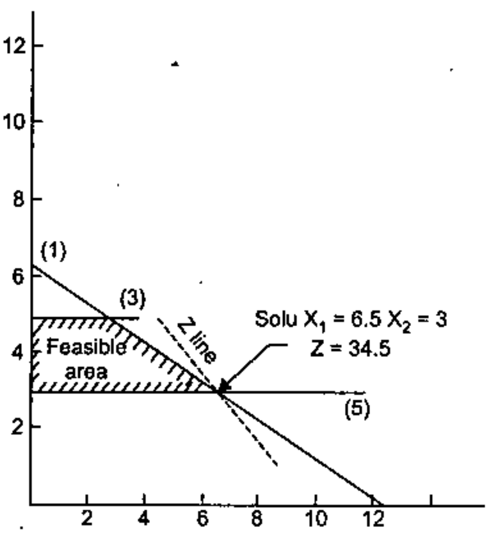

Fig. 3.9 Graphical solution to LP-3 dotted line is Z line

The optimal solution of LP-2 (Fig. 3.8) is $x_1 = 8$, $x_2 = 2$ and $Z_{max} = 34$. Hence Z = 34 is lower bound of the maximum value of Z for ILP. We cannot call the LP-2 solution as the optimal integer solution without examining the optimal solution to LP-3 (Fig. 3.9), which is $x_1 = 6.5$ and $x_2 = 3$ and Z = 34.5.

The optimal solution to LP-3 is not feasible solution since x_1 is taking a fractional value. But the max Z value (34.5) is larger than the lower bound (34). Hence it is necessary to examine whether there exists an integer solution in the feasible region of LP-3 whose

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value of Z is larger than 34. To determine this we add the constraints either $x_1 \le 6$ or $x_1 \ge 7$ to LP-3. This gives two new linear programmes LP-4 and LP-5.

LP-4

Maximize $Z = 3x_1 + 5x_2$

Subject to $2x_1 + 4x_2 = 25$ ...(1)

$x_1 \le 8$ (Redundant) ...(2)

$2x_2 \le 10$ ...(3)

$x_2 \ge 3$ ...(5)

$x_1 \le 6$ ...(6)

$x_1, x_2 \ge 0$

LP-5

Maximize $Z = 3x_1 + 5x_2$

Subject to $2x_1 + 4x_2 \le 25$ ...(1)

$x_1 \le 8$ ...(2)

$2x_2 \le 10$ ...(3)

$x_2 \ge 3$ ...(5)

$x_1 \ge 7$ ...(7)

$x_1, x_2 \ge 0$

The feasible region of LP-4 is shown in Fig. 3.9 while LP-5 shows infeasible solution because constraints $x_1 \ge 7$, $x_2 \le 3$, do not satisfy the first constraint.

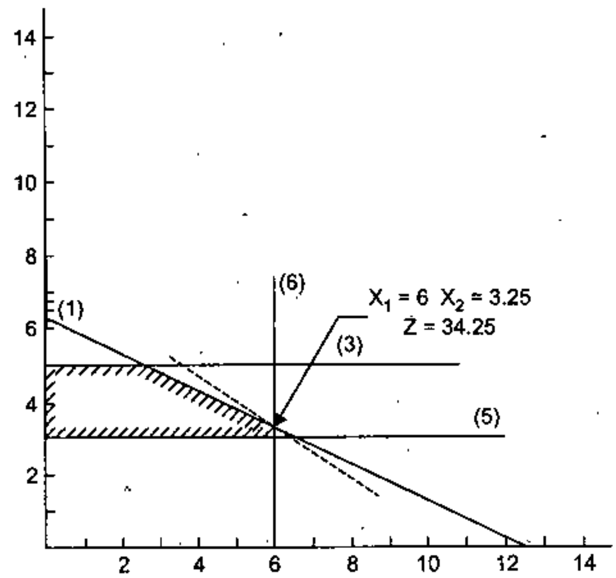

Fig. 3.10 Graphical solution to LP-4

The optimal solution to LP-4 is $x_1 = 6$, $x_2 = 3.25$, Z = 34.25. This is not feasible problem since x_2 is taking a fractional value. But the max Z value (34.25) is larger than the lower bound (34). Hence it is necessary to examine whether there exists an integer solution in the feasible region of LP-4. Whose value of Z is larger than 34. To determine we add the constraints either $x_2 \le 3$ or $x_2 \ge 4$ to LP-4. This gives two new linear programs LP-6 and LP-7.

LP-6

Maximize $Z = 3x_1 + 5x_2$

Subject to $2x_1 + 4x_2 \le 25$ ...(1)

$x_1 \le 8$ (Redundant) ...(2)

$2x_2 \le 10$ (Redundant) ...(3)

$x_2 \ge 3$ ...(5)

$x_1 \le 6$ ...(6)

$x_2 \le 3$ ...(8)

$x_1, x_2 \ge 0$

LP-7

Maximize $Z = 3x_1 + 5x_2$

Subjected to$2x_1 + 4x_2 \le 25$ ...(1)

$x_1 \le 8$ (Redundant) ...(2)

$2x_2 \le 10$ ...(3)

$x_2 \ge 3$ (Redundant) ...(5)

$x_1 \le 6$ ...(6)

$x_2 \ge 4$ ...(9)

$x_1, x_2 \ge 0.$

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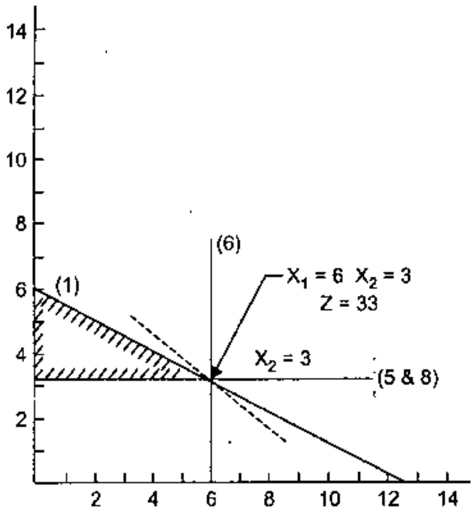


Fig. 3.11 Graphical solution to LP-6

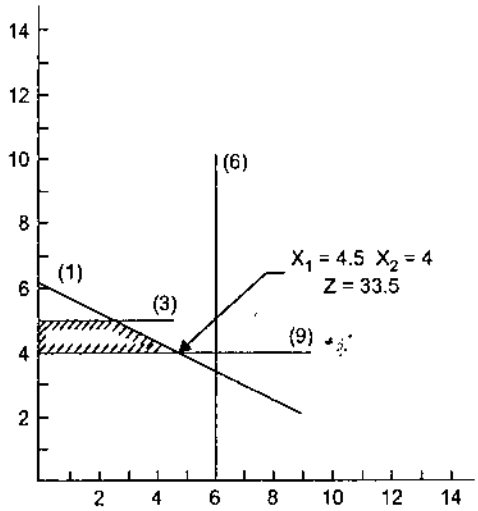


Fig. 3.12 Graphical solution to LP-7

Now since the Z_{max} value of LP-7 (33.5) is less than 34, the branching process is terminated. Solution for LP-6 (33), though integer, is less than the solution for LP-2 (34).

The branch and bound algorithm thus terminates and the optimal integer solution is $x_1 = 8$, $x_2 = 2$ and $Z = 34$, yielded at node 2.

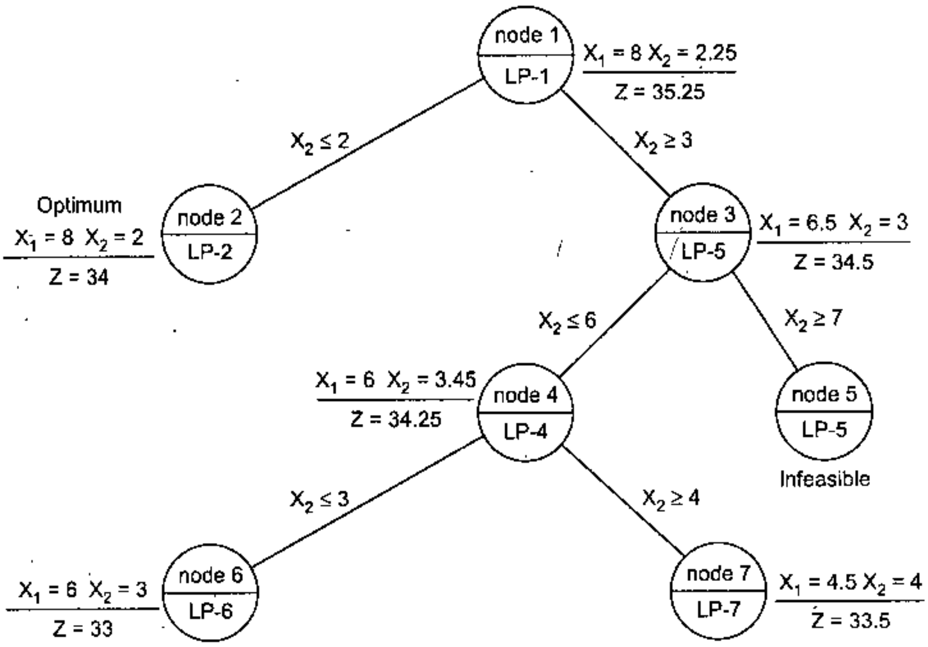


Fig. 3.13 Network representation of the branch and bound method for Example 3.5

3.6 ZERO ONE ALGORITHM

A special type of integer programming problem is a case where the values of the decision variables are limited to two logical variables, like yes or no, match or no match and so on, which are symbolized by the values zero and one. An IPP where all the variables must equal to zero or one is called as zero-one integer programming problem. There are a large number of real world problems such as assignment problem, capital budgeting problem, matching problem, location problem, travelling salesman problem *etc*, which give rise to 0 – 1 IPPs. A few of them are discussed.

NOTES

Assignment Problem

Formulation of assignment problem as an integer programming problem is as shown below.

Maximize : $$Z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij}$$

Subject to $$\sum_{j=1}^{n} x_{ij} = 1 \quad \text{for} \quad i = 1, 2, \ldots n$$

$$\sum_{j=1}^{n} x_{ij} = 1 \quad \text{for} \quad j = 1, 2, \ldots n$$

$$X_{ij} = 0 \quad \text{or} \quad 1 \quad \text{for all } i \text{ and } j$$

Example 3.6. *There is an assignment problem where there are three workers and an equal number of jobs and the cost matrix is as given below.*

Workers i \ Jobs j	1	2	3
1	3	6	7
2	5	4	9
3	6	6	8

Solution.

Let x_{ij} represent assignment of i^{th} worker to j^{th} job.

The problem is formulated as

Maximize $\quad Z = 3x_{11} + 6x_{12} + 7x_{13} + 5x_{21} + 4x_{22} + 9x_{23} + 6x_{31} + 6x_{32} + 8x_{33}$

Subject to $\quad z = x_{11} + x_{12} + x_{13} = 1 \qquad x_{11} + x_{21} + x_{31} = 1$

$x_{21} + x_{22} + x_{23} = 1 \qquad x_{12} + x_{22} + x_{32} = 1$

$x_{31} + x_{32} + x_{33} = 1 \qquad x_{13} + x_{23} + x_{33} = 1$

$x_{ij} = 0$ or 1 for $i = 1, 2, 3, j = 1, 2, 3$

$x_{ij} = 0$ (if assignment is not made) or

$= 1$ otherwise.

The Knapsack Problem

This is an IPP which has only one constraint.

Example 3.7. *Four items are considered for loading on a truck, which has a capacity to load upto 20 tones. The weights and values of the items are indicated below.*

Item	A	B	C	D
Weight (tones)	2	4	5	3
Per unit value	20	25	30	22

Which items and what quantities should be loaded on the truck so as to maximize the value of the items transported ?

Solution.

Let x_1, x_2, x_3, x_4 be the number of items A, B, C, D respectively, that are loaded on the truck.

The problem is

Maximize $Z = 20x_1 + 25x_2 + 30x_3 + 20x_4$

Subject to $2x_1 + 4x_2 + 5x_3 + 3x_4 \le 20$

$x_i = 0 \text{ or } 1 \ (i = 1, 2, 3, 4)$

The decision variables are 0 (if not loaded) or 1 (if loaded).

Capital Budgeting

NOTES

A company is considering four possible investment opportunities. The following table gives information about the investment (in ₹ Thousands) and profits.

Project	*Present value of expected return*	*Capital required year wise by projects*		
		Year 1	*Year 2*	*Year 3*
1	650	700	550	400
2	700	850	550	350
3	225	300	150	100
4	250	350	200	—
Capital available for investment	1200	700	400	400

Conditions

(*i*) The company should invest in project 1 if it invests in project 2.

(*ii*) If the company invests in project 3, then it should not invest in project 4.

Formulate an integer programming model with the conditions mentioned above to determine which projects should be accepted and which should be rejected to maximize the present value from accepted projects.

Formulation

Let x_j (1, 2, 3, 4) be investment made to projects 1, 2, 3, 4.

$x_j = 1$ if project j is accepted 0 otherwise .

Linear IP Model

Maximize $Z = 650x_1 + 700x_2 + 225x_3 + 250x_4$

Subject to $700x_1 + 850x_2 + 300x_3 + 350x_4 \le 700$

$550x_1 + 550x_2 + 150x_3 + 200x_4 \le 400$

$400x_1 + 350x_2 + 100x_3 \le 400$

$-x_1 + x_2 \le 0$

$x_3 + x_4 \le 1$

$x_j \le 0 \quad \text{or} \quad 1$

Fixed Charge Problem

Three telephone companies have offered telephone service to the subscriber. They will charge a flat fee plus a charge per minute of talk as shown below.

Telephone company	*Flat fee/ month (Rs.)*	*Charge per minute of talk(Rs.)*
1	1600	25.00
2	2500	21.00
3	1800	22.00

A subscriber usually makes an average of 200 minutes of long distance calls a month. How should the subscriber use the three companies to minimize his monthly bill. Formulate the problem as an integer (0 – 1) programme.

NOTES

Let x_1, x_2, x_3 represent the long distance calls made in minutes through the three companies' telephone service and y_1, y_2, $y_3 = 0$ or 1 depending on whether the services of the company is used or not.

Formulation :

Minimize : $Z = 25x_1 + 21x_2 + 22.1x_3 + 1600y_1 + 2500y_1 + 1800y_3$.

Subject to $x_1 + x_2 + x_3 \geq 200$

$x_1 \geq 200y_1$

$x_2 \leq 200y_2$

$x_3 \leq 200y_3$

$x_1, x_2, x_3 \geq 0$

$y_1, y_2, y_3 = (0, 1)$

(The optimum solution yields $x_3 = 200$ $y_3 = 1$)

(The concept of "flat fee" is typical of what is known in the literature as the fixed charge problem)

Set Covering Problem

To promote on campus safety, the security department is in the process of installing emergency telephones at selected locations. The department wants to install the minimum number of telephones provided that each of the campus main street is served by at least one telephone Fig. 3.14. Maps the principal streets (A to K) on campus. Formulate the LIP Model.

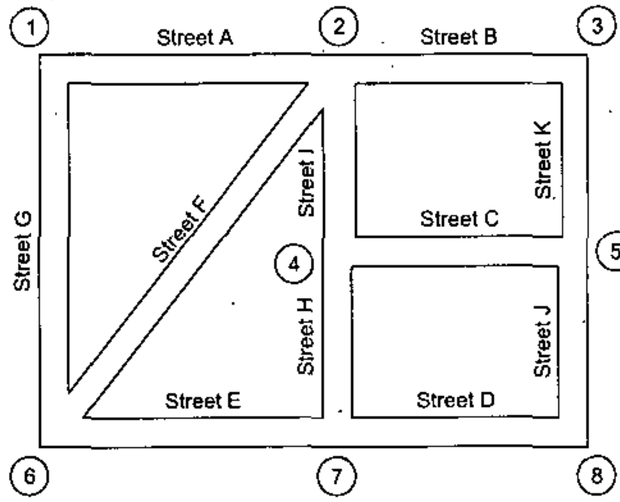


Fig. 3.14

It is logical to place the telephone at the intersections of streets so that each telephone will serve at least two streets. Fig. 3.14, shows that the layout of the streets requires a maximum of eight telephone locations.

Define $x_j = 1$ if telephone is installed in location j

$= 0$ other wise

The constraints of the problem require installing at least one telephone on each of the 11 streets (A to K). Thus the model becomes

Minimize $Z = x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8$

Subjected to

$$\begin{aligned}
x_1 + x_2 &\geq 1 \text{ (Street A)}\\
x_2 + x_3 &\geq 1 \text{ (Street B)}\\
x_4 + x_5 &\geq 1 \text{ (Street C)}\\
x_7 + x_8 &\geq 1 \text{ (Street D)}\\
x_6 + x_7 &\geq 1 \text{ (Street E)}\\
x_2 + x_6 &\geq 1 \text{ (Street F)}\\
x_1 + x_6 &\geq 1 \text{ (Street G)}\\
x_4 + x_7 &\geq 1 \text{ (Street H)}\\
x_2 + x_4 &\geq 1 \text{ (Street I)}\\
x_5 + x_8 &\geq 1 \text{ (Street J)}\\
x_3 + x_5 &\geq 1 \text{ (Street K)}\\
xj &= (0, 1), j = 1, 2 \ldots 8
\end{aligned}$$

(The optimum solution of the problem is to install telephones at inter sections 1, 2, 5 and 7. The problem has alterative optima.)

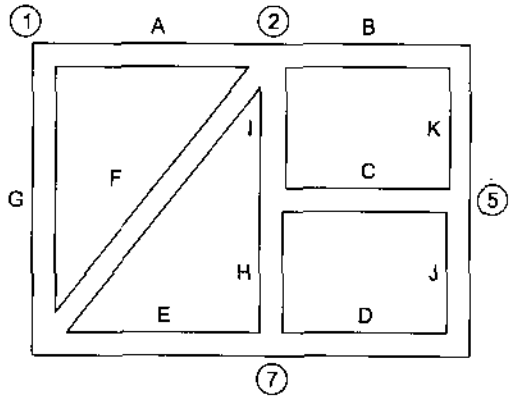


Fig. 3.15

NOTES

PART II: TRANSPORTATION PROBLEMS

3.7 INTRODUCTION AND MATHEMATICAL FORMULATION OF TRANSPORTATION PROBLEM

Transportation problem (T.P.) is generally concerned with the distribution of a certain commodity/product from several origins/sources to several destinations with minimum total cost through single mode of transportation. If different modes of transportation considered then the problem is called 'solid T.P'. In this unit, we shall deal with simple T.P.

Suppose there are m factories where a certain product is produced and n markets where it is needed. Let the supply from the factories be $a_1, a_2, \ldots, a_m$ units and demands at the markets be $b_1, b_2, \ldots, b_n$ units.

Also consider

c_{ij} = Unit of cost of shipping from factory i to market j.

x_{ij} = Quantity shipped from factory i to market j.

Then the LP formulation can be started as follows :

Minimize Z = Total cost of transportation

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}\, x_{ij}$$

Subject to, $\sum_{j=1}^{n} x_{ij} \le a_i,\ i = 1, 2, ..., m.$

NOTES

(Total amount shipped from any factory does not exceed its capacity)

$$\sum_{i=1}^{m} x_{ij} \ge b_j,\ j = 1, 2, ..., n.$$

(Total amount shipped to a market meets the demand of the market)

$$x_{ij} \ge 0 \text{ for all } i \text{ and } j.$$

Here the market demand can be met if

$$\sum_{i=1}^{m} a_i \ge \sum_{j=1}^{n} b_j.$$

If $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$ *i.e.,* total supply = total demand, the problem is said to be "Balanced T.P." and all the constraints are replaced by equality sign.

Minimize $z = \sum \sum c_{ij}\, x_{ij}$

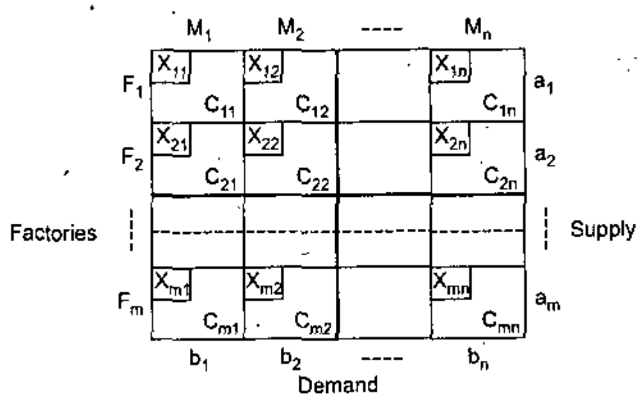
Subject to, $\sum_{j=1}^{n} x_{ij} = a_i,\ i = 1, 2, ..., m.$

$$\sum_{i=1}^{m} x_{ij} = b_j,\ j = 1, 2, ..., n.$$

$$x_{ij} \ge 0 \text{ for all } i \text{ and } j.$$

(Total $m + n$ constraints and mn variables)

The T.P. can be represented by *table form* as given below :



In the above, each cell consists of decision variable x_{ij} and per unit transportation cost c_{ij}.

Theorem 1. A necessary and sufficient condition for the existence of a feasible solution to a T.P. is that the T.P. is balanced.

Proof. (Necessary part)

NOTES

Total supply from an origin $\sum_{j=1}^{n} x_{ij} = a_i, i = 1, 2, ..., m.$

Overall supply, $\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij} = \sum_{i=1}^{m} a_i$

Total demand met of a destination

$$\sum_{i=1}^{m} x_{ij} = b_j, j = 1, 2, ..., n.$$

Overall demand, $\sum_{j=1}^{n}\sum_{i=1}^{m} x_{ij} = \sum_{j=1}^{n} b_j.$

Since overall supply exactly met the overall demand.

$$\sum_i \sum_j x_{ij} = \sum_j \sum_i x_{ij}$$

$\Rightarrow$ $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j.$

(Sufficient part) Let $\sum_i a_i = \sum_j b_i = l$ and $x_{ij} = a_i b_j / l$ for all i and j.

Then $\sum_{j=1}^{n} x_{ij} = \sum_{j=1}^{n} (a_i b_j)/l = a_i \left(\sum_{j=1}^{n} b_j\right) \Big/ l = a_i = 1, 2, ..., m.$

$$\sum_{i=1}^{m} x_{ij} = \sum_{i=1}^{m} (a_i b_j)/l = b_j \left(\sum_{i=1}^{m} a_i\right) \Big/ l = b_j, j = 1, 2, ..., n.$$

$x_{ij} \geq 0$ since a_i and b_j are non-negative.

Therefore x_{ij} satisfies all the constraints and hence x_{ij} is a feasible solution.

Theorem 2. The number of basic variables in the basic feasible solution of an $m \times n$ T.P. is $m + n - 1$.

Proof. This is due to the fact that the one of the constraints is redundant in balanced T.P.

We have overall supply, $\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij} = \sum_{i=1}^{m} a_i$

and overall demand $\sum_{j=1}^{n}\sum_{i=1}^{m} x_{ij} = \sum_{j=1}^{n} b_j$

Since $\sum_i a_i = \sum_j b_i$, the above two equations are identical and we have only $m + n - 1$ independent constraints. Hence the theorem is proved.

Note. 1. If any basic variable takes the value zero then the basic feasible solution (BFS) is said to be degenerate. Like LPP, all non-basic variables take the value zero.

2. If a basic variable takes either positive value or zero, then the corresponding cell is called 'Basic cell' or 'Occupied cell'. For non-basic variable the corresponding cell is called 'Non-basic cell' or 'Non-occupied cell' or 'Non-allocated cell'.

NOTES

Loop. This means a closed circuit in a transportation table connecting the occupied (or allocated) cells satisfying the following :

(*i*) It consists of vertical and horizontal lines connecting the occupied (or allocated) cells.

(*ii*) Each line connects only two occupied (or allocated) cells.

(*iii*) Number of connected cells is even.

(*iv*) Lines can skip the middle cell of three adjacent cells to satisfy the condition (*ii*).

The following are the examples of loops.

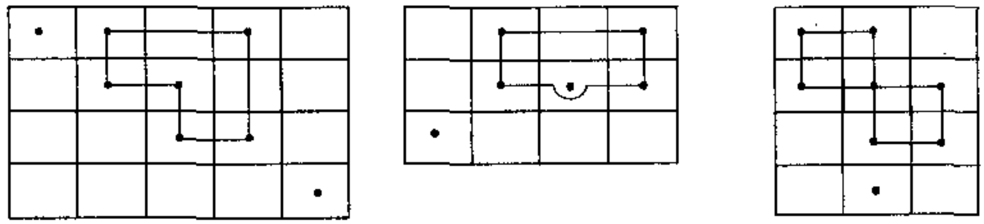


Fig. 3.1

Note. A solution of a T.P. is said to be basic if it does not consist of any loop.

3.8 FINDING INITIAL BASIC FEASIBLE SOLUTION

In this section three methods are to be discussed to find initial BFS of a T.P. In advance, it can be noted that the above three methods may give different initial BFS to the same T.P. Also allocation = minimum (supply, demand).

1. North-West Corner Rule (NWC)

Step 1. Select the north west corner cell of the transportation table.

Step 2. Allocate the min (supply, demand) in that cell as the value of the variable.

If supply happens to be minimum, cross-off the row for further consideration and adjust the demand.

If demand happens to be minimum, cross-off the column for further consideration and adjust the supply.

Step 3. The table is reduced and go to step (*i*) and continue the allocation until all the supplies are exhausted and the demands are met.

Example 3.8. *Find the initial BFS of the following T.P. using NWC rule.*

To

From	M_1	M_2	M_3	M_4	Supply
F_1	3	2	4	1	20
F_2	2	4	5	3	15
F_3	3	5	2	6	25
F_4	4	3	1	4	40
Demand	30	20	25	25	

Solution. Here, total supply = 100 = total demand. So the problem is balanced T.P.

The northwest corner cell is (1, 1) cell. So allocate min. (20, 30) = 20 in that cell. Supply exhausted. So cross-off the first row and demand is reduced to 10. The reduced table is

	M_1	M_2	M_3	M_4	
F_2	2	4	5	3	15
F_3	3	5	2	6	25
F_4	4	3	1	4	40
	10	20	25	25	

NOTES

Here the northwest corner cell is (2, 1) cell. So allocate min. (15, 10) = 10 in that cell. Demand met. So cross-off the first column and supply is reduced to 5. The reduced table is

	M_2	M_3	M_4	
F_2	4	5	3	5
F_3	5	2	6	25
F_4	3	1	4	40
	20	25	25	

Here the northwest corner cell is (2, 2) cell. So allocate min. (5, 20) = 5 in that cell. Supply exhausted. So cross-off the second row (due to F2) and demand is reduced to 15. The reduced table is

	M_2	M_3	M_4	
F_3	5	2	6	25
F_4	3	1	4	40
	15	25	25	

Here the northwest corner cell is (3, 2) cell. So allocate min. (25, 15) = 15 in that cell. Demand met. So cross off the second column (due to M_2) and supply is reduced to 10. The reduced table is

	M_3	M_4	
F_3	2	6	10
F_4	1	4	40
	25	25	

Here the northwest corner cell is (3, 3) cell. So allocate min. (10, 25) = 10 in that cell. Supply exhausted. So cross off the third row (due to F_3) and demand is reduced to 15. The reduced table is

	M_3	M_4	
F_4	1	4	40
	25	25	

continuing we obtain the allocation 15 to (4, 3) cell and 25 to (4, 4) cell so that supply exhausted and demand met. The **complete allocation** is shown below :

	M_1	M_2	M_3	M_4
F_1	[20] 3	2	4	1
F_2	[10] 2	[5] 4	5	3
F_3	3	[15] 5	[10] 2	6
F_4	4	3	[15] 1	[25] 4

Thus the initial BFS is

$$x_{11} = 20, x_{21} = 10, x_{22} = 5, x_{32} = 15, x_{33} = 10, x_{43} = 15, x_{44} = 25.$$

The transportation cost

$$= 20 \times 3 + 10 \times 2 + 5 \times 4 + 15 \times 5 + 10 \times 2 + 5 \times 1 + 25 \times 4 = ₹\ 310.$$

2. Least Cost Entry Method (LCM or Matrix Minimum Method)

NOTES

Step 1. Find the least cost from transportation table. If the least value is unique, then go for allocation.

If the least value is not unique then select the cell for allocation for which the contributed cost is minimum.

Step 2. If the supply is exhausted cross-off the row and adjust the demand.

If the demand is met cross-off the column and adjust the supply.

Thus the matrix is reduced.

Step 3. Go to step (*i*) and continue until all the supplies are exhausted and all the demands are met.

Example 3.9. *Find the initial BFS of Example 1 using least cost entry method :*

	M_1	M_2	M_3	M_4	
F_1	3	2	4	1	20
F_2	2	4	5	3	15
F_3	3	5	2	6	25
F_4	4	3	1	4	40
	30	20	25	25	

Solution. Here the least value is 1 and occurs in two cells (1, 4) and (4, 3). But the contributed cost due to cell (1, 4) is 1 × min (20, 25) *i.e.*, 20 and due to cell (4, 3) is 1 × min. (40, 25) *i.e.*, 25. So we selected the cell (1, 4) and allocate 20. Cross-off the first row since supply exhausted and adjust the demand to 5. The reduced table is given below :

2	4	5	3	15
3	5	2	6	25
4	3	1	4	40
30	20	25	5	

The least value is 1 and unique. So allocate min. (40, 25) = 25 in that cell. Cross-off the third column (due to M_3) since the demand is met and adjust the supply to 15. The reduced table is given below :

2	4	3	15
3	5	6	25
4	3	4	15
30	20	5	

The least value is 2 and unique. So allocate min. (15, 30) = 15 in that cell. Cross-off the second row (due to F_2) since the supply exhausted and adjust the demand to 15. The reduced table is given below :

3	5	6	25
4	3	4	15
15	20	5	

The least value is 3 and occurs in two cells (3, 1) and (4, 2). The contributed cost due to cell (3, 1) is $3 \times \min. (25, 15) = 45$ and due to cell (4, 2) is $3 \times \min. (15, 20) = 45$. Let us select the (3, 1) cell for allocation and allocate 15. Cross-off the first column (due to M_1) since demand is met and adjust the supply to 10. The reduced table is given below :

NOTES

5	6	10
3	4	15
20	5	

Continuing the above method and we obtain the allocations in the cell (4, 2) as 15, in the cell (3, 2) as 5 and in the cell (3, 4) as 5. The complete allocation is shown below :

	M_1	M_2	M_3	M_4
F_1	3	2	4	[20] 1
F_2	[15] 2	4	5	3
F_3	[15] 3	[5] 5	2	[5] 6
F_4	4	[15] 3	[25] 1	4

The initial BFS is

$$x_{14} = 20, x_{21} = 15, x_{31} = 15, x_{32} = 5, x_{34} = 5, x_{42} = 15, x_{43} = 25.$$

The transportation cost

$$= 20 \times 1 + 15 \times 2 + 15 \times 3 + 5 \times 5 + 5 \times 6 + 15 \times 3 + 25 \times 1 = \text{Rs. } 220.$$

Note. If the least cost is only selected columnwise then it is called 'column minima' method. If the least cost is only selected row wise then it is called 'row minima' method.

3. Vogel's Approximation Method (VAM)

Step 1. Calculate the row penalties and column penalties by taking the difference between the lowest and the next lowest costs of every row and of every column respectively.

Step 2. Select the largest penalty by encircling it. For tie cases, it can be broken arbitrarily or by analyzing the contributed costs.

Step 3. Allocate in the least cost cell of the row/column due to largest penalty.

Step 4. If the demand is met, cross off the corresponding column and adjust the supply.

If the supply is exhausted, cross-off the corresponding row and adjust the demand.

Thus the transportation table is reduced.

Step 5. Go to Step (*i*) and continue until all the supplies exhausted and all the demands are met.

Example 3.10. *Find the initial BFS of example 1 using Vogel's approximation method.*

NOTES

Solution.

	M_1	M_2	M_3	M_4	Row penalties
F_1	3	2	4	[20] 1	20 (1)
F_2	2	4	5	3	15 (1)
F_3	3	5	2	6	25 (1)
F_4	4	3	1	4	40 (2)
	30	20	25	25	
Column penalties	(1)	(1)	(1)	(2)	

Since there is a tie in penalties, let us break the tie by considering the contributed costs. Due to M_4, the contributed cost is $1 \times \min. (20, 25) = 20$. While due to F_4, the contributed cost is $1 \times \min. (40, 25) = 25$. So select the column due to M_4 for allocation and we allocate min. (20, 25) *i.e.*, 20 in (1, 4) cell. Then cross-off the first row as supply is exhausted and adjust the corresponding demand as 5. The reduced table is

	M_1	M_2	M_3	M_4	Row penalties
F_2	2	4	5	3	15 (1)
F_3	3	5	2	6	25 (1)
F_4	4	3	1	4	40 (2)
	30	20	25	5	
Column penalties	(1)	(1)	(1)	(1)	

Here the largest penalty is 2 which is due to F_4. Allocate in (4, 3) cell as min. (40, 25) = 25. Cross off the third column due to M_3 since demand is met and adjust the corresponding supply to 15. The reduced table is

	M_1	M_2	M_4	Row penalties
F_2	2	4	3	15 (1)
F_3	3	5	6	25 (2)
F_4	4	3	4	15 (1)
	30	20	5	
Column penalties	(1)	(1)	(1)	

Here the largest penalty is 2 which is due to F_3. Allocate in (3, 1) cell as min. (25, 30) = 25. Cross-off the third row due to F_3 since supply is exhausted and adjust the corresponding demand to 5. The reduced table is

	M_1	M_2	M_4	Row penalties
F_2	2	4	3	15 (1)
F_4	4	3	4	15 (1)
	5	20	5	
Column penalties	(2)	(1)	(1)	

Here the largest penalty is 2 which is due to M_1. Allocate in (2, 1) cell as min. (15, 5) = 5. Corss off the first column due to M_1 since demand is met and adjust the supply to 10. The reduced table is

	M_2	M_4	Row penalties
F_2	4	3	10 (1)
F_4	3	4	15 (1)
	20	5	
Column penalties	(1)	①	

NOTES

Here tie has occurred. The contributed cost is minimum due to (2, 4) cell which is 3 × min. (10, 5) = 15. So allocate min. (10, 5) = 5 in (2, 4) cell. Cross-off the fourth column which is due to M_4 since demand is met and adjust the corresponding supply to 5. On continuation we obtain the allocation of 5 in (2, 2) cell and 15 in (4, 2) cell. The complete allocation in shown below :

	M_1	M_2	M_3	M_4
F_1	3	2	4	[20] 1
F_2	[5] 2	[5] 4	5	[5] 3
F_3	[25] 3	5	2	6
F_4	4	[15] 3	[25] 1	4

The initial BFS is

$$x_{14} = 20, x_{21} = 5, x_{22} = 5, x_{24} = 5, x_{31} = 25, x_{42} = 15, x_{43} = 25.$$

The transportation cost

$$= 1 \times 20 + 2 \times 5 + 4 \times 5 + 3 \times 5 + 3 \times 25 + 3 \times 15 + 1 \times 25 = \text{Rs. } 210.$$

3.9 FINDING OPTIMA BASIC FEASIBLE SOLUTION

1. The Row Minima Method

Row Minima method takes into account the minimum cost of transportation for obtaining the initial basic feasible solution and can be summarized as follows:

Step 1. In the transportation table, determine the smallest cost (let it be C_{ij} in the first row and allocate the maximum amount

i.e., $x_{1j} = \min (a_i, b_j)$ in the cell (1, *j*), so that either the capacity at first origin or the requirement at jth destination or both of them are satisfied.

Step 2. (*a*) If $x_{1j} = a_1$ then the capacity at first origin is completely exhausted and cross out the first row of the table and move to the second row.

(*b*) If $x_{ij} = b_j$, then the requirement at j^{th} destination is satisfied and cross out the j^{th} column.

But in this case, the first row is again considered with the remaining availability at first origin.

(*c*) If $x_{ij} = a_1 = b_j$, then the capacity at first origin as well as the requirement at j^{th} destination is completely satisfied. Then choose arbitrarily as a tie appears there.

Step 3. Repeat step 1 and step 2 until all the requirements are satisfied.

NOTES

2. The Column Minima Method

The column minima method takes into account the minimum cost of transportation for obtaining the initial basic feasible solution and can be summarized as follows:

Step 1. In the transportation table, determine the smallest cost (C_{i1}) *in the first column* and allocated the maximum amount.

i.e., $x_{i1} = \min(a_i, b_1)$

Step 2. (*a*) If $x_{i1} = b_1$, then the requirement at first destination is satisfied and move towards right to the second column of the transportation table after crossing the first column.

(*b*) If $x_1 = a_i$, then cross the i^{th} row of the transportation table as the first row capacity is completely filled.

But in this case, the first column is again considered with the remaining demand.

(*c*) If $b_1 = a_i$, then the capacity of the first origin as well as the requirement at j^{th} destination is completely satisfied. Cross the i^{th} row and make the second allocation $x_{k1} = 0$ in the cell (*k*, 1) *as* C_{k1} *will become the new minimum cost* in the first column. Cross the column and move towards right to the second column.

Step 3. Repeat step 1 and step 2 until all the requirements are satisfied.

Remark. The row-minima and the column-minima method are not generally preferred.

3. UV-Method/Modi Method

Taking the initial BFS by any method discussed above, this method find the optimal solution to the transportation problem. The steps are given below :

(*i*) For each row consider a variable u_i and for each column consider another variable v_j.

Find u_i and v_j such that

$u_i + v_j = c_{ij}$ for every basic cells.

(*ii*) For every non-basic cells, calculate the net evaluations as follows :

$$\bar{c}_{ij} = u_i + v_j - c_{ij}$$

If all $\bar{c}_{ij}$ are non-positive, the current solution is optimal.

If at least one $\bar{c}_{ij} > 0$, select the variable having the largest positive net evaluation to enter the basis.

(*iii*) Let the variable x_{rc} enter the basis. Allocate an unknown quantity θ to the cell (*r*, *c*). Identify a loop that starts and ends in the cell (*r*, *c*).

Subtract and add θ to the corner points of the loop clockwise/anticlockwise.

(*iv*) Assign a minimum value of θ in such a way that one basic variable becomes zero and other basic variables remain non-negative. The basic cell which reduces to zero leaves the basis and the cell with θ enters into the basis.

If more than one basic variables become zero due to the minimum value of θ, then only one basic cell leaves the basis and the solution is called degenerate.

(*v*) Go to step (*i*) until an optimal BFS has been obtained.

Note. In step (*ii*), if all $\bar{c}_{ij} < 0$, then the optimal solution is unique. If at least one $\bar{c}_{ij} < 0$, then we can obtain alternative solution. Assign θ in that cell and repeat one iteration (from step (*iii*)).

NOTES

Example 3.11. *Consider the initial BFS by LCM of Example 2, find the optimal solution of the T.P.*

Solution. Iteration 1.

	M_1	M_2	M_3	M_4	
F_1	3	2	4	[20] 1	$u_1 = -5$
F_2	[15] 2	4	5	3	$u_2 = -1$
F_3	[15] 3	[5] 5	2	[5] 6	$u_3 = 0$ (Let)
F_4	4	[15] 3	[25] 1	4	$u_4 = -2$
	$V_1=3$	$V_2=5$	$V_3=3$	$V_4=6$	

For non-basic cells : $\bar{c}_{ij} = u_i + v_j - c_{ij}$

$$\bar{c}_{11} = -5, \bar{c}_{12} = -2, \bar{c}_{13} = -6, \bar{c}_{22} = 0, \bar{c}_{23} = -3, \bar{c}_{24} = 2, \bar{c}_{33} = 1, \bar{c}_{41} = -3, \bar{c}_{44} = 0.$$

Since all $\bar{c}_{ij}$ are not non-positive, the current solution is not optimal.

Select the cell (2, 4) due to largest positive value and assign an unknown quantity θ in that cell. Identify a loop and subtract and add θ to the corner points of the loop which is shown below :

3	2	4	[20] 1
[15] −θ 2	4	5	3
[15] +θ 3	[5] 5	2	[5] −θ 6
4	[15] 3	[25] 1	4

Select θ = min. (5, 15) = 5. The cell (3, 4) leaves the basis and the cell (2, 4) enters into the basis. Thus the current solution is updated.

Iteration 2.

3	2	4	[20] 1	$u_1 = -2$
[10] 2	4	5	[5] 3	$u_2 = 0$ (Let)
[20] 3	[5] 5	2	6	$u_3 = 1$
4	[15] 3	[25] 1	4	$u_4 = -1$
$V_1 = 2$	$V_2 = 4$	$V_3 = 2$	$V_4 = 3$	

For non-basic cells : $\bar{c}_{ij} = u_i + v_j - c_{ij}$

$$\bar{c}_{11} = -3, \bar{c}_{12} = 0, \bar{c}_{13} = -4, \bar{c}_{22} = 0, \bar{c}_{23} = -3, \bar{c}_{33} = 1, \bar{c}_{34} = -2, \bar{c}_{41} = -3, \bar{c}_{44} = -2.$$

Since all $\bar{c}_{ij}$ are not non-positive, the current solution is not optimal.

Select the cell (3, 3) due to largest positive value and assign an unknown quantity θ in that cell. Identify a loop and subtract and add θ to the corner points of the loop which is shown below :

NOTES

3	2	4	[20] 1
[10] 2	4	5	[5] 3
[20] 3	[5]−θ 5	θ [5] 2	6
4	[15]+θ 3	[25]−θ 1	4

Select θ = min. (5, 25) = 5. The cell (3, 2) leaves the basis and the cell (3, 3) enters into the basis. Thus the current solution is updated.

Iteration 3.

3	2	4	[20] 1	$u_1 = -2$
[10] 2	4	5	[5] 3	$u_2 = 0$ (Let)
[20] 3	5	[5] 2	6	$u_3 = 1$
4	[20] 3	[20] 1	4	$u_4 = 0$
$V_1 = 2$	$V_2 = 3$	$V_3 = 1$	$V_4 = 3$	

For non-basic cells : $\bar{c}_{ij} = u_i + v_j - c_{ij}$

$$\bar{c}_{11} = -3, \bar{c}_{12} = -1, \bar{c}_{13} = -5, \bar{c}_{22} = -1, \bar{c}_{23} = -5, \bar{c}_{32} = -1, \bar{c}_{34} = -2, \bar{c}_{41} = -2, \bar{c}_{44} = -1.$$

Since all $\bar{c}_{ij}$ are non-positive, the current solution is optimal. Thus the optimal solution is

$$x_{14} = 20, x_{21} = 10, x_{24} = 5, x_{31} = 20, x_{33} = 5, x_{42} = 20, x_{43} = 20.$$

The optimal transportation cost

$= 1 \times 20 + 2 \times 10 + 3 \times 5 + 3 \times 20 + 2 \times 5 + 3 \times 20 + 1 \times 20 =$ Rs. 205.

Example 3.12. *Consider the initial BFS by VAM of Example 3, find the optimal solution of the T.P.*

Solution. Iteration 1.

3	2	4	[20] 1	$u_1 = -2$
[5] 2	[5] 4	5	[5] 3	$u_2 = 0$ (Let)
[25] 3	5	2	6	$u_3 = 1$
4	[15] 3	[25] 1	4	$u_4 = -1$
$V_1 = 2$	$V_2 = 4$	$V_3 = 2$	$V_4 = 3$	

For non-basic cells : $\bar{c}_{ij} = u_i + v_j - c_{ij}$

$$\bar{c}_{11} = -3, \bar{c}_{12} = 0, \bar{c}_{13} = -4, \bar{c}_{23} = -3, \bar{c}_{32} = 0, \bar{c}_{33} = 1, \bar{c}_{34} = -2, \bar{c}_{41} = -3, \bar{c}_{44} = -2.$$

Since all $\bar{c}_{ij}$ are not non-positive, the current solution is not optimal.

Select the cell (3, 3) due to largest positive value and assign an unknown quantity θ in that cell. Identify a loop and subtract and add θ to the corner points of the loop which is shown below :

3	2	4	[20] 1
[5]+θ 2	[5]−θ 4	5	[5] 3
[25]−θ 3	5	θ 2	6
4	[15]+θ 3	[25]−θ 1	4

NOTES

Select θ = min. (5, 25, 25) = 5. The cell (2, 2) leaves the basis and the cell (3, 3) enters into the basis. Thus the current solution is updated.

Iteration 2.

3	2	4	[20] 1	$u_1 = -2$
[10] 2	4	5	[5] 3	$u_2 = 0$ (Let)
[20] 3	5	[5] 2	6	$u_3 = 1$
4	[20] 3	[20] 1	4	$u_4 = 0$
$V_1 = 2$	$V_2 = 3$	$V_3 = 1$	$V_4 = 3$	

For non-basic cells : $\bar{c}_{ij} = u_i + v_j - c_{ij}$

$$\bar{c}_{11} = -3, \bar{c}_{12} = -1, \bar{c}_{13} = -5, \bar{c}_{22} = -1, \bar{c}_{23} = -5, \bar{c}_{32} = -1, \bar{c}_{34} = -2, \bar{c}_{41} = -2, \bar{c}_{44} = -1.$$

Since all $\bar{c}_{ij}$ are non-positive, the current solution is optimal. Thus the optimal solution is

$$x_{14} = 20, x_{21} = 10, x_{24} = 5, x_{31} = 20, x_{33} = 5, x_{42} = 20, x_{43} = 20.$$

The optimal transportation cost = Rs. 205.

Note. To find optimal solution to a T.P., the number of iterations by uv-method is always more if we consider the initial BFS by NWC.

3.10 DEGENERACY IN TRANSPORTATION PROBLEMS

A *BFS of a T.P. is said to be degenerate if one or more* basic variables assume a zero value. This degeneracy may occur in initial BFS or in the subsequent iterations of *uv*-method. An initial BFS could become degenerate when the supply and demand in the intermediate stages of any one method (NWC/LCM/VAM) are equal corresponding to a selected cell for allocation. In *uv*-method it is identified only when more than one corner points in a loop vanishes due to minimum value of θ.

For the degeneracy in initial BFS, arbitrarily we can delete the row due to supply adjusting the demand to zero or delete the column due to demand adjusting the supply to zero whenever there is a tie in demand and supply.

For the degeneracy in *uv*-method, arbitrarily we can make one corner as non-basic cell and put zero in the other corner.

NOTES

Example 3.13. *Find the optimal solution to the following T.P. :*

Source	Destination 1	2	3	Available
1	50	30	190	10
2	80	45	150	30
3	220	180	50	40
Requirement	40	20	20	80

Solution. Let us find the initial BFS using VAM :

	1	2	3	Row penalties
1	50	30	190	10 (20)
2	80	45	150	30 (35)
3	220	180	50	40 (130)
Column penalties	40 (30)	20 (15)	20 (100)	

Select (3, 3) cell for allocation and allocate min (40, 20) = 20 in that cell. Cross-off the third column as the requirement is met and adjust the availability to 20. The reduced table is given below :

	1	2	Row penalties
1	50	30	10 (20)
2	80	45	30 (35)
3	220	180	20 (40)
Column penalties	40 (30)	20 (15)	

Select (3, 2) cell for allocation. Now there is a tie in allocation. Let us allocate 20 in (3, 2) cell and cross-off the second column and adjust the availability to zero. The reduced table is given below :

	1	
1	50	10
2	80	30
3	220	0
	40	

On continuation we obtain the remaining allocations as 0 in (3, 1) cell, 30 in (2, 1) cell and 10 in (1, 1) cell. The complete initial BFS is given below and let us apply the first iteration of *uv*-method :

Iteration 1.

[10] 50	30	190	$u_1 = -170$
[30] 80	45	150	$u_2 = -140$
[0] 220	[20] 180	[20] 50	$u_3 = 0$ (Let)
$V_1 = 220$	$V_2 = 180$	$V_3 = 50$	

For non-basic cells : $\bar{c}_{ij} = u_i + v_j - c_{ij}$

$$\bar{c}_{12} = -20, \bar{c}_{13} = -310, \bar{c}_{22} = -5, \bar{c}_{23} = -240.$$

Since all $\bar{c}_{ij} < 0$, the current solution is optimal. Hence the optimal solution is

$$x_{11} = 10, x_{21} = 30, x_{31} = 0, x_{32} = 20, x_{33} = 20.$$

The transportation cost

$$= 50 \times 10 + 80 \times 30 + 0 + 180 \times 20 + 50 \times 20 = \text{Rs. } 7500.$$

NOTES

PART III: ASSIGNMENT PROBLEMS

3.11 INTRODUCTION AND MATHEMATICAL FORMULATION OF ASSIGNMENT PROBLEMS

Consider n machines $M_1, M_2, ..., M_n$ and n different jobs $J_1, J_2, ..., J_n$. These jobs to be processed by the machines one to one basis *i.e.,* each machine will process exactly one job and each job will be assigned to only one machine. For each job the processing cost depends on the machine to which it is assigned. Now we have to determine the assignment of the jobs to the machines one to one basis such that the total processing cost is minimum. This is called an *assignment problem*.

If the number of machines is equal to the number of jobs then the above problem is called *balanced* or *standard* assignment problem. Otherwise, the problem is called *unbalanced* or *non-standard* assignment problem. Let us consider a balanced assignment problem.

For linear programming problem formulation, let us define the decision variables as

$$x_{ij} = \begin{cases} 1, \text{ if job } j \text{ is assigned to machine } i \\ 0, \text{ otherwise} \end{cases}$$

and the cost of processing job j on machine i as c_{ij}. Then we can formulate the assignment problem as follows :

Minimize $$z = \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij} \qquad ...(1)$$

subject to, $$\sum_{j=1}^{n} x_{ij} = 1, i = 1, 2, ..., n$$

(Each machine is assigned exactly to one job)

$$\sum_{i=1}^{n} x_{ij} = 1, j = 1, 2, ..., n.$$

(Each job is assigned exactly to one machine)

$$x_{ij} = 0 \text{ or } 1 \text{ for all } i \text{ and } j$$

In **matrix form**, Minimize $z = Cx$

subject to, $Ax = 1,$

$$x_{ij} = 0 \text{ or } 1,\ i, j = 1, 2, \ldots, n.$$

NOTES

where A is a $2n \times n^2$ matrix and total unimodular *i.e.*, the determinant of every sub square matrix formed from it has value 0 or 1. This property permits us to replace the constraint $x_{ij} = 0$ or 1 by the constraint $x_{ij} \geq 0$. Thus we obtain

Minimize $z = Cx$

subject to, $Ax = 1, x \geq 0$

The **dual** of (1) with the non-negativity restrictions replacing the 0-1 constraints can be written as follows :

Maximize
$$W = \sum_{i=1}^{n} u_i + \sum_{j=1}^{n} v_j$$

subject to, $u_i + v_j \leq c_{ij},\ i, \quad j = 1, 2, \ldots, n.$

u_i, v_j unrestricted in signs $i, j = 1, 2, \ldots, n.$

Example 3.14. *A company is facing the problem of assigning four operators to four machines. The assignment cost in rupees in given below :*

Machine

Operator	M_1	M_2	M_3	M_4
I	5	7	–	4
II	7	5	3	2
III	9	4	6	–
IV	7	2	7	6

In the above, operators I and III can not be assigned to the machines M_3 and M_4 respectively. Formulate the above problem as a LP model.

Solution. Let
$$x_{ij} = \begin{cases} 1, \text{ if the } i\text{th operator is assigned to } j\text{th machine} \\ 0, \text{ otherwise} \end{cases}$$
$$i, j = 1, 2, 3, 4.$$

be the decision variables.

By the problem, $x_{13} = 0$ and $x_{34} = 0.$

The LP model is given below :

Minimize
$$z = 5x_{11} + 7x_{12} + 4x_{14} + 7x_{21} + 5x_{22} + 3x_{23} + 2x_{24} + 9x_{31} + 4x_{32} + 6x_{33} + 7x_{41} + 2x_{42} + 7x_{43} + 6x_{44}$$

subject to,

(Operator assignment constraints)

$$x_{11} + x_{12} + x_{14} = 1$$
$$x_{21} + x_{22} + x_{23} + x_{24} = 1$$
$$x_{31} + x_{32} + x_{33} = 1$$
$$x_{41} + x_{42} + x_{43} + x_{44} = 1$$

(Machine assignment constraints)

$$x_{11} + x_{21} + x_{31} + x_{41} = 1$$
$$x_{12} + x_{22} + x_{32} + x_{42} = 1$$
$$x_{23} + x_{33} + x_{43} = 1$$
$$x_{14} + x_{24} + x_{44} = 1$$
$$x_{ij} \geq 0 \text{ for all } i \text{ and } j.$$

3.12 HUNGARIAN ALGORITHM

NOTES

This is an efficient algorithm for solving the assignment problem developed by the Hungarian mathematician **König**. Here the optimal assignment is not affected if a constant is added or subtracted from any row or column of the balanced assignment cost matrix. The **algorithm** can be started as follows :

(*a*) Bring at least one zero to each row and column of the cost matrix by subtracting the minimum of the row and column respectively.

(*b*) Cover all the zeros in cost matrix by *minimum* number of horizontal and vertical lines.

(*c*) If number of lines = order of the matrix, then select the zeros as many as the order of the matrix in such a way that they cover all the rows and columns.

(Here $A_{n \times n}$ means *n*th order matrix)

(*d*) If number of lines ≠ order of the matrix, then perform the following and create a new matrix :

- (*i*) Select the minimum element from the uncovered elements of the cost matrix by the lines.
- (*ii*) Subtract the uncovered elements from the minimum element.
- (*iii*) Add the minimum element to the junction (*i.e.*, crossing of the lines) elements.
- (*iv*) Other elements on the lines remain unaltered.
- (*v*) Go to Step (*b*).

Example 3.15. *A construction company has four engineers for designing. The general manager is facing the problem of assigning four designing projects to these engineers. It is also found that Engineer 2 is not competent to design project 4. Given the time estimate required by each engineer to design a given project, find an assignment which minimizes the total time.*

		Projects			
		P1	*P2*	*P3*	*P4*
	E1	6	5	13	2
Engineers	*E2*	8	10	4	–
	E3	10	3	7	3
	E4	9	8	6	2

Solution. Let us first bring zeros rowwise by subtracting the respective minima from all the row elements respectively.

4	3	11	0
4	6	0	–
7	0	4	0
7	6	4	0

Let us bring zero columnwise by subtracting the respective minima from all the column elements respectively. Here the above operations is to be performed only on first column, since at least one zero has appeared in the remaining columns.

0	3	11	0
0	6	0	–
3	0	4	0
3	6	4	0

(This completes Step-*a*)

Now (Step-*b*) all the zeros are to be covered by minimum number of horizontal and vertical lines which is shown below. It is also to be noted that this covering is not unique.

NOTES

0	3	11	0
0	6	0	–
3	0	4	0
3	6	4	0

It is seen that no. of lines = 4 = order of the matrix. Therefore by Step-*c*, we can go for assignment *i.e.*, we have to select 4 zeros such that they cover all the rows and columns which is shown below :

[0]	3	11	0
0	6	[0]	–
3	[0]	4	0
3	6	4	[0]

Therefore the optimal assignment is

E1→P1, E2→P3, E3→P2, E4→P4

and the minimum total time required = 6 + 4 + 3 + 2 = 15 units.

Example 3.16. *Solve the following job machine assignment problem. Cost data are given below :*

		Machines					
		1	*2*	*3*	*4*	*5*	*6*
	A	*21*	*35*	*20*	*20*	*32*	*28*
	B	*30*	*31*	*22*	*25*	*28*	*30*
	C	*28*	*29*	*25*	*27*	*27*	*21*
Jobs	*D*	*30*	*30*	*26*	*26*	*31*	*28*
	E	*21*	*31*	*25*	*20*	*27*	*30*
	F	*25*	*29*	*22*	*25*	*30*	*21*

Solution. Let us first bring zeros first rowwise and then columnwise by subtracting the respective minima elements from each row and each column respectively and the cost matrix, thus obtained, is as follows :

0	11	0	0	7	8
7	5	0	3	1	8
6	4	4	6	1	0
3	0	0	0	0	2
0	7	5	0	2	10
3	4	1	4	4	0

By Step-*b*, all the zeros are covered by minimum number of horizontal and vertical lines which is shown below :

0	11	0	0	7	8
7	5	0	3	1	8
6	4	4	6	1	0
3	0	0	0	0	2
0	7	5	0	2	10
3	4	1	4	4	0

NOTES

Here no. of lines ≠ order of the matrix. Hence we have to apply Step-*d*. The minimum uncovered element is 1. By applying Step-*d* we obtain the following matrix :

0	11	1	0	7	9
6	4	0	2	0	8
5	3	4	5	0	0
3	0	1	0	0	3
0	7	6	0	2	11
2	3	1	3	3	0

Now, by Step-*b*, we cover all the zeros by minimum number of horizontal and vertical straight lines.

0	11	1	0	7	9
6	4	0	2	0	8
5	3	4	5	0	0
3	0	1	0	0	3
0	7	6	0	2	11
2	3	1	3	3	0

Now the no. of lines = order of the matrix. So we can go for assignment by Step-*c*. The assignment is shown below :

[0]	11	1	0	7	9
6	4	[0]	2	0	8
5	3	4	5	[0]	0
3	[0]	1	0	0	3
0	7	6	[0]	2	11
2	3	1	3	3	[0]

The optimal assignment is A→1, B→3, C→5, D→2, E→4, F→6. An alternative assignment is also obtained as A→4, B→3, C→5, D→2, E→1, F→6. For both the assignments, the minimum cost is 21 + 22 + 27 + 30 + 20 + 21 *i.e.*, Rs. 141.

3.13 UNBALANCED ASSIGNMENTS

NOTES

For unbalanced or non-standard assignment problem no. of rows ≠ no. of columns in the assignment cost matrix *i.e.*, we deal with a rectangular cost matrix. To find an assignment for this type of problem, we have to first convert this unbalanced problem into a balanced problem by adding dummy rows or columns with zero costs so that the defective function will be unaltered. For machine-job problem, if no. of machines (say, m) > no. of jobs (say, n), then create m-n dummy jobs and the processing cost of dummy jobs as zero. When a dummy job gets assigned to a machine, that machine stays idle. Similarly the other case *i.e.*, $n > m$, is handled.

Example 3.17. *Find an optimal solution to an assignment problem with the following cost matrix :*

	M1	*M2*	*M3*	*M4*	*M5*
J1	13	5	20	5	6
J2	15	10	16	10	15
J3	6	12	14	10	13
J4	13	11	15	11	15
J5	15	6	16	10	6
J6	6	15	14	5	12

Solution. The above problem is unbalanced. We have to create a dummy machine M6 with zero processing time to make the problem as balanced assignment problem. Therefore we obtain the following :

	M1	M2	M3	M4	M5	M6 (dummy)
J1	13	5	20	5	6	0
J2	15	10	16	10	15	0
J3	6	12	14	10	13	0
J4	13	11	15	11	15	0
J5	15	6	16	10	6	0
J6	6	15	14	5	12	0

Let us bring zeros columnwise by subtracting the respective minima elements from each column respectively and the cost matrix, thus obtained, is as follows :

7	0	6	0	0	0
9	5	2	5	9	0
0	7	0	5	7	0
7	6	1	6	9	0
9	1	2	5	0	0
0	10	0	0	6	0

Let us cover all the zeros by minimum number of horizontal and vertical lines and is given below :

NOTES

7	0	6	0	0	0
9	5	2	5	9	0
0	7	0	5	7	0
7	6	1	6	9	0
9	1	2	5	0	0
0	10	0	0	6	0

Now the number of lines ≠ order of the matrix. The minimum uncovered element by the lines is 1. Using Step-*d* of the Hungarian algorithm and covering all the zeros by minimum no. of lines we obtain as follows :

7	0	6	0	1	1
8	4	1	4	9	0
0	7	0	5	8	1
6	5	0	5	9	0
8	0	1	4	0	0
0	10	0	0	7	1

Now the number of lines = order of the matrix and we have to select 6 zeros such that they cover all the rows and columns. This is done in the following :

7	[0]	6	0	1	1
8	4	1	4	9	[0]
[0]	7	0	5	8	1
6	5	[0]	5	9	0
8	0	1	4	[0]	0
0	10	0	[0]	7	1

Therefore, the optimal assignment is

J1→M2, J2→M6, J3→M1, J4→M3, J5→M5, J6→M4 and the minimum cost = Rs. (5 + 0 + 6 + 15 + 6 + 5) = Rs. 37.

In the above, the job J2 will not get processed since the machine M6 is dummy.

3.14 MAX-TYPE ASSIGNMENT PROBLEMS

When the objective of the assignment is to maximize, the problem is called 'Max-type assignment problem'. This is solved by converting the profit matrix to an opportunity loss matrix by subtracting each element from the highest element of the profit matrix. Then the minimization of the loss matrix is the same as the maximization of the profit matrix.

NOTES

Example 3.18. *A company is faced with the problem of assigning 4 jobs to 5 persons. The expected profit in rupees for each person on each job are as follows :*

	Job			
Persons	*J1*	*J2*	*J3*	*J4*
I	*86*	*78*	*62*	*81*
II	*55*	*79*	*65*	*60*
III	*72*	*65*	*63*	*80*
IV	*86*	*70*	*65*	*71*
V	*72*	*70*	*71*	*60*

Find the assignment of persons to jobs that will result in a maximum profit.

Solution. The above problem is unbalanced max.-type assignment problem. The maximum element is 86. By subtracting all the elements from it obtain the following opportunity loss matrix.

0	8	24	5
31	7	21	26
14	21	23	6
0	16	21	15
14	16	15	26

Now a dummy job J5 is added with zero losses. Then bring zeros in each column by sub-tracting the respective minimum element from each column we obtain the following matrix.

0	1	9	0	0
31	0	6	21	0
14	14	8	1	0
0	9	6	10	0
14	9	0	21	0

Let us cover all the zeros by minimum number of lines and is given below :

0	1	9	0	0
31	0	6	21	0
14	14	8	1	0
0	9	6	10	0
14	9	0	21	0

Since the no. of lines = order of the matrix, we have to select 5 zeros such that they cover all the rows and columns. This is done in the following :

0	1	9	[0]	0
31	[0]	6	21	0
14	14	8	1	[0]
[0]	9	6	10	0
14	9	[0]	21	0

The optimal assignment is

I→J4, II→J2, III→J5, IV→J1, V→J3 and maximum profit = Rs. (81 + 79 + 86 + 71) = Rs. 317. Here person III is idle.

Note. The max.-type assignment problem can also be converted to a minimization problem by multiplying all the elements of the profit matrix by – 1. Then the Hungarian method can be applied directly.

SUMMARY

- The integer solution to a problem can be obtained by rounding off the optimum value of the variable to the nearest integer value. The approach is easy in terms of efforts involved in deriving an integer solution, but this may not satisfy all the given constraints. Moreover the value of the objective function so obtained may not be optimum value.
- In the optimum solution, if any basic variable is not integer, an additional linear constrainst called the Gomory constraint (or cut) is generated. After having generated a linear constraint it is added to the bottom of the optimal simplex table so that the solution no longer remains feasible. This is then solved by using dual simplex method until integer solution is obtained. *The information in the optimum table can be written explicitly as*

$$Z - \frac{63}{22}s_1 - \frac{31}{22}s_2 = 66\frac{1}{2}$$

$$x_2 + \frac{7}{22}s_1 + \frac{1}{22}s_2 = 3\frac{1}{2}$$

$$x_1 - \frac{1}{22}s_1 + \frac{3}{22}s_2 = 4\frac{1}{2}$$

- The branch and bound algorithm is the most widely used method for solving both pure and mixed integer programming problems in practice. Basically the branch and bound algorithm is just an efficient enumeration procedure for examining all possible integer feasible solutions.
- Transportation problem (T.P.) is generally concerned with the distribution of a certain commodity/product from several origins/sources to several destinations with minimum total cost through single mode of transportation. If different modes of transportation considered then the problem is called 'solid T.P'.
- Consider n machines $M_1, M_2, ..., M_n$ and n different jobs $J_1, J_2, ..., J_n$. These jobs to be processed by the machines one to one basis *i.e.,* each machine will process exactly one job and each job will be assigned to only one machine. For each job the processing cost depends on the machine to which it is assigned. Now we have to determine the assignment of the jobs to the machines one to one basis such *that the total processing cost is* minimum. This is called an *assignment problem.*
- When the objective of the assignment is to maximize, the problem is called 'Max-type assignment problem'. *This is solved by converting the profit matrix to an opportunity* loss matrix by subtracting each element from the highest element of the profit matrix. Then the minimization of the loss matrix is the same as the maximization of the profit matrix.

NOTES

GLOSSARY

- **Gomory Constraint** : It is an additional linear constraint, generated when any basic variable is not an integer in optimal solutions.
- **Feasible Solution** : Non-negative values of x_{ij} where $i = 1, 2 \ldots m$ and $j = 1, 2 \ldots n$, which satisfy the constraints of availability (supply) and requirement (demand) in transportation problems.
- **Cell** : In the matrix used in transportation problems the squares are called 'cell'. These cells form 'columns' vertically and 'row' horizontally unit costs are written in the cells.
- **Optimal Solution** : A feasible solution is said to be optimal solution when the transportation cost in minimum.

REVIEW QUESTIONS

1. Solve the following all integer programming problem using the branch and bound method.

   Minimize $z = 3x_1 + 2.5x_2$

   Subjected to $x_1 + 2x_2 \geq 20$ ...(*i*)

   $3x_1 + 2x_2 \geq 50$ ...(*ii*)

   $x_1, x_2 \geq 0$ and integers
2. Write short notes on :
   (*i*) Gomary's cutting plane method
   (*ii*) Branch and bound method
3. What is a transportation problem ? How is it useful in business and industry ?
4. What do you understand by
   (*a*) Feasible solution
   (*b*) North-west solution
   (*c*) Vogel's approximation method (VAM).
5. Discuss various steps involved in finding initial feasible solution of a transportation problem.
6. Discuss any two methods of solving a transportation problem. State the advantages and disadvantages of these methods.
7. XYZ Ltd. has three manufacturing plants P_1, P_2 and P_3 which are stepping the production to three warehouses W_1, W_2 and W_3. The following data is available :

Plant	Production (units)	Warehouse	Requirement (units)
P_1	150,000	W_1	160,000
P_2	120,000	W_2	130,000
P_3	130,000	W_3	80, 00

   The rate of freight charges/unit is as shown below.

		To W_1	W_2	W_3
From	P_1	1.50	1.60	1.80
	P_2	2.0	1.80	2.50
	P_3	1.60	1.40	3.00

   Determine the initial basic feasible solution using north-west comer method and VAM.
8. Plant location of a firm manufacturing a single product has three plants located at A, B and C. Their production during week has been 60, 40 and 50 units respectively. The company has firm commitment orders for 25, 40, 20, 20 and 30 units of the product to customers C-1, C-2, C-3, C-4 and C-5 respectively. Unit cost of transporting from the three plants to the five customers is given in the table below :

		C-1	C-2	C-3	C-4	C-5
Plant location	A	6	1	3	4	6
	B	4	4	3	3	2
	C	2	6	4	4	6

Use VAM to determine the cost of shipping the product from plant locations to the customers.

9. Find the initial basic feasible solution to the following transportation problem by

(*a*) Least cost method. (*b*) North-west corner rule.

State which of the methods is better

Table

	To			Supply
From	2	7	4	4
	3	3	1	8
	5	4	7	7
	1	6	2	14
Demand	7	9	18	

10. Solve the following transportation problem by VAM.

Consumers

	A	B	C	**Available**
Suppliers I	6	7	4	14
II	4	3	1	12
III	1	4	7	5
Required	6	10	15	31

Use VAM to find an initial BFS.

11. Solve the following problem in which cell entries represent unit costs :

	D1	D2	D3	Available
Q1	2	7	4	5
Q2	3	3	1	8
Q3	5	4	7	7
Q4	1	6	2	14
Required	7	9	18	44

Apply MODI method to test optimality.

12. Solve the following transportation problem

To From	D1	D2	D3	D4	Available
S1	4	3	1	2	80
S2	5	2	3	4	60
S3	3	5	6	3	40
Requirement	50	60	20	50	180 Total

Apply MODI method to test its optimality.

13. Determine the optimal solution to the p.ubiem given below and find the minimum cost of transportation.

To From	E	F	G	H	I	ai
A	4	7	3	8	2	4
B	1	4	7	3	8	7
C	7	2	4	7	7	9
D	4	8	2	4	2	2
bi	8	3	7	2	2	

14. (*a*) Show that assignment model is a special case of transportation model.

(*b*) Consider the problem of assigning five operators to five machines. The assignment costs are given below.

NOTES

		Operators				
		I	II	III	IV	V
Machines	A	10	5	13	15	16
	B	3	9	18	3	6
	C	10	7	2	2	2
	D	5	11	7	7	12
	E	7	9	4	4	12

Assign the operators to different machines so that total cost is minimised.

15. Six machines M_1, M_2, M_3, M_4, M_5 and M_6 are to be located in six places P_1, P_2, P_3, P_4, P_5 and P_6. C_{ij} the cost of locating machine M_i at place P_j is given in the matrix below.

	P_1	P_2	P_3	P_4	P_5	P_6
M_1	20	23	18	10	16	20
M_2	50	20	17	16	15	11
M_3	60	30	40	55	8	7
M_4	6	7	10	20	25	9
M_5	18	19	28	17	60	70
M_6	9	10	20	30	40	55

Formulate an LP model to determine an optimal assignment. Write the objective function and the constraints in detail. Define any symbol used. Find an optimal layout by assignment techniques of linear programming.

16. Solve the following assignment problem.

	I	II	III	IV	V
1	11	17	8	16	20
2	9	7	12	6	15
3	13	16	15	12	16
4	21	24	17	28	26
5	14	10	12	11	15

17. Assign four trucks 1, 2, 3 and 4 to vacant spaces 7, 8, 9, 10, 11 and 12 so that the distance travelled is minimised. The matrix below shows the distance.

	1	2	3	4
7	4	7	3	7
8	8	2	5	5
9	4	9	6	9
10	7	5	4	8
11	6	3	5	4
12	6	8	7	3

18. A Company has five jobs to be done. The following matrix shows the return in rupees of assigning ith machine ($i = 1, 2, \ldots\ldots 5$) to the job ($j = 1, 2, \ldots\ldots 5$). Assign the five jobs to the five machines so as to maximise the total expected profit.

		Job				
		1	2	3	4	6
Machine	1	5	11	10	12	4
	2	2	4	6	3	5
	3	3	12	5	14	6
	4	6	14	4	11	7
	5	7	9	8	12	5

FURTHER READINGS

- *Operational Research,* by col. D.S. Cheema, University Science Press.
- *Statistics and Operational Research– A unified Approach,* by Dr. Debashis Dutta, Laxmi Publications (P) Ltd.

NOTES

NOTES

UNIT IV: NLP AND DYNAMIC PROGRAMMING

★ STRUCTURE ★

4.0 LEARNING OBJECTIVES

After going through this unit, you should be able to:

- define NLP and dynamic programming.
- explain convex programming problems, quadratic programming problems, Wolfe's method for quadratic programming problems and Kuhn-Tucker conditions in reference to NLP.
- describe Bellman's principle of optimality of dynamic programming.
- enumerate solution of linear programming problems as a dynamic programming problem.

NOTES

PART I: NON-LINEAR PROGRAMMING

4.1 INTRODUCTION TO NON-LINEAR PROGRAMMING

The general linear programming problem can be stated as

Minimize $f(X)$ subject to constraints

$$g_i(X) \leq 0 \; ; \; i = 1, 2, \ldots\ldots, m$$
$$X \geq 0$$

where $f(X)$, $g_i(X)$ are real valued functions.

If the constraints and the functions to be minimized are linear, then the above problem is linear programming problem, otherwise it is said to be non-linear.

Linear programming has been successfully used in many business and economic problems. However, it suffers from many limitations, the most significant being its linearity assumption. The linear objective function requires the use of constant marginal return, constant returns to scale fixed input and output prices etc. However, we must rely on non-linear programming methods when returns to scale are increasing or decreasing in the objective function or constraints. Non-linearity can also occur when any of the cost or profit coefficients in a linear programming model is the random variable.

We solve linear programming problem with the help of simplex method and this method is based upon the property that the optimal solution lies at one or more extreme points of the feasible region. Thus, we try to find the solution at the corner points only and it is obtained in a finite number of steps. Unfortunately, this is not the case for non-linear programming problems. In such problems, the optimal solution can be located at any point along the boundaries of the feasible region or even within the region.

Since the objective function and the constraints are supposed to be non-linear, it becomes more difficult to distinguish between local and global solution. Moreover, sometimes it becomes more difficult to test the optimality of the non-linear programming problems, especially when the feasible region is not convex. Therefore, the solution of non-linear programming becomes more difficult than the linear programming and there is no single method to solve such problems. A number of algorithms have been developed by various authors to tackle special types of non-linear programming problems.

Below, we discuss some situations where non-linear programming can be successfully applied.

I. The Product Mix Problems

While dealing with product mix problem in linear programming, our objective is to determine the product mix so as to maximize the profits subject to constraints on availability of resources. The objective function is linear and we assume that there is fixed profit associated with each product. But this situation may not always arise. In case of large manufacturers, the price of the product depends upon the quantity demanded. Thus there is a price elasticity, that is, more the volume of sales, lesser is the price per unit. Thus the price-demand curve does not remain to be linear but non-linear. If the quantity demanded x is small, the price $p(x)$ is very high and as x increases, the price drops rapidly and ultimately tends to stabilise.

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If we assume that the unit production and distribution cost of the product is fixed at c then the profit from x units is given by:

$$P(x) = x[p(x) - c] = x\,p(x) - cx$$

which is non-linear function and has been presented in the following figure.

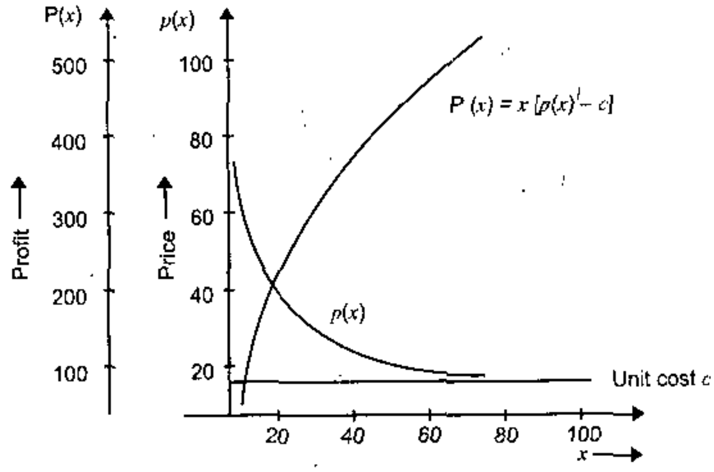

Fig. 4.1

If the manufacturing firm produces x products x_j ($j = 1, 2, \ldots\ldots, n$) with identical profit functions $P_j(x_j)$, then the overall objective function is the sum of n non-linear functions.

$$f(X) = \sum_{j=1}^{n} P_j(x_j)$$

In addition to price elasticity, there can be a number of reasons for the objective function to be non-linear. The unit production cost may decrease with increase in volume of production or it may increase if some special steps are required to be taken to increase the production level. The constraint functions may also be non-linear when the use of resources is not strictly proportional to the production levels of respective products.

II. The Transportation Problem

In the usual transportation problems, whatever quantity we transport, it is assumed that the per unit transportation cost from a given source to the given destination is always fixed. But in actual practice, if we transport more quantity, then discount are often available. Thus we see that as the volume increases, the unit transportation cost decreases as shown in the following Fig. 4.2 (*a*). The resulting total cost c (x) of transporting x units is shown in the following Fig. 42 (*b*).

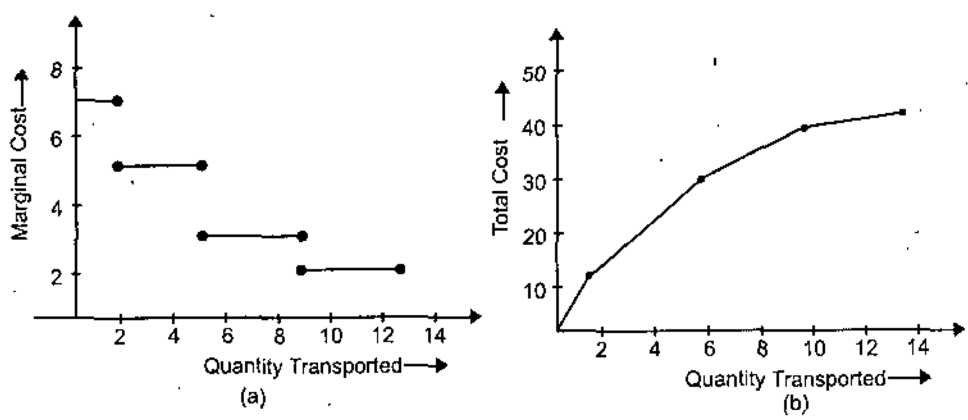

Fig. 4.2

The above curve represents a non-linear function but it is piece wise linear function with slope at a point giving the marginal cost at that point. Thus if each combination of m sources and n definitions has a similar cost function, that is, the cost of transporting x_{ij} units from source i to destination j is given by a non-linear function $c_{ij}(x_{ij})$, then the overall objective function is

$$\text{maximize} \quad f(X) = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}(x_{ij}) \; ; \; i = 1, 2, \ldots\ldots, m \; ; \; j = 1, 2, \ldots\ldots, n$$

4.2 FORMULATION OF NON-LINEAR PROGRAMMING PROBLEM

We explain the formulation of non-linear programming problem with the help of following examples:

Example 4.1. *A manufacturing firm produces two products: Radios and TV's. The sales-price relationship for these two product is given below:*

Product	*Quantity Demanded*	*Unit Price*
Radios	$1500 - 5\,p_1$	p_1
TV's	$3800 - 10p_2$	p_2

The total cost functions for these two products are given by $200\,x_1 + 0.1\,x_1^2$ *and* $300\,x_2 + 0.\,x_2^2$ *respectively. The production takes place on two assembly lines. Radio sets are assembled on assembly line 1 and TV's are assembled on assembly line II. Because of the limitations of the assembly-line capacities, the daily production is limited to no more than 80 radio sets and 60 TV sets. The production of both types of products requires electronic components. The production of each of these sets requires five units and six units of electronic equipment components respectively. The electronic components are supplied by another manufacturer and the supply is limited to 600 units per day. The company has 160 employees, i.e., the labour supply amounts to 160 man-days. The production of one unit of radio set requires 1 man-day of labour, whereas 2 man-days of labour are required for TV set. How many units of radio and TV sets should the firm produce in order to maximize the total profit ? Formulate the problem as a non-linear programming problem.*

Formulation:

Let x and x_2 quantities of radio sets and TV sets be produced by the firm respectively. Then, we are given that

$$x_1 = 1500 - 5\,p_1 \quad \text{or} \quad p_1 = 300 - 0.2\,x_1$$
$$x_2 = 3800 - 10\,p_2 = 380 - 0.1\,x_2$$

If C_1, C_2 stand for the total cost of production of these amounts of radio sets and TV sets respectively, then it is also given that

$$C_1 = 200\,x_1 + 0.1\,x_1^2$$
$$C_2 = 300\,x_2 + 0.1\,x^2$$

Thus, the revenue on radio sets is p_1x_1 and on TV sets is p_2x_2. Therefore, the total revenue R is given by

$$R = p_1x_1 + p_2x_2$$

which can be written as

$$R = (300 - 0.2\,x_1)\,x_1 + (380 - 0.1\,x_2)\,x_2$$
$$= 300\,x_1 - 0.2\,x_1^2 + 380\,x_2 - 0.1\,x_2^2$$

Now, the total profit Z is measured by the difference between the total revenue R and the total cost $(C_1 + C_2)$.

Thus $\quad Z = R - C_1 - C_2$

or $\quad Z = (300\, x_1 - 0.2\, x_1^2 + 380\, x_2 - 0.1\, x_2^2) - (200\, x_1 + 0.1\, x_1^2 + 300\, x_2 + 0.1\, x_2^2)$

$$= 100\, x_1 - 0.3\, x_1^2 + 80\, x_2 - 0.2\, x_2^2$$

Thus, we observe that the objective function obtained above is non-linear.

In this problem the available resources affect the production. Since more than 80 radio sets cannot be assembled on assembly line I and 60 TV sets on assembly line II per day, we have the restrictions :

$$x_1 \le 80 \quad \text{and}$$
$$x_2 \le 60$$

Another constraint of daily requirement of the electronic components is $5x_1 + 6x_2 \le 600$. The number of available employees is restricted to 160 man-days. Therefore, we have one more constraint

$$x_1 + 2x_2 \le 160.$$

Since the production of negative quantities has no meaning, we must have the non-negativity restrictions :

$$x_1 \ge 0 \text{ and } x_2 \ge 0$$

Thus, finally, the complete formulation of the problem becomes :

Maximize $\quad Z = 100\, x_1 - 0.3\, x_1^2 + 80\, x_2 - 0.2\, x_2^2$

subject to the constraints

$$5\, x_1 + 6x_2 \le 600$$
$$x_1 + 2x_2 \le 160$$
$$x_1 \le 80$$
$$x_2 \le 60$$
$$x_1, x_2 \ge 0$$

The problem is a non-linear programming problem because of the non-linearity of the objective function.

Example 4.2. *A partnership firm sells two types of items P and Q. Item P sells for Rs. 25 per unit. No quantity discount is given. The sales revenue for item Q, decreases as the number of its units sold increases and is given by*

Sales revenue $= (30 - 0.3\, x_2)x_2 = 30x_2 - 0.30\, x_2^2$

where x_2 *is the number of units sold of item Q.*

The marketing department has only 1200 hours available for distributing these items in the next year. Further, the firm estimates the sales time function is non-linear and is given by

Sales time $= x_1 + 0.2\, x_1^2 + 3\, x_2 + 0.35\, x_2^2$

The firm can only procure 6000 units of item P and Q for sales in the next year.

Given the above information, how many units of P and Q should the company procure in order to maximize its total revenue ?

Formulation:

The given problem can be put in the following mathematical format:

$$\text{Maximize } Z = 25x_1 + 30x_2 - 0.30x_2^2$$

subject to constraints

$$x_1 + 0.2\, x_1^2 + 3x_2 + 0.35\, x_2^2 \le 1200$$
$$x_1 + x_2 \le 6000$$
$$x_1, x_2 \ge 0$$

This problem is a non-linear programming problem, since the objective function and one of the constraints is non-linear.

4.3 GENERAL NON-LINEAR PROGRAMMING PROBLEM

NOTES

The general non-linear programming problem can be stated mathematically in the following form:

Maximize (or minimize) $Z = C(x_1, x_2, \ldots\ldots\ldots, x_n)$

subject to the constraints :

$$a_1(x_1, x_2, \ldots\ldots, x_n) \{\le = \text{or} \ge\} b_1$$
$$a_2(x_1, x_2, \ldots\ldots, x_n) \{\le = \text{or} \ge\} b_2$$
$$\ldots\ldots\ \ldots\ \ldots\ldots\ldots\ \ldots\ldots\ldots\ \ldots\ldots$$
$$a_m(x_1, x_2, \ldots\ldots, x_n) \{\le = \text{or} \ge\} b_m,$$

and $\quad x_j \ge 0,\ j = 1, 2, \ldots, n$

where either $C(x_1, x_2, \ldots, x_n)$ or some $a_i(x_1, x_2, \ldots, x_n)$, $i = 1, \ldots, m$; or both are non-linear.

In matrix notation, the general non-linear programming problem may be stated as below :

Maximize or minimize $Z = C(x)$

subject to the constraints

$$a_i(x) \{\le = \text{or} \ge\}\ b_i,\ i = 1, 2, \ldots, m$$

and $\quad x \ge 0$

where either $C(x)$ or some $a_i(x)$ or both are non-linear in x.

4.4 CONSTRAINED OPTIMIZATION WITH EQUALITY CONSTRAINTS (LAGRANGE'S METHOD)

In case the non-linear programming problem is composed of some differentiable objective function and equality side conditions or constraints, the optimization may be achieved by the use of **Lagrange's Multipliers** method which provides a necessary condition for an optimum when constraints are equations.

Consider the problem of maximizing or minimizing

$$Z = f(x_1, x_2)$$

subject to the constraints

$$g(x_1, x_2) = C$$

and $\quad x_1, x_2 \ge 0$

where C is a constant.

Let us assume that $f(x_1, x_2)$ and $g(x_1, x_2)$ are differentiable with respect to x_1 and x_2. Introduce differentiable function $h(x_1, x_2)$ differentiable with respect to x_1 and x_2 and defined by $h(x_1, x_2) = g(x_1, x_2) - C$.

Then the problem can be rewritten as

Maximize $\quad Z = f(x_1, x_2)$

subject to the constraints :

$$h(x_1, x_2) = 0$$

and $\quad x_1, x_2 \ge 0$

NOTES

In order to find the necessary condition for a maximum (or minimum) value of Z, a new function is formed by introducing a Lagrange multiplier λ, as

$$L(x_1, x_2, \lambda) = f(x_1, x_2) - \lambda\, h(x_1, x_2).$$

Here the number λ is an unknown constant, and the function $L(x_1, x_2, \lambda)$ is called the Lagrangian function with Lagrange multiplier λ. The necessary conditions for a maximum or minimum value of $f(x_1, x_2)$ subject to $h(x_1, x_2) = 0$ are as given below

$$\frac{\partial L(x_1,x_2,\lambda)}{\partial x_1} = 0,\ \frac{\partial L(x_1,x_2,\lambda)}{\partial x_2} = 0 \text{ and } \frac{\partial L(x_1,x_2,\lambda)}{\partial \lambda} = 0$$

These partial derivatives are given by

$$\frac{\partial L}{\partial x_1} = \frac{\partial f}{\partial x_1} - \lambda \frac{\partial h}{\partial x_1}$$

$$\frac{\partial L}{\partial x_2} = \frac{\partial f}{\partial x_2} - \lambda \frac{\partial h}{\partial x_2}$$

and $$\frac{\partial L}{\partial \lambda} = -h$$

where L, f and h denote the functions $L(x_1, x_2, \lambda)$, $f(x_1, x_2)$ and $h(x_1, x_2)$ respectively, or in other words

$$L_1 = f_1 - \lambda h_1, L_2 = f_2 - \lambda h_2 \text{ and } L_1 = -h$$

The necessary conditions for maximum or minimum of $f(x_1, x_2)$ are therefore given by

$$f_1 = \lambda h_1, f_2 = \lambda h_2 \quad \text{and} - h\ (x_1, x_2) = 0$$

These necessary conditions become sufficient conditions for a maximum/minimum if the objective functions is concave/convex and the side constraints are in the form of equalities.

Example 4.3. *Obtain the necessary and sufficient conditions for the optimum solution of the following non-linear programming problem :*

Minimize $$Z = f(x_1, x_2) = 3e^{2x_1+1} + 2e^{x_2+5}$$

subject to the constraints :

$$x_1 + x_2 = 7$$

and $$x_1, x_2 \geq 0$$

Solution. Let us have a new differentiable Lagrangian function $L(x_1, x_2, \lambda)$ defined by

$$L(x_1, x_2, \lambda) = f(x_1, x_2) - \lambda(x_1 + x_2 - 7)$$

$$= 3e^{2x_1+1} + 2e^{x_2+5} - \lambda(x_1 + x_2 - 7)$$

where λ is the Lagrangian multiplier

Since the objective function $Z = f(x_1, x_2)$ is convex and the side constraint is an equality, the necessary and sufficient conditions for the minimum of $f(x_1, x_2)$ are given by

$$\frac{\partial L}{\partial x_1} = 6e^{x_2+1} - \lambda = 0 \quad \text{or} \quad 1 = 6e^{x_2+1}$$

$$\frac{\partial L}{\partial x_2} = 2e^{x_2+5} - \lambda = 0 \quad \text{or} \quad 1 = 2e^{x_2+5}$$

$$\frac{\partial L}{\partial \lambda} = -(x_1 + x_2 - 7) = 0 \quad \text{or} \quad x_1 + x_2 = 7$$

From these, we have

$$6e^{2x_1+1} = 2e^{x_2+5} = 3e^{7-x_2+5}$$

or $\log 3 + 2x_1 + 1 = 7 - x_1 + 5$

or $$x_1 = \frac{1}{3}(11 - \log 3)$$

or $$x_2 = 7 - \frac{1}{3}(11 - \log 3).$$

Example 4.4. *Obtain the set of necessary conditions for the non-linear programming problem :*

NOTES

Maximize $Z = x_1^2 + 3x_2^2 + 5x_3^2$

subject to the constraints

$$x_1 + x_2 + 3x_3 = 2$$
$$5x_1 + 2x_2 + x_3 = 5$$

and $x_1, x_2, x_3 \geq 0$

Solution. We are given

$$x = (x_1, x_2, x_3) f(x) = x_1^2 + 3x_2^2 + 5x_3^2$$
$$g_1(x) = x_1 + x_2 + 3x_3 - 2 = 0$$
$$g_2(x) = 5x_1 + 2x_2 + x_3 - 5 = 0$$

Now we construct the Lagrangian function for maximizing $f(x)$

$$L(x, \lambda) = f(x) - \lambda_1 g_1(x) - \lambda_1 g_2(x)$$

Thus we get the following necessary conditions

$$\frac{\partial L}{\partial x_1} = 2x_1 - \lambda_1 - 5\lambda_2 = 0$$

$$\frac{\partial L}{\partial x_2} = 6x_1 - \lambda_1 - 2\lambda_2 = 0$$

$$\frac{\partial L}{\partial x_3} = 10x_3 - 3\lambda_1 - \lambda_3 = 0$$

$$\frac{\partial L}{\partial \lambda_1} = -(x_1 + x_2 + 3x_3 - 2) = 0$$

$$\frac{\partial L}{\partial \lambda_2} = -(5x_1 + 2x_2 + x_3 - 5) = 0.$$

4.5 NECESSARY CONDITIONS FOR A GENERAL NON-LINEAR PROGRAMMING PROBLEM

Let the general NLPP be:

Maximize (or Minimize) $Z = f(x_1, x_2, \ldots, x_n)$

subject to the constraints

$$g^i(x_1, \ldots\ldots, x_n)$$

and $x_i \geq 0$; $i = 1, 2, \ldots, m\ (<n)$

The constraints can be reduced to

$$h^i(x_1, \ldots, x_n) = 0 \text{ for } i = 1, 2, \ldots\ldots, m,$$

by the transformation $h^i(x_1, \ldots\ldots, x_n) = g^i(x_1, \ldots, x_n) - C_i$ for all $i = 1, 2, \ldots, m\ (<n)$

Now above problem can be written in the matrix form as

Maximize (or minimize) $Z = f(x), x \in R^n$

subject to constraints

$$h^i(x) = 0,\ x \geq 0$$

To find the necessary conditions for a maximum (or minimum) of $f(x)$, the Lagrangian function L (x, λ) is formed by introducing m Lagrangian multipliers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$

Above function is defined by

$$L(x, \lambda) = f(x) = \sum_{i=1}^{m} \lambda_i h^i(x)$$

NOTES

Supposing that L, f and h^i are all differentiable partially with respect to $x_1, x_2, \ldots, x_n$ nd $\lambda_1, \lambda_2, \ldots, \lambda_m$, the necessary conditions for maximum (minimum) of $f(x)$ are

$$\frac{\partial L}{\partial x_j} = \frac{\partial f}{\partial x_j} - \sum_{i=1}^{m} \lambda_i \frac{\partial h^i(x)}{\partial x_j} = 0,\ j = 1, 2, \ldots, n$$

$$\frac{\partial L}{\partial \lambda_i} = 1 - h^i(x) = 0,\ i = 1, 2, \ldots, m.$$

The following abbreviated form represent above $m + n$ necessary conditions:

$$L_j = f_j - \sum_{i=1}^{m} \lambda_i h^i_j = 0$$

or $$f_j = \sum_{i=1}^{m} \lambda_i h^i_j \qquad j = 1, 2, \ldots\ldots, n$$

and $$L\lambda_j = -h^i = 0$$

or $$h^i = 0 \qquad i = 1, 2, \ldots\ldots, m$$

where $$f_j = \frac{\partial f(x)}{\partial x_j},\ h^i = h^i(x)$$

and $$h^i j = \frac{\partial h^i(x)}{\partial x_j}$$

4.6 SUFFICIENT CONDITIONS FOR A GENERAL NLPP (WITH ONE CONSTRAINT)

In case the concavity/convexity of the objective function is unknown, the method of Lagrange multipliers can be generalized to obtain a set of sufficient conditions for a maximum/minimum of the objective function.

Let the Lagrangian function for a general NLPP involving n variables and single constraint be:

$$L(x, \lambda) = f(x) - \lambda h(x)$$

The necessary conditions for a stationary point to be maximum or minimum are:

$$\frac{\partial L}{\partial x_i} = \frac{\partial f}{\partial x_j} - \lambda \frac{\partial h}{\partial x_j} = 0 \qquad j = 1, 2, \ldots, n$$

and $$\frac{\partial L}{\partial \lambda} = -h(x) = 0.$$

The value of λ is obtained by

$$\lambda = \frac{\dfrac{\partial f}{\partial x_j}}{\dfrac{\partial h}{\partial x_j}} \quad j = 1, 2, \ldots, n.$$

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The sufficient conditions for a maximum or minimum need the computation of $(n-1)$ principal minors of the determinant for each stationary point, as given below:

$$\Delta_{n+1} = \begin{vmatrix} 0 & \frac{\partial h}{\partial x_1} & \frac{\partial h}{\partial x_2} & \cdots & \frac{\partial h}{\partial x_n} \\ \frac{\partial h}{\partial x_1} & \frac{\partial^2 f}{\partial x_1^2} - \lambda\frac{\partial^2 h}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} - \lambda\frac{\partial^2 h}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} - \lambda\frac{\partial^2 h}{\partial x_1 \partial x_n} \\ \frac{\partial h}{\partial x_2} & \frac{\partial^2 f}{\partial x_2 \partial x_1} - \lambda\frac{\partial^2 h}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} - \lambda\frac{\partial^2 h}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} - \lambda\frac{\partial^2 h}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \frac{\partial h}{\partial x_n} & \frac{\partial^2 f}{\partial x_n \partial x_1} - \lambda\frac{\partial^2 h}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} - \lambda\frac{\partial^2 h}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 h}{\partial x_n^2} - \lambda\frac{\partial^2 h}{\partial x_n^2} \end{vmatrix}.$$

If $\Delta_3 > 0, \Delta_4 < 0, \Delta_5 > 0, \ldots$, the signs are alternatively positive and negative, the stationary point is a local maximum. If $\Delta_3 < 0$, $\Delta_4 < 0$, $\Delta_5 < 0$, ..., $\Delta_{n+1} < 0$ the sign being always negative, the stationary point is a local minimum.

Example 4.5. *Solve the non-linear programming problem:*

Minimize $Z = 2x_1^2 - 24x_1 + 2x_2^2 - 8x_2 + 2x_3^2 - 12x_3 + 200$

subject to the constraints

$$x_1 + x_2 + x_3 = 11$$

and $\quad x_1, x_2\, x_3 \geq 0$

Solution. The Lagrangian function can be formulated as below:

$$L(x_1, x_2, \lambda) = 2x_1^2 - 24x_1 + 2x_2^2 - 8x_2 + 2x_3^2 - 12x_3 + 200 - \lambda(x_1 + x_2 + x_3 - 11)$$

The necessary conditions for the stationary point are :

$$\frac{\partial L}{\partial x_1} = 4x_1 - 24 - \lambda = 0,\ \frac{\partial L}{\partial x_2} = 4x_2 - 8 - \lambda = 0,$$

$$\frac{\partial L}{\partial x_3} = 4x_3 - 12 - \lambda = 0,\ \frac{\partial L}{\partial \lambda} = -(x_1 + x_2 + x_3 - 11) = 0$$

By solving these simultaneous equations, stationery point is obtained as

$$x_0 = (x_1, x_2, x_3) = (6, 2, 3)\ ;\ \lambda = 0$$

The sufficient condition for the stationary point to be a minimum is that the minors Δ_3 and Δ_4 must both be negative. To verify this, we have

$$\Delta_3 = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 4 & 0 \\ 1 & 0 & 4 \end{vmatrix} = -8, \text{ and } \Delta_4 = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 4 & 0 & 0 \\ 1 & 0 & 4 & 0 \\ 1 & 0 & 0 & 4 \end{vmatrix} = -48$$

which are both negative. Thus, $x_0 = (6, 2, 3)$ is the solution to the given NLPP.

4.7 SUFFICIENT CONDITIONS FOR A GENERAL NLP PROBLEM WITH m ($< n$) CONSTRAINTS

First introducing the m Lagrange multipliers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$, let the Lagrangian function for a general NLPP having more than one constraint be

$$L(x, \lambda) = f(x) - \sum_{i=1}^{m} \lambda_i h^i(x) \quad (m < n)$$

It may be verified that the equations

$$\frac{\partial L}{\partial x_j} = 0 \qquad (j = 1, 2, ..., n)$$

NOTES

$$\frac{\partial L}{\partial \lambda_i} = 0 \qquad (i = 1, 2, ..., m)$$

provide the necessary conditions for stationary point of $f(x)$. So the optimization of $f(x)$ subject to $h(x) = 0$ is equivalent to the optimization of $L(x, \lambda)$. The Lagrange multiplier method for a stationary point of $f(x)$ to be a maxima or minima is stated here without proof. For this we assume that the function $L(x, \lambda)$, $f(x)$ and $h(x)$ all possess partial derivatives of first and second order with respect to the decision variables.

Let $$V = \left[\frac{\partial^2 L(x, \lambda)}{\partial x_i \partial x_j}\right]_{n \times n} \quad \text{for all } i \text{ and } j$$

be the matrix of second order partial derivatives of $L(X, \lambda)$ with respect to decision variables

$$U = [h_j^i(x)]_{m \times n}$$

where $$h_j^i(X) = \frac{\partial h^i(x)}{\partial x_j}, i = 1, 2, ..., m; j = 1, 2, ..., n.$$

Now define the square matrix

$$H_B = \begin{bmatrix} O & | & U \\ \cdots & & \cdots \\ U^T & | & V \end{bmatrix}_{(m+n) \times (m+n)}$$

where O is an $m \times m$ null matrix. The matrix H_B is called the bordered Hessian matrix. Then, the sufficient conditions for maximum and minimum stationary points can be stated as below.

Let (x_0, λ_0) be the stationary point for the Lagrangian function $L(x, \lambda)$ and H_B^0 be the value of corresponding bordered Hessian matrix computed at this stationary point. Then,

(*i*) x_0 is a maximum point, if starting with principal minor of order $(2m + 1)$, the last $(n - m)$ principal minors of H_B^0 from an alternating sign pattern starting with $(-1)^{m+n}$; and

(*ii*) x_0 is a minimum point, if starting with principal minor of order $(2m + 1)$, the last $(n - m)$ principal minors of H_B^0 have the sign of $(-1)^m$.

Note : *It may be found that the above conditions are only sufficient for identifying an extreme point, but not necessary. In other words, a stationary point may be an extreme point without satisfying the above conditions.*

Example 4.6. *Solve the non-linear programming problem*

Optimize $Z = 4x_1^2 + 2x_2^2 + x_3^2 - x_1 x_2$

NOTES

subject to constraints

$$x_1 + x_2 + x_3 = 15$$
$$2x_1 - x_2 + 2x_3 = 20$$

and $x_1, x_2, x_3 \geq 0$

Solution. We are given that

$$f(x) = 4x_1^2 + 2x_2^2 + x_3^2 - x_1x_2$$
$$h^1(x) = x_1 + x_2 + x_3 - 15$$
$$h^2(x) = 2x_1 - x_2 + 2x_3 - 20$$

The Lagrangian function is given by

$$L(x, \lambda) = f(x) - \lambda_1 h^1(x) - \lambda_2 h^2(x)$$
$$= \left(4x_1^2 + 2x_2^2 + x_3^2 - x_1x_2\right) - \lambda^1 (x_1 + x_2 + x_3 - 15) - \lambda_2 (2x_1 - x_2 + 2x_3 - 20)$$

The stationary point (x_0, l_0) can be obtained by the following necessary conditions :

$$\frac{\partial L}{\partial x_1} = 8x_1 - 4x_2 - \lambda_1 - 2\lambda_2 = 0 \quad \frac{\partial L}{\partial x_2} = 4x_2 - 4x_1 - \lambda_1 + \lambda_2 = 0$$
$$\frac{\partial L}{\partial x_3} = 2x_3 - \lambda_1 - 2\lambda_2 = 0 \quad \frac{\partial L}{\partial \lambda_1} = -(x_1 + x_2 + x_3 - 15 = 0$$
$$\frac{\partial L}{\partial \lambda_2} = -(2x_1 - x_2 + 2x_3 - 20) = 0$$

Solving these simultaneous equations, we get

$$x_0 = (x_1, x_2, x_3) = (33/9, 10/3, 8) \text{ and } \lambda_0 = (\lambda_1, \lambda_2) = (40/9, 52/9)$$

For this stationary point (x_0, λ_0), the bordered **Hessian matrix** is given by

$$H_B^0 = \left[\begin{array}{cc|ccc} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & -1 & 2 \\ \hline 1 & 2 & 8 & -4 & 0 \\ 1 & -1 & -4 & 4 & 0 \\ 1 & 2 & 0 & 0 & 2 \end{array}\right]$$

Since $m = 2$ and $n = 3$ here, so $n - m = 1$, $2m + 1 = 5$. This means that we need only to check the determinant of H_B^0 and *it must have the positive sign [i.e. the sign of* $(-1)^2$]

Now, since $\left|H_B^0\right| = 72$ which is positive, x_0 is a minimum point.

4.8 CONSTRAINED OPTIMIZATION WITH INEQUALITY CONSTRAINTS

In this section we shall derive the necessary and sufficient conditions for identifying the stationary points of the general inequality constrained optimization problem. These conditions known as the **Kuhn-Tucker conditions,** named after the men who developed them. These conditions are sufficient under certain limitations which are discussed below.

Consider the general NLPP

Optimize $Z = f(x_1, x_2, ..., x_n)$

subject to constraints

$$g(x_1, ..., x_n) \leq C$$

and $\quad x_1, x_2, ..., x_n \geq 0$

where C is a constant.

Introduce the function $h(x_1,..., x_n) = g - C$, the constraint reduces to $h(x_1,..., x_n) \leq 0$.

Now, the problem can be stated as

Optimize $\quad Z = f(x)$

subject to $\quad h(x) \leq 0$

and $\quad x \geq 0$, where $x \in R^n$

Now, we slightly modify the problem by introducing new variable S,

where $\quad S^2 = -h(x)$, or $h(x) + S^2 = 0$

Here, S is called a slack variable and appears as its square in the constraint equation so as to ensure its being non-negative. This avoids an additional constraint $S \geq 0$.

Now the problem can be restated as

Optimize $\quad Z = f(x),\ x \in R^n$

subject to the constraints

$$h(x) + S^2 = 0$$

and $\quad x \geq 0$

This is a problem of constrained optimization in $n + 1$ variables and a single equality constraint and thus can be solved by the Lagrangian multiplier method.

In order to determine the stationary points, consider the Lagrangian function defined by

$$L(x, S, \lambda) = f(x) - \lambda[h(x) + S^2].$$

where λ is the Lagrange multiplier. The necessary conditions for stationary points are

$$\frac{\partial L}{\partial x_j} = \frac{\partial L}{\partial x_j} - 1\frac{\partial h}{\partial x_j} = 0 \text{ for } j = 1, 2, ..., n \qquad ...(i)$$

$$\frac{\partial L}{\partial \lambda} = -[h(x) + S^2] = 0 \qquad ...(ii)$$

$$\frac{\partial L}{\partial S} = -2S\lambda = 0 \qquad ...(iii)$$

Equation (*iii*) states that $\frac{\partial L}{\partial S} = 0$, which needs either $l = 0$ or $S = 0$. If $S = 0$, (*ii*) implies that $h(x) = 0$. So (*ii*) and (*iii*) together imply $lh(x) = 0$.

Now S variable may be discarded as it was introduced just to convert the inequality constraint into equality. Moreover, as $S^2 \geq 0$, (*ii*) provides $h(x) \leq 0$. Whenever $h(x) < 0$, we get $l = 0$ and whenever $l > 0$, $h(x) = 0$. But, l is unrestricted in sign whenever $h(x) = 0$.

Thus the necessary conditions for the point x to be a point of maximum is restated below in the abbreviated form

$$f_j - \lambda h_j = 0 \qquad (j = 1, 2,..., n)$$

$$\lambda h = 0$$

$$h \leq 0$$

$$\lambda \geq 0$$

The set of these necessary conditions is called **Kuhn-Tucker conditions.** A similar reason holds for the minimization NLPP.

Minimize $\quad Z = f(x)$

subject to the constraints

$$g(x) \geq C$$

and $\quad x \geq 0$

Introduction of $h(x) = g(x) - C$ reduces the first constraint to $h(x) \geq 0$. The fresh surplus variable S_0 can be introduced in $h(x) \geq 0$ so that we get equality constraint $h(x) - S_0^2 = 0$. The relevant Lagrangian function is

$$L(x, S_0, \lambda) = f(x) - \lambda\,[h(x) - S_0^2].$$

So, we get the following set of Khun-Tucker conditions :

NOTES

$$f_j - \lambda h_j = 0 \qquad (j = 1, 2, ..., n)$$

$$\lambda h = 0$$

$$h \geq 0$$

$$\lambda \geq 0$$

Example 4.7. *Determine x_1, x_2 and x_3 so as to*

Maximize $\quad Z = -x_1^2 - x_2^2 - x_3^2 + 4x_1 + 6x_2$

subject to the constraints

$$x_1 + x_2 \leq 2$$

$$2x_1 + 3x_2 \leq 12$$

and $\quad x_1, x_2 \geq 0$

Solution. Here $f(x) = -x_1^2 - x_2^2 - x_3^2 + 4x_1 + 6x_2 \quad ; \quad x \in R^n$

$$h^1(x) = x_1 + x_2 - 2$$

$$h^2(x) = 2x_1 + 3x_2 - 12$$

First we decide about the concavity-convexity of $f(x)$. For this we compute the bordered Hessian matrix

$$H_B = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}, n = 3, m = 2, n - m = 1$$

$\therefore \quad f_0\, |H_B| = -8 < 0$

The objective function $f(x)$ is concave if the principal minors of matrix H_B alternate in sign, starting with the negative sign. If the principal minors are positive, the objective function is convex. So in this case $f(x)$ is concave.

Clearly $h^1(x)$, $h^2(x)$ are convex in x. Thus the Kuhn-Tucker conditions will be the necessary and sufficient conditions for a maximum. These conditions are obtained by partial derivatives of Lagrangian function.

$$L(x, S, l) = f(x) - \lambda_1\,[h^1(x) + S_1^2] - \lambda_2\,[h_2(x) + S_2^2]$$

where $S = (S_1, S_2)$, $\lambda = (\lambda_1, \lambda_2)$, S_1, S_2 being slack variables and λ_1, λ_2 are Lagrangian multipliers.

The Kuhn-Tucker conditions are given by

1. (*i*) $-2x_1 + 4 = \lambda_1 + 2x_2$
   (*ii*) $-2x_2 + 6 = \lambda_1 + 3x_2$
   (*iii*) $-2x_3 = 0$
2. (*i*) $\lambda_1(x_1 + x_2 - 2) = 0$
   (*ii*) $\lambda_2(2x_1 + 3x_2 - 12) = 0$
3. (*i*) $x_1 + x_2 - 2 \leq 0$
   (*ii*) $2x_1 + 3x_2 - 12 \leq 0$
4. $\lambda_1 \geq 0,\ \lambda_2 \geq 0.$

Now, four different cases may arise :

Case 1. $\lambda_1 = 0$ and $\lambda_2 = 0$. In this case (*i*), (*ii*) and (*iii*) yield $x_1 = 2, x_2 = 3, x_3 = 0$. However, this solution violates (3) [(*i*) and (*ii*) both], and it must therefore be discarded.

Case 2. $\lambda_1 = 0$, and $\lambda_2 \neq 0$. In this case (2) yield $2x_1 + 3x_2 = 2$ and (1) (*i*) and (*ii*) yield $-2x_1 + 4 = 2\lambda_2$, $-2x_2 + 6 = 3\lambda_2$. The solution of these simultaneous equations gives $x_1 = 2/13, x_2 = 3/13, \lambda_2 = 24/13 > 0$; also (1) (*iii*) gives $x_3 = 0$. However, this solution violates (3) (*i*). This solution is also discarded.

Case 3. $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$. In this case (2) (*i*) and (*ii*) give $x_1 + x_2 = 2$ and $2x_1 + 3x_2 = 12$. These equations give $x_1 = -6$ and $x_2 = 8$. Thus (1) (*i*), (*ii*) and (*iii*) yield $x_3 = 0, \lambda_1 = 68, \lambda_2 = -26$. Since $\lambda_2 = -26$ violates the condition (4), so this solution is also discarded.

Case 4. $\lambda_1 \neq 0, \lambda_2 \neq 0$. In this case (2) (*i*) gives $x_1 + x_2 = 2$. This together with (1) (*i*) and (*ii*) gives $x_1 = 1/2$ and $x_2 = 3/2$, $\lambda_1 = 3 > 0$. Further from (1) (*iii*) $x_3 = 0$. This solution does not violate any of the **Kuhn-Tucker conditions.**

Hence the optimum (maximum) solution to the given NLPP is

$$x_1 = 1/2, x_2 = 3/2, x_3 = 0 \text{ with } \lambda_1 = 3, \lambda_2 = 0$$

the maximum value of the objective function is $Z_0 = 17/2$.

4.9 GRAPHICAL SOLUTION OF NLPP

In a linear programming problem, the optimal solution lies at one of the extreme points of the convex feasible region. But, it is not necessary to find the solution at a corner or edge of the feasible region of a non-linear programming problem. In other words, solution may be found anywhere on the boundary of the feasible region or even at some interior point of it. The following sample problems will make the method clear.

Example 4.8. *Solve graphically the following NLP problem*

Maximize $Z = 2x_1 + 3x_2$

subject to constraints

$$x_1^2 + x_2^2 \leq 20$$

$$x_1 . x_2 \leq 8$$

and $\quad x_1, x_2 \geq 0$

Verify that Khun-Tucker conditions hold for the maxima you obtain.

Solution. In this non-linear programming problem, the objective function is linear whereas constraints are non-linear. Plot the given constraints on the graph by the usual method as shown in Fig 4.3.

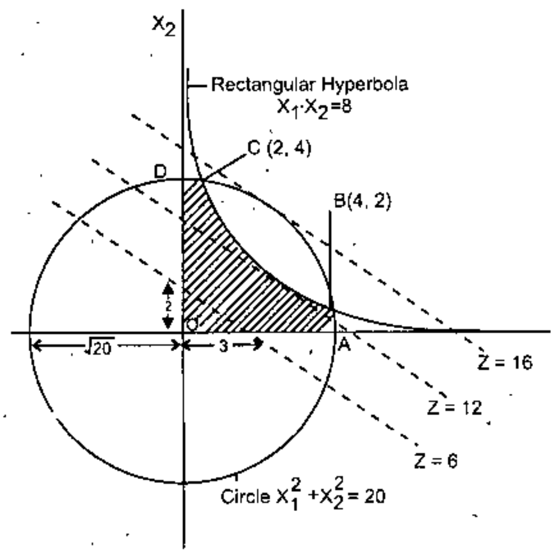


Fig. 4.3 Graphical Solution

Let OX_1 and OX_2 be the set of rectangular cartesian coordinate axes in the plane. Obviously, the feasible region will lie in the first quadrant only, because $x_1 \geq 0, x_2 \geq 0$.

The constraint $x_1^2 + x_2^2 = 20$ represents a circle of radius $\sqrt{(20)}$ with its centre at the origin *i.e.* (0, 0), and $x_1 . x_2 = 8$ represents a rectangular hyperbola whose asymptotes are represented by the X-axis and Y-axis.

Solving the two equations :

$x_1^2 + x_2^2 = 20$ and $x_1 . x_2 = 8$, we get $(x_1, x_2) = (4, 2)$ and $(x_1, x_2) = (2, 4)$ at points B and C respectively. These solution points which also satisfy both the constraints may be obtained within the shaded non-convex region OABCD. So, it may also be called the feasible region.

Now we are in search of such a point (x_1, x_2) within the region OABCD which maximize the value of the given objective function $Z = 2x_1 + 3x_2$ and lies in the convex part of the region. Such a point can be located by the iso-profit function approach. That is, draw parallel objective function $2x_1 + 3x_2 = c$ lines for different constant values of c_j and stop the process when a line touches the extreme boundary point of the feasible region for some value of c. Starting with $c = 6$ and so on we find that the iso-profit line with $c = 16$ touches the extreme boundary point C (2, 4) where the value of Z is maximum. Hence the graphical solution of the problem is finally obtained as $x_1 = 2$, $x_2 = 4$, max. Z = 16.

NOTES

4.10 VERIFICATION OF KUHN-TUCKER CONDITIONS

We can also verify that the above optimum solution satisfies the Kuhn-Tucker conditions

Here we are given that

$$f(x) = x_1 + 3x_2$$
$$h^1(x) = x_1 . x_2 - 8$$
$$h^2(x) = x_1^2 + x_2^2 = -20$$

and the problem is that of maximizing $f(x)$ subject to the constraints $h^1(x) \leq 0$, $h^2(x) \leq 0$ and $x \geq 0$

The Kuhn-Tucker conditions for this maximizing NLPP are

$$\frac{\partial f(x)}{\partial x_j} = \lambda_1 \frac{\partial h^1(x)}{\partial x_j} + \lambda_2 \frac{\partial h^2(x)}{\partial x_j}, \text{ for } j = 1, 2$$
$$\lambda_i h^i(x) = 0 \quad i = 1, 2$$
$$h_i(x) \leq 0 \quad i = 1, 2$$
$$h_i \geq 0 \quad i = 1, 2$$

where λ_1 and λ_2 are Lagrangian multipliers.

These conditions are thus written as

(a) $\begin{cases} 2 = \lambda_1 x_2 + 2\lambda_2 x_1 \\ 3 = \lambda_1 x_1 + 2\lambda_2 x_2 \end{cases}$

(b) $\begin{cases} \lambda_1 [x_1 . x_2 - 8] = 0 \\ \lambda_2 [x_1^2 + x_2^2 - 20] = 0 \end{cases}$

(c) $\begin{cases} x_1 \cdot x_2 - 8 \leq 0 \\ x_1^2 + x_2^2 - 20 \leq 0 \end{cases}$

(d) $\lambda_1 \geq 0, \lambda_2 \geq 0$

If the point (2, 4) satisfies these conditions, then we must have from (a) $\lambda_1 = 1/6$ and $x_1, x_2 = 1/3$. From $(x_1, x_2) = (2, 4)$ and $(\lambda_1, \lambda_2) = (1/6, 1/3)$, it is clear that the conditions (b), (c) and (d) are satisfied. Hence the optimum solution obtained by graphical method satisfies the Kuhn-Tucker conditions for a maxima.

NOTES

Example 4.9. Solve graphically the following NLPP

Maximize $Z = 8x_1 - x_1^2 + 8x_2 - x_2^2$

subject to constraints

$$x_1 + x_2 \le 12$$

$$x_1 - x_2 \ge 4$$

and $x_1, x_2 \ge 0$.

Solution. Let OX_1 and OX_2 be the set of rectangular cartesian coordinate axes in the plane of the paper. Because of the non-negativity restrictions $x_1 \ge 0$, $x_2 \ge 0$, the feasible region will lie in the first quadrant only.

The feasible region is shown by the shaded region in Fig. 4.4. Thus, the optimal point (x_1, x_2) must be somewhere in the convex region PQR. However, the desired point will be that at which a side of the convex region is tangent to the circle, $Z = 8x_1 - x_1^2 + 8x_2 - x_2^2$.

The gradient of the tangent to this circle can be obtained by differentiating the equation $Z = 8x_1 - x_1^2 + 8x_2 - x_2^2$ with respect to x_1,

that is $$8 - 2x_1 + 8\frac{dx_2}{dx_1} - 2x_2\frac{dx_2}{dx_1} = 0$$

or $$\frac{dx_2}{dx_1} = \frac{2x_1 - 8}{8 - 2x_2} \quad ...(1)$$

The gradient of the line $x_1 + x_2 = 12$ and $x_1 - x_2 = 4$ is

$$\left.\begin{aligned} dx_1 + dx_2 = 0 \text{ or } \frac{dx_2}{dx_1} = -1 \\ dx_1 - dx_2 = 0 \text{ or } \frac{dx_2}{dx_1} = 1 \end{aligned}\right] \text{...(2) respectively.}$$

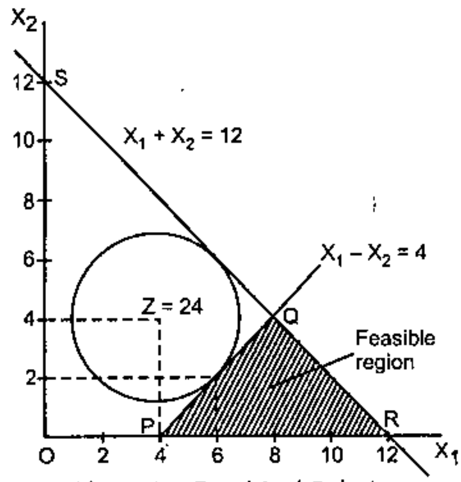


Fig. 4.4 Graphical Solution

If the line $x_1 + x_2 = 12$ is the tangent to the circle, then substituting $\frac{dx_2}{dx_1} = -1$ from equation (2) in equation (1), we have

$$\frac{2x_1 - 8}{8 - 2x_2} = -1 \quad i.e.\ x_1 = x_2$$

and hence for $x_1 = x_2$, the equation $x_1 + x_2 = 12$ yields $(x_1, x_2) = (6, 6)$. This means the tangent of the line $x_1 + x_2 = 12$ is at (6, 6). But it does not satisfy the given constraints.

Similarly, if the line $x_1 - x_2 = 4$ is the tangent to the circle, then substituting $\frac{dx_2}{dx_1} = 1$ from equation (2) in equation (1), we have

$$\frac{2x_1 - 8}{8 - 2x_2} = 1 \quad i.e.\ x_1 + x_2 = 8$$

and hence for $x_1 + x_2 = 8$, the equation $x_1 - x_2 = 4$ yields $(x_1, x_2) = (6, 2)$. This means the tangent of the circle to the line $x_1 - x_2 = 4$ is at (6, 2). This point lies in the feasible region and satisfies both the constraints. Thus, the optimal solution is : Max $Z = 24$, $x_1 = 6$, $x_2 = 2$

(b) If $x^T Qx$ is positive-definite (or negative definite), then it is strictly convex (or strictly concave) in X over all of R^n.

These results help in determining whether the quadratic objective function $f(x)$ is concave (convex) and the implication of the same on the sufficiency of the Kuhn-Tucker conditions for constrained maxima (minima) of $f(x)$.

Kuhn-Tucker Conditions for Quadratic Programming Problem

NOTES

The necessary and sufficient Kuhn-Tucker conditions to get an optimal solution to the problem of maximizing the given quadratic objective function subject to linear constraints can be derived as below:

Let us consider a QPP in the form:

$$\text{Maximize } Z = f(x) = \sum_{j=1}^{n} C_j x_j = + \frac{1}{2}\sum_{j=1}^{n} \sum_{k=1}^{n} C_j k x_j x_k$$

$$\text{subject to the constraints } \sum_{j=1}^{n} a_{ij} x_j \le b_i \qquad (i = 1, 2, \ldots\ldots m)$$

$$x_j \ge 0 \qquad (j = 1, 2, \ldots\ldots, n)$$

where $C_{jk} = C_{kj}$ for all j and k and where $b_i \ge 0$

Step 1. Introduce slack variables a_i^2 and r_j^2, the problem becomes,

$$\text{Maximize } Z = f(x) = \sum_{j=1}^{n} C_j x_j + 1/2 \sum_{j=1}^{n} \sum_{k=1}^{n} C_{jk} x_j x_k$$

subject to the constraints

$$\sum_{j=1}^{n} a_{ij} x_j + q_i^2 = b_i \qquad i = 1, 2 \ldots\ldots m$$

$$-x_j + r_j^2 = 0 \qquad j = 1, 2, \ldots\ldots, n$$

Step 2. Forming the Lagrange function as follows :

$$L(x, q, \mu, \lambda, r) = f(x) - \left[\sum_{i=1}^{m} \lambda i, \sum_{j=1}^{n} (a_{ij} x_j + q_i^2 - b_i)\right] - \sum_{j=1}^{n} \mu_i (-x_j + r_j^2)$$

Forming the necessary conditions, we have

$$\frac{\partial L(\cdot)}{\partial xj} = \frac{\partial f}{dxj} - \sum_{i=1}^{m} \lambda_i a_{ij} + \mu_j = 0, \quad j = 1, 2, \ldots n, \qquad \ldots(1)$$

$$\sum_{j=1}^{n} a_{ij} x_j + q_i^2 - b_i = 0$$

$$m_j x_j = 0$$

$$Ax \le b$$

and finally, x, λ, and μ must all be non-negative.

Equation (1) can be rewritten as

$$\frac{\partial L}{\partial x_j} = \left[C_j + \frac{1}{2}\left(2\sum_{k=1}^{n} C_{jk} x_k\right)\right] - \sum_{i=1}^{m} \lambda_i a_{ij} + \mu_j = 0, \; j = 1, 2 \ldots, n$$

NOTES

Let $\sum_i^2 = S_i \geq 0$ the above equation becomes

$$\mu_j + C_j + \sum_{k=1}^{n} C_{jk}\, x_k - \sum_{i=1}^{m} \lambda_i a_{ij} = 0 \qquad j = 1, 2, \ldots, n$$

$$Ax + Is = b$$

$$x \geq 0, s \geq 0, \lambda \geq 0, \mu \geq 0$$

and lastly

$$\lambda_i s_i = 0, \qquad i = 1, 2, \ldots, m$$

$$x_j = 0, \qquad j = 1, 2, \ldots, n$$

It should be noted that except for the final conditions $\lambda_i s_i = 0 = m_j x_j$, the remaining equations are linear functions in X, λ, μ and S. Thus, the problem becomes equivalent to determing the solution to a set of linear equations which also satisfies the additional conditions $\lambda_i s_i = 0 = \mu_j x_j$. As $f(x)$ is strictly concave and the solution space is convex, the feasible solution which satisfies all these conditions must give the optimum solution directly.

Wolfe suggested a solution for this problem which is summarized below.

Wolfe's Modified Simplex Method

The iterative procedure for the solution of a quadratic programming problem by Wolfe's modified simplex method can be summarized in the following steps:

Let the quadratic programming problem be

$$\text{Maximize } Z = f(x) = \sum_{j=1}^{n} C_j x_j + 1/2 \sum_{j=1}^{n} \sum_{k=1}^{n} C_{jk} x_j x_k$$

Subject to the constraints

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i x_k {}^3 0 \quad (i = 1, 2, \ldots, m; j = 1, 2, \ldots n)$$

where $C_{jk} = C_{kj}$ for all j and k, $b_i \geq 0$, for all $i = 1, 2, \ldots, m$

Also suppose that the quadratic form $\sum_{j=1}^{n} \sum_{k=1}^{n} C_{jk} x_j x_k$ be negative semi definite.

Step 1. Convert the inequality constraints into equations by introducing the slack variables q_i^2 in the ith constraint $i = 1, 2, \ldots, m$ and the slack variables r_j^2 in the jth non-negativity constraint, $j = 1, 2, \ldots n$.

Step 2. Construct the Lagrangian function

$$L(x, q, r, \lambda, \mu) = f(x) - \sum_{i=1}^{m} \lambda_i \left[\sum_{j=1}^{n} a_{ij}\, x_j - b_i + q_i^2 \right] - \sum_{j=1}^{n} \mu_i \left[-x_j + r_j^2 \right]$$

where $x = (x_1, x_2, \ldots x_n)$, $q = (q_1^2, \ldots q_m^2)$, $r = (r_1^2, r_2^2, \ldots r_n^2)$

and $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$, $\mu = (\mu_1, \mu_2 \ldots, \mu_n)$.

Differentiate $L(x, q, r, \lambda, \mu)$ partially with respect to the components of $x, q, r, \lambda\, \mu$ and equate the first order partial derivatives equal to zero. Derive the Kuhn-Tucker conditions from the resulting equations.

Step 3. Introduce the non-negative artificial variables A_j, $j = 1, 2, \ldots n$ in the Kuhn-Tucker conditions

$C_j \sum_{k=1}^{n} C_{jk}x_k - \sum_{i=1}^{m} \lambda_i a_{ij} + \mu_j = 0$ for $j = 1, 2, \ldots n$, and construct an objective function

$$Z = A_1 + A_2 + \ldots\ldots + A_n$$

NOTES

Step 4. Obtain an initial basic feasible solution to the following linear programming problem.

Minimize $\quad Z = A_1 + A_2 + \ldots\ldots + A_n$

Subject to the constraints

$$\sum_{k=1}^{n} C_{jk}x_k - \sum_{i=1}^{m} \lambda_i a_{ij} + \mu_j + A_j = -c_j \qquad (j = 1, 2, \ldots\ldots, n)$$

$$\sum_{j=1}^{n} a_{ij}x_j + q_i^2 = b_i \qquad (i = 1, 2, \ldots\ldots, m)$$

$$A_j, \lambda_i, \mu_j, x_j \geq 0 \qquad (i = 1,2, \ldots\ldots, m; j = 1,2, \ldots\ldots n)$$

and satisfying the complementary slackness conditions

$$\sum_{j=1}^{n} \mu_j x_j + \sum_{i=1}^{m} \lambda_i s_i = 0 \text{ (where } s_i = q_i^2)$$

or $\quad \lambda_i s_i = 0$ and $m_j x_j = 0 \quad$ (for $i = 1, 2, \ldots, m; j = 1, 2, \ldots, n$)

Step 5. Use two phase simplex method to obtain an optimum solution to the LP problem of step 4, the solution satisfying the complementary slackness condition.

Step 6. The optimum solution obtained in step 5 is an optimum solution to the given QPP also.

Note : If the given QPP is in the minimization form, convert it into that of maximization by suitable modifications in $f(x_1, x_2, \ldots, x_n)$

Example 4.10. *Use Wolfe's method to solve the quadratic*

Maximize $Z = 4x_1 + 6x_2 - 2x_1^2 - 2x_1 x_2 - 2x_2^2$

Subject to the constraints

$$x_1 + 2x_2 \leq 2$$

and $\quad x_1, x_2 \geq 0$

Solution. First, write all the constraint inequalities with ≤ sign as follows:

$$x_1 + 2x_2 \leq 2$$
$$-x_1 \leq 0$$
$$-x_2 \leq 0$$

Add slack variables q_1^2, r_1^2, r_2^2 all inequality constraints to express them as equations, our problem be comes.

Max $\quad Z = 4x_1 + 6x_2 - 2x_1^2 - 2x_1 x_2 - 2x_2^2$

subject to the constraints

$$x_1 + 2x_2 + q_1^2 = 2$$
$$-x_1 + r_1^2 = 0$$
$$-x_2 + r_2^2 = 0$$

NOTES

To obtain the Kuhn-Tucker conditions, we construct, the Lagrange function as follows:

$$L(X_1, X_2, q_1, r_1, r_2, \lambda_1, \mu_1, \mu_2) = (4x_1 + 6x_2 - 2x_1^2 - 2x_1x_2 - 2x_2^2) - \lambda_1(x_1 + 2x_2 + q_1^2 - 2) - \mu_1(-x_1 + r_1^2) - \mu_2(-x_2 + r_2^2)$$

The necessary and sufficient conditions for the maximum of L and hence of Z are:

$$\frac{\partial L}{\partial x_1} = 4 - 4x_1 - 2x_2 - \lambda_1 + \mu_1 = 0$$

$$\frac{\partial L}{\partial x_2} = 6 - 2x_1 - 4x_2 - 2\lambda_1 + \mu_2 = 0$$

Defining $s_1 = q_1^2$, we get $\lambda_1 s_1 = 0, \mu_1 x_1 = 0, \mu_2 x_2 = 0$

Also $x_1 + 2x_2 + s_1 = 2$ and

lastly $x_1, x_2, s_1, \lambda, \mu_1, \mu_2 \geq 0$

Introduce artificial variables A_1 and A_2, the modified linear programming problem becomes

Maximize $Z = -A_1 - A_2$

subject to the constraints

$$4x_1 + 2x_2 + \lambda_1 - \mu_1 + A_1 = 4$$
$$2x_1 + 4x_2 + 2\lambda_1 - \mu_1 + A_2 = 6$$
$$x_1 + 2x_2 + s_1 = 2$$

and $x_1, x_2, \lambda_1, \mu_1, \mu_2, A_1, A_2 \geq 0$

and $\mu_1 x_1 = 0, \mu x_2 = 0, \lambda_1 s_1 = 0$

Now all the above constraint-equations can be represented in matrix form as follows

$$\begin{bmatrix} 4 & 2 & 1 & -1 & 0 & 1 & 0 & 0 \\ 2 & 4 & 2 & 0 & -1 & 0 & 1 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \lambda_1 \\ \mu_1 \\ \mu_2 \\ A_1 \\ A_2 \\ s_1 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 2 \end{bmatrix}$$

The simplex iterations leading to an optimum solution are

Initial Solution Table 1

$C_j \rightarrow$			0	0	0	0	0	0	–1	–1	
↓	Basic Variables	Qty.	X_1	X_2	λ_1	μ_1	μ_2	S_1	A_1	A_2	
–1	A_1	4	(4)	2	1	–1	0	0	1	0	→
–1	A_2	6	2	4	2	0	–1	0	0	1	
0	S_1	2	1	2	0	0	0	1	0	0	
Z = –10	$C_j - Z_j$		+6	+6	+3	–1	–1	0	0	0	
			↑								

Iteration 1. In table, 1, the largest positive values among $C_j - Z_j$ values is + 6 corresponding to X_1 and X_2 columns. This means either of these two variables can be entered into the basis. Since $\mu_1 = 0$, X_1 is considered to enter into the basis. It will replace A_1 already in the basis. The new solution is shown in Table 2.

First Iteration Table 2

$C_j \rightarrow$			0	0	0	0	0	0	−1	
↓	Basic Variables	Qty.	X_1	X_2	λ_1	μ_1	μ_2	S_1	A_2	
0	X_1	1	1	1/2	¼	−1/4	0	0	0	
−1	A_2	4	0	3	3/2	1/2	−1	0	1	
0	S_1	1	0	(3/2)	−1/4	1/4	0	1	1	→
Z = −4	$C_j - Z_j$		0	+3	+3/2	+1/2	−1	0	0	
				↑						

Iteration 2. In table 2, $\mu_2 = 0$, not in the basis, therefore X_2 can be introduced into the basis to replace s_1 already in basis.

The new solution is shown in table 3.

Second Iteration Table 3

$C_j \rightarrow$			0	0	0	0	0	0	−1	
↓	Basic Variables	Qty.	X_1	X_2	λ_1	μ_1	μ_2	S_1	A_2	
0	X_1	2/3	1	0	1/3	−1/3	0	−1/3	0	
−1	A_2	2	0	0	(2)	.	−1	−2	1	→
0	X_2	2/3	0	1	−1/6	1/6	0	2/3	0	
Z = −2	$C_j - Z_j$		0	0	2	0	−1	−2	0	
					↑					

Iteration 3. In table 3, $S_1 = 0$ not in the basis, therefore, λ_1 can be entered into the basis to replace A_2. The new solution is shown in Table 4.

Third Iteration Table 4

$C_j \rightarrow$			0	0	0	0	0	0
↓	Basic Variables	Qty.	X_1	X_2	λ_1	μ_1	μ_2	S_1
0	X_1	2/3	1	0	0	−1/3	0	0
0	λ_1	1	0	0	1	0	−1/2	−1
0	X_2	5/6	0	1	0	1/6	−1/12	1/2
Z = 0	$C_j - Z_j$		0	0	0	0	0	0
				↑				

In table 4, since all $C_j - Z_j = 0$, an optimal solution for phase I is reached. The optimal solution is :

$$X_1 = 1/3, X_2 = 5/6, \lambda_1 = 1, \lambda_2 = 0, \mu_1 = \mu_2 = 0, S_1 = 0$$

This solution also satisfies the complementary conditions:

$$\lambda_1 S_1 = 0, \mu_1 X_1 = \mu_2 X_2 = 0$$

and the restriction on the signs of Lagrange multipliers, λ_1, μ, and μ_2.

Further as Z = 0, this implies that the current solution is also feasible. Thus the maximum value of the given quadratic programming problem is

Maximize $Z = 4x_1 + 6x_2 + x_1^2 - 2x_1x_2 - 2x_2^2$

$$= 4(1/3) + 6\,(5/6) - 2\,(1/3)^2 - 2\,(1/3)\,(5/6) - 2\,(5/6)^2 = 25/6$$

Example 4.11. *Apply Wolfe's method to solve the quadratic programming problem.*

Maximize $Z = 2x_1 + x_2 - x_1^2$

Subject to the constraints

$$2x_1 + 3x_2 \le 6$$

$$2x_1 + x_2 \le 4$$

and $\quad x_1, x_2 \ge 0$

Solution. Writing all the constraints inequalities with ≤ sign we get.

$$2x_1 + 3x_2 \le 6$$
$$2x_1 + x_2 \le 4$$
$$-x_1 \le 0$$
$$-x_2 \le 0$$

NOTES

Now, adding slack variables, $q_1^2, q_2^2, r_1^2, r_2^2$, the problem becomes

Maximize $Z = 2x_1 + x_2 - x_1^2$

Subject to the constraints

$$2x_1 + 3x_2 + q_1^2 = 6$$
$$2x_1 + x_2 + q_2^2 = 4$$
$$-x_1 + r_1^2 = 0$$
$$-x_2 + r_2^2 = 0$$

Construct the Lagrangian function

$$L(x_1, x_2, q_1, q_2, r_1, r_2, \lambda_1, \lambda_2, \mu_1, \mu_2) = (2x_1 + x_2 - x_1^2) - \lambda_1 (2x_1 + 3x_2 + q_1^2 - 6) - \lambda_2 (2x_1 + x_2 + q_2^2 - 4) - \mu_1 (-x_1 + r_1^2) - \mu_2(-x_2 + r_2^2)$$

The necessary and sufficient conditions for maximum of L and hence of Z are

$$\frac{\partial L}{\partial X_1} = 2 - 2x_1 - 2\lambda_1 - 2\lambda_2 + \mu_1 = 0\,; \quad \frac{\partial L}{\partial X_2} = 1 - 3\lambda_1 - \lambda_2 + \mu_2 = 0,$$

Now defining : $s_1 = q_1^2$, $s_2 = q_2^2$ we have $\lambda_1 s_1 = 0, \lambda_2 s_2 = 0$

$$\mu_1 x_1 = 0; \mu_2 x_2 = 0$$

Also, $\quad 2x_1 + 3x_2 + s_1 = 6$

$$2x_1 + x_2 + s_2 = 4$$

and lastly $x_1, x_2, s_1, s_2, \lambda_1, \lambda_2, \mu_1, \mu_2 \ge 0$

Introduce the artificial variables A_1 and A_2, the modified problem becomes

Maximize $Z = -A_1 - A_2$

Subject to the constraints

$$2x_1 + 2\lambda_1 + 2\lambda_2 - \mu_1 + A_1 = 2$$
$$3\lambda_1 + \lambda_2 - \mu_2 + A_2 = 1$$
$$2x_1 + 3x_2 + s_1 = 6$$
$$2x_1 + x_2 + s_2 = 4$$

with all variables non-negative and $\mu_1 x_1 = 0, \mu_2 x_2 = 0, \lambda_1 s_1 = 0, \lambda_2 s_2 = 0$

The initial basic feasible solution to this LP problem is shown in table 5.

Initial Solution Table 5

Cj→			0	0	0	0	0	0	0	0	−1	−1
↓	Basic Variables	Qty.	x_1	x_2	λ_1	λ_2	μ_1	μ_2	s_1	s_2	A_1	A_2
−1	A_1	2	(2)	0	2	2	−1	0	0	0	1	0 →
−1	A_2	1	0	0	3	1	0	−1	0	0	0	1
0	s_1	6	2	3	0	0	0	0	1	0	0	0
0	s_2	4	2	1	0	0	0	0	0	1	0	0
Z = −3	$C_j - Z_j$		2	0	5	3	−1	−1	0	0	0	0
			↑									

Iteration 1. In table 5, the largest positive value among $C_j - Z_j$ values is + 5, but we cannot enter λ_1 or λ_2 in the basis because of the complementary conditions $\lambda_1 s_1 = \lambda_1 s_2 = 0$. Since $\mu_1 = 0$, x_1 can be entered into the basis with A_1 as the leaving variable. The new solution is shown in table 6.

First Iteration Table 6

$C_j \rightarrow$			0	0	0	0	0	0	0	0	–1
↓	Basic Variables	Qty.	x_1	x_2	λ_1	λ_2	μ_1	μ_2	s_1	s_2	A_2
0	x_1	1	1	0	1	1	1/2	0	0	0	0
–1	A_2	1	0	0	(3)	1	0	–1	0	0	1
0	s_1	4	0	3	– 2	–2	1	0	1	0	0 →
0	s_2	2	0	1	– 2	– 2	1	0	0	1	0
Z = –1	$C_j - Z_j$		0	0	3	1	0	–1	0	0	0
				↑							

Iteration 2. Again we cannot enter λ_1, λ_2 and μ_1 in the basis in the table 6 because s_1, s_2 *and* x_1 *respectively are already in the basis. So enter* x_2 *into the basis with* s_1 as the leaving variable because $\mu_2 = 0$. The new solution is shown in table 7.

Second Iteration Table 7

$C_j \rightarrow$			0	0	0	0	0	0	0	0	–1
↓	Basic Variables	Qty.	x_1	x_2	λ_1	λ_2	μ_1	μ_2	s_1	s_2	A_2
0	x_1	1	1	0	1	1	–1/2	0	0	0	0
1	A_2	1	0	0	(3)	1	0	–1	0	0	1 →
0	X_2	4/3	0	1	–2/3	–2/3	1/3	0	1/3	0	0
0	S_2	2/3	0	0	–4/3	–4/3	2/3	0	–1/3	1	0
Z = –1	$C_j - Z_j$		0	0	3	1	0	–1	0	0	0
					↑						

Iteration 3. Since $s_1 = 0$, λ_1 can be entered into the basis in table 7 with A_2 as the leaving variable. The new solution is shown in table 8.

Third Iteration Table 8

$C_j \rightarrow$			0	0	0	0	0	0	0	0
↓	Basic Variables	Qty.	x_1	x_2	λ_1	λ_2	μ_1	μ_2	s_1	s_2
0	x_1	2/3	1	0	0	2/3	–1/2	1/3	0	0
0	λ_2	1/3	0	0	1	1/3	0	–1/3	0	0
0	x_2	14/9	0	1	0	–4/9	1/3	–2/9	1/3	0
0	s_2	10/9	0	0	0	–8/9	2/3	–4/9	–1/3	1
Z = –1	$C_j - Z_j$		0	0	0	0	0	0	0	0

Since both A_1 and A_2 are out of the basic solution, the computation is now complete. The optimal solution is $x_1 = 2/3$, $x_2 = 14/9$, $\lambda_1 = 1/3$, $\lambda_2 = 0$, $\mu_1 = \mu_2 = 0$, $s_1 = 0$, $s_2 = 10/9$

This solution also satisfies the complementary slackness conditions : $\lambda_1 s_1 = \lambda_2 s_2 = 0$; $\mu_1 X_1 = \mu_1 X_2 = 0$ and the restriction on the signs of Lagrange multipliers : $\lambda_1, \lambda_2, \mu_1$ and μ_2.

NOTES

The maximum value of the objective function can be computed from the original objective function not from modified. Thus

Maximize $Z = 2x_1 + x_2 - x_1^2 = 2(2/3) + 14/9 - (2/3)^2 = 22/9$

4.13 CONVEX PROGRAMMING PROBLEM

NOTES

Convex optimization, a subfield of mathematical optimization, studies the problem of minimizing convex functions. Given a real vector space X together with a convex, real-valued function:

$$f: X \to R$$

defined on a convex subset X, the problem is to find a point x^* in X for which the number $f(x)$ is smallest, *i.e.*, a point x such that

$$f(x) \leq f(x)\ \forall x \in X.$$

Convex minimization has applications in a wide range of disciplines, such as automatic control systems, estimation and signal processing, communications and networks, electronic circuit design, data analysis and modeling, statistics (optimal design), and finance. With recent improvements in computing and in optimization theory, convex minimization is nearly as straightforward as linear programming.

Definition

A convex programming problem consists of a convex feasible set X and a convex cost function

$$c: X \to R.$$

The optimal solution is the solution that minimizes the cost.

Convex programming unifies and generalizes least squares (LS), linear programming (LP), and quadratic programming (QP). It has received considerable attention recently for a number of reasons: its attractive theoretical properties; the development of efficient, reliable numerical algorithms; and the discovery of a wide variety of applications in both scientific and non-scientific fields. For these reasons, convex programming has the potential to become a numerical technology alongside LS, LP, and QP. Nevertheless, there remains a significant impediment to its more widespread adoption: the high level of expertise in both convex analysis and numerical algorithms required to use it. For potential users whose focus is the application, this prerequisite poses a formidable barrier, especially if it is not yet certain that the outcome will be better than with other methods.

As its name suggests, disciplined convex programming imposes a set of conventions to follow when constructing problems. Compliant problems are called, appropriately, disciplined convex programs, or DCPs. The conventions are simple and teachable, taken from basic principles of convex analysis, and inspired by the practices of experts who regularly study and apply convex optimization. Disciplined convex programming also provides a framework for collaboration between users with different levels of expertise. In short, disciplined convex programming allows applications-oriented users to focus on modeling and as these would with LS, LP, and QP to leave the underlying mathematical details to experts.

4.14 INTRODUCTION TO DYNAMIC PROGRAMMING

Dynamic programming is a mathematical technique dealing with the optimization of multistage decision problems. Dynamic programming was developed by Richard

NOTES

Bellman and G.B. Dantzig in 1950s. In the beginning this technique was termed as stochastic linear programming or linear programming dealing with uncertainty. Dynamic programming can be given a more significant name as recursive optimization. In dynamic programming the term 'dynamic' stands for its usefulness in problems where decisions have to be taken at several distinct stages and 'programming' is used in a mathematical sense of selecting an optimum allocation of resources. In dynamic programming a large problem having n variables is divided into n subproblems (stages), each of which involves one variable. The optimum solution is obtained in an orderly manner starting from one stage to the next and continued till the final stage is reached.

There are two terms stage and state in dynamic programming. Each point in the problem where a decision must be made is known as stage. Each stage has a number of states associated with it. The states are various possible conditions in which the system might find itself at that stage of the problem.

Dynamic programming technique is applicable in solving a wide variety of problems including inventory control, allocation, longterm corporate planning, replacement, etc.

Dynamic programming works on the Bellman's principle of optimality.

Bellman's Principle of Optimality

It states that "An optimal policy (set of decisions) has the property that whatever be the initial state and decisions are, the remaining decisions must constitute an optimal policy regardless of the policy adopted in previous stages."

Forward and Backward Recursion

If the computational procedure proceed from stage–1 to stage–n then it is called forward recursion. If the computational procedure proceed from stage–n to stage–1 then it is called backward recursion.

Both the forward and backward recursions gives the same solution. Generally, backward recursion is used, as it is more convenient.

Characteristics of Dynamic Programming

The basic features which characterize dynamic programming problem are as follows:

1. The problem can be sub divided into stages with a policy decision required at each stage.
2. Each stage has a number of states associated with it.
3. The effect of the policy decision at each stage is to transform the current state into a state associated with the next stage.
4. The state of the system at a stage is described by a set of variables, called state variables.
5. Remaining decisions constitutes an optimal policy regardless of the policy adopted in previous stage, *i.e.,* given the current stage, an optimal policy for the remaining stages is independent of the policy adopted in previous stage.
6. A recursive equation is formed that connect the optimal solution of the previous stage and the contribution of the current stage, to identify the optimal policy for each state with (n –1) stages.
7. Using this recursive equation, the solution procedure moves backward stage by stage to get the optimal policy for each state of that stage until the optimal policy for the initial stage.

NOTES

Basic Steps of Dynamic Programming

The basic steps for the solution of problem by dynamic programming are as follows :

1. Identify the decision variables and specify the objective function to be optimized.
2. Divide the given problem into a number of sub-problems (stages) and identify the state variables at each stage.
3. Deveolp the recursive equation for computing the optimal policy. Decide whether forward or backward recursion approach is to be used for the solution of the problem.
4. Construct appropriate stages to show the required values of the return function at each stage.
5. Move backward stage by stage to find the overall optimal solution and its value. There may be more than one optimal solution of the problem.

Applications of Dynamic Programming

Dynamic programming can be used to solve many real-life problems. Some of the important applications of dynamic programming are as follows :

1. Linear programming problem
2. Inventory control problem
3. Replacement problem
4. Capital budgeting problem
5. Shortest route problem
6. Cargo loading problem
7. Production planning and scheduling
8. Investment analysis
9. Reliability problem

4.15 MODEL–I: SINGLE ADDITIVE CONSTRAINT, MULTIPLICATIVE SEPARABLE RETURN

The general form of the recursive equation to solve this type of problem by dynamic programming can be illustrated by considering following problem.

Suppose we have separable return functions $f_j(x_j)$; $j = 1, 2, ..., n$ and we want

Maximize $\quad Z = f_1(x_1).f_2(x_2) ... f_n(x_n)$

subject to, $\quad a_1x_1 + a_2x_2 + ... + a_nx_n = b$

$$x_j, a_j, b \geq 0 \text{ for all } j = 1, 2, ..., n.$$

Where $\quad j = j^{th}$ number of stage $(j = 1, 2, ..., n)$.

x_j = decision variable at j^{th} stage

Now define state variables $s_1, s_2, ..., s_n$ such that

$$s_n = a_1x_1 + a_2x_2 + ... + a_nx_n = b$$

$$s_{n-1} = a_1x_1 + a_2x_2 + ... + a_{n-1}x_{n-1} = s_n - a_nx_n$$

$$\vdots$$

$$s_{j-1} = s_j - a_jx_j$$

$$\vdots$$

$$s_1 = s_2 - a_2x_2$$

At the n^{th} stage, s_n is expressed as the function of decision variables. Thus the maximum value of Z denoted by $f_n^*(s_n)$ for any feasible value of s_n is given by

$$f_n^*(s_n) = \text{Maximum } \{f_1(x_1), .f_2(x_2) \ldots, .f_n(x_n)\}$$
$$x_j > 0$$

subject to, $\quad s_n = b$

Now keeping a particular value of x_n fixed, the maximum value of Z will be given by

$f_n(x_n)$. Max $\{f_1(x_1).f_2(x_2) \ldots, .f_{n-1}(x_{n-1})\}; j = 1, 2, \ldots, n-1$

$$x_j > 0$$
$$= f_n(x_n), .f_{n-1}^*(s_{n-1})$$

NOTES

The maximum value $f_{n-1}^*(x_{n-1})$ of Z due to decision variables x_j ($j = 1, 2, \ldots, n-1$) depends upon the state variables s_{n-1}. The maximum of Z for any feasible value of all decision variables will be given by

$$f_j^*(s_j) = \underset{x_j > 0}{\text{Max}} \; \{f_j(x_j), .f_{j-1}^*(s_{j-1})\}; j = n, n-1, \ldots, 2$$

$$f_1^*(s_1) = f_1(x_1)$$

where $\quad s_{j-1} = f_j(s_j, x_j)$

The value of $f_j^*(s_j)$ represents the general recursive equation.

Example 4.12. *Use dynamic programming to solve the following problem:*

Maximize $\quad Z = x_1 . x_2 . x_3$

subject to $\quad = x_1 + x_2 + x_3 = 20$

$$x_1, x_2, x_3 \geq 0$$

Solution. Let the state variable s_j ($j = 1, 2, 3$) such that

$$s_3 = x_1 + x_2 + x_3 = 20 \qquad \text{(stage 3)}$$
$$s_2 = s_3 - x_3 . s_3 = x_1 + x_2 \qquad \text{(stage 2)}$$
$$s_1 = s_2 - x_2 = x_1 \qquad \text{(stage 1)}$$

The maximum value of Z for any feasible value of state variable is given by

$$f_3(s_3) = \underset{x_3}{\text{Max}} \; \{x_3 . f_2(s_2)\}$$

$$f_2(s_2) = \underset{x_2}{\text{Max}} \; \{x_2 . f_1(s_1)\}$$

$$f_1(s_1) = x_1 = s_2 - x_2$$

Thus $\quad f_2(s_2) = \underset{x_2}{\text{Max}} \; \{x_2 . (s_2 - x_2)\}$

$$= \underset{x_2}{\text{Max}} \; \{x_2 s_2 - x_2^2)\}$$

Now differentiating $f_2(s_2)$ with respect to x_2 and then equating to zero (using the condition for maxima or minimia of a function), we have

$$s_2 - 2x_2 = 0 \text{ or } x_2 = \frac{s_2}{2}$$

Now, $\quad f_2(s_2) = \left(\frac{s_2}{2}\right) . s_2 - \left(\frac{s_2}{2}\right)^2 = \frac{s_2^2}{4}$

and $\quad f_3(s_3) = \underset{x_3}{\text{Max}} \; \{x_3 . f_2(s_2)\}$

$$= \underset{x_3}{\text{Max}} \left\{x_3 . \left(\frac{s_2^2}{4}\right)\right\}$$

$$= \text{Max} \left\{x_3 . \frac{(s_3 - x_3)^2}{4}\right\}$$

Now differentiating $f_3(s_3)$ with respect to x_3 and equating to zero, we have

$$\frac{1}{4}\left\{x_3 \cdot 2(s_3 - x_3)(-1) + (s_3 - x_3)^2 \cdot 1\right\} = 0$$

$$(s_3 - x_3)(-2x_3 + s_3 - x_3) = 0$$

$$(s_3 - x_3)(s_3 - 3x_3) = 0$$

Now either $x_3 = s_3$ which is trivial as $x_1 + x_2 + x_3 = s_3$

NOTES

or $\quad s_3 - 3x_3 = 0$ or $x_3 = \frac{s_3}{3} = \frac{20}{3}$.

Therefore, $\quad x_2 = \frac{s_2}{2} = \frac{s_3 - x_3}{2} = \frac{1}{2}\left(20 - \frac{20}{3}\right) = \frac{20}{3}$

$$x_1 = s_2 - x_2 = \frac{40}{3} - \frac{20}{3} = \frac{20}{3}$$

So, $\quad x_1 = x_2 = x_3 = \frac{20}{3}$ and Max. $Z = \left(\frac{20}{3}\right)^3 = \frac{8000}{27}$ **Ans.**

4.16 MODEL–II: SINGLE ADDITIVE CONSTRAINT, ADDITIVE SEPARABLE RETURN

The general form of the recursive equation to solve this type of problem by dynamic programming can be illustrated by considering following problem:

Suppose we want to

Minimize $\quad Z = f_1(x_1) + f_2(x_2) + \ldots + f_n(x_n)$

Subject to $\quad = a_1 x_1 + a_2 x_2 + \ldots + a_n x_n \geq b$

$$x_j, a_j = b \geq 0 \text{ for all } j, 1, 2, \ldots, n$$

where $\quad j = j^{th}$ number of stage $(j = 1, 2, \ldots, n)$.

x_j = decision variable at j^{th} stage.

Now define state variables $s_1, s_2, \ldots, s_n$ such that

$$s_n = a_1 x_1 + a_2 x_2 + \ldots + a_n x_n \geq b$$

$$s_{n-1} = a_1 x_1 + a_2 x_2 + \ldots + a_{n-1} x_{n-1} = s_n - a_n x_n$$

$$\vdots$$

$$s_{j-1} = s_j - a_j x_j$$

$$\vdots$$

$$s_1 = s_2 - a_2 x_2$$

At the n^{th} stage, s_n is expressed as the function of decision variables. Thus the minimum value of Z denoted by $f_n^*(s_n)$ for any feasible value of s_n is given by

$$f_n^*(s_n) = \underset{x_j > 0}{\text{Minimize}} \sum_{j=1}^{n} f_j(x_j)$$

subject to $\quad s_n \geq b$

Now keeping a particular value of x_n fixed, the minimum value of Z will be given by

$$f_n(x_n) + \underset{x_j > 0}{\text{Minimize}} \sum_{j=1}^{n-1} f_j(x_j) = f_n(x_n) + f_{n-1}^*(s_{n-1})$$

The minimum value $f_{n-1}^*(x_{n-1})$ of Z due to decision variables x_j $(j = 1, 2, \ldots, n-1)$ depends upon the state variables s_{n-1}. The minimum value of Z for any feasible value of all decision variables will be given by

$$f_j^*(s_j) = \underset{x_j>0}{Min} \{f_j(x_j) + f_{j-1}^*(s_{j-1})\}; j = n, n-1, ..., 2$$

$$f_1^*(s_1) = f_1(x_1)$$

where $\quad s_{j-1} = t_j(s_j, x_j)$

The value of $f_j^*(s_j)$ represents the general recursive equation.

NOTES

Example 4.13. *Use dynamic programming to solve the following problem.*

Minimize $\quad Z = x_1^2 + x_2^2 + x_3^2$

subject to $\quad x_1 + x_2 + x_3 = 20$

$$x_1, x_2, x_j \geq 0$$

Solution. Let the state variables s_j (j = 1, 2, 3) such that

$$s_3 = x_1 + x_2 + x_3 = 20 \quad \text{(stage 3)}$$

$$s_2 = s_3 - x_3 = x_1 + x_2 \quad \text{(stage 2)}$$

$$s_1 = s_2 + x_2 = x_1 \quad \text{(stage 1)}$$

The minimum value of Z for any feasible value of state variable is given by

$$f_3(s_3) = \underset{x_3}{\text{Min}} \{x_2^3 + f_2(s_2)\}$$

$$f_2(s_2) = \underset{x_2}{\text{Min}} \{x_2^2 + f_1(s_1)\}$$

$$f_1(s_1) = \underset{x_1}{\text{Min}} \{x_1^2\} = {x_1}^2 = (s_2 - x_2)^2$$

Thus, $\quad f_2(s_2) = \underset{x_2}{\text{Min}} \{x_2^2 + (s_2 - {x_2}^2)\}$

Now differentiating $f_2(s_2)$ with respect to x_2 and then equating to zero (using the conditions for maxima or minima of a function), we have

$$2x_2 + 2(s_2 - x_2)(-1) = 0 \text{ or } x_2 = \frac{s_2}{2}$$

Now, $\quad f_2(s_2) = \left(\frac{s_2}{2}\right)^2 + \left(s_2 - \frac{s_2}{2}\right)^2 = \frac{s_2^2}{2}$

and $\quad f_3(s_3) = \underset{x_3}{\text{Min}} \{x_3^2 + f_2(s_2)\}$

$$= \underset{x_3}{\text{Min}} \{x_3^2 + \frac{s_2^2}{2}\}$$

$$= \underset{x_3}{\text{Min}} \{{x_3}^2 + \frac{(s_3 - x_3)^2}{2}\}$$

Now, differentiating $f_3(s_3)$ with respect to x_3 and equating to zero, we have

$$2x_3 + \frac{2(s_3 - x_3)}{2}(-1) = 0 \text{ or } x_3 = \frac{s_3}{3}$$

Thus, $\quad f_3(s_3) = \left(\frac{s_3}{3}\right)^2 + \frac{1}{2}\left(s_3 - \frac{s_3}{3}\right)^2 = \frac{s_3^2}{3}$

$$f_3(s_3) = \frac{(20)^3}{3} = \frac{400}{3} \quad (\because \; s_3 = 20)$$

$$x_3 = \frac{s_3}{3} = \frac{20}{3}$$

$$x_2 = \frac{s_2}{2} = \frac{(s_3 - x_3)}{2} = \frac{\left(20 - \frac{20}{3}\right)}{2} = \frac{20}{3}$$

$$x_1 = s_2 - x_2 = \frac{40}{3} - \frac{20}{3} = \frac{20}{3}$$

NOTES

So $$x_1 = x_2 = x_3 = \frac{20}{3} \text{ and Minimize } Z = \frac{(20)^2}{3} = \frac{400}{3} \quad \textbf{Ans.}$$

4.17 MODEL–III: SINGLE MULTIPLICATIVE CONSTRAINT, ADDITIVELY SEPARABLE RETURN

The general form of the recursive equation to solve this type of problem by dynamic programming can be illustrated by considering following problem

Suppose we want to

Minimize $$Z = f_1(x_1) + f_2(x_2) + \ldots + f_n(xn)$$

subject to, $$x_1 . x_2 \ldots, . x_n \geq b$$

$$x_j, b \geq 0 \text{ for all } j = 1, 2, \ldots, n$$

where $j = j^{th}$ number of stage $(j = 1, 2, \ldots, n)$

x_j = decision variable at jth stage

Now define state variables $s_1, s_2, \ldots, s_n$ such that

$$s_n = x_n . x_{n-1} \ldots, . x_2 . x_1 \geq b$$

$$s_{n-1} = x_{n-1} . x_{n-2} \ldots, . x_2 . x_1 = \frac{s_n}{x_n}$$

$$\vdots$$

$$s_{j-1} = \frac{s_j}{x_j}; j = 2, 3, \ldots, n$$

The minimum value of Z denoted by $f_n^*(s_n)$ for any feasible value of s_n is given by

$$f_n^*(s_n) = \underset{x_j > 0}{\text{Min}} \sum_{j=1}^{n} f_j(x_j)$$

subject to, $$s_n \geq b$$

The minimum value $f_{n-1}^*(x_{n-1})$ of Z due to decision variables x_j $(j = 1, 2, \ldots, n-1)$ depends upon the state variables s_{n-1}. The minimum of Z for any feasible value of all decision variables will be given by

$$f_j^*(s_j) = \underset{x_j > 0}{\text{Min}} \{f_j(x_j) + f_{j-1}^*(s_{j-1})\}; j = 2, 3, \ldots, n$$

$$f_1^*(s_1) = f_1(x_1)$$

where $$s_{j-1} = t_j(s_j, x_j)$$

The value of $f_j^*(s_j)$ represents the general recursive equation.

Example 4.14. *Use dynamic programming to solve the following problem:*

Minimize $$Z = x_1^2 + x_2^2 + \ldots + x_n^2$$

subject to $$x_1 . x_2 \ldots, . x_n = c$$

$$x_j \geq o; j = 1, 2, \ldots, n$$

Solution. Let $f_n(c)$ be the minimum attainable sum of $x_1^2 + x_2^2 + ... + x_n^2$ when c is divided into n factors.

For $n = 1$, we have $x_1 = c$

So $$f_1(c) = \underset{x_1 = c}{\text{Min}} \{x_1^2\} = c^2.$$

For $n = 2$, Let $x_1 = y$ and $x_2 = \frac{c}{x_1} = \frac{c}{y}$, then

$$f_2(c) = \underset{0 \le y \le c}{\text{Min}} \{x_1^2 + x_2^2\}$$

$$= \underset{0 \le y \le c}{\text{Min}} \left\{y^2 + \left(\frac{c}{y}\right)^2\right\}$$

$$= \underset{0 \le y \le c}{\text{Min}} \left\{y^2 + f_1\left(\frac{c}{y}\right)\right\} \left(\because f_1(c) = c^2, \text{so } f_1\left(\frac{c}{y}\right) = \left(\frac{c}{y}\right)^2\right)$$

For $n = 3$, Let $x_1 = y$ and $x_2 . x_3 = \frac{c}{y}$, then

$$f_3(c) = \underset{0 \le y \le c}{\text{Min}} \{x_1^2 + x_2^2 + x_3^2\}$$

$$= \underset{0 \le y \le c}{\text{Min}} \left\{y^2 + f_2\left(\frac{c}{y}\right)\right\}$$

Proceeding in the same way, we have

$$f_n(c) = \underset{0 \le y \le c}{\text{Min}} \left\{y^2 + f_{n-1}\left(\frac{c}{y}\right)\right\}$$

Now, the solution to the recursive equation to get the optimal policy can be obtained by using the principle of maxima and minima of function.

Let $$f(y) = y_2 + \left(\frac{c}{y}\right)^2 \frac{df}{dy}$$

$$= 2y - \frac{2c^2}{y^2} = 0 \Rightarrow y = (c)^{1/2}$$

Therefore, $x_1 = (c)^{1/2}$ and $x_2 = \frac{c}{y} = \frac{c}{c^{1/2}} = (c)^{1/2}$

Since the second derivative of $f(y)$ with respect to y is positive, so $f(y)$ minimum. Also

$$f_2(c) = \underset{0 \le y \le c}{\text{Min}} \left\{y^2 + \left(\frac{c}{y}\right)^2\right\}$$

$$= \left(c^{1/2}\right)^2 + \left(\frac{c}{c^{1/2}}\right)^2 = 2c$$

Thus, $f_2(c/y) = 2(c/y)$

Hence, the optimal policy is $(c^{1/2}, c^{1/2})$ and $f_2(c) = 2c$.

Again,

$$f_3(c) = \underset{0 \le y \le c}{\text{Min}} \left\{y^2 + f_2\left(\frac{c}{y}\right)\right\}$$

$$= \underset{0 \le y \le c}{\text{Min}} \left\{y^2 + 2\left(\frac{c}{y}\right)\right\}$$

$$= \left(c^{1/3}\right)^2 + 2\left(\frac{c}{c^{1/3}}\right)^2 = 3c^{2/3}$$

Since minimum of $f(y) = y^2 + \frac{c}{y}$ is attained at $y = c^{1/3}$.

Hence the optimal policy is $(c^{1/3}, c^{1/3}, c^{1/3})$ and $f_3(c) = 3c^{2/3}$

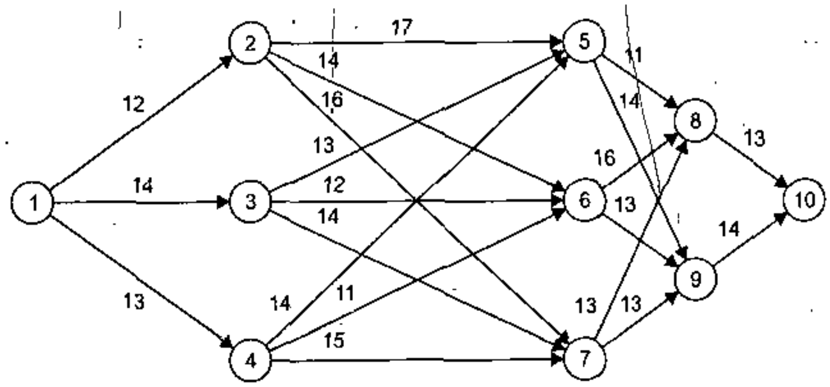
NOTES

Now, on continuing in the same manner, we can get the optimal policy for an n stage problem as

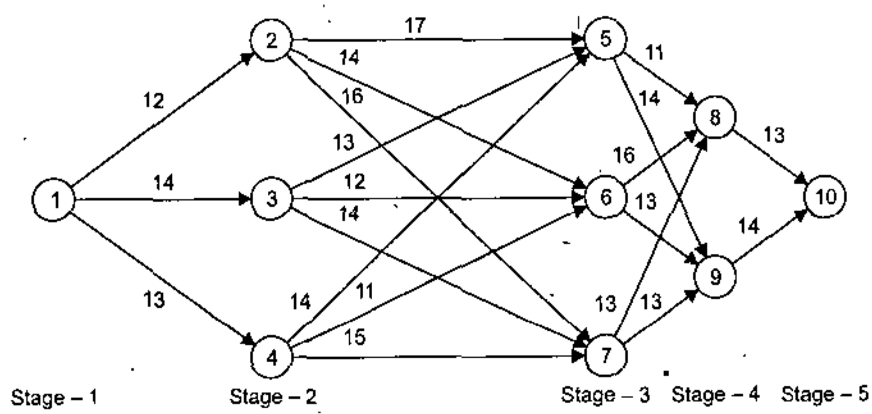
$(c^{1/n}, c^{1/n}, ..., c^{1/n})$ and $f_n(c) = n\,c^{2/n}$.

4.18 MODEL–IV SHORTEST ROUTE PROBLEM

Example. Consider the following diagram where circles denote cities, and lines between two such circles represent highways connecting the cities. The numbers inside the circles represents city numbers and *those given beside the lines denote the distance between* the cities connected by the lines. Suppose a salesman start from city 1 and would like to go to city 10. What is the shortest route from city 1 to city 10 and what is the minimum distance ?



Solution. The given problem is 5 stage problem as.



The backward recursive equation for the problem is :

$$f_i(x_i) = \underset{\substack{\text{all feasible} \\ \text{routes }(x_i,\, x_{i+1})}}{\text{Min}} \{d\,(x_i, x_{i+1}) + f_{i+1}(x_{i+1})\}\ ;\ i = 1, 2, 3, 4$$

where x_i = the state of the system at stage i

$f_i(x_i)$ = the shortest distance to node x_i at stage i

Here, $f_5(x_5) = 0$ for $x_5 = 10$. The associated order of computations is $f_4 \longrightarrow f_3 \longrightarrow f_2 \longrightarrow f_1$.

Stage–4. Because node 10 ($x_5 = 10$) is connected to nodes 8 and 9 ($x_4 = 8$ and 9) with exactly one route each, there are no alternatives to choose from and stage–4 results can be represented as

x_4	$d(x_4, x_5)$	Optimum solution	
	$x_5 = 10$	$f_4(x_4)$	x_5^*
8	13	13	10
9	14	14	10

NOTES

Stage–3. Here, we have two alternatives each from nodes 5, 6 and 7 respectively. Given $f_4(x_4)$ from stage–4. Now stage 3 result can be represented as

x_3	$d(x_3, x_4) + f_4(x_4)$		Optimum solution	
	$x_4 = 8$	$x_4 = 9$	$f_3(x_3)$	x_3^*
5	11 + 13 = 24	14 + 14 = 28	24	8
6	16 + 13 = 29	13 + 14 = 27	27	9
7	13 + 13 = 26	13 + 14 = 27	26	8

Stage–2. Here, we have three alternatives each from nodes 2, 3 and 4 respectively. Given $f_3(x_3)$ from stage–3. Now Stage–2 result can be represented as

x_2	$d(x_3, x_4) + f_4(x_4)$			Optimum solution	
	$x_3 = 5$	$x_3 = 6$	$x_3 = 7$	$f_2(x_2)$	x_3^*
2	17 + 24 = 41	14 + 27 = 41	16 + 26 = 42	41	5 or 6
3	13 + 24 = 37	12 + 27 = 39	14 + 26 = 40	37	5
4	14 + 24 = 38	11 + 27 = 38	15 + 26 = 41	38	5 or 6

Stage–1. Here, we have three alternatives from node 1. Given $f_2(x_2)$ from stage–2. Now stage–1 result can be represented as

x_1	$d(x_1, x_2) + f_2(x_2)$			Optimum solution	
	$x_2 = 2$	$x_2 = 3$	$x_2 = 4$	$f_1(x_1)$	x_2^*
1	12 + 41 = 53	14 + 37 = 51	13 + 38 = 51	51	3 or 4

The optimum solution of the problem, using optimum solutions of stage–1, stage–2, stage–3 and stage–4 is

$1 \to 3 \to 5 \to 8 \to 10$ or $1 \to 4 \to 5 \to 8 \to 10$ or $1 \to 4 \to 6 \to 9 \to 10$ and the minimum distance = 51 units

4.19 SOLUTION OF SOME OTHER PROBLEMS BY USING DYNAMIC PROGRAMMING

Example 4.15. *(Cargo Loading Problem) A 40 ton vessel is to be loaded with one or more of three items. The following table gives the unit weight w_i (in tonnes) and the unit revenue r_i (in thousands of Rs.) for items i.*

Item (i)	w_i	r_i
1	*20*	*31*
2	*30*	*47*
3	*10*	*14*

How should the vessel be loaded to maximize the total return?

NOTES

Solution. First, we formulate the problem as:

Maximize $Z = 31y_1 + 47y_2 + 14y_3$

subject to, $20y_1 + 30y_2 + 10y_3 \le 40$

$y_1, y_2, y_3 \ge 0$ and integer

Now, the backward recursive equation for the optimization is

$$f_i(x_i) = \text{Max}\ \{r_i y_i + f_{i+1}(x_{i+1})\};\ i = 1, 2, 3$$

$$y_i = 0, 10, ..., \left[\frac{W}{w_i}\right]$$

$$x_i = 0, 10, ..., W$$

Where $f_{n+1} \cdot (x_{n+1}) = 0$, $x_{i+1} = x_i - w_i y_i$ and, W = capacity of the vessel

There are 3 stages in the given problem.

Stage–3. The weight to be allocated in stage–3 must take one of the values 0,10, ...,40 ($\because$ W = 40 tonnes).

The maximum number of units of item–3 that can be loaded is $\left[\frac{40}{10}\right] = 4$, which means the possible values of y_3 are 0, 1, 2, 3 and 4. An alternative y_3 is feasible only if $w_3 y_3 \le x_3$. Thus, all the infeasible alternatives (those for which $w_3 y_3 > x_3$) are excluded. So,

$$f_3(x_3) = \underset{y_3}{\text{Max}}\ \{14\ y_3\};\ \text{maximum}\ y_3 = \left[\frac{40}{10}\right] = 4$$

The stage–3 result can be represented as

x_3	$14y_3$					Optimum solution	
	$y_3 = 0$	$y_3 = 1$	$y_3 = 2$	$y_3 = 3$	$y_3 = 4$	$f_3(x_3)$	y_3^*
0	0	—	—	—	—	0	0
10	0	14	—	—	—	14	1
20	0	14	28	—	—	28	2
30	0	14	28	42	—	42	3
40	0	14	28	42	56	56	4

Stage–2. Here,

$$f_2(x_2) = \underset{y_2}{\text{Max}}\ \{47y_2 + f_3(x_2 - 30y_2)\};\ \text{maximum}\ y_2 = \left[\frac{40}{30}\right] = 1$$

The stage–2 result can be represented as

x_2	**$47y_2 + f_3(x_3 - 30y_2)$**		**Optimum solution**	
	$y_2 = 0$	$y_2 = 1$	$f_2(x_2)$	y_2^*
0	0 + 0 = 0	—	0	0
10	0 + 14 = 14	—	14	0
20	0 + 28 = 28	—	28	0
30	0 + 42 = 42	47 + 0 = 47	47	1
40	0 + 56 = 56	47 + 14 = 61	61	1

Stage–1. Here,

$$f_1(x_1) = \underset{y_1}{\text{Max}}\ \{31y_1 + f_2(x_1 - 20y_1)\};\ \text{maximum}\ y_1 = \left[\frac{40}{20}\right] = 2$$

The stages –1 result can be represented as

x_1	$31\, y_1 + f_2(x_1 - 20\, y_1)$			Optimum solution	
	$y_1 = 0$	$y_1 = 1$	$y_1 = 2$	$f_1(x_1)$	y_1^*
0	0 + 0 = 0	—	—	0	0
10	0 + 14 = 14	—	—	14	0
20	0 + 28 = 28	31 + 0 = 31	—	31	1
30	0 + 47 = 47	31 + 14 = 45	—	47	0
40	0 + 61 = 61	31 + 28 = 59	62 + 0 = 62	62	2

NOTES

The optimum solution is now determined as follows:

Given W = 40 tonnes, from stage – 1 table $x_1 = 40$ gives the optimum alternative $y_1^* = 2$, which means that 2 units of item – 1 will be loaded on the vessel. This allocation leaves $x_2 = x_1 - 20\, y_1^* = 40 - 20 \times 2 = 0$, From stage–2 table $x_2 = 0$ gives the optimum alternative $y_2^* = 0$. This allocation leaves $x_3 = x_2 - 30\, y_2 = 0 - 30 \times 0 = 0$, from stage – 3 table $x_3 = 0$ gives the optimum alternative $y_3^* = 0$.

Thus, the optimum solution of the problem is $y_1 = 2$, $y_2 = 0$, $y_3 = 0$ and maximum return is Rs. 62,000.

Example 4.16. ***(Inventory control)*** *A man is engaged in buying and selling identical items. He operates from a warehouse that can hold 500 items. Each month he can sell any quantity that he chooses upto the stock at the beginning of the month. Each month he can buy as much as he wishes for delivery at the end of the month so as his stock does not exceed 500 items. For the next four months, he has the following error free forecasts of cost, sales prices:*

Month	*(i)*	*1*	*2*	*3*	*4*
Cost	(c_i)	*27*	*24*	*26*	*28*
Sales price	(p_i)	*28*	*25*	*25*	*27*

If he currently has a stock of 200 units, what quantities should he sell and buy in the next four months? Find the solution using dynamic programming.

Solution. Let x_i = amount of sell during the month i

y_i = amount of ordered during the month i

b_i = stock level at the beginning of the month i

Let $f_n(b)$ be the maximum possible return when there are n months left with the initial stock level b at the beginning of this month.

Here, we use the backward recursive approach. The problem can be solved in four stages as follows:

Stage–4. Let us be in the 4th month starting with the stock level $b = b_4$ at the beginning of this month.

So, $$f_1(b_4) = \underset{x_4,\, y_4}{\text{Max}} \; \{p_4 x_4 - c_4 y_2\}$$

where $$0 \le x_4 \le b_4,\; y_4 \ge 0 \text{ and } b_4 - x_4 + y_4 \le 0$$

or $$f_1(b_4) = \underset{\substack{0 \le x_4 \le b_4 \\ 0 \le y_4 \le x_4 - b_4}}{\text{Max}} \; \{27x_4 - 28y_4\}$$

It is obvious that Max. $(27x_2 - 28y_4)$ occurs at $x_4 = b_4$, $y_4 = 0$

$\therefore$ $$f_1(b_4) = 27b_4 - 28 \times 0 = 27b_4$$

$\therefore$ Optimal decisions are $x_4 = b_4$, $y_4 = 0$ and $f_1(b_4) = 27b_4$ ...(1)

Stage–3. Let us be in the 3rd month, *i.e.*, two months are left with the initial stock b_3 at the beginning of this month. Since stock $b_4 = b_3 - x_3 + y_3$ will remain available at the beginning of next month.

NOTES

So $$f_2(b_3) = \max_{x_3, y_3} \{p_3x_3 - c_3y_3 + f_1(b_3 - x_3 + y_3)\}$$

where $0 \le x_3 \le b_3, y_3 \ge 0$ and $b_3 - x_3 + y_3 \le 500$

or $$f_2(b_3) = \max_{\substack{0 \le x_3 \le b_3 \\ 0 \le y_3 \le 500 - b_3 - x_3}} \{25x_3 - 26y_3 + 27(b_3 - x_3 + y_3)\} \text{ [From (i)]}$$

$$= \max_{\substack{0 \le x_3 \le b_3 \\ 0 \le y_3 \le 500 - b_3 + x_3}} \{27b_3 - 2x_3 + y_3\}$$

$$= \max_{0 \le x_3 \le b_3} \{27b_3 - 2x_3 + (500 - b_3 + x_3)\}$$

$$= \max_{0 \le x_3 \le b_3} \{26b_3 - x_3 + 500\}$$

which occurs at $x_3 = 0$

$\therefore$ Optimal decisions are $x_3 = 0$, $y_3 = 500 - b_3 + x_3 = 500 - b_3$ and $f_2(b_3) = 26b_3 + 500$ ...(2)

Stage–2. Let us be in the 2^{nd} month, *i.e.*, 3 months are left with the initial stock level b_2 at the beginning of this month. Since stock $b_3 = b_2 - x_2 + y_2$ will remain available at the beginning of the next month.

So, $$f_3(b_2) = \max_{x_2, y_2} \{p_2 x_2 - c_2 y_2 + f_2(b_2 - x_2 + y_2)\}$$

where $0 \le x_2 \le b_2, y_2 \ge 0$ and $b_2 - x_2 + y_2 \le 500$

or $$f_3(b_2) = \max_{\substack{0 \le x_2 \le b_2 \\ 0 \le y_2 \le 500 - b_2 + x_2}} \{25x_2 - 24y_2 + 26(b_2 - x_3 + y_2) + 500\} \text{[from (2)]}$$

$$= \max_{\substack{0 \le x_2 \le b_2 \\ 0 \le y_2 \le 500 - b_2 + x_2}} \{- x_2 + 2y_2 + 26b_2 + 500\}$$

$$= \max_{0 \le x_2 \le b_2} \{- x_2 + 2\,(500 - b_2 + x_2) + 26b_2 + 500\}$$

$$= \max_{0 \le x_2 \le b_2} \{24b_2 + x_2 + 1500\}$$

which occurs at $x_2 = b_2$

$\therefore$ Optimal decisions are $x_2 = b_2$, $y_2 = 500 - b_2 + x_2 = 500$

and $f_3(b_2) = 24b_2 + b_2 + 1500 = 25b_2 + 1500$ ...(3)

Stage– 1. Let us be in the 1st month. *i.e.*, 4 months are left with the initial stock level b_1 at the beginning of this month. Since stock $b_2 = b_1 - x_1 + y_1$ will remain available at the beginning of the next month.

So. $$f_4(b_1) = \max_{x_1, y_1} \{p_1x_1 - c_1y_1 + f_3(b_1 - x_1 + y_1)\}$$

where $0 \le x_1 \le b_1, y_1 \ge 0$ and $b_1 - x_1 + y_1 \le 500$

or $$f_4(b_1) = \max_{\substack{0 \le x_1 \le b_1 \\ 0 \le y_1 \le 500 - b_1 + x_1}} \{28x_1 - 27y_1 + 25\,(b_1 - x_1 + y_1) + 1500\}$$

$$= \max_{0 \le x_1 \le b_1} \{3x_1 - 2y_1 + 25b_1 + 1500\}$$

which occurs at $x_1 = b_1, y_1 = 0$

$\therefore$ Optimal decisions are $x_1 = b_1$, $y_1 = 0$ and $f_4(b_1) = 28b_1 + 1500$ ...(4)

But the stock at the beginning of Ist month is 200 units.

$\therefore$ $b_1 = 200$, so that $x_1 = 200\ y_1 = 0$

$b_2 = b_1 - x_1 + y_1 = 200 - 200 + 0 = 0$, $x_2 = b_2 = 0$, $y_2 = 500$

$b_3 = b_2 - x_2 + y_2 = 0 - 0 + 500 = 500$, $x_3 = 0$, $y_3 = 500 - b_3 = 0$

$b_4 = b_3 - x_3 + y_3 = 500 - 0 + 0 = 500$, $x_4 = b_4 = 500$, $y_4 = 0$

Hence, the optimal solution for 4 years is

Month	(i)	1	2	3	4
Purchase	(yi)	0	500	0	0
Sale	(xi)	200	0	0	500

and maximum profit = $f_4(b_1) = 28 \times 200 + 1500$ = Rs. 7100.

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4.20 SOLUTION OF LINEAR PROGRAMMING PROBLEM BY DYNAMIC PROGRAMMING

Consider the general linear programming problem

Maximize $$Z = \sum_{j=1}^{n} c_j x_j$$

subject to, $$= \sum_{j=1}^{n} a_{ij} x_j \le b_i \,;\, i = 1, 2, ..., m$$

$$x_j \ge 0 \,;\, i = 1, 2, ..., n.$$

We consider determination of variable x_j (activity j), ($j = 1, 2, ..., n$) as a stage. So it is a n stage problem. The value of x_j at several stages can be obtained by the forward computational procedure or the backward computational procedure. The state variables at each stage are the amount of resources available for allocation to the current stage and subsequent stages. The constants $b_1, b_2, ..., b_m$ are the amounts of the available resources.

Let $f_n(b_1, b_2, ..., b_m)$ be the optimal value of the objective function defined above for stages $x_1, x_2, ..., x_n$ for states $b_1, b_2, ..., b_m$. We Shall use backward computational procedure here.

The recursive equation for optimization is

$$f_j(b_1, b_2, ..., b_m) = \max_{0 \le a_{ij} x_j \le b_i} \{c_j x_j + f_{j+1}(b_1 - a_{1j} x_j, b_2 - a_{2j} x_j, ..., b_m - a_{mj} x_j)\}$$

Example 4.17. *Solve the following linear programming problem by dynamic programming:*

Maximize $Z = 5x_1 + 7x_2$

subject to, $= x_1 + x_2 \le 4$

$$3x_1 + 8x_2 \le 24$$

$$10x_1 + 7x_2 \le 35$$

$$x_1, x_2 \ge 0$$

Solution.

1. Stage i corresponds to $i = 1, 2$
2. Alternative x_i for $i = 1, 2$
3. State (u_2, v_2, w_2) represents the amounts of resoures 1, 2 and 3 used in stage 2.
4. State (u_1, v_1, w_1) represents the amounts of resourecs 1, 2 and 3 used in stages 1 and 2.

Stage 2. Define $f_2(u_2, v_2, w_2)$ as the maximum profit for stage 2 given the state (u_2, v_2, w_2). Then

$$f_2(u_2, v_2, w_2,) = \max_{\substack{0 \le x_2 \le u_2 \\ 0 \le 8x_2 \le v_2 \\ 0 \le 7x_2 \le w_2}} \{7x_2\}$$

Thus max $\{7x_2\}$ occurs at $x_2 = \min\{u_2, v_2, w_2\}$ and the solution for stage 2 is

State	Optimum solution	
	$f_2(u_2, v_2, w_2)$	x_2
(u_2, v_2, w_2)	$7 \min (u_2, v_2, w_2)$	$\text{Min } (u_2, v_2, w_2)$

Stage 1.

$$f_1(u_1, v_1, w_1) = \max_{\substack{0 \le x_1 \le u_1 \\ 0 \le 3x_1 \le v_1 \\ 0 \le 10x_1 \le w_1}} \{5x_1 + f_2(u_1 - x_1, \frac{v_1 - 3x_1}{8}, \frac{w_1 - 10x_1}{7})\}$$

$$= \max_{\substack{0 \le x_1 \le u_1 \\ 0 \le 3x_1 \le v_1 \\ 0 \le 10x_1 \le w_1}} \{5x_1 + 7 \min (u_1 - x_1, \frac{v_1 - 3x_1}{8}, \frac{w_1 - 10x_1}{7})\}$$

The optimization of stage 1 requires the solution of a minimax problem. For this problem, we get $u_1 = 4$, $v_1 = 24$ and $w_1 = 35$ which gives $0 \le x_1 \le 4$, $0 \le 3x_1 \le 24$ and $0 \le 10x_1 \le 35$.

Because $\min\left(4 - x_1, \frac{24 - 3x_1}{8}, \frac{35 - 10x_1}{7}\right)$ is the lower envelope of the two intersecting lines it follows that

$$\min\left(4 - x_1, \frac{24 - 3x_1}{8}, \frac{35 - 10x_1}{7}\right) = \begin{cases} \frac{24 - 3x_1}{8} & ; \quad 0 \le x_1 \le 8/5 \\ 4 - x_1 & ; \quad 8/5 \le x_1 \le 7/3 \\ \frac{35 - 10x_1}{7} & ; \quad 7/3 \le x_1 \le 7/2 \end{cases}$$

[To check minimum put $x_1 = 0$ and select minimum one so it will be lower limit of minimum (Ist) one and for upper limit of Ist and lower limit of IInd solve Ist and IInd for x_1. Now for upper limit of IInd and lower limit of IIIrd solve IInd and IIIrd for x_1 and for upper limit of IIIrd one take

$$x_1 = \min(u_1, v_1, w_1) = \min\left(4, \frac{24}{3}, \frac{35}{10}\right) = 3.5 = \frac{7}{2}$$]

Now $$f_1(4, 24, 35) = \max_{x_1} \begin{cases} 5x_1 + \frac{7}{8}(24 - 3x_1) & ; \quad 0 \le x_1 \le 8/5 \\ 5x_1 + 7(4 - x_1) & ; \quad 8/5 \le x_1 \le 7/3 \\ 5x_1 + (35 - 10x_1) & ; \quad 7/3 \le x_1 \le 7/2 \end{cases}$$

$$= \max_{x_1} \begin{cases} \frac{19}{8} x_1 + 21 & ; \quad 0 \le x_1 \le 8/5 \\ -2x_1 + 28 & ; \quad 8/5 \le x_1 \le 7/3 \\ -5x_1 + 35 & ; \quad 7/3 \le x_1 \le 7/2 \end{cases}$$

$$= \max \begin{cases} 21 & \text{for} \quad x_1 = 0 \\ \frac{124}{5} & \text{for} \quad x_1 = 8/5 \\ \frac{70}{3} & \text{for} \quad x_1 = 7/3 \\ \frac{35}{2} & \text{for} \quad x_1 = 7/2 \end{cases}$$

$$= \frac{124}{5} \text{ at } x_1 = \frac{8}{5}$$

State	Optimum solution	
	$f_1(u_1, v_1, w_1)$	x_1
(4, 24, 35)	$\frac{124}{5}$	$\frac{8}{5}$

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To determine the optimum value of x_2, we note that

$$u_2 = u_1 - x_1 = 4 - \frac{8}{5} = \frac{12}{5}\,;\ v_2 = \frac{v_1 - 3x_1}{8} = \frac{24 - 24/5}{8} = \frac{12}{5}\,;$$

$$w_2 = \frac{w_1 - 10x_1}{7} = \frac{35 - \frac{80}{5}}{7} = \frac{19}{7}$$

So $$x_2 = \min(u_2, v_2, w_2) = \min\left(\frac{12}{5}, \frac{12}{5}, \frac{19}{7}\right) = \frac{12}{5}$$

Hence, the optimum solution is

$$x_1 = \frac{8}{5},\ x_2 = \frac{12}{5} \text{ and Maximize } Z = \frac{124}{5}$$

SUMMARY

- Since the objective function and the constraints are supposed to be non-linear, it becomes more difficult to distinguish between local and global solution. Moreover, sometimes it becomes more difficult to test the optimality of the non-linear programming problems, especially when the feasible region is not convex. Therefore, the solution of non-linear programming becomes more difficult than the linear programming and there is no single method to solve such problems.
- In case the non-liner programming problem is composed of some differentiable objective function and equality side conditions or constraints, the optimization may be achieved by the use of Lagrange's Multipliers method which provides a necessary condition of an optimum when contraints are equations.

  consider the problem of maxximizing or minimizing

  $$Z = f(x_1, x_2)$$

  subject to the contraints

  $$g(x_1, x_2) = C$$

  and $$x_1, x_2 \geq 0$$

  where C is a constant.
- Let x^T and $C \in R^n$ and Q be a symmetric $n \times n$ real matrix. Then, the problem of maximizing (determining x) so as to maximize.

  $$f(x) = Cx + \frac{1}{2} x^T Qx$$

  subject to the constraints

  $$Ax \leq b$$

  and $$x \geq 0$$

  where $b^T \in R^m$ and A is an $m \times n$ real matrix, is called a general quadratic programming problem (GQPP.)
- A convex programming problem consists of a convex feasible set X and a convex cost function c:

  $$X \rightarrow R.$$

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The optimal solution is the solution that minimizes the cost.

- In dynamic programming the term 'dynamic' stands for its usefulness in problems where decisions have to be taken at several distinct stages and 'programming' is used in a mathematical sense of selecting an optimum allocation of resources. In dynamic programming a large problem having n variables is divided into n subproblems (stages), each of which starting from one stage to the next and continued till the final stage is reched.

GLOSSARY

- **Dynamic** : Stands for usefulness in problems where decisions have to be taken at several distict stages.
- **Programming** : Is used in a mathematical sence of selecting an optimum allocation of resources.
- **Stage** : In a problem, decision must be made on a point, which is known as stage.
- **States** : States are various possible conditions in which the system might find itself at the stage of the problem.

REVIEW QUESTIONS

1. What is a non-linear programming problem ?
2. Give the general form of NLPP. Also explain unconstrained optimization.
3. Derive the Kuhn-Tucker conditions for the non-linear programming problem:

   Maximize $f(x)$

   Subject to constraints

   $$g_i(x) \le b_i$$
   $$x \ge 0 \qquad i = 1, 2, \ldots, m.$$
4. State and prove Kuhn-Tucker necessary and sufficient conditions in non-linear programming.
5. Write a short note on Kuhn-Tucker conditions.
6. Solve the following non-linear programming problem, using the method of Lagrangian multipliers.

   $$\text{Minimize} = 6x_1^2 + 5x_2^2$$

   Subject to constraints

   $$x_1 + 5x_2 = 3$$
   $$x_1, x_2 \ge 0$$

   [**Ans.** $x_1 = 3/31$, $x_2 = 18/31$, minimum $Z = 54/31$]
7. Use the Kuhn-Tucker conditions to solve the NLPP.

   $$\text{Minimize } Z = x_1^2 + x_2^2 + x_3^2$$

   Subject to constraints

   $$2x_1 + x_2 \le 5$$
   $$x_1 + x_2 \le 2$$
   $$x_1 \ge 1$$
   $$x_2 \ge 2$$
   $$x_3 \ge 0$$

   [**Ans.** $x_1 = 1$, $x_2 = 2$, $x_3 = 0$, min. $Z = 5$]
8. Solve the NLPP graphically:

   $$\text{Maximize } Z = x_1 + 2x_2$$

   Subject to constraints

   $$x_1^2 + x_2^2 \le 1$$
   $$2x_1 + x_2 \le 2$$

$x_1, x_2 \geq 0.$ [**Ans.** $x_1 = 0.6, x_2 = 0.8$, max. Z = 2.20]

9. Maximize $Z = x_1$

Subject to constraints

$$(3 - x_1)^3 - (x_2 - 2) \geq 0$$
$$(3 - x_2)^3 + (x_2 - 2) \geq 0$$
$$x_1, x_2 \geq 0$$

Also show that the Kuhn-Tucker necessary conditions for the maxima do not hold in this case. [**Ans.** $x_1 = 3, x_2, 2$, max. Z = 3, constraint qualification is not satisfied]

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10. What is meant by quadratic programming? How does quadratic programming problem differ from the linear programming problem ?

11. What is quadratic programming ? Explain Wolfe's method of solving it.

12. Mention briefly the Wolfe's algorithm for solving a quadratic programming problem given in the usual notations.

$$\text{Maximize } Z = f(x) + \frac{1}{2} X^T QX$$

Subject to $AX \leq b$

and $X \geq 0$

13. Use Wolfe's method in solving the following QPP.

Maximize $Z = 8X_1 + 10X_2 - X_1^2 - X_2^2$

Subject to the constraints

$$3X_1 + 2X_2 \leq 6$$
$$X_1, X_2 \geq 0$$

[**Ans.** $X_1 = 4/13$, $X_2 = 33/13$, max. Z = 267/13]

14. Use Wolfe's method to solve the following problem.

Minimize $Z = X_1^2 + X_2^2 + X_3^2$

Subject to constraints

$$X_1 + X_2 + 3X_3 = 2$$
$$5X_1 + 2X_2 + X_3 = 5$$
$$X_1, X_2, X_3 = 0$$

[**Ans.** $X_1 = 0.81$, $X_2 = 0.35$, $X_3 = 0.35$ min. Z = 0.857]

15. Explain dynamic programming. What are applications of dynamic programming?

16. State the 'Bellman's Principle of optimality' in dynamic programming and give a mathematical formulation of dynamic programming.

17. Define dynamic programming problem. List and explain the terminologies of dynamic programming problem, also applications of this technique.

18. What do you understand from multistage decision process ? Give example. State the 'Principle of optimality' and explain it giving examples. What do you understand from the 'concept of sub-optimization'?

19. State the essential characteristics of dynamic programming problems.

20. State the various steps involved for solving the multistage problem by dynamic programming.

21. Use dynamic programming to solve the following problem:

Maximize $Z = x_1 . x_2 . x_3$

subject to, $x_1 + x_2 + x_3 = 5$

$x_1, x_2, x_3 \geq 0$

[**Ans.** $x_1 = x_2 = x_3 = \frac{5}{3}$, max $Z = \frac{125}{27}$]

22. *Use dynamic programming to solve the following problem:*

Minimize $Z = x_1^2 + x_2^2 + x_3^2$

subject to, $x_1 + x_2 + x_3 = 5$

$x_1, x_2, x_3 \geq 0$

[**Ans.** $x_1 = x_2 = x_3 = \frac{5}{3}$, max $Z = \frac{25}{3}$]

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23. Use dynamic programming to solve the following problem:

Minimize $Z = x_1^2 + x_2^2 + x_3^2$

subject to, $x_1 + x_2 + x_3 = 27$

$x_1, x_2, x_3 \geq 0$ [**Ans.** $x_1 = x_2 = x_3$, max $Z = 27$]

24. Use dynamic programming to solve the following linear programming problem:

Maximize $Z = 8x_1 + 7x_2$

subject to, $2x_1 + x_2 \leq 8$

$5x_1 + 2x_2 \leq 15$

$x_1, x_2 \geq 0$ [**Ans.** $x_1 = 3, x_2 = 0$; max $Z = 24$]

25. Use dynamic programming to solve the following linear programming problem:

Maximize $Z = 50x_1 + 100x_2$

subject to, $2x_1 + 3x_2 \leq 48$

$x_1 + 3x_2 \leq 42$

$x_1 + x_2 \leq 21$

$x_1, x_2 \geq 0$ [**Ans.** $x_1 = 6, x_2 = 12$; max $Z = 1500$]

26. Use dynamic programming to solve the following linear programming problem:

Maximize $Z = 3x_1 + 4x_2$

subject to, $2x_1 + x_2 \leq 40$

$2x_1 + 5x_2 \leq 180$

$x_1, x_2 \geq 0$ [**Ans.** $x_1 = \frac{5}{2}, x_2 = 35$; max $Z = 147.50$]

FURTHER READINGS

- *Operational Research,* by col. D.S. Cheema, University Science Press.
- *Statistics and Operational Research – A Unified Approach,* by Dr. Debashis Dutta, Laxmi Publications (P) Ltd.

UNIT V: QUEUING THEORY

NOTES

★ STRUCTURE ★

5.0 LEARNING OBJECTIVES

After going through this unit, you should be able to:

- explain queues and basic elements of queuing models.
- describe role of exponential and Poisson distributions.
- define Markovian process and Erlang distribution, distribution of arrivals, distribution of service times and steady and transient state .

5.1 INTRODUCTION

Queues of customers arriving for service of one kind or another arise in many different fields of activity. Businesses of all types, government, industry, telephone exchanges, and airports, large and small, all have queuing problems. Many of these congestion situations could benefit from OR analysis, which employs to this *purpose* a variety of queuing models, referred to as queuing systems or simply queues.

A queuing system involves a number of servers (or serving facilities) which we will also call *service channels* (in deference to the source field of the theory telephone communication system). The serving channels can be communications links, work stations, check-out counters, retailers, elevators, buses, to mention but a few. According to the number of servers, queuing systems can be of single and multi-channel type.

Customers arriving at a queuing system at random intervals of time are serviced generally for random times too. When a service is completed, the customer leaves the servicing facility rendering it empty and ready and get next arrival. The random nature of arrival and service times may be a cause of congestion at the input to the system at some periods when the incoming customers either queue up for service or leave the system unserved ; in other periods the system might not be completely busy because of the lack of customers, or even be idle altogether.

A queuing system operation is a random process with discrete states and continuous time. *The state changes jump-wise at the instant some events occurs : an arrival occurs,* a service is completed, or a customer unable to wait any longer leaves the queue.

The subject matter of queuing theory is to build mathematical models, which relate the specified operating conditions for the system (number of channels, their servicing mechanism, distribution of arrivals) to the concerned characteristics of value-measures of effectiveness describing the ability of the system to handle the incoming demands. Depending on the circumstances and the objective of the study, such measures may be : the expected (mean) number of arrivals served per unit time, the expected number of busy channels, the expected number of customers in the queue and the mean waiting time for service, the probability that the number in queue is above some limit, and so on. We do not single out purposely among intended for the given operating conditions, those intended for decision variables, since they may be either of these characteristics, for example, the number of channels, their capacity, service mechanism, etc. The most important part of a study is model establishment (primal problems) while its optimisation (inverse problem) is indeed depending on which parameters are selected to work with or to change. We are not going to consider optimization of queuing models in this text with the exception made only for the simplest queuing situations.

The mathematical analysis of a queuing system simplifies considerably when the process concerned is Markovian. Markov process may be defined as , 'A random process is referred to as Markov, if for any moment of time, its probability characteristics in the future 'depend only on *its state at time and are to independent of when and how this* state was acquired.' As we already know a sufficient condition for this is that all the process changing system's states (arrival intervals, service intervals) are Poisson. If this property does not hold, the mathematical description of the process complicates substantially and acquires an explicit analytical form only in seldom cases. However, the simplest mathematics of Markov queues may prove of value for approximate handling even of those queuing problems whose arrivals are distributed not in a Poisson process. In many situations a reasonable decision on queuing system organization suffices with an approximate model.

All of the queuing systems have certain common basic characteristics. They are (*a*) *input process (arrival pattern)* which may be specified by the source of arrivals, type of arrivals and the inter-arrival times, (*b*) *service mechanism which* is the duration and mode of service and may be characterized by the service-time distribution, capacity of the *system, and service availability, and (c)* queue discipline which includes all other factors regarding the rules of conduct of the queue.

We start illustrating the classification breakdown with a *loss* and *delay system*. In a purely loss system, customers arriving when all the servers are busy are denied service and are lost to the system. Examples of the loss system may be met in telephony : an incoming call arrived at an instant when all the channels are busy cannot be placed and leaves the exchange unserved. In a *delay system* an arrival incoming when all the channels are busy does not leave the system but joins the queue and waits (if there is enough waiting room) until a server is free. These latter situations more often occur in applications and are of great importance, which can be readily inferred from the name of the theory.

According to the type of the source-supplying customers to the system, the models are divided into those of a finite population size, when the customers are only few, and the infinite-population systems. The length of the queue is subject to further limitations *imposed by allowable waiting time or handling of impatient customers* which are liable to be lost to the system.

The queue discipline, that is the rule followed by the server in taking the customers in service, may be according to such self-explanatory principles as "first-come, first-served", "last-come, first-served", or chain "random selection for serve". In some situations priority disciplines need be introduced to allow for realistic queues with high priority arrivals. To illustrate, in extreme cases the server may stop the service of a customer of lower priority in order to deal with a customer of high priority ; this is called *pre-emptive* priority. For example a gantry crane working on a container ship may stop the unloading halfway and shift to another load to unload perishable goods of a later arrived ship. The situation when a service of a low priority customer started prior to the arrival of a high priority customer is completed and the high priority customer receives only a better position in the queue is called *non-pre-emptive* priority. This situation can be exemplified by an airplane which enters a queue of a few other aircraft circling around an airport and asks a permission for an emergency landing ; the ground control issues the permission on the condition that it lands next to the airplane being on a runway at the moment.

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Turning over to the service mechanism, we may find systems whose servicing channels are placed *in parallel* or *in series*. When in series, a customer leaving a previous server enters a queue for the next channel in the sequence. For example, a work piece being through the operation with one robot on a conveyor is stacked to wait when the next robot in the process is free to handle it. These operation stages of a series-channel queuing system are called *phases*. The arrival pattern may and may not correlate with the other aspects of the system. Accordingly, the system can be loosely divided into "Open" and "Close". In an open system, the distribution of arrivals does not depend on the status of the system. say on how many channel are busy. To contrast, in a close system, in does. For example, if a single operator tends a few similar machines each of which has a chance of stopping *i.e.*, arriving for serve, at random, then the arrival rate of stopping depends on how many mechanics have been already adjusted and put on the yet served.

An optimisation of a queuing system may be attempted from either of two standpooints, the first in favour of "queuers" or owners of the queue, the second to favour the "queuers", *i.e.*, the customers. The first stand makes a point of the efficiency of the system and would tend to load all the channels as high as possible, *i.e.*, to cut down ideal times. The customers on the contrary would like to cut down waiting time in a queue. Therefore, any optimisation of congestion necessitates a "system approach" with the intrinsic complex evaluation and assessment of all consequences for each possible decision. The need for optimality over conflicting requirements may be illustrated with the viewpoint of the customer wishing to increase the number of channels, which, however, would increase the total servicing cost. The development of a reasonable model may help solving the optimisation problem by choosing the number of channels which account for all pros and cons. This is the reason why we do not suggest a single measure of effectiveness for all queuing problems, formulating them instead as multiple objective problems.

All the mentioned forms of queues (and many others for which we give no room here) are being studied by queuing theory where there is a huge literature. The discussion, though, almost nowhere tunes an appropriate methodological level : the derivations are often too complicated, and deduced not in the very best way.

5.2 BENEFITS OF QUEUING THEORY

Queuing theory has been used for many real life applications to a great advantage. It is so because many problems of business and industry can be assumed/simulated to be arrival-departure or queuing problems. In any practical life situations, it is not possible to accurately determine the arrival and departure of customers when the number and types of facilities as also the requirements of the customers are not known. Queuing theory techniques, in particular, can help us to determine suitable number and type of

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service facilities to be provided to different types of customers. Queuing theory techniques can be applied to problems such as :

(*a*) Planning, scheduling and sequencing of parts and components to assembly lines in a mass production system.

(*b*) Scheduling of workstations and machines performing different operations in mass production.

(*c*) Scheduling and dispatch of war material of special nature based on operational needs.

(*d*) Scheduling of service facilities in a repair and maintenance workshop.

(*e*) Scheduling of overhaul of used engines and other assemblies of aircrafts, missile systems, transport fleet etc.

(*f*) Scheduling of limited transport fleet to a large number of users.

(*g*) Scheduling of landing and take off from airports with heavy duty of air traffic and limited facilities.

(*h*) Decision of replacement of plant, machinery, special maintenance tools and other equipment based on different criteria.

Special benefit which this technique enjoys in solving problems such as above are:

(*i*) Queuing theory attempts to solve problems based on a scientific understanding of the problems and solving them in optimal manner so that facilities are fully utilised and waiting time is reduced to minimum possible.

(*ii*) Waiting time (or queuing) theory models can recommend arrival of customers to be serviced, setting up of workstations, requirement of manpower *etc.*, based on probability theory.

5.3 LIMITATION OF QUEUING THEORY

Though queuing theory provides us a scientific method of understanding the queues and solving such problems, the theory has certain limitations which must be understood while using the technique, some of these are :-

(*a*) Mathematical distributions, which we assume while solving queuing theory problems, are only a close approximation of the behaviour of customers, time between their arrival and service time required by each customer.

(*b*) Most of the real life queuing problems are complex situation and very difficult to use the queuing theory technique, even then uncertainty will remain.

(*c*) Many situations in industry and service are multi-channel queuing problems. When a customer has been attended to and the service provided, it may still have to get some other service from another service and may have to fall in queue once again. Here the departure of one channel queue becomes the arrival of the other channel queue. In such situations, the problem becomes still more difficult to analyse.

(*d*) Queuing model may not be the ideal method to solve certain very difficult and complex problems and one may have to resort to other techniques like Monte-Carlo simulation method.

5.4 IMPORTANT TERMS USED IN QUEUING THEORY

1. **Arrival Pattern.** It is the pattern of the arrival of a customer to be serviced. The pattern may be regular or at random. Regular interval arrival patterns are rare, in most of the cases of the customers cannot be predicted. Remainder pattern of arrival of customers follows Poisson's distribution.

2. **Poisson's Distribution.** It is discrete probability distribution which is used to determine the number of customers in a particular time. It involves allotting probability of occurrence of the arrival of a customer. Greek letter λ (lamda) is used to denote mean arrival rate. A special feature of the Poisson's distribution is that its mean is equal to the variance. It can be represented with the notation as explained below.

$P(n)$ = Probability of n arrivals (customers)

l = Mean arrival rate

e = Costant = 2.71828

$$P(n) = \frac{e^{-\lambda} (\lambda)^n}{\underline{|n}} \text{ where } n = 0, 1, 2, \ldots$$

Notation $\underline{|\ }$ or ! is called the factorial and it mean that

$$\underline{|n} \text{ or } n! = n(n-1)(n-2)(n-3) \ldots 2, 1$$

Poisson's distribution tables for different values of n and λ are available and can be used for solving problems where Poisson's distribution is used. However, It has certain limitations because of which its used is restricted. It assumes that arrivals are random and independent of all other variables or other variables or parameters. Such can never be the case.

3. **Exponential Distribution.** This is based on the probability of completion of a service and is the most commonly used distribution in queuing theory. In queuing theory, our effort is to minimise the total cost of queue and it includes cost of waiting and cost of providing service. A queue model is prepared by taking different variables into consideration. In this distribution system, no maximisation or minimisation is attempted. Queue models with different alternatives are considered and the most suitable for a particular is attempted. Queue models with different alternatives are considered and the most suitable for a particular situation is selected.

4. **Service Pattern.** We have seen that arrival pattern is random and poissons distribution can be used for use in queue model. Service pattern are assumed to be exponential for purpose of avoiding complex mathematical problem.

5. **Channels.** A service system has a number of facilities positioned in a suitable manner. These could be

   (*a*) **Single Channel – Single Phase System.** This is very simple system where all the customers wait in a single line in front of a single service facility and depart after service is provided. In a shop if there is only one person to attend to a customer is an example of the system.

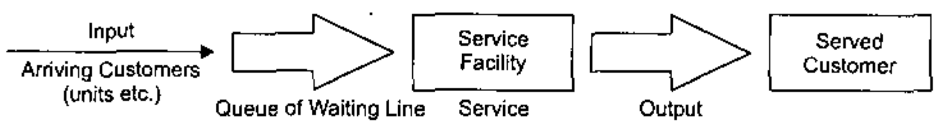

Fig. 5.1

   (*b*) **Service in Series.** Here the input gets serviced at one service station and then moves to second and or third and so on before going out. This is the case when a raw material input has to undergo a number of operations like cutting, turning drilling etc.

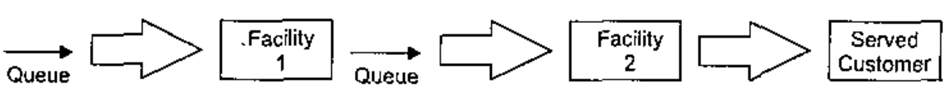

Fig. 5.2

5.5 MULTI PARALLEL FACILITY WITH A SINGLE QUEUE

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Here the service can be provided at a number of points to one queue. This happens when in a grocery store, there are 3 persons servicing the same queue or a service station having more than one facility of washing cars.

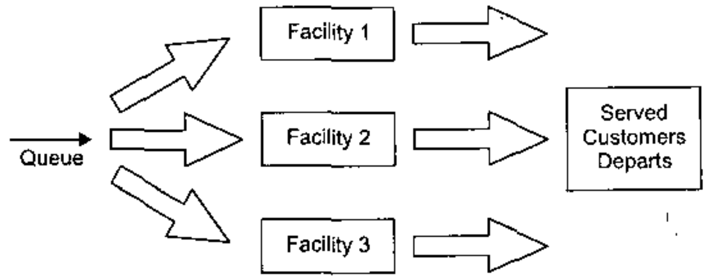

Fig. 5.3

Multiple parallel facilities with multiple queue. Here there are a number of queues and separate facility to service each queue. Booking of tickets at railway stations, bus stands *etc*, is a good example of this. This is shown in Fig. 5.4.

6. **Service Time.** Service time *i.e.*, the time taken by the customer when the facility is dedicated to it for serving depends upon the requirement of the customer and what needs to be done as assessed by the facility provider. The arrival pattern is random so also is the service time required by different customers. For the sake of simplicity the time required by all the customers is considered constant under the distribution. If the assumption of exponential distribution is not valid, Erlang Distribution is applied to the queuing model.
7. **Erlang Distribution.** It has been assumed in the queuing process we have seen that service is either constant or it follows negative exponential distribution in which case the standard deviation s (sigma) is equal to its mean. This assumption makes the use of the exponential distribution simple. However, in cases where σ and mean are not equal, Erlang distribution developed by AK Erlang is used. In this method, the service time is divided into number of phases assuming that total service can be provided by different phases of service. It is assumed that service time of each phase follows the exponential distribution *i.e.*, σ = mean.
8. **Traffic Intensity or Utilisation Rate.** This is the rate of at which the service facility is utilised by the components.

   If λ = mean arrival rate and

   (Mue) μ = Mean service rate, then utilisation rate $(p) = \lambda/\mu$ it can be easily seen from the equation that $p > 1$ when arrival rate is more than the service rate and new arrivals will keep increasing the queue. $p < 1$ means that service rate is more than the arrival rate and the waiting time will keep reducing as μ keeps increasing. This is true from the commonsense.

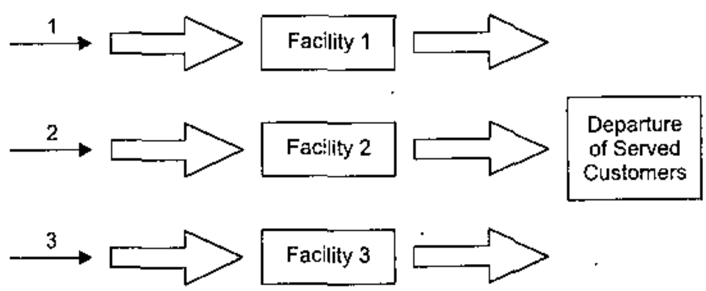

Fig. 5.4

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9. **Idle Rate.** This is the rate at which the service facility remains unutilised and is lying idle.

$$\text{Idle rate} = 1 - \text{utilisation rate} = 1 - p = \left(1 - \frac{\lambda}{\mu}\right) \times \text{total service facility} = \left(1 - \frac{\lambda}{\mu}\right) \times \frac{\lambda}{\mu}.$$

10. **Expected number of customers in the system.** This is the number of customers in queue plus the number of customers being serviced and is denoted by $\text{En} = \frac{\lambda}{(\mu - \lambda)}$.

11. **Expected number of customers in queue (Average queue length.).** This is the number of expected customers minus the number being serviced and is denoted by Eq.

$$\text{Eq} = \text{En} - p = \frac{\lambda}{(\mu - \lambda)} - \frac{\lambda}{\mu} = \frac{\lambda^2}{\mu(\mu - \lambda)}.$$

12. **Expected time spent by customer in system.** It is the time that a customer spends waiting in queue plus the time it takes for servicing the customer and is denoted by Et

$$\text{Et} = \frac{\text{En}}{\lambda} = \frac{\frac{\lambda}{(\mu - \lambda)}}{\lambda} = \frac{1}{(\mu - \lambda)}.$$

13. **Expected waiting time in queue.** It is known that Et = expected waiting time in queue + expected service time, therefore expected waiting time in queue (Ew) = $\text{Et} - \frac{1}{\mu}$.

14. **Average length of non-empty queue.** $\text{El} = \frac{\mu}{(\mu - \lambda)} = \frac{1}{(\mu - \lambda)} - \frac{1}{\mu} = \frac{\lambda}{\lambda(\mu - \lambda)}$.

15. **Probability that customer wait is zero.** It means that the customer is attended to for servicing at the point of arrival and the customer does not wait at all. This depends upon the utilisation rate of the service or idle rate of the system, $p_0 = 0$ persons waiting in the queue $= 1 - \frac{\lambda}{\mu}$ and the probability of 1, 2, 3 ... n persons waiting in the queue will be given by $p_1 = p_0\left(\frac{\lambda}{\mu}\right)^1$, $p_2 = p_1\left(\frac{\lambda}{\mu}\right)^2$, $p_n = p_0\left(\frac{\lambda}{\mu}\right)^n$.

16. **Queuing Discipline.** All the customers get into a queue on arrival and are then serviced. The order in which the customer is selected for servicing is known as queuing discipline. A number of systems are used to select the customer to be served. Some of these are :
    (*a*) **First in First Served (FIFS).** This is the most commonly used method and the customers are served in the order of their arrival.
    (*b*) **Last in First Served (LIFS).** This is rarely used as it will create controversies and ego problems amongst the customers. Any one who comes first expects to be served first. It is used in store management, where it is convenient to issue the store last received and is called (LIFO) *i.e.,* Last In First Out
    (*c*) **Service in Priority (SIP).** The priority in servicing is allotted based on the special requirement of a customer like a doctor may attend to a serious patient out of turn, so may be the case with a vital machine which has broken down. In such cases the customer being serviced may be put on hold

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and the priority customer attended to or the priority may be put on hold and the priority customer attended to or the priority may be on hold and the priority customer waits till the servicing of the customer already being serviced is over.

17. **Customer Behaviour.** Different types of customers behave in different manner while they are waiting in queue, some of the patterns of behaviour are :
    (*a*) **Collusion.** Some customers who do not want to wait they make one customer as their representative and he represents a group of customers. Now only the representative waits in queue and not all members of the group.
    (*b*) **Balking.** When a customer does not wait to join the queue at the correct place which he warrants because of his arrival. They want to jump the queue and move ahead of others to reduce their waiting time in the queue. This behaviour is called balking.
    (*c*) **Jockeying.** This is the process of a customer leaving the queue which he had joined and goes and joins another queue to get advantage of being served earlier because the new queue has lesser customers ahead of him.
    (*d*) **Reneging** Some customers either do not have time to wait in queue for a long time or they do not have the patience to wait, they leave the queue without being served.
18. **Queuing Cost Behaviour.** The total cost a service provider system incurs is the sum of cost of providing the services and the cost of waiting of the customers. Suppose the garage owner wants to install another car washing facility so that the waiting time of the customer is reduced. He has to manage a suitable compromise in his best interest. If the cost of adding another facility is more than offset by reducing the customer waiting time and hence getting more customers, it is definitely worth it. The relationship between these two costs is shown below.

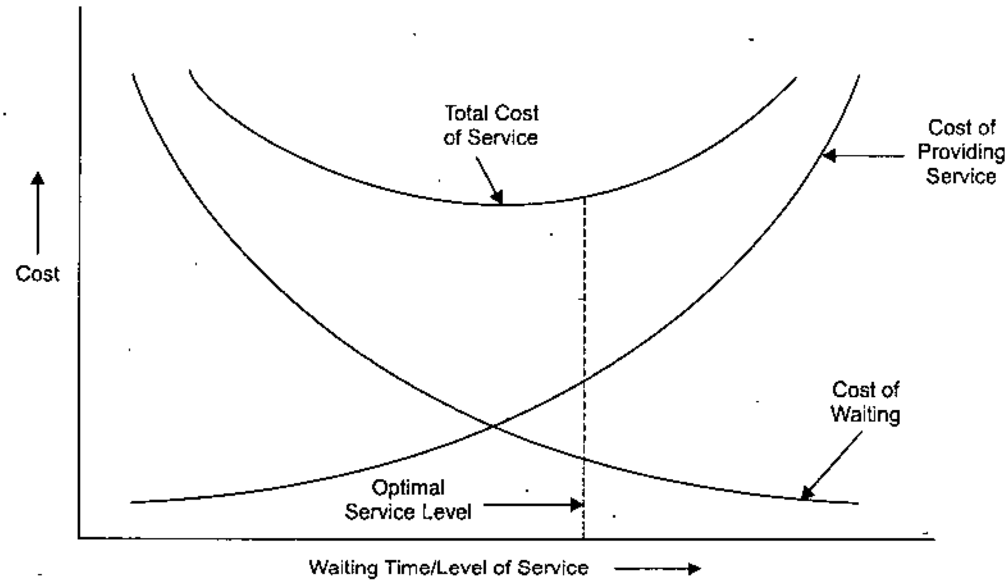

Fig. 5.5

5.6 TYPES OF QUEUING MODELS

Different types of models are in use. The three possible types of categories are :

(*a*) **Deterministic Model.** Where the arrival and service rates are known. This is rarely used as it is not a practical model.

(*b*) **Probabilitistic Model.** Here both the parameters *i.e.*, the arrival rate as also the service rate are unknown and are assumed random in nature probability distribution *i.e.*, Poissons, Exponential or Erlang distributions are used.

(*c*) **Mixed Model.** Where one of the parameters out of the two is known and the other is unknown

Single Channel Queuing Model

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(Arrival – Possion and Service time Exponential)

This is the simplest queuing model and is commonly used. It makes the following assumptions :

(*a*) Arriving customers are served on First Come First Serve (FCFS) basis.

(*b*) There is no Balking or Reneging. All the customers wait the queue to be served, no one jumps the queue and no one leaves it.

(*c*) Arrival rate is constant and does not change with time.

(*d*) New customers arrival is independent of the earlier arrivals.

(*e*) Arrivals are not of infinite population and follow Poisson's distribution.

(*f*) Rate of serving is known.

(*g*) All customers have different service time requirements and are independent of each other.

(*h*) Service time can be described by negative exponential probability distribution.

(*i*) Average service rate is higher than the average arrival rate and over a period of time the queue keeps reducing.

Example 5.1. *Assume a single channel service system of a library in a school. From past experiences it is known that on an average every hour 8 students come for issue of the books on an average rate of 10 per hour. Determine the following.*

(*a*) *Probability of the assistant librarian being idle.*

(*b*) *Probability that there are at least 3 students in system.*

(*c*) *Expected time that a student is in queue.*

Solution.

(*a*) Probability that server is idle = $\left(\frac{\lambda}{\mu}\right)\left(1-\frac{\lambda}{\mu}\right)$ in this example $\lambda = 8$, $\mu = 10$

$$p_0 = \frac{8}{10}\left(1-\frac{8}{10}\right) = 16\% = 016.$$

(*b*) Probability that at least 3 students are in the system

$$\text{En} = \left(\frac{\lambda}{\mu}\right)^{3+1} = \left(\frac{8}{10}\right)^4 = 0.4$$

(*c*) Expected time that a students is in queue

$$= \frac{\lambda^2}{\mu(\mu-\lambda)} = \frac{64}{(10\times 2)} = 3.2 \text{ hours.}$$

Example 5.2. *At a garage, car owners arrive at the rate of 6 per hour and are served at the rate of 8 per hour. It is assumed that the arrival follows Poisson's distribution and the service pattern is exponentially distributed. Determine.*

(*a*) *Average queue length.*

(*b*) *Average waiting time.*

Solution. Average arrival (mean arrival rate) $\lambda = 6$ per hour.

Average (mean) service rate $\mu = 8$ per hour.

Utilisation rate (traffic intensity) $p = \frac{\lambda}{\mu} = \frac{6}{8} = 0.75$

(a) Average queue length $El = \frac{\lambda^2}{\lambda(\mu - \lambda)} = \frac{36}{[8(8-6)]} = 2.25$ cars.

(b) Average weighting time $Et = \frac{1}{(\mu - \lambda)} = \frac{1}{2} = 30$ minutes.

Example 5.3. *An emergency facility in a hospital where only one patient can be attended to at any one time receives 96 patients in 24 hours. Based on past experiences, the hospital knows that one such patient, on an average needs 10 minutes of attention and this time would cost Rs. 20 per patient treated. The hospital wants to reduce the queue of patients from the present number to $\frac{1}{2}$ patients. How much will it cost the hospital ?*

Solution. Using the usual notations $\lambda = \frac{96}{24} = 4$ patients/hour.

$$\mu = \frac{1}{10} \times 60 = 6 \text{ patients/hour.}$$

Average expected number of patients in the queue $= Eq = \frac{\lambda^2}{\mu(\mu - \lambda)} = \frac{16}{6(6-4)} = \frac{16}{12} = \frac{4}{3}$ patients

This number is to reduced to $\frac{1}{2}$ therefore $\frac{1}{2} = \frac{16}{\mu_1(\mu_1 - 4)}$ or $\mu_1^2 - 4\mu_1 = 32$

Or

$$\mu_1^2 - 4\mu_1 - 32 = 0 \text{ or } (\mu_1 - 8)(\mu_1 + 4) = 0 \text{ or } \mu_1 = 8 \text{ patient per hours.}$$

For $\mu_1 = 8$ Average time required to attend to a patient $= \frac{1}{8} \times 60 = \frac{15}{2}$ minutes Decrease in time $= 10 - \frac{15}{2} = \frac{5}{2}$ minutes.

Budget required for each patient $= 100 + \frac{5}{2} \times 20 =$ Rs 150

Thus to decrease the queue from $\frac{4}{3}$ to $\frac{1}{2}$, the budget per patient will have to be increased from Rs 100 to Rs. 150.

Example 5.4. *A bank plans to open a single server derive-in banking facility at a particular centre. It is estimated that 28 customers will arrive each hour on an average. If on an average, it requires 2 minutes to process a customer transaction, determine*

(a) *The probability of time that the system will be idle.*

(b) *On the average how long the customer will have to wait before receiving the server.*

(c) *The length of the drive way required to accommodate all the arrivals. On the average 20 feet of derive way is required for each car that is waiting for service.*

Solution. $\lambda = 28$ per hour

$$\mu = \frac{60}{2} = 30 \text{ per hour}$$

Traffic intensity $p = \frac{\lambda}{\mu} = \frac{28}{30} = 0.93$

(a) System idle $p_0 = 1 - P = 1 - 0.93 = 0.07$
7% of the time the system will be idle.

(b) Average time a customer is waiting in the queue $Et = \frac{\lambda}{\mu(\mu - \lambda)} = \frac{28}{60} = 28$ minutes.

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(c) Average number of customers waiting $Eq = \frac{\lambda^2}{\mu(\mu - \lambda)} = 28 \times \frac{28}{60} = 13$

Length of drive way = 13 × 20 = 260 feet.

Example 5.5. *Customer arrive at a one-windows-drive-in bank according to Poisson's distribution with mean 10 per hour. Service timer per customer is exponential with mean 5 minutes. The space in front of the window, including for the service car accommodates a maximum of three cars. Other cars can wait outside the space.*

(a) *What is the probability that an arriving customer can drive directly to the space in front of the window ?*

(b) *What is the probability that an arriving customer will have to wait outside the indicated space ?*

(c) *How long is an arriving customer expected to wait before starting service ?*

Solution. Using the usual notations

Here $\lambda = 10/\text{hour}$

$$\mu = \frac{60}{5} = 12/\text{hour}$$

(a) Probability that an arriving customer can directly drive to the space in front of the window. Since a maximum of three cars can be accommodated, we must determine the total probability *i.e.*, of p_0, p_1, and p_2.

$$p_0 = \frac{(\mu - \lambda)}{\lambda} = \frac{2}{12}$$

$$p_1 = \frac{\lambda}{\mu}(\mu - \lambda)\mu = \frac{10}{12} \times \frac{2}{12} = \frac{20}{144}$$

$$p_2 = \left(\frac{\lambda}{\mu}\right)^2 (\mu - \lambda)\mu = \frac{100}{144} \times \frac{2}{12} = \frac{200}{12 \times 144}$$

Total probability $= \frac{2}{12} + \frac{20}{144} + \frac{200}{12 \times 144} = \frac{728}{144 \times 12} = 0 \times 42$

(b) Probability that an arriving customer has to wait = 1 – 0.42 = 0.58

(c) Average waiting time of a customer in the queue = Ew

$$= \frac{\lambda}{\mu(\mu - \lambda)} = \frac{10}{12(12 - 10)} = 0.417 \text{ hours}$$

= 25 minutes.

Multi Channel Queuing Model (Arrival Poisson and Service Time Exponential)

This is a common facilities system used in hospitals or banks where there are more than one service facilities and the customers arriving for service are attended to by these facilities on first come first serve basis. It amounts to parallel service points in

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front of which there is a queue. This shortens the length of the queue if there was only one service station. The customer has the advantage of shifting from a longer queue where he has to spend more time to shorter queue and can be serviced in lesser time. Following assumptions are made in this model :

(*a*) The input population is infinite *i.e.*, the customers arrive out of a large number and follow Poisson's distribution.

(*b*) Arriving customers form one queue.

(*c*) Customer are served on First come First served (FCFS) basis.

(*d*) Service time follows an exponential distribution.

(*e*) There are a number of service station (K) and each one provides exactly the same service.

(*f*) The service rate of all the service stations put together is more than arrival rate.

In this analysis we will use the following notations.

λ = Average rate of arrival

μ = Average rate of service of each of the service stations

K = Number of service stations.

Kμ = Mean combined service rate of all the service stations.

Hence ρ(row) the utilisation factor for the system = $\frac{\lambda}{K\mu}$.

(*a*) Probability that system will be idle $p_0 = \left[\sum_{n=0}^{K-1} \frac{\left(\frac{\lambda}{\mu}\right)^n}{\lfloor n} + \frac{\left(\frac{\lambda}{\mu}\right)^k}{\lfloor k(1-p)}\right]^{-1}$

(*b*) Probability of n customers in the system.

$$p_n = \frac{\left(\frac{\lambda}{\mu}\right)^n}{\lfloor n} \times p_0 n \le k$$

$$p_n = \frac{\left(\frac{\lambda}{n}\right)^n}{\lfloor k} K^{n-k} \times p_0 \; n > k$$

Expected number of customer in queue or queue length

$$Eq = \frac{\left(\frac{\lambda}{\mu}\right)^k p}{\lfloor k(1-l)^2} \times p_0.$$

(*d*) Expected number of customers in the system = En = Eq + $\frac{\lambda}{\mu}$

(*e*) Average time a customer spends in queue.

$$Ew = \frac{Eq}{\lambda}$$

(*f*) Average time a customer spends in waiting line

$$= Ew + \frac{1}{\mu}.$$

Example 5.6. *A workshop engaged in the repair of cars has two separate repair lines assembled and there are two tools stores one for each repair line. Both the stores keep in identical type of tools. Arrival of vehicle mechanics has a mean of 16 per hour and follows a Poisson distribution. Service time has a mean of 3 minutes per machine and follows an exponential distribution. Is it desirable to combine both the tool stores in the interest of reducing waiting time of the machine and improving the effecicney ?*

Solution. $\lambda = 16/\text{hours}$

$$\mu = 1\frac{1}{3} \times 60 = 20 \text{ hours}$$

Expected waiting time in queue, $Ew = \frac{\lambda}{\mu(\mu - \lambda)} = \frac{16}{20(20-16)} = 0.2 \text{ hours} = 12 \text{ minutes.}$

If we combine the two tools stores.

λ = Mean arrival rate = 16 + 16 = 32/hour K = 2, $n = 1$.

μ = Mean service rate = 20/hour

Expected waiting time in queue, $Ew = \frac{Eq}{\lambda} = \frac{\lambda\mu\left(\frac{\lambda}{\mu}\right)^k}{\lfloor k-1 (K\mu - \lambda)^2} \times p_0$.

where
$$p_0 = \left[\sum_{n=0}^{k-1} \frac{\left(\frac{\lambda}{\mu}\right)^k}{\lfloor k \left\{1 - \frac{\lambda}{k\mu}\right\}}\right]^{-1}$$

$$= \left[\sum_{n=0}^{1} \frac{\left(\frac{32}{20}\right)^n}{\lfloor n} + \frac{\left(\frac{32}{20}\right)^2}{\lfloor 2 - \left\{1 - \frac{32}{2 \times 24}\right\}}\right]^{-1}$$

$$= 0.182$$

$$Ew = \frac{Eq}{\lambda} \times p_0$$

$$\frac{Eq}{\lambda} = \frac{32\left(\frac{32}{20}\right)}{\lfloor 2 - 1 (40 - 32)^2} = \frac{32}{25}$$

Hence
$$Ew = \frac{32}{25} \times 0.182 = 14 \text{ minutes.}$$

Since the waiting time in queue has increased, it is not desirable to combine both the tools stores. Present system is more efficient.

Example 5.7. *XYZ is a large corporate house having two independent plants A and B working next to each other. Its production manager is concerned with increasing the overall output and so has suggested the two plants being combined with facilities in both the plants. The maintenance manager has indicated that at least 6 breakdown occur in plants A and B each in 12 hours shift and it follows the Poisson's distribution. He feels that when both the plants are combined on an average 8 breakdowns per shift will take place following Poisson's distribution. The existing service rate per shift is 9 and follows exponential distribution. The company management is considering tow options, one combining the two plants. This will increase the average service rate to 12, second retaining the two plants A and B and the capacity of serving in this will be 10 servicing per shift in each of the plants. Servicing/repair time follows exponential distribution. Which alternative will reduce the customer waiting time ?*

Solution. (First alternative) combining two plants.

$$\lambda = 8$$
$$\mu = 12$$

$$p_0 = 1 - \frac{\lambda}{\mu} = 1 - \frac{8}{12} = \frac{1}{3} = 0.33$$

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Expected number of machines waiting for service (in queue) $Eq = \frac{\lambda^2}{\mu(\mu - \lambda)} = \frac{64}{48} = 1.33$

Expected time before a machine is repaired or

(Expected time spent by machine in a system) $\frac{1}{(\mu - \lambda)} = \frac{1}{12 - 8} = 0.25$ hours $= 15$ minutes

Second alternative (two channels)/having two plants,.

$$p_0 = \left[\sum_{n=0}^{k-1} \frac{\left(\frac{\lambda}{\mu}\right)}{\underline{|n}} + \frac{\left(\frac{\lambda}{\mu}\right)^k}{\underline{|k}\left\{1 - \left(\frac{\lambda}{k\mu}\right)\right\}} \right]^{-1}$$

Here $k = 2, \lambda = 6$

$\mu = 10$

$$p_0 = \left[1 + \frac{\frac{6}{10}}{\underline{|1}} + \frac{\frac{36}{100}}{2\left(1 - \frac{6}{20}\right)} \right]^{-1}$$

$$p_0 = \left[1 + \left(\frac{6}{10}\right) + \left(\frac{36}{100} \times \frac{20}{28}\right) \right]^{-1} = \left(1 + \frac{6}{10} + \frac{36}{140}\right)^{-1}$$

$$= \left[\frac{140 + 84 + 36}{140}\right]^{-1} = \left(\frac{26}{14}\right)^{-1} = \frac{14}{26} = 0.54$$

$$Ew = \frac{Eq}{\lambda} = \frac{\mu\left(\frac{\lambda}{\mu}\right)^k}{\underline{|k-1}(k\mu - \lambda)^2} \times p_0 = \frac{10 \times \frac{36}{100}}{196} \times 0.54$$

$$= \frac{36}{19600} \times 0.54$$

or $\frac{10\left(\frac{6}{10}\right)^2}{\underline{|2} - 1(2 \times 10 - 6)^2} \times 0.54 = \frac{36}{100 \times 196} \times 0.54$

Expected number of machines waiting for services $= \frac{36}{19600} \times 0.54 = 0.0018$

Expected time before a machine is repaired $= 0.0018$ hours $+ \frac{1}{\mu} = 0 \times 108$ hours

In 8 hours $= 8 \times 0108$ hours $= 52$ minutes

Single channel or combined facility has less waiting time as compared to having two plants hence, combining the two plants is preferable.

Example 5.8. *At a polyclinic three facilities of clinical laboratories have been provided for blood testing. Three Lab technicians attend to the patients. The technicians are equally qualified and experienced and they take 30 minutes to serve a patient. This average time follows exponential distribution. The patients arrive at an average rate of 4 per hour and this follows Poissons's distribution. The management in interested in finding out the following :*

(a) Expected number of patients waiting in the queue.

(b) Average time that a patient spends at the polyclinic.

(c) *Probability that a patient must wait before being served.*

(d) *Average percentage idle time for each of the lab technicians.*

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Solution. In this example

$$\lambda = 4/\text{hour}$$

$$\mu = \frac{1}{30} \times 60 = 2/\text{hour}$$

$$K = 3$$

p_0 = Probability that there is no patient in the system.

$$= \left[\sum_{n=1}^{k-1} \frac{1}{\lfloor n} \left(\frac{\lambda}{\mu}\right)^n + \frac{1}{\lfloor n} \frac{\left(\frac{\lambda}{\mu}\right)^k}{\left(1 - \frac{\lambda}{k\mu}\right)} \right]^{-1}$$

$$= \left[\frac{1}{\lfloor 0} + \frac{2}{\lfloor 2} + \frac{2^2}{\lfloor 2} + \frac{1}{16}(2)^3 \times \frac{1}{1} - \frac{4}{6} \right]^{-1} = \left[1 + \frac{2^1}{\lfloor 2} + \frac{2^2}{\lfloor 2} + \frac{(2)^3}{2 \times \frac{2}{6}} \right]^{-1}$$

$$= \left[1 + 1 + 2\,\frac{8 \times 6}{4} \right]^{-1} = (26)^{-1} = 0.038$$

(a) Expected number of patients waiting in the queue

$$Eq = \frac{1}{\lfloor k-1} \left(\frac{\lambda}{\mu}\right)^k \frac{\lambda\mu}{(k\mu - k)^2} \times p_0$$

$$= \left[\frac{1}{2} \times 8 \times \frac{8}{4} \right] \times 0.038 = 8 \times 0.038 = 0.304 \text{ or one patient}$$

(b) Average time a patient spends in the system

$$= \frac{Eq}{\lambda} + \frac{1}{\mu} = \frac{0.304}{4} + \frac{1}{2} = 0.076 + 0.5 = 0576 \text{ hours} = 35 \text{ minutes}$$

(c) Probability that a patient must wait

$$p(n \geq k) = \frac{1}{\lfloor k} \left(\frac{\lambda}{\mu}\right)^k \frac{1}{\left(\frac{1-\lambda}{k\mu}\right)} \times p_0$$

$$= \frac{1}{6} \times 8 \times 8 \times 0.038$$

$$= 0.40$$

(d) $p\,(\text{idle technician}) = \frac{3}{3} p_0 + \frac{2}{3} p_1 + \frac{1}{3} p_2$ when $p_n = \frac{1}{\lfloor n} \left(\frac{\lambda}{\mu}\right)^n p_0$

p_0 = when all the 3 technician are idle (no one is busy)

p_1 = when only one technician is idle (two are busy)

p_2 = when two technicians are idle (only one busy)

$$p\,(\text{idle technician}) = \frac{3}{3} \times 0.038 + \frac{2}{3} \times \left(\frac{4}{2}\right) \times 0.038 + \frac{1}{3} \times \frac{1}{\lfloor 2} (2)^2 \times 0.038$$

$$= 0.038 + 0.05 + 0.025$$

$$= 0.113$$

NOTES

Example 5.9. *A general insurance company handles the vehicle accident claims and employs three officers for this purpose. The policy holders make on an average 24 claims during 8 hours working day and it follows the Poisson's distribution. The officers attending the claims of policy holders spend on an average 30 minutes per claim and this follows the exponential distribution. Claims of the policy holders are processed on first come first served basis. How many hours does the claim officers spend with the policy holder per day ?*

Solution.

Arrival rate $\lambda = \frac{24}{8} = 3$ claims/hours.

Service rate $\mu = \frac{60}{30} = 2$ claims/hours.

Probability that no policy holder is with bank officer

$$p_0 = \left[\sum_{n=0}^{k-1} \frac{1}{\lfloor n} \left(\frac{\lambda}{\mu}\right)^n + \frac{1}{\lfloor \mu} \left(\frac{\lambda}{\mu}\right)^k \frac{1}{\left(\frac{1-\lambda}{k\mu}\right)}\right]^{-1}$$

$$= \left[\frac{1}{\lfloor 0} + \frac{1}{\lfloor 1} \frac{3}{2} + \frac{1}{\lfloor 2}\left(\frac{3}{2}\right)^2 + \frac{1}{6}\left(\frac{3}{2}\right)^3 \frac{1}{\left(1-\frac{3}{6}\right)}\right]^{-1}$$

$$= \left[1 + \frac{3}{2} + \frac{8}{9} + \frac{8}{9}\right]^{-1} = \left(\frac{(8+12+9+9)}{8}\right)^{-1} = \frac{8}{38} = 0.21$$

Probability that one policy holder is with bank officer

$$p_1 = \frac{1}{\lfloor n}\left(\frac{\lambda}{\mu}\right)^1 \times p_0 = 1 \times \frac{3}{2} \times 0.21 = 0.315$$

Probability that two policy holders are with bank officer

$$p_2 = \frac{1}{\lfloor n}\left(\frac{\lambda}{\mu}\right)^2 \times p_0 = 1 \times \left(\frac{3}{2}\right)^2 \times 0.21 = \frac{1.89}{8} = 0.236$$

Expected number of bank officers being idle

= All three idle + any two idle + one idle

$= 3\,p_0 + 2\,p_1 + 1\,p_2$

$= 3 \times 0.21 + 2 \times 0.315 + 1 \times 0.236 = 0.63 + 0.630 + 0.236 = 0.866$

Probability of any bank officer not remaining idle = 1 – 0.21 = 0.79

Time bank officers will spend with the policy holder per day = 0.79 × 8 = 6.02 hours (Assuming 8 hours working day)

Example 5.10. *You have been asked to consider three systems of providing service when customers arrive with a mean arrival rate of 24 per hour.*

(i) Single channel with a mean service rate of 30 per hour at Rs. 5 per customer with a fixed cost of Rs. 50 per hour.

(ii) 3 channels in parallel each with a mean service rate of 10 per hour at Rs. 3 per customer and fixed cost of Rs. 25 per hour per channel. It is confirmed that the system are identical in all other aspects with a simple queue. Average time a customer is in the system is given by

$$\frac{(pc)^c}{\lfloor c(1-p)^2 C\mu} \times p_0 + \frac{1}{\mu}$$ ×*(where symbols have usual meaning)*

and $p_0 = 0.2$ when $c = 1$

$p_0 = 0.111$ when $c = 2$

$p_0 = 0.056$ when $c = 3$

You are required to calculate

(i) *The average time a customer is in the system when 1, 2, 3 channels are in use.*

(ii) *The most economical system to adopt if the value of the customer's time is ignored and to state the total cost per hour of this system.*

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Solution. Average time in system

Arrival rate $\lambda = 24$/hours

Service rate $\mu = 30$/hour ($c = 1$)

$= 15$/hour ($c = 2$)

$= 10$/hour ($c = 3$)

(a) Average time in system $= \dfrac{\mu\left(\frac{\lambda}{\mu}\right)^1 \times p_0 + \mu}{\lfloor(1-1)\,(\mu-\lambda)^2}$

for $c = 1$

$$p_0 = 1 - \text{traffic intensity} = \frac{1-c}{c\mu} = \frac{1-24}{30} = 1.8 = 0.2$$

$$\text{Average time in system} = \frac{30 \times 24}{30} \times 0.2 + \frac{1}{30}\,30 - 24$$

$$= \frac{24}{6} \times \frac{2}{10} + \frac{1}{30} = \frac{48}{60} + \frac{1}{30} = \frac{50}{60} = 50 \text{ minutes}$$

(b) Average time in the system $\left[\dfrac{15\left(\frac{24}{15}\right)^2 \times 0.111 + \frac{1}{15}}{36}\right]$ for $c = 2$

$$= \frac{\left(24 \times 24 \times 0.11 \times \frac{1}{15}\right)}{15 \times 36}$$

$$= \left(1.06 \times 0.11 + \frac{1}{15}\right)$$

$$= (0.118 + 0.06) = 0.184 \times 60 = 11 \text{ minutes}$$

(c) Average time in the system $10\left(\dfrac{24}{10}\right)^3 \times 0.056 + \dfrac{1}{10}$

for $c = 3\ \lfloor 2(20-24)^2$

$$= \frac{\left(24 \times 24 \times 24 \times 0.056 + \frac{1}{10}\right)}{2 \times 16 \times 100} = 0.34 \text{ hours.}$$

$$= 20.5 \text{ minutes.}$$

Total cost/hour

One channel $= (24 \times 5) + (50 \times 1) =$ Rs. 170

As cost @ Rs . 5 per hour for 24 customers + fixed cost.

Two channels $= (24 \times 4) + (30 \times 2) =$ Rs 156

Three channels $= (24 \times 3) + (25 \times 3) =$ Rs 147

5.7 POISSON ARRIVAL AND ERLANG DISTRIBUTION FOR SERVICE

We have assumed in our earlier problems that the two service pattern distribution follows exponential distribution in a manner that its standard deviation is equal to its mean. But there are many situations where these two will vary, we must use a model which is more relevent and applicable to real life situations. In this method the service

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is considered in a number of phases each with a service time $\frac{1}{\mu}$ and time taken in each phase is exponentially distributed. With same mean time of $\frac{1}{\mu}$, with different channels we get different distribution. The method makes the follows assumptions.

(*a*) The arrival pattern follows Poisson distribution.

(*b*) One unit completes service in all the phases and only then the other unit is served.

(*c*) In each phase the service follows exponential distribution.

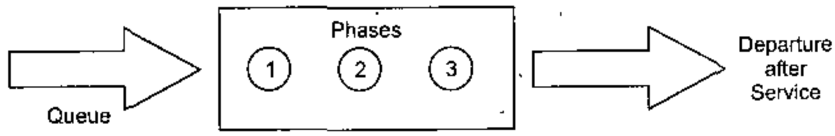

Fig. 5.6

The following formulae are used in this method.

1. Expected number of customer in the system

$$\text{En} = k + \frac{1}{2k} \times \frac{\lambda^2}{\mu(\mu - k)} + \frac{\lambda}{\mu} = \text{Eq} + \frac{\lambda}{\mu}$$

2. Expected number of customers in the queue (or Average queue length)

$$\text{Eq} = k + \frac{1}{2k} \times \frac{\lambda^2}{\mu(\mu - \lambda)}$$

3. Average waiting time of a customer in queue

$$\text{Et} = k + \frac{1}{2k} \times \frac{\lambda^2}{\mu(\mu - \lambda)}$$

4. Expected waiting time of a customer in the system

$$\text{Et} = \frac{k+1}{2k} \times \frac{\lambda^2}{\mu(\mu - \lambda)} + \frac{1}{\mu}$$

Example 5.11. *Maintenance of machine can be carried out in 5 operations which have to be performed in a sequence. Time taken for each of these operations has a mean time of 5 minutes and follows exponential distribution. The break down of machine follows Poisson distribution and the average rate of break down is 3 per hour. Assume that there is only one mechanic available, find out the average idle time for each machine break down.*

Solution. K = 3

Arrival $\lambda = \frac{3}{60} = 1/20$ machines/hours

Total service time for one machine = 5 × 3 = 15 minutes

Service rate $\mu = 1/15$ machines/hour

$$p = \text{Utilisation rate/traffic intensity} = \frac{\lambda}{k\mu} = \left(\frac{1}{20} \times 3\right) \times 15 = \frac{1}{4} = 0.25$$

$$\text{Expected idle time for machine} = k + \frac{1}{2}k = \frac{\lambda^2}{\mu(\mu - \lambda)} + \frac{1}{\mu}$$

$$= \frac{4}{6}\frac{1}{20} \times \frac{1}{20} \times 15\left(\frac{1}{15} - \frac{1}{20}\right) + \frac{\frac{1}{1}}{15}$$

$$= \frac{1}{600} + 15 \times 60 + 15 = 1.5 + 15 = 16.5 \text{ minutes.}$$

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Example 5.12. *A servicing garage carries out the servicing in two stages. Service time at each state is 40 minutes and follows exponential distribution. The arrival pattern is one car every 2 hours and it follows Poisson's distribution. Determine.*

(a) Expected number of customer in the queue.

(b) Expected number of vehicles in the system.

(c) Expected waiting time in the system.

(d) Expected time in the system.

Solution. We have $\lambda = \frac{1}{2}$ vehicles/hour

$$\mu = \frac{40}{60} \text{ vehicles/hours} = \frac{2}{3} \text{ vehicles/hour}$$

$$k = 2$$

(a) Expected number of customer s in a queue

$$\text{Eq} = k + \frac{1}{2k} \times \frac{\lambda^2}{\mu(\mu - \lambda)} = \frac{3}{4} \times \frac{1}{4} \times \frac{3}{2}\left(\frac{2}{3} - \frac{1}{2}\right) = \frac{9}{32} \times 6 = \frac{54}{32} = \frac{27}{16} \text{ per hour.}$$

(ii) Expected number of vehicles in the system

$$\text{En} = \text{Eq} + \frac{\lambda}{\mu} = \frac{27}{16} + \frac{1}{2} \times \frac{3}{2} = \frac{27}{16} + \frac{3}{4} = \frac{(27+12)}{16} = \frac{39}{16} \text{ vehicles/hour.}$$

(iv) Expected time in the system

$$= \text{Et} = k + \frac{1}{2k} \times \frac{\lambda^2}{\mu(\mu - \lambda)} = \frac{1}{\lambda} = \frac{54}{32} + \frac{3}{2} = 54 + \frac{48}{32} = \frac{102}{32} \text{ hours.}$$

Example 5.13. *In a restaurant, the customers are required to collect the coupons after making the payment at one counter, after which he moves to the second counter where he collects the snacks and then to the third counter, where he collects the cold drinks. At each counter he spends $1 \times \frac{1}{2}$ minutes on an average and this time of service at each counter is exponentially distributed. The arrival of customer is at the rate of 10 customers per hour and it follows Poisson's distribution. Determine.*

(a) Average time a customer spends waiting in the restaurant.

(b) Average time the customer is in queue

Solution. $\lambda = 10$ customer/hours

$\mu =$ Total service time for one customer

$$= \frac{3}{2} \times 3 = \frac{9}{4} \text{ customers}$$

$$= \frac{4}{9} \times 60 = \frac{80}{3} \text{ hours.}$$

(a) Average time a customer spends waiting in the restaurant $\text{Et} = k + \frac{1}{2k} \times \frac{\lambda}{\mu(\mu - \lambda)}$

$$\frac{4}{9} = 10 \times \frac{3}{80} \times \frac{80}{3} - 10 = \frac{1}{4} \times \frac{3}{50} = \frac{3}{200} \text{ minutes or } \frac{3}{200} \times 600 = 0.9 \text{ minutes.}$$

(*b*) Average time the customer is in queue

$$= \frac{1}{\mu} = \frac{1}{\frac{80}{3}} = \frac{3}{80} \times 60 = \frac{9}{4} \text{ minutes.}$$

NOTES

SUMMARY

- A queuing system involves a number of servers (or serving facilities) which we will also call *service channels*. The serving channels can be communications links, work stations, check-out counters, retailers, elevators, buses, to mention but a few. According to the number of servers, queuing systems can be of single and multi-channel type.
- The mathematical analysis of a queuing system simplifies considerably when the process concerned is Markovian. Markov process may be defined as , 'A random process is referred to as Markov, if for any moment of time, its probability characteristics in the future 'depend only on its state at time and are to independent of when and how this state was acquired.
- The queue discipline, that is the rule followed by the server in taking the customers in service, may be according to such self-explanatory principles as "first-come, first-served", "last-come, first-served", or chain "random selection for serve". In some situations priority disciplines need be introduced to allow for realistic queues with high priority arrivals.
- **Poisson's Distribution.** It is discrete probability distribution which is used to determine the number of customers in a particular time. It involves allotting probability of occurrence of the arrival of a customer. Greek letter λ (lamda) is used to denote mean arrival rate. A special feature of the Poisson's distribution is that its mean is equal to the variance. It can be represented with the notation as explained below.

  $P(n)$ = Probability of n arrivals (customers)

  λ = Mean arrival rate

  e = Costant = 2.71828

  $$P(n) = \frac{e^{-\lambda} (\lambda)^n}{\lfloor n} \text{ where } n = 0, 1, 2, \ldots$$
- Service time *i.e.*, the time taken by the customer when the facility is dedicated to it for serving depends upon the requirement of the customer and what needs to be done as assessed by the facility provider. The arrival pattern is random so also is the service time required by different customers.
- Erlang distribution developed by AK Erlang is used. In this method, the service time is divided into number of phases assuming that total service can be provided by different phases of service. It is assumed that service time of each phase follows the exponential distribution *i.e.*, σ = mean.

GLOSSARY

- **Arrival pattern :** It is the pattern of the arrival of a customer to be serviced, which is may be regular or at random.
- **Idle rate:** This is the rate at which the service facility remain unutilized and is lying idle.
- **FIFS:** It is stand for 'First in First Served', where customers are served in the order of their arrival.
- **SIP:** Service in priority (SIP) is the priority in servicing is allotted based on the special requirement of a customer.

REVIEW QUESTIONS

NOTES

1. What is a queue ? Give an example and explain the basic concept of queue.
2. Define a queue. Give a brief description of the type of queue discipline commonly faced.
3. (*a*) Explain the single channel and multi-channel queuing models.
   (*b*) Draw a diagram showing the physical layout of a queuing system with a multi server, multi-channel service facility.
4. (*a*) Give some applications of queuing theory.
   (*b*) State three applications of waiting line theory in business enterprises.
5. With respect to the queue system, explain the following :
   (*i*) Input process, (*ii*) Queue discipline, (*iii*) Capacity of the system, (*iv*) Holding time, (*v*) Balking and (*vi*) Jockeying.
6. Consider the pure birth process, where the number of departures in some time interval follows a Poisson distribution. Show that the line between successive departures is exponential.
7. If $\lambda\Delta t$ is the probability of a single arrival during a small interval of time Δt, and if the probability of more than one arrival is negligible, prove that the arrivals follows the Poisson's law.
8. (*a*) Derive Poisson's process assuming that the number of arrival, in non-overlapping intervals, are statistically independent and then apply the binomial distribution.
   (*b*) What are the various queuing models available ?
9. Explain (*i*) Single queue, single server queuing system, and (*ii*) Single queue, multiple servers in series queues.
   [**Hint.** GD indicates that discipline is general, *i.e.*, it may be FCFS or LCFS or SIRO]
10. For a (M/M/1) : (∞/F/FO) queuing model in the steady-state case, show that
    (*a*) The expected number of units in the system and in the queue are given by
    $$E(n) = \frac{\lambda}{(\mu - \lambda)} \text{ and } E(m) = \frac{\lambda^2}{\mu(\mu - \lambda)}$$
    (*b*) (*i*) Expected waiting time of an arrival in the queue is $\frac{\rho}{\mu(1 - \rho)}$.
    (*ii*) Expected waiting time the customer spends in the system (including services) is $\frac{1}{(\mu - \lambda)}$
11. Define busy period of a queuing system. Obtain the busy period distribution for the simple (M/M/1) : (∞/FCFS) queue.
    What is the condition that the busy period will terminate eventually ?
12. Discuss the queuing model $M / E_k / 1$: (∞/FCFS) under state regarding number of units in the system.
13. Explain Erlangian model, find the steady-state distribution of the queue size. Derive the average waiting time in the queue.
14. Define M/G/1 queuing system and state when such a system is to be considered. With usual notation find out $E(n)$ when 'n' is the queue length.

FURTHER READINGS

- *Operational Research*, by col. D.S. Cheema, University Science Press.
- *Statistics and Operational Research– A Unified Approach*, by Dr. Debashis Dutta, Laxmi Publications (P) Ltd.