CONTENTS

SYLLABUS

MATHEMATICS AND GRAPH THEORY

SECTION – A

Sets and elements, universal set and empty set, subsets, Venn diagrams, Set operations, Algebra of sets, Cartesian product, Relations, mappings, Countable and uncountable sets, Domain and range, prepositional logic, FOPL, Logical equivalences, Quantifiers.

SECTION – B

Partially ordered sets, Extremal elements of partial ordered sets, Least upper bound and greatest lower bound, Finite Boolean algebra, Functions on Boolean algebra, Lattices, Bounded lattices, Distributive lattices, Complemented lattices.

SECTION – C

Matrices, Matrix addition and scalar multiplication, Matrix multiplication, Transpose, Inverse, Determinants, Eigen values and Eigen vectors.
Permutations, Combinations, Pigeon hole principle, Elements of Probability, Conditional probability, Baye's Theorem.

SECTION – D

Tree, Binary tree, Traversals, Huffman's algorithm, Minimum spanning trees, Euler graph, Hamiltonian cycle, Cutsets, Matching, Coloring.

SECTION A

1. Set Theory, Relations and Functions
2. Quantification Theory

NOTES

CHAPTER

1

SET THEORY, RELATIONS AND FUNCTIONS

LEARNING OBJECTIVES

- Set and its Concept
- Different Types of Sets
- Subset
- Proper Subset
- Countable and Uncountable Sets
- Venn Diagrams
- Disjoint Sets
- Complement of a Set
- Symmetric Difference
- Some Applications
- Cartesian Product
- Domain and Range of a Relation
- Properties of Binary Relations
- Equivalence Classes
- Partitions of a Set
- Mapping or Function
- Set Notations
- Singleton Set
- Equality of Sets
- Power Set
- Universal Set
- Basic Operations on Sets
- Difference of Two Sets
- Theorems (De-Morgan's Laws)
- Duality Principle
- Ordered Pairs
- Relations
- Inverse Relation
- Equivalence Relation
- Properties of Equivalence Classes
- Product of Equivalence Relations
- Functions Defined as Sets of Ordered Pairs
- Operator
- Diagrammatic Representation of Function
- Constant Function
- Various Kinds of Mapping
- Diagrammatic Representation of Different Type of Mappings
- Inclusion Map
- Identity Map or Identity Function
- Equality of Functions
- Cardinally Equivalent Sets
- Inverse Image of an Element
- Inverse Image of a Subset
- Inverse Mapping (or Function)
- Product of Mapping of Composite Functions

1. SET AND ITS CONCEPT

Nature has given us a variety of things. Most of these things are found in groups. We have around us assembly of men, flock of birds, bunch of grapes, aggregate of points, and so on. Although in literary language we use different words for different types of groups such as assembly, flock, bunch aggregate, etc. but in mathematics, we use the word collection or set for all types of groups. In mathematical language all the groups referred to above could be written as set of men, set of birds, set of grapes, set of points, and so on. A set is accepted as an undefined term because any effort towards defining it would lead to a circular definition. The words used to define a set, whether group or collection or aggregate or an assemblage are synonymous. Therefore, the concept of a set is understood intuitively. The following explanation will make the idea clear as to what a set actually means.

For the objects to be included in a set, it is necessary that they must have some quality common to all yet each element should be distinct from the other. Thus, a formal definition may be given as follows :

A set or a collection is, therefore, a group of objects having some well-defined and common property or properties. The individual object of the collection or set is called an element or a member of the set.

A few examples of the sets are given below :

1. The set of states in the Indian Union.
2. The set of vowels in the alphabets of a language.
3. The set of all positive integers *i.e.*, 1, 2, 3, ...
4. The set of integers less than zero *i.e.*, – 1, – 2, – 3, ...
5. The set of points on a straight line.
6. The set of circles drawn from a given center.

2. SET NOTATIONS

A set is usually denoted by a capital letter (say A, B, C, ..., X, Y, Z etc.) and its elements by small letters (say $a, b, c, \ldots\ldots, x, y, z$ etc.). If an element x belongs to a set X, then we express it by writing

$$x \in X$$

which can also be read as "x is an element of the set X" or "X contains x as one of its elements." If, on the contrary, an element x is not a member of a set X, then we write

$$x \notin X$$

A set can be specified in following two ways. We can make use of any of the two notations according to our convenience.

(*a*) **Roster Method (Tabular form).** In this method a set is represented by listing all its elements within braces { }. For example, the set of vowels in the English alphabets would be written as $\{a, e, i, o, u\}$.

(*b*) **Rule Method.** In this method, a set is represented by describing its elements in terms of one or several characteristic properties which enable us to decide whether a given object is an element of the set under consideration or not. For example, the set of all natural numbers *i.e.*, N = {1, 2, 3, ...} can be denoted as follows :

$$N = \{ x : x \text{ is a natural number}\} \text{ or } \{ x \mid x \text{ is a natural number}\}$$

which is a read as the set of all x such that x belongs to N.

The symbol : or | stands for 'such that' and ∈ for 'belongs to' as explained earlier. A few examples of set notations by Roster and Rule methods are given below :

(*i*) Set of all odd numbers.

(*a*) O = { 1, 3, 5, ...} (Roster method)

(*b*) O = { x : x is an odd number}. (Rule method)

(*ii*) Set of all odd numbers between 2 and 12.

(*a*) B = { 3, 5, 7, 9, 11} (Roster method)

(*b*) B = { $x : x \in O, 2 < x < 12$}. (Rule method)

It is worth mentioning that a set is not just any collection. For a collection to be a set it is essential that given any element, one should be able to decide whether it does or does not belong to the set. The assumption that every collection is a set leads to a paradox known as Russel Paradox after Bertrand Russel, a famous philosopher.

The following example will illustrate the basic idea underlying this Paradox.

A barber in a certain town shaved all those and only those, who did not shave themselves. If A is the collection of all those whom the barber shaved, is the barber a member of A or not ?

The two answers are possible :

(1) The barber is a member of A.

(2) The barber is not a member of A.

But these two are contradictory statements for if the barber is a member of A, then barber has shaved himself and since he has shaved himself, he cannot be a member of A.

3. DIFFERENT TYPES OF SETS

(*i*) **Finite Set.** *A set is finite if it consists of a finite number of different elements*, *i.e.*, if in counting the different members of the set of the counting process can come to an end. For example, if A is the set of the week days, it is finite as its elements are seven in number.

(*ii*) **Infinite Set.** A set is infinite if it contains infinite number of elements *i.e.*, it is not possible to count all members of the set. For example, if B = {2, 4, 6, 8, ...}, then B is infinite.

Note. The number of elements in a set is called **cardinal number** of the set.

Null Set. A set which contains no element is known as Null, Empty or Void set and is denoted by symbol F or { }. For example, following are the null sets.

(*i*) A set of even numbers greater than 4 and less than 6.

(*ii*) *A set of unmarried women Presidents of India since independence to this date.*

(*iii*) Let B = {$x^2 = 4$, x is odd}. Then B is the empty set because $x^2 = 4 \Rightarrow x = \pm 2$ and 2 is not odd.

NOTES

4. SINGLETON SET

A set which has only one member is known as a Singleton Set or Simply Singlet.

For example, (*i*) A set of positive integers between 1 and 3 *i.e.*, {2} will be a singleton set, consisting of one element 2.

(*ii*) {Φ} is a set whose only element is a null set therefore, {Φ} is Singleton set.

5. SUBSET

If every element in a set A is also a member of a set B, then A is called a subset of B. More specifically, A is a subset of B if $x \in$ A implies $x \in$ B. We denote this relationship by writing.

$$A \subset B$$

which can also be read as "A is contained in B". Also B is called the super set of A and can be written as B ⊃ A. Thus if $x \in A \;\Rightarrow\; x \in B$ then A ⊂ B.

Example 1. Let G = {x : x is even} *i.e.*, G = {2, 4, 6, ...} and let F = {x : x is a positive power of 2} *i.e.*, let ϕ = {2, 4, 8, 16}. Then F ⊂ G, *i.e.*, is contained in G.

Example 2. If P be the set of all parallelograms and T is the set of all squares in a plane *i.e.*, P = {all parallelograms in the plane} and T = {all squares in the plane}.

Then T is subset of P *i.e.*, T ⊂ P.

Example 3. The set X = {1, 7, 11} is a subset of Y = {11, 1, 7} as each member of X belongs to Y.

Note. The symbol ⇒ stands for 'implies that' or 'means that'. Some people use the symbol ⊆ to express a subset. If A is not a subset of B, we write it as A ⊄ B or A ⊈ B.

Remark 1. If we have to prove A ⊂ B, then we should prove that $x \in A \Rightarrow x \in B$.

Symbolically, A ⊂ B iff ($x \in A \Rightarrow x \in B$).

Remark 2. If we have to prove that A ⊄ B then we should show that there exist at least one element x such that $x \in$ A but $x \notin$ B.

Symbolically, A ⊄ B iff (∃ $x \in$ A s.t. $x \notin$ B).

The symbol ∃ stands for 'there exists' and s.t. for 'such that'.

Remark 3. A set is always a subset of itself.

Remark 4. Null set Φ always a subset of any other set.

Example Write the subsets of the following set.

$$A = \{a, b, c\},$$

The required subsets are

$$\Phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}.$$

6. EQUALITY OF SETS

Set A is said to be equal to set B if both of them have the same members *i.e.*, every element which belongs to A also belongs to B($A \subset B$) and every element which belongs to B also belongs to A($B \subset A$). We denote the equality of set A and B by writing

$$A = B$$

Symbolically, $A \subset B$ and $B \subset A \Rightarrow A = B$.

7. PROPER SUBSET

Let A and B be two sets.

If $B \subset A$ and $B \neq A$, then B is said to be proper subset of A.

For example, if A = {1, 3, 5} and B = {1, 3} then B is proper subset of A because all the elements of B are in A but one element 5 of A is not in B.

Theorem 1. *If $A \subset B$ and $B \subset C$ then $A \subset C$.*

Proof. We must show that every element in A is also an element in C.

Let x be an element of A *i.e.*, $x \in A$

But it is given that $A \subset B$

$\therefore$ $x \in A \Rightarrow x \in B$ ...(1)

Also $B \subset C$

Therefore, $x \in B \Rightarrow x \in C$ ...(2)

Hence from (1) and (2) $x \in A \Rightarrow x \in C$

$\therefore$ $A \subset C$.

Theorem 2. *Null set Φ is a subset of every set.*

Proof. Let A be any given set. We shall show that $\Phi \subset A$.

Let us suppose, on the contrary, that $\Phi \not\subset A$.

$\Phi \not\subset A \Rightarrow$ there is at least one element x such that

$x \in \Phi$ and $x \notin A$ ...(1)

But since Φ is a null set, therefore,

$x \notin \Phi$ ...(2)

Therefore (1) and (2) $\Rightarrow$ $x \in \Phi$ and $x \notin \Phi$, which is absurd and therefore, our assumption is wrong, Consequently $\Phi \subset A$. Since A is an arbitrary set, Φ is a subset of every set.

Number of Subsets of a Finite Set. Consider a set {1}. It has two possible subsets {1} and Φ, again for the possible subsets are Φ, {1}, {7}, {1, 7} which are 4 subsets in number. A set {1, 7, 11} has an many as the following subsets :

Φ, {1}, {7}, {11}, {1, 7}, {1, 11}, {7, 11}, {1, 7, 11}

which are eight in number.

Thus we can generalize the result as follows :

A set with 1 element has 2 subsets

A set with 2 elements has 2^2 subsets

A set with 3 elements has 2^3 subsets

....

NOTES

Theorem 3. *A set with n elements has 2^n subsets.*

Proof. The above fact can be proved as follows :

Let $A = \{a_1, a_2, a_3, ..., a_n\}$.

We can think of subsets of above set by taking one element, two elements, three elements ...... and finally n elements out of n elements of A in succession. Thus the above selection can be made in ${}^nC_1, {}^nC_2, {}^nC_3, ..., {}^nC_n$ ways. Also null set is a subset of every set. Thus the subsets of a finite set A may contain, no element, one element, two elements, ..., and n elements. Hence, required number of subsets

$$= {}^nC_0 + {}^nC_1 + {}^nC_2 + {}^nC_3 + ... + {}^nC_n + ... + {}^nC_n$$
$$= 1 + {}^nC_1 + {}^nC_2 + {}^nC_3 + ... + {}^nC_n$$
$$= (1 + 1)^n = 2^n.$$

Remark. Every non-void set has at least two subsets.

8. POWER SET

The set of all subsets of a given set A is called the power set of A and is denoted by symbol P(A).

i.e., $P(A) = \{T : T \subset A\}$

Φ and A are both members of P(A).

Example. Let $S = \{a, b\}$

then $P(S) = \{\{a, b\}, \{a\}, \{b\}, \{\Phi\}\}$.

9. COUNTABLE AND UNCOUNTABLE SETS

One-one Correspondence. If each element of a set A corresponds to one and only one element of another set B and vice-versa then we say that there is one-one correspondence between the elements of A and B. The symbol $\leftrightarrow$ is used to express one-one correspondence between the elements of two sets. For example, $A = \{a, b, c\}$ and B = {pen, pencil, book} then one-one correspondence is shown as given below :

$a \leftrightarrow$ pen

$b \leftrightarrow$ pencil

$c \leftrightarrow$ book

Remark 1. Showing one-to-one correspondence between elements of two sets means pairing of the elements of two sets such that no elements of either set is paired with more than one element in the other.

Remark 2. If the elements of two sets A and B can put in one-one correspondence they are called *cardinally equivalent.*

Countable and Uncountable Sets. A set whose elements can be put in one-to-one correspondence with the elements of the set N, the set of positive integers, is called a *countable or enumerable* set, otherwise it is *uncountable.*

Note. When a one-to-one correspondence exists between two sets they are said to be matching sets also.

Comparability of Sets. Two sets A and B are said to be comparable if $A \subset B$ or $B \subset A$. If neither $A \subset B$ nor $B \subset A$ then A and B are called non-comparable. For example

(*i*) A = {2, 4} and B = {2, 4, 6} are comparable as $A \subset B$

(*ii*) If A = {x : x is an odd integer}, and B = {x : x is an even integer}

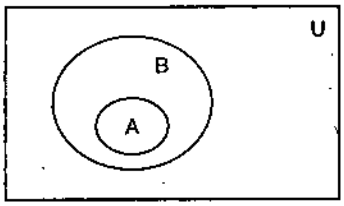
then A and B are non-comparable because $A \not\subset B$ and $B \not\subset A$.

10. UNIVERSAL SET

In any discussion in set theory, all the sets under consideration are likely to be subsets of a fixed set. Such a set is called the "*universal set*" or "*universe of discourse*". This set is generally denoted by capital letter U. For example, the set of natural numbers is universal set for the set of odd numbers, the set of even numbers and the set of prime numbers.

11. VENN DIAGRAMS

Swiss mathematician Euler, first of all gave an idea to represent a set by the points in a closed curve (usually a circle but not necessarily circle). Later on British mathematician Venn brought this idea to practice. That is why the diagrams drawn for this purpose are called *Venn-Euler diagrams* or simply *Venn-diagrams*. These diagrams are very useful for the beginners to understand the set theoric ideas, though they are not of much importance as the reader advances.

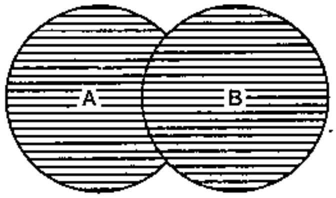


Note. The universal set U is represented by points within a rectangle. The subsets A, B,...... of U are represented by points in closed curves in rectangle.

The idea of $A \subset B$ and $A \neq B$ can be represented by the adjoining diagram.

12. BASIC OPERATIONS ON SETS

1. Union of Sets. Let A and B be two given sets. The set which contains every element contained in A or B or both A and B is called the union (or join) of A and B. In a bit different language it is described that the union is as 'either' 'or' idea. The symbol $\cup$ is used to denote the union of sets. Thus $A \cup B$ is read as 'A union B' or 'A join B' or 'A cup B'. Symbolically $A \cup B$ = {x : $x \in$ A or/ and $x \in$ B.} or simply



$A \cup B$ (Shaded)

$$A \cup B = \{x : x \in A \text{ or } x \in B\}.$$

The shaded part represents $A \cup B$ in the adjoining Venn diagram.

Example. Let $A = \{1, 2, 3\}$ and $B = \{3, 5, 7\}$ then

$$A \cup B = \{1, 2, 3, 5, 7\}.$$

Note 1. Each element in a set is listed once only because the repetition of elements is meaningless in a set. If A is a set of hockey players, B is a set of tennis players and some players are common to both the teams, then

$$A \cup B = \text{Set of all players in the two teams.}$$

Note 2. The union of a finite number of sets $A_1, A_2, \ldots\ldots A_n$ is denoted by

$$A_1 \cup A_2 \cup A_3 \ldots\ldots \cup A_n$$

or by $\bigcup_{i=1}^{n} A_i$.

Note 3. From the definition it is clear that A and B are the subsets of $A \cup B$. Symbolically, we write as $A \subset (A \cup B)$ and $B \subset (A \cup B)$.

SOME LAWS OF UNION OPERATION

Commutative Law for Union of Sets. *If A and B be two sets, then*

$$A \cup B = B \cup A.$$

Proof. In order to prove this result we need to show that

(*a*) $A \cup B \subset B \cup A$; (*b*) $B \cup A \subset A \cup B$.

Let $x \in A \cup B$ then $x \in A \cup B \Rightarrow x \in A$ or $x \in B$

$\Rightarrow$ $x \in B$ or $x \in A$

$\Rightarrow$ $x \in B \cup A$ ...(1)

$\therefore$ $A \cup B \subset B \cup A$

Similarly, let $x \in B \cup A$ then

$x \in B \cup A \Rightarrow x \in B$ or $x \in A$

$\Rightarrow$ $x \in A$ or $x \in B$

$\Rightarrow$ $x \in A \cup B$

$\therefore$ $B \cup A \subset A \cup B$ ...(2)

Consequently from (1) and (2)

$$A \cup B = B \cup A$$

Associative Law for Union of Sets. *If A, B, and C are any three sets, then*

$$(A \cup B) \cup C = A \cup (B \cup C).$$

Proof. To prove this result we have to show that

(*a*) $(A \cup B) \cup C \subset A \cup (B \cup C)$; (*b*) $A \cup (B \cup C) \subset (A \cup B) \cup C$.

Let $x \in (A \cup B) \cup C$, then

$x \in (A \cup B) \cup C \Rightarrow x \in (A \cup B)$ or $x \in C$

$\Rightarrow$ $[x \in A$ or $x \in B]$ or $x \in C$

$\Rightarrow$ $x \in A$ or $[x \in B$ or $x \in C]$

$\Rightarrow$ $x \in A \cup (B \cup C)$

$\therefore$ $(A \cup B) \cup C \subset A \cup (B \cup C)$ ...(1)

Similarly, let $x \in A \cup (B \cup C)$, then

$x \in A \cup (B \cup C) \Rightarrow x \in A$ or $x \in (B \cup C)$

$\Rightarrow$ $x \in A$ or $[x \in B$ or $x \in C]$

$\Rightarrow \quad [x \in A \text{ or } x \in B] \text{ or } x \in C$

$\Rightarrow \quad x \in (A \cup B) \text{ or } x \in C$

$\Rightarrow \quad x \in (A \cup B) \cup C$

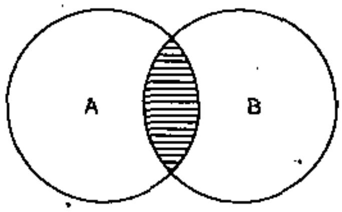
$\therefore \quad A \cup (B \cup C) \subset (A \cup B) \cup C.$ ...(2)

NOTES

Consequently,

$$(A \cup B) \cup C = A \cup (B \cup C).$$

2. Intersection of Sets. Let A and B be two given sets. The set containing all the elements which are contained in A as well as in B is called the intersection of A and B. This is why it is said that intersection is an 'and' idea. The symbol $\cap$ is used to denote the intersection of sets. Thus $A \cap B$ is read as the intersection of A and B.



$A \cap B$ (Shaded)

Symbolically $\quad A \cap B = \{x : x \in A \text{ and } x \in B\}$

In the adjoining Venn diagram the shaded part represents $A \cap B$.

Example. Consider $\quad A = \{a, b, i, 2\}$

$B = \{b, c, i, 3, d\}$

then $\quad A \cap B = \{b, i\}$.

Note 1. If A and B are any two sets then

$$A \cap B \subset A \quad \text{and} \quad A \cap B \subset B.$$

Note 2. The intersection of a finite number of sets $A_1, A_2, \ldots\ldots, A_n$ is denoted by

$$A_1 \cap A_2 \ldots\ldots \cap A_n \quad \text{or by} \quad \bigcap_{i=1}^{n} A_i.$$

SOME LAWS OF INTERSECTION OPERATION

Commutative law. *If A and B be two sets then $A \cap B = B \cap A$.*

Proof. Let x be any element of the set $A \cap B$, then

$X \in A \cap B \Rightarrow x \in A \text{ and } x \in B$

$\Rightarrow \quad x \in B \text{ and } x \in A$

$\Rightarrow \quad x \in B \cap A$

$A \cap B \subset B \cap A$ ...(1)

Again let x be any element of the set $B \cap A$, then

$x \in B \cap A \Rightarrow x \in B \text{ and } x \in A$

$\Rightarrow \quad x \in A \text{ and } x \in B$

$\Rightarrow \quad x \in A \cap B$

$B \cap A \subset A \cap B$ ...(2)

Hence from (1) and (2), we have $A \cap B = B \cap A$.

Associative Law. *If A, B and C be three sets, then*

$$(A \cap B) \cap C = A \cap (B \cap C).$$

Proof. Let $x \in (A \cap B) \cap C$, then

$x \in (A \cap B) \cap C \Rightarrow x \in (A \cap B) \text{ and } x \in C$

$\Rightarrow \quad x \in A \text{ and } x \in B \text{ and } x \in C$

NOTES

$\Rightarrow \qquad x \in A$ and $x \in (B \cap C)$

$\Rightarrow \qquad x \in A \cap (B \cap C)$

i.e., $\qquad (A \cap B) \cap C \subset A \cap (B \cap C)$.

Similarly, it can be proved that

$$A \cap (B \cap C) \subset (A \cap B) \cap C$$

Hence $\qquad (A \cap B) \cap C = A \cap (B \cap C)$.

ILLUSTRATIVE EXAMPLES

Example 1. *Prove that*

(*a*) $A \cup \Phi = A$, $\qquad$ (*b*) $A \cup A = A$,

(*c*) $(A \cap A) = A$ $\qquad$ (*d*) $A \cap \Phi = \Phi$.

Solution. (*a*) $\qquad A \cup \Phi = A$.

We know that $A \subset A \cup \Phi$ (by definition) ...(1)

We shall now show that

$$A \cup \Phi \subset A.$$

Consider $x \in A \cup \Phi$, then

$$x \in A \cup \Phi \;\Rightarrow\; x \in A \;\text{ or }\; x \in \Phi$$

but since Φ contains no element,

$\therefore \qquad x \in A \cap \Phi \;\Rightarrow\; x \in A$

$\therefore$ *i.e.*, $\qquad A \cup \Phi \subset A$. ...(2)

(1) and (2) $\Rightarrow \quad A \cup \Phi = A$

This is called identity property for union operation.

(*b*) $\qquad A \cup A = A$.

We know that $\quad A \subset A \cup A$ (by definition) ...(1)

We shall show that $\quad A \cup A \subset A$

Let $x \in A \cup A$, then

$$x \in A \cup A \;\Rightarrow\; x \in A \;\text{ or }\; x \in A$$

$\Rightarrow \qquad x \in A$ *i.e.*, $A \cup A \subset A$ ...(2)

(1) and (2) $\Rightarrow A \cup A = A$

It is called idempotent property for union operation.

(*c*) $\qquad A \cap A = A$.

We know that $\quad A \cap A \subset A \quad$ (by definition) ...(1)

Now let $x \in A$ then $x \in A \;\Rightarrow\; x \in A$ and $x \in A$

$\Rightarrow \qquad x \in A \cap A$ *i.e.*, $A \subset A \cap A$ ...(2)

(1) and (2) $\Rightarrow \quad A \cap A = A$.

This is known as idempotent property of intersection operation.

(*d*) $\qquad A \cap \Phi = \Phi$.

We know that, $A \cap \Phi \subset \Phi$ (by definition) ...(1)

Also, we know that Φ is a subset of every set, therefore,

$$\Phi \subset A \cap \Phi \qquad \ldots(2)$$

(1) and (2) $\Rightarrow$ $A \cap \Phi = \Phi$.

This is known as identity property of intersection operation.

Example 2. *The necessary and sufficient condition for a set Y to be a subset of X is that*

$$X \cup Y = X.$$

Solution. Let $Y \subset X$, then

$$x \in Y \Rightarrow x \in X \qquad \ldots(1)$$

Now $x \in X \cup Y \Rightarrow x \in X$ or $x \in Y$

$\Rightarrow$ $x \in X$ [using (1)]

$\therefore$ $X \cup Y \subset X$ ...(2)

Also, we know that

$$X \subset X \cup Y \qquad \ldots(3)$$

(2) and (3) $\Rightarrow$ $X \cup Y = X$.

Conversely, if $X \cup Y = X$, we have to prove that $Y \subset X$.

Now $X \cup Y = X \Rightarrow X \cup Y \subset X$ and $X \subset X \cup Y$

$\Rightarrow$ $X \cup Y \subset X$

$\Rightarrow$ $X \subset X$ and $Y \subset X$

$\Rightarrow$ $Y \subset X$.

Example 3. *Prove that*

$$X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z).$$

Solution. Let $x \in X \cup (Y \cap Z)$, then

$x \in X \cup (Y \cap Z) \Rightarrow x \in X$ or $x \in (Y \cap Z)$

$\Rightarrow$ $x \in X$ or $[x \in Y$ and $x \in Z]$

$\Rightarrow$ $[x \in X$ or $x \in Y]$ and $[x \in X$ or $x \in Z]$

$\Rightarrow$ $x \in (X \cup Y)$ and $x \in X \cup Z$

$\Rightarrow$ $x \in (X \cup Y) \cap (X \cup Z)$

$\therefore$ $X \cup (Y \cap Z) \subset (X \cup Y) \cap (X \cup Z)$ ...(1)

Again let $x \in (X \cup Y) \cap (X \cup Z)$, then

$x \in (X \cup Y) \cap (X \cup Z) \Rightarrow x \in X \cup Y$ and $x \in X \cup Z$

$\Rightarrow$ $[x \in X$ or $x \in Y]$ and $[x \in X$ or $x \in Z]$

$\Rightarrow$ $x \in X$ or $[x \in Y$ and $x \in Z]$

$\Rightarrow$ $x \in X$ or $x \in Y \cap Z$

$\Rightarrow$ $x \in X \cup (Y \cap Z)$

$\therefore$ $(X \cup Y) \cap (X \cup Z) \subset X \cup (Y \cap Z)$ ...(2)

(1) and (2) $\Rightarrow$ $X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z)$.

Aliter. We have

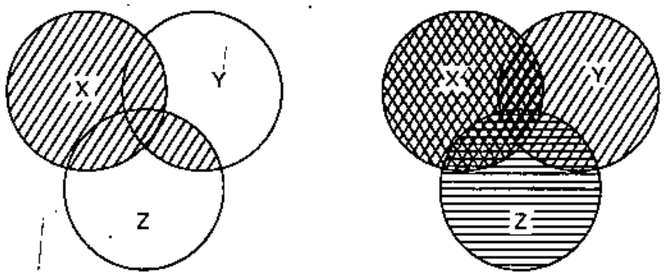
$$(X \cup (Y \cap Z) = \{x : x \in X \text{ or } x \in Y \cap Z\}$$
$$= \{x : x \in X \text{ or } (x \in Y \text{ and } x \in Z)\}$$

$= \{x : (x \in X \text{ or } x \in Y) \text{ and } (x \in X \text{ or } x \in Z)\}$

$= \{x : x \in X \cup Y \text{ and } x \in X \cup Z\}$

$= (X \cup Y) \cap (X \cup Z)$

This has been represented by Venn Diagrams as given below :



$$X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z)$$

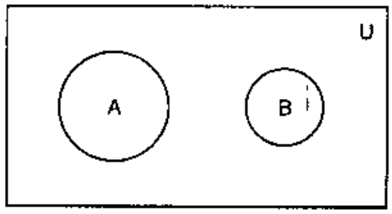
Thus union is distributive over intersection. Similarly it can be proved that intersection is distributive over union.

i.e., $X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z)$.

13. DISJOINT SETS

Two sets A and B are said to be disjoint sets, if they have no element in common *i.e.*, if their intersection is a null set *i.e.*,

$$A \cap B = \Phi$$



Thus $A \cap B = \Phi \Rightarrow$ A and B are disjoint. For example if

$$E = \{x, y, z\} \text{ and } F = \{r, s, t\}$$

Then E and F are disjoint sets as there is no element common to both of them.

The idea will be clear from the above Venn diagram.

14. DIFFERENCE OF TWO SETS

Let A and B be two sets, then the difference of the sets A and B is the set of all those elements x such that x belongs to A and x does not belong to B and is denoted by A – B to be read as 'A difference B'

i.e., $A - B = \{x : x \in A \text{ and } x \notin B\}$.

For example, if $A = \{2, 4, 6, 8\}$ and $B = \{4, 6, 9, 11\}$, then

$$A - B = \{2, 8\}$$

and $B - A = \{9, 11\}$.

Here $A - B \neq B - A$.

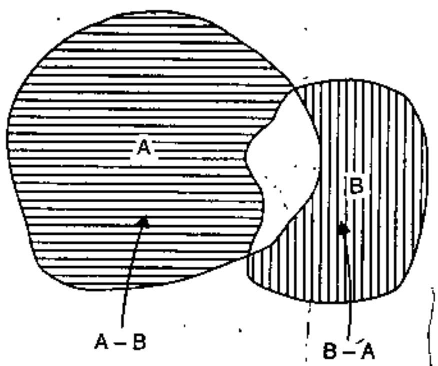


Fig. (*i*)

Note. It is obvious from the definition of the difference that.

$A - B \subset A$ and $(A - B) \cap B = \Phi$

The two sets A – B and B – A can be represented by Venn diagrams as shown in the figure.

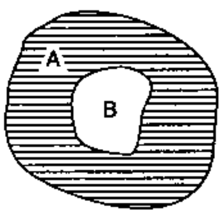


Fig. (*ii*)

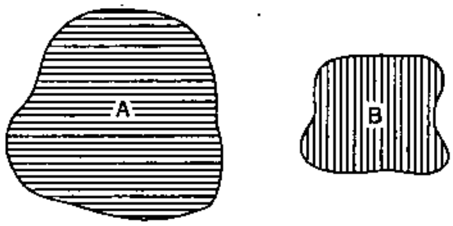
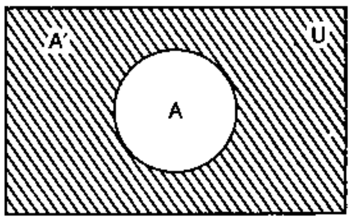


Fig. (*iii*)

In figures (*i*), (*ii*), and (*iii*) A – B is shown by shaded parts with horizontal lines and B – A is shown by shaded parts with vertical lines.

15. COMPLEMENT OF A SET



The complement of a given set A is defined as the set consisting of those elements of the universal set which are not contained in the given set A. It is denoted by the symbol A′ or $\overline{A}$ or A or – A

Symbolically $A' = \{x : x \in U, x \notin A\}$.

In the adjoining Venn diagram shaded portion is A′.

For example, Let N, the set of all natural numbers, be taken as the universal set and let $A = \{x : x$ is an even number and $x \in N\}$, then $A' = \{x : x$ is an odd number and $x \in N\}$

Note 1. The set A and its complement A′ are disjoint sets.

Note 2. Let A and B be two sets. The set $X = \{x : x \in A, x \notin B\}$ (*i.e.*, A – B) is called the complement of the set B with respect to A.

Example. *If U = {1, 2, 3, 4, 5, 6, 7, 8, 9}, then find the complement of the set*

A = {1, 3, 5,7}.

Solution. A′ = the complement of A = {2, 4, 6, 8, 9}.

16. THEOREMS (DE-MORGAN'S LAWS)

Let U be the universal set and $A \subset U$, $B \subset U$, then

(i) the complement of the union of two sets A and B is equal to intersection of their complements i.e., $(A \cup B)' = A' \cap B'$.

(ii) the complement of the intersection of two sets is equal to the union of their complements i.e., $(A \cap B)' = A' \cup B'$.

Proof. (*i*) In order to prove (*i*) we need to prove that

$$(A \cup B)' \subset A' \cap B' \quad \text{and} \quad A' \cap B' \subset (A \cup B)'$$

Let $x \in (A \cup B)'$ then

$$x \in (A \cup B)' \Leftrightarrow x \notin A \cup B$$

$\Leftrightarrow$ $x \notin A$ and $x \notin B$

$\Leftrightarrow$ $x \in A'$ and $x \in B'$

$\Leftrightarrow$ $x \in A' \cap B'$

$\therefore$ $x \in (A \cup B)' \Leftrightarrow x \in A' \cap B'$

$\Leftrightarrow$ $(A \cup B)' \subset A' \cap B'$ and $A' \cap B' \subset (A \cup B)'$

$\Rightarrow$ $(A \cup B)' = A' \cap B'$

(*ii*) In order to prove (*ii*) we need to prove that

$$(A \cap B)' \subset A' \cup B' \quad \text{and} \quad A' \cup B' \subset (A \cap B)'$$

Let $x \in (A \cap B)'$ then

$$x \in (A \cap B)' \Leftrightarrow x \notin A \cap B$$

$\Leftrightarrow$ $x \notin A$ or $x \notin B$

$\Leftrightarrow$ $x \in A'$ or $x \in B'$

$\Leftrightarrow$ $x \in A' \cup B'$

$\therefore$ $x \in (A \cap B)' \Leftrightarrow x \in A' \cup B'$

$\Rightarrow$ $(A \cap B)' \subset A' \cup B'$ and $(A \cap B)' \supset A' \cup B'$

$\Rightarrow$ $(A \cap B)' = A' \cup B'$.

ILLUSTRATIVE EXAMPLES

Example 1. *Prove that $(A')' = A$.*

Solution. We have to prove,

$$(A')' \subset A \quad \text{and} \quad A \subset (A')'.$$

Let $x \in (A')'$ then $x \in (A')' \Leftrightarrow x \notin A'$

$\Leftrightarrow$ $x \in A$

Thus $x \in (A')' \Leftrightarrow x \in A \Rightarrow (A')' \subset A$ and $A \subset (A')'$

$\Rightarrow$ $(A')' = A$.

Example 2. *If A and B are any two sets such that $A \subset B$, then $B \subset A$ and conversely, i.e.,*

$$A \subset B \Leftrightarrow B' \subset A'$$

Solution. First we will prove that if

$$A \subset B \text{ then } B' \subset A'$$

Let $x \in B' \Rightarrow x \notin B$

$\Rightarrow$ $x \notin A$ $\quad [\because A \subset B]$

$\Rightarrow$ $x \in A'$

$\therefore$ $x \in B' \Rightarrow x \in A' \Rightarrow B' \subset A'$. ...(1)

Conversely if $B' \subset A'$ then $A \subset B$. Since $B' \subset A'$ therefore we have from above result

$$(A')' \subset (B')'$$

$$\Rightarrow \quad A \subset B \qquad [(A')' = A \text{ etc.}]$$

$$\therefore \quad B' \subset A' \quad \Rightarrow \quad A \subset B. \qquad ...(2)$$

From (1) and (2) $\quad A \subset B \ \Leftrightarrow \ B \subset A.$

Example 3. *Let A, B, C be three sets then* $(A - B) \cap (A - C) = A - (B \cup C)$

Solution. We have to prove that

$$(A - B) \cap (A - C) \subset A - (B \cup C) ;$$

$$A - (B \cup C) \subset (A - B) \cap (A - C).$$

Let $x \in (A - B) \cap (A - C)$, then

$$x \in (A - B) \cap (A - C) \ \Leftrightarrow \ x \in (A - B) \ \text{ and } \ x \in (A - C)$$

$$\Leftrightarrow \ x \in A, x \notin B \ \text{ and } \ x \in A, x \notin C$$

$$\Leftrightarrow \ x \in A, x \notin (B \cup C)$$

$$\Leftrightarrow \ x \in A - (B \cup C)$$

which implies that $\quad (A - B) \cap (A - C) = A - (B \cup C).$

17. SYMMETRIC DIFFERENCE

The symmetric difference of two sets A and B is the set

$$(A - B) \cup (B - A)$$

and is denoted by $A \Delta B$.

For example, if $\quad A = \{1, 2, 3\}, B = \{3, 4, 5\}$, then

$$A - B = \{1, 2\}, B - A = \{4, 5\}.$$

$$\therefore \quad A \Delta B = \{1, 2, 4, 5\}.$$

Example. *Prove the following :*

(a) $A \Delta \Phi = A$ *(b)* $A \Delta A = \Phi$, *(c)* $A \Delta B = \Phi \ \Leftrightarrow A = B$.

Solution. *(a)* $\quad A \Delta \Phi = (A - \Phi) \cup (\Phi - A)$

$$= A \cup \Phi \qquad [\because \ A - \Phi = A, \Phi - A = \Phi]$$

$$= A.$$

(b) $\quad A \Delta A = (A - A) \cup (A - A)$

$$= \Phi \cup \Phi = \Phi.$$

(c) $\quad A \Delta B = \Phi \ \Rightarrow \ (A - B) \cup (B - A) = \Phi$

$$\Rightarrow \quad A - B = \Phi \ \text{ and } \ B - A = \Phi$$

$$\Rightarrow \quad A = B.$$

and $\quad A = B \ \Rightarrow \ A - B = \Phi \ \text{ and } \ B - A = \Phi$

$$\Rightarrow \quad (A - B) \cup (B - A) = \Phi$$

$$\Rightarrow \quad A \Delta B = \Phi$$

Hence $\quad A \Delta B = \Phi \ \Leftrightarrow \ A = B.$

NOTES

18. DUALITY PRINCIPLE

It may be noted that if in any law of the algebra of the sets, universal set U is replaced by Φ, Φ by U, ∪ by ∩ and ∩ by ∪ wherever these occur, the new statement thus obtained is also a law of the algebra of these sets. This fact is known as the duality principle, and any law obtained as a result of its application is called the dual of the original law. Original law is sometimes called primal law.

For example, the dual of the law

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

is

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

It will be observed that this principle is always true. However we do not attempt to give a proof of the same.

EXERCISE 1(A)

1. Define a set and give examples to illustrate the difference between a collection and a set. What are the different ways to specify a set ? Give examples.

2. (*a*) Find out which of the following sets are null or singleton :

(*i*) A = {x : x is a letter before a in alphabets},

(*ii*) A = {x : $x + 16 = 16$}

(*iii*) A = {x : $x^2 = 4$, x is an odd integer.}

(*b*) Which of the following sets are different from others :

(*i*) Φ (*ii*) {0} (*iii*) {Φ}.

3. Show that the set of even integers is a subset of the set of integers.

4. If A ⊂ B, B ⊂ C and C ⊂ A, prove that B = C.

5. Prove that every set has a proper subset.

6. If A = {a, b, c, d}, B = {a, e, g}, C = [e, g, m, n, p}, find

(*a*) A ∪ B, (*b*) B ∪ C, (*c*) A ∩ C, (*d*) B ∩ C, (*e*) (A ∩ B) ∩ C, (*f*) A ∪ (B ∩ C), (*g*) (A ∪ B) ∩ C, (*h*) (A ∩ B) ∪ C, (*i*) (A ∪ B) ∩ (A ∪ C).

7. Prove that A – (B ∩ C) = (A – B) ∪ (A – C).

8. If A = {$a, b, c, d, e,$ }, B = {a, c, e, g} and c = (b, e, f, g), prove that (A ∪ B) ∩ C = (A ∩ C) ∪ (B ∩ C)

9. U = {a, b, c, d, e, f}, A = {a, b, e} and B = {b, e, f}.

Find (*a*) B′ (*b*) A′ ∩ B (*c*) A ∪ B′ (*d*) (A ∪ B)′ (*e*) B′ – A′.

10. If A, B, C are the subsets of a set S, then show that

(*a*) A ∩ B ⊂ (A ∩ C) ∪ (B ∩ C) (*b*) A – (B ∩ C) = (A – B) ∪ (A – C).

11. Complete the following statements by inserting ⊂, ⊃. Here A and B are arbitrary sets :

(*a*) A ...... A – B, (*b*) A ...... (A ∩ B),

(*c*) A ...... A ∪ B, (*d*) A′ ...... B – A.

12. Prove that

(*a*) A ∩ (B Δ C) = (A ∩ B) Δ (A ∩ C), (*b*) A – B = A Δ (A ∩ B),

(*c*) A ∪ B = (A Δ B) Δ (A ∩ B), (*d*) A ∩ (B – C) = (A ∩ B) – (A ∩ C).

NOTES

13. Under what condition would each of the following be true ?

(*a*) $A \cup B = \Phi$ (*b*) $A \cup \Phi = \Phi$ (*c*) $A \cap U = U$

(*d*) $A \cup \Phi = U$ (*e*) $A \cap B = A$ (*f*) $A \cup A' = U$

(*g*) $A \cup B = A \cap B$ (*h*) $A \cap B = \Phi$.

14. If A and B are any two sets, then prove that $A - B$, $A \cap B$ and $B - A$ are pair wise disjoint.

15. Prove that $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$.

16. If A and B are subsets of a universal set U, show that

$$A \cup B = (A - B) \cup B.$$

17. Prove that $(A \cap B) \cap B' = A$, if and only if $A \cap B = \Phi$.

18. Prove that $(A - B) \cup B = A \Leftrightarrow B \subset A$.

19. Prove that $B \subset A \Rightarrow (B - C) \subset (A - C)$.

20. Prove that $(A \cup C) \cap (B \cup C') \subset A \cup B$.

Answers

2. (*a*) (*i*) $A = \Phi$, (*ii*) A is singleton (*iii*) $A = \Phi$

(*b*) Each is different from other.

6. (*a*) $\{a, b, c, d, e, g\}$ (*b*) $\{a, e, g, m, n, p\}$, (*c*) Φ

(*d*) $\{e, g\}$

(*e*) Φ (*f*) $\{a, b, c, d, e, g\}$ (*g*) $\{e, g\}$

(*h*) $\{a, e, g, m, n, p\}$

(*i*) $\{a, b, c, d ; e, g\}$.

9. (*a*) $\{a, c, d\}$ (*b*) $\{f\}$ (*c*) $\{a, b, c, d, e\}$

(*e*) $\{e\}$

11. (*a*) $\supset$ (*b*) $\supset$ (*c*) $\subset$ (*d*) $\supset$

13. (*a*) $A = \Phi$, $B = \Phi$ (*b*) $A = \Phi$ (*c*) $A = U$

(*d*) $A = U$ (*e*) $A \subset B$

(*f*) A may be any set (*g*) $A = B$ (*h*) $A \neq B$.

19. SOME APPLICATIONS

Use of symbols in logical statements.

Example 1. (*a*) *Suresh is a first class student.*

(*b*) *all the first class students are industrious.*

Prove that Suresh is industrious.

Solution. Let $A = \{x : x \text{ is a first class student}\}$

and $B = \{x : x \text{ is industrious}\}$.

According to the statement (*a*) Suresh $\in$ A, and $A \subset B$.

$\therefore$ Suresh $\in A \Rightarrow$ Suresh $\in B$

$\therefore$ Suresh is industrious.

Example 2. *If A = set of all students,*

B = set of all students offering mathematics,

C = set of all women,

D = set of all industrious persons,

E = set of first class students.

NOTES

Write the following statement symbolically.

Some industrious women students are the students of mathematics but they are not first class.

Solution. Set of women students = $A \cap C$.

$\because$ Set of industrious women students $A \cap C \cap D$.

Set of industrious women students offering mathematics = $A \cap C \cap D \cap B$.

Set of all those industrious women students offering mathematics who are not first class

$$= A \cap C \cap D \cap B - E.$$

According to the statement there are students of this type.

$$A \cap C \cap D \cap B - E \neq \phi.$$

Example 3. *In a class containing 50 students, 15 play Tennis, 20 play Cricket and 20 play Hockey. 3 play Tennis and Cricket, 6 play Cricket and Hockey, and 5 play Tennis and Hockey. 7 play no game at all. How many play Cricket, Tennis and Hockey ?*

Solution. Let the sets of Tennis, Cricket and Hockey players be denoted by T, C and H respectively then $n(U) = 50$, $n(T) = 15$, $n(C) = 20$, $n(H) = 20$, $n(T \cap C) = 3$, $n(C \cap H) = 6$, $n(T \cap H) = 5$, $n(T \cup C \cup H)' = 7$.

We want to find $n(T \cap C \cap H)$.

Now $n(T \cup C \cup H) = n(T) + n(C) + n(H) - n(T \cap C) - n(C \cap H)$

$$- n(T \cap H) + n(T \cap C \cap H).$$

$\therefore$ $n(T \cup C \cup H) = 15 + 20 + 20 - 3 - 6 - 5 + n(T \cap C \cap H).$

or $n(T \cup C \cup H) = 41 + n(T \cap C \cap H)$

but $n(T \cup C \cup H)' = n(U) - n(T \cup C \cup H)$

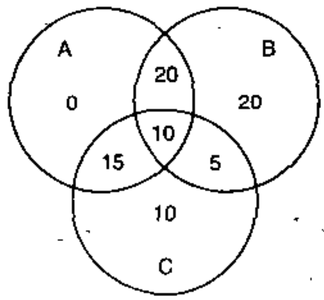
$\therefore$ $7 = 50 - [41 + n(T \cap C \cap H)].$

or $n(T \cap C \cap H) = 2.$

Note. n before the set has been used to indicate the cardinal number of a set.

Example 4. *In a town 45% read magazine A, 55% read magazine B, 40% read magazine C, 30% read magazines A and B, 15% read magazines B and C, 25% read C and A, 10% read all the three magazines. Find what percentage does not read any magazine ? What percentage reads exactly two of the magazines ?*

Solution. Let A, B and C represent the set of all those who read magazines A, B and C, respectively.



$$N(A) = 45, n(B) = 55, n(C) = 40,$$
$$n(A \cap B) = 30, n(B \cap C) = 15,$$
$$n(C \cap A) = 25, n(A \cap B \cap C) = 10.$$

The diagram given here may be helpful to understand the following :

The number of persons who read only A and B but not C = 30 – 10 = 20.

The number of persons who read B and C but not A = 15 – 10 = 5.

The number of persons who read only C and A but not B = 25 – 10 = 15.

The number of persons who read only A = 45 – (20 + 10 + 15) = 0.

The number of persons who read only B = 55 – (20 + 10 + 5) = 20.

The number of persons who read only C = 40 – (15 + 10 + 5) = 10.

Hence the number of persons who read at least one magazine

$$= 0 + 20 + 10 + 20 + 15 + 5 + 10 = 80.$$

Hence those who do not read any magazine are 20 in number *i.e.*, 20% do not read any magazine. The number of persons who read exactly two magazines = 20 + 5 + 15 = 40 *i.e.*, 40% read exactly two of the magazines.

EXERCISE 1(B)

1. Suresh in industrious. All the first class students are industrious. Is Suresh first class student ?
2. In a town, a survey discloses that 45% smoke cigarette A, 55% smoke cigarette B, 40% smoke cigarette C, 35% smoke cigarette A and B, 20% smoke A and C, 15% smoke B and C, 10% smoke all the three.
   (*a*) What percentage does not smoke ?
   (*b*) What percentage smokes exactly two types of cigarettes ?
3. A market research group conducted a survey of 1000 consumers and reported that 720 consumers liked product A and 450 consumers liked product B. What is the least number that must have liked both products ?
4. If $n(A) = 20$, $n(B) = 25$, $n(A \cap B) = 10$ then find $n(A \cup B)$.

Answers

2. (*a*) 20% (*b*) 40% **3.** 170

4. 35.

20. ORDERED PAIRS

If x and y be any two elements, then (x, y) is called ordered pair. The element x is said to be the first member or the first coordinate of (x, y) and the element y is called the second member or second coordinate of ordered pair (x, y). It is clear from here that

$$(a, b) = (c, d) \Leftrightarrow a = c, b = d.$$

Example 1. *Find x and y if (x, y) = (3, 4).*

Solution. From definition $x = 3, y = 4$.

Example 2. *Find x and y if ordered pairs (2x – 1, – 5) and (x, y + 1) are equal.*

Solution. From definition,

$$(2x - 1, -5) = (x, y + 1) \Leftrightarrow 2x - 1 = x \text{ so that}$$
$$x = 1 \text{ and } -5 = y + 1 \text{ so that } x = -6.$$

EXERCISE 1(C)

Find the numbers x and y if ordered pairs are equal :

1. $(x, y) = (2, 3)$
2. $(x + 3, y) = (0, 5)$
3. $(x + 5, y - 3) = (1, 0)$
4. $(x - 2, y + 1) = (3, 3y + 2)$
5. $(x + y, 0) = (2, x - y)$.

NOTES

Answers

1. $x = 2, y = 3$ **2.** $x = -3, y = 5$ **3.** $x = -4, y = -3$

4. $x = 5, y = -\frac{1}{2}$ **5.** $x = 1, y = 1.$

21. CARTESIAN PRODUCT

If A and B are any two sets, then set of all distinct ordered pairs whose first coordinate is an element of A and whose second coordinate is an element of B is called the Cartesian product of A and B and is denoted by A × B.

Symbolically $A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$.

For example, if $A = \{a, b, c\}$ and $B = \{x, y, z\}$ then

$$A \times B = \{(a, x), (a, y), (a, z), (b, x), (b, y) (b, z), (c, x), (c, y), (c, z)\}$$

and $$B \times A = \{(x, a), (x, b), (x, c), (y, a), (y, b), (y, c), (z, x), (z, b), (z, c)\}.$$

Notes 1. $A \times B \neq B \times A$ *i.e.*, Cartesian product is not commutative.

2. If the set A and B have m and n elements respectively, then the set A × B has mn elements.

3. If either A or B is a null set then the set A × B is also a null set.

4. If either A or B is an infinite set and other is a non empty set, then (A × B) is also an infinite set.

5. $A \times B = B \times A \Rightarrow A = B$.

22. RELATIONS

The world relation is not unfamiliar even to a child. He is aware of his various relations with the family members. As he grows up he comes across the other types of relation also *e.g.*, relation between money and its purchasing power, relation of education with school, relation between numbers in mathematical operations. The description of relation in above language is very vague. Here we shall try to define and describe relation in the precise language of mathematics. For this we shall use the concept of ordered pairs and cartesian product which have been already defined.

First we shall restrict ourselves only to the relations between only two objects. The relations which associate only two objects are known as 'Binary Relations'. We shall formally discuss the properties of binary relations in the coming sections. Below we give a mathematical definition of a relation.

Definition. *Let A and B be two sets. A relation from A to B is a subset of A × B. Symbolically, R is a relation from A to B iff* $R \subset A \times B$.

If (x, y) be a member of a relation set R we express it by writing xRy and say that 'x is in the relation R to y'. Thus

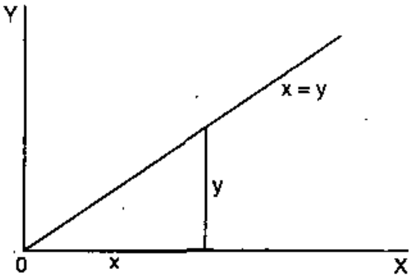
$$(x, y) \in R \iff xRy.$$

For example, If A = {2, 3, 5, 6} and R means 'divides' then 2R2, 2R6, 3R3, 3R6, 5R5, 6R6 and as such relation set R = {(2, 2), (2, 6), (3, 3), (3, 6), (5, 5), (6, 6)}.

Furthermore if I be the set of all integers, the statement "x is less than y where $x, y \in I$" determines a relation in I. If we denote this relation by R, then we may describe the set R in the set builder notation as given below :

$$R = \{(x, y) : x, y \in I, x < y\}.$$

Binary relation. A relation R between pairs of elements of a set A is called a binary relation. In the discussions hereafter the word relation will mean binary relation.



Let A be any set and R be the set of all those pairs (x, y) of $A \times A$ for which $x = y$. Then the relation R is called the equality relation in the set A. It is also called the *diagonal relation* in A and is denoted by Δ.

Thus $$\Delta = \{(x, y) : x, y \in x = y\}.$$

23. DOMAIN AND RANGE OF A RELATION

Let R be a relation from A to B. Then the set of all first coordinates of the members of the relation set R is called the domain of R and the set of all second members of R is called the range of R. Thus

$$\text{Domain } R = \{x : (x, y) \in R\}$$
$$\text{Range } R = \{y : (x, y) \in R\}.$$

For example, if $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5\}$, then

$$A \times B = \{(1, 3), (1, 4), (1, 5), (2, 3), (2, 4), (2, 5), (3, 3), (3, 4), (3, 5), (4, 3), (4, 4), (4, 5)\}.$$

Now if R stands for 'less than' then relation set

$$R = \{(1, 3), (1, 4), (1, 5), (2, 3), (2, 4), (2, 5), (3, 4), (3, 5), (4, 5)\}.$$

Domain R = set of first coordinate of ordered pairs in the relation set = {1, 2, 3, 4}, and Range R = set of second coordinates of ordered pairs in the relation set R = {3, 4, 5}.

24. INVERSE RELATION

Let R be a relation from A to B. The inverse relation of R, denoted by R^{-1}, is a relation from B to A defined by

$$R^{-1} = \{(y, x) : y \in B, x \in A, (x, y) \in R\}.$$

In other words, the inverse relation R^{-1} consists of those ordered pairs which when reversed belong to R. Obviously xRy iff $yR^{-1}x$.

Thus $$(x, y) \in R \iff (y, x) \in R^{-1}.$$

For example, if $A = \{a, b, c\}$, $B = \{1, 2, 3\}$

and $R = \{(a, 1), (a, 3), (b, 3), (c, 3)\}$

then $R^{-1} = \{(1, a), (3, a), (3, b), (3, c)\}$.

Likewise the inverse of relation 'x is the son of y' in the set of all living people is the relation 'y is father of x'.

NOTES

25. PROPERTIES OF BINARY RELATIONS

1. Reflexive. A relation R in a set A is a said to be reflexive, if $(x, x) \in R \ \forall x \in A$. In other words a relation is reflexive if every element of A related to itself in the sense of R

i.e., $$xRx \ \forall x \in A$$

Example 1. Let S be the set of all straight lines in a plane. The relation R in S defined by "x is || to y" is reflexive, since every straight line is || to itself.

Example 2. The relation '<' defined on a set of real numbers is not reflexive because x is not less than x.

Note. The necessary and sufficient condition for a relation to be reflexive is $\Delta \subset R$.

2. Symmetric. A relation R on a set A is said to be symmetric,

if $x R y \Rightarrow yRx$ *i.e.*, $(x, y) \in R \Rightarrow (y, x) \in R$, or if x is related to y then y is related to x for $x, y \in A$.

Example 1. Let N be the set of all natural numbers. The relation R in N defined by "x is equal to y" is symmetric because if $x = y$ then $y = x$.

Example 2. Let P be the set of all straight lines in a plane. The relation R defined by "a is perpendicular to b" is a symmetric relation because $a \perp^r b \Rightarrow b \perp^r a$.

Note. The necessary and sufficient condition for a relation to be symmetric is $R = R^{-1}$.

3. Transitive. A relation R in a set A is said to be transitive if for $x, y, z \in A$,

$$xRy \text{ and } yRz \Rightarrow xRz$$

i.e., $$(x, y) \in R \text{ and } (y, z) \in R \Rightarrow (x, z) \in R.$$

In other words, R is said to be transitive if x related to y and y related to z in the sense of R imply that x is related to z in the sense of R.

Example 1. The relation '>' defined on the set of natural numbers N is transitive because for $x, y, z \in N$,

if $x > y, y > z$ then $x > y > z$ *i.e.*, $xRy, yRz \Rightarrow xRz$ for $x, y, z \in N$.

Example 2. *The relation R of parallelism in the set S of straight liners in a plane* is a transitive relation because

$$x \parallel y \text{ and } y \parallel z \Rightarrow x \parallel z$$

i.e., $$xRy, yRz \Rightarrow xRz, \text{ for } x, y, z \in S.$$

Note. The necessary and sufficient condition that a relation be transitive is "R operated on $R \subset R$".

4. Anti-symmetric. A relation R in a set A is said to be anti-symmetric if

$$xRy \text{ and } yRx \Rightarrow x = y, \forall x, y \in A.$$

Example 1. Let N be the set of natural numbers and let R be the relation defined by "a divides b" for $a, b \in N$. Then R is an anti-symmetric relation as a divides b and b divides $a \Rightarrow a = b$.

Example 2. Let P be a family of sets ; then the relation R in P defined by "A is a subset of B" is anti-symmetric because

$$A R B \text{ and } B R A \Rightarrow A \subset B \text{ and } B \subset A$$
$$\Rightarrow A = B.$$

Note. The necessary and sufficient conditions that R is anti-symmetric is that $R \cap R^{-1} = \Delta$.

ILLUSTRATIVE EXAMPLES

Example 1. *Prove that the relation defined by 'is perpendicular to' in the set of straight lines in a plane is symmetric but neither reflexive nor transitive.*

Solution. Let R stand for "is perpendicular to". Let L be the set of straight lines in a plane.

1. Reflexivity. Since no straight line is perpendicular to itself, hence relation is not reflexive *i.e.*, $a\text{R}a$ for $a \in \text{L}$ is not true.

2. Symmetry. If straight line a is $\perp^r$ to straight line a then b is also $\perp^r$ to a *i.e.*,

$$a\text{R}b \Rightarrow b\text{R}a, \text{ for } a, b \in \text{L}.$$

Hence relation is symmetric.

3. Transitivity. If for $a, b, c \in \text{L}$, $a \perp^r b$ and $b \perp^r c$ then it is not necessary that $a \perp^r c$ (it may be parallel). Hence $a\text{R}b$, $b\text{R}c$ does not imply $a\text{R}c$ $\forall a, b, c \in \text{L}$.

Example 2. *Show that in the set of all real numbers, the relation 'less than' is transitive and anti-symmetric but not reflexive.*

Solution. The relation '<' defined on **R**, the set of real numbers is

(*i*) not reflexive because no real number is less than itself,

(*ii*) not symmetric since for $a, b \in \mathbf{R}$, $a < b$ does not imply $b < a$.

For example, $1 < 2$ but $2 \nless 1$

(*iii*) transitive because $a < b$ and $b < c \Rightarrow a < c$ for $a, b, c \in \mathbf{R}$.

26. EQUIVALENCE RELATION

A relation R in a set A is said to be an equivalence relation if

(*i*) R is reflexive *i.e.*, $x\text{R}x \ \forall x \in \text{R}$;

(*ii*) R is symmetric *i.e.*, $x\text{R}y \Rightarrow y\text{R}x$ for $x, y \in \text{R}$; and

(*iii*) R is transitive *i.e.*, $x\text{R}y$ and $y\text{R}z \Rightarrow x\text{R}z$ for $x, y, z \in \text{R}$.

Example 1. *If I be the set of integers and if R be define over I by "aRb iff a – b is an even integer" where a, b ∈ I, then show that the relation R is an equivalence relation.*

Solution. (1) **Reflexivity.** Since $0 = a - a$ is even.

Hence $\qquad a\text{R}a \ \forall a \in \text{I}$.

Therefore, R is reflexive over I.

(2) **Symmetric.** Since $a - b$ is even then $b - a = -(a - b)$ is also even and hence $a\text{R}b \Rightarrow b\text{R}a$. Therefore, R is symmetric in I.

(3) **Transitivity.** $a\text{R}b$ and $b\text{R}c$

$\Rightarrow$ $a - b$ and $b - c$ are even numbers

$\Rightarrow$ $(a - b) + (b - c)$ is even

$\Rightarrow$ $a - c$ is even

$\therefore$ $a\text{R}b$ and $b\text{R}c \Rightarrow a\text{R}c$.

Therefore, R is transitive in I

Hence R is an equivalence relation.

NOTES

Example 2. *In the set of all triangles in a plane, show that the relation of congruence (or similarity) is an equivalence relation.*

Solution. Let S be the set of all triangles and R the relation defined by $x\text{R}y$ iff $x \equiv y$, $x, y \in \text{S}$.

(*a*) $x\text{R}x \ \forall \ x \in \text{S}$, since every triangle is congruent to itself. Hence R is reflexive.

(*b*) For $x, y \in \text{S}$, $x\text{R}y \Rightarrow y\text{R}x$ because if x is congruent to y, then y is surely congruent to x. Hence the relation is symmetric.

(*c*) For $x, y, z \in \text{S}$, $x\text{R}y$ and $y\text{R}z \Rightarrow x\text{R}z$ because if x is congruent to y and y is congruent to z, then x is also congruent to z. Thus the relation is transitive also.

Hence it is an equivalence relation.

Example 3. *If I be the set of integers then show that relation R in I such that aRb iff $a - b$ is divisible by m (positive integer) is an equivalence relation.*

Solution. (*i*) For each $a \in \text{I}$, $a - a = 0$ which is divisible by m, so $a\text{R}a$.

Therefore R is reflexive

(*ii*) Let $a\text{R}b$ for $a, b \in \text{I}$, *i.e.*, let $a - b$ divisible by m. Then as

$$a - b = -(b - a),$$

$(b - a)$ is also divisible by m.

Hence $a\text{R}b \Rightarrow b\text{R}a$ so that R is symmetric.

(*iii*) For $a, b, c \in \text{I}$ let $a\text{R}b$, $b\text{R}c$ *i.e.*, let $(a - b)$, $(b - c)$ be divisible by m. Then as

$$(a - b) + (b - c) = a - c,$$

$a - c$ is also divisible by m.

Hence $a\text{R}b$, $b\text{R}c \Rightarrow a\text{R}c$, *i.e.*, R is transitive, showing that E is an equivalence relation.

EXERCISE 1(D)

1. Define a relation. When a relation R on a set A is known as symmetric, reflexive transitive and anti-symmetric ? Give an example for each.
2. Show that in the set of all real numbers, the relation 'greater than' is transitive but not reflexive.
3. Show that the relation R in the set of natural numbers N defined by $a\text{R}b$ if a divides b, is reflexive and transitive but not symmetric.
4. Prove that the relation R in the set of natural numbers N defined by $a\text{R}b$ if
   $a^2 - 4ab + 3b^2 = 0$, $(a, b \in \text{N})$, is reflexive, but neither symmetric nor transitive.
5. Discuss the nature of the following relations in N,
   (*a*) $x > y$, (*b*) $x \geq y$
   (*c*) $x + 3y = 12$, (*d*) x times y is a square
   (*e*) x is a multiple of y.
6. Give an example of the relation which is :
   (*a*) reflexive, symmetric but not transitive,
   (*b*) symmetric, transitive but not reflexive,
   (*c*) symmetric but neither transitive nor reflexive,
   (*d*) neither symmetric nor reflexive, nor transitive.

7. Define an equivalence relation, if

$$S = \{(a, b) : a, b \text{ are real numbers}\}$$

and $(a, b) R(c, d)$ is $a + d = b + c$.

Then determine whether R is an equivalence relation on S.

8. If R and S are equivalence relations in a set X, prove that $R \cap S$ is an equivalence relation in X.

9. Which of the following relations in the set of real numbers are equivalence relations :

(a) aRb if $|a| = |b|$ (b) aRb if $|a| \geq |b|$

(c) aRb if $a - b \geq 0$?

10. Is the relation 'brother of' an equivalence relation on a set of human beings ?

[**Hint.** 'brother of' is to be understood by blood relation' *i.e.*, one of the parents is common.]

11. Consider the set N × N the set of ordered pairs of natural numbers. Let R be relation in N × N which is defined by

$$(a, b) R (c, d)$$

if and only if $ad = bc$

Prove that R is an equivalence relation.

12. Show that if R and R′ are symmetric relations in a set A, then $R \cap R'$ and $R \cup R'$ are also symmetric relations in A.

Answers

5. (a) Anti-symmetric (b) Reflexive, anti-symmetric, transitive

(c) No-relation (d) Reflexive and symmetric

(e) Reflexive and transitive.

9. (a) an equivalence (b) not an equivalence

(c) not an equivalence.

10. No as it is not transitive.

27. EQUIVALENCE CLASSES

Let X be any non-empty set and let R be an equivalence relation in X. Further let x be an arbitrary element of X then the set of all those elements y in X for which xRy, is known as equivalence class of x. We shall denote this equivalence class by $[x]$ or $\bar{x}$. Evidently an equivalence class is a subset of the given set, such that any two elements of it are equivalent to each other.

Some authors denote the equivalence class of x determined by a relation R in X by symbol $x \mid R$. The set of all equivalence classes will be denoted by $X \mid R$ and read as "X modulo R". This is called the Quotient set of X by R.

Thus the set of all disjoint equivalent classes defined by an equivalence relation R over a set X is called the Quotient set of X relative to R and also denoted by $\overline{X}$.

Example. Let A be the set of all triangles in a plane and let R be an equivalence relation in A defined by "x is congruent to y", $x, y \in A$. When $x \in A$ we shall mean by the equivalence class $[x]$ the set of all triangles of A congruent to the triangle x. Similarly when $y \in A$ we shall mean by the equivalence class $[y]$ the set of all triangles of A congruent to the triangle y.

NOTES

28. PROPERTIES OF EQUIVALENCE CLASSES

Let A be non-empty set and let R be an equivalence relation in A. Let x and y be arbitrary elements in A. Then

(1) $x \in [x]$

(2) *If* $y \in [x]$, *then* $[y] = [x]$,

(3) $[x] = [y] \Leftrightarrow (x, y) \in R$ *i.e.,* xRy,

(4) *Either* $[x] = [y]$ *or* $[x] \cap [y] = \phi$ *i.e., two equivalence classes are either disjoint or identical.*

Proof. (1) Since R is reflexive, $x\text{R}x \ \forall \ x \in x$.

But $[x] = \{a : a \in \text{A and } a\text{R}x\}$

Hence $x\text{R}x \Rightarrow x \in [x]$.

(2) $y \in [x] \Rightarrow y\text{R}x$ for $x, y \in \text{A}$.

Let a be any arbitrary element of $[y]$, then $a \in [y] \Rightarrow a\text{R}y$.

But R is transitive, hence $a\text{R}y$ and $y\text{R}x$

$\Rightarrow a\text{R}x \Rightarrow a \in [x]$

Thus $a \in [y] \Rightarrow a \in [x]$. $\therefore [y] \subset [x]$.

Again let b be any arbitrary element of $[x]$, then

$$b \in [x] \Rightarrow b\text{R}x.$$

Since R is symmetric, $y\text{R}x \Leftrightarrow x\text{R}y$

Now $b\text{R}x$ and $x\text{R}y \Rightarrow b\text{R}y$ [$\because$ R is transitive]

$\Rightarrow b \in [y]$

Thus $b \in [x] \Rightarrow b \in [y]$

$\therefore [x] \subset [y]$

Ultimately, $[x] \subset [y]$ and $[y] \subset [x] \Rightarrow [x] = [y]$.

(3) Let $[x] = [y]$ then we have to prove $x\text{R}y$ first. As R is reflexive, $x\text{R}x$ and, therefore

$$x\text{R}x \Rightarrow x \in [x]$$

$\Rightarrow x \in [y] \quad \because [x] = [y]$

$\Rightarrow x\text{R}y$.

Thus $[x] = [y] \Leftrightarrow x\text{R}y$

Conversely, suppose $x\text{R}y$, then we have to prove $[x] = [y]$

Let a be any arbitrary element of $[x]$, then $a\text{R}x$ but it is given that $x\text{R}y$, therefore,

$a\text{R}x$ and $x\text{R}y \Rightarrow a\text{R}y$ [$\because$ R is transitive]

$\Rightarrow a \in [y]$

Hence $a \in [x] \Rightarrow a \in [y]$ *i.e.,* $[x] \subset [y]$ ...(1)

Again let b be any arbitrary element of $[y]$, then

$$b \in [y] \Rightarrow b\text{R}y$$

Since R is symmetric $x\text{R}y \Rightarrow y\text{R}x$

$\therefore b\text{R}y$ and $y\text{R}x \Rightarrow b\text{R}x$ [$\because$ R is transitive]

$\Rightarrow \qquad b \in [x]$

$\therefore \qquad b \in [y] \Rightarrow b \in [x]$ *i.e.,* $[y] \subset [x]$ ...(2)

Hence (1) and (2) $\Rightarrow [x] = [y]$

Finally, $[x] = [y] \Rightarrow x\text{R}y$ and $x\text{R}y \Rightarrow [x] = [y]$

$\therefore \qquad [x] = [y]$ iff $x\text{R}y$.

(4) If $[x] \cap [y] = \phi$, then the result is obvious.

Hence let us take $[x] \cap [y]$ non-empty and then we have to prove $[x] = [y]$.

$$[x] \cap [y] \neq \phi \Rightarrow \exists\, x \in [x] \text{ s.t. } x \in [x] \cap [y].$$

Now $\qquad a \in [x] \cap [y] \Rightarrow a \in [x]$ and $a \in [y]$.

$\Rightarrow x\text{R}a$ and $a\text{R}y$ (as R is symmetric, $a\text{R}x \Rightarrow x\text{R}a$)

$\Rightarrow x\text{R}y$ [$\because$ R is transitive]

$\Rightarrow \qquad [x] = [y]$ (by 3)

Thus $\qquad [x] \cap [y] \neq \phi \Rightarrow [x] = [y]$

or $\qquad [x] \neq [y] \Rightarrow [x] \cap [y] = \phi$.

29. PARTITIONS OF A SET

Let X be non-empty set. A set P = {A, B, C} of non-empty subsets of X will be called a partition of X if

(*i*) A ∪ B ∪ C ...... = X *i.e.*, the set X is the union of the sets in P, and

(*ii*) The intersection of every pair of distinct subsets of X ∈ P is the null set *i.e.*, if A and B ∈ P then either A = B or A ∩ B = ϕ.

Example. Consider the set X = {1, 2, ..., 9, 10} and its subset $B_1 = \{1, 3\}$, $B_2 = \{7, 8, 10\}$, $B_3 = \{2, 5, 6\}$, $B_4 = \{4, 9\}$.

The set $P = \{B_1, B_2, B_3, B_4\}$ is such that

(*i*) B_1, B_2, B_3, B_4 are all non-empty subsets of X.

(*ii*) $B_1 \cup B_2 \cup B_3 \cup B_4 = X$.

(*iii*) For any sets B_i, $B_i \cap B_j = \phi$.

Hence the set $\{B_1, B_2, B_3, B_4\}$ is a partition of X.

Theorem 1. *An equivalence relation defined in a set decomposes the set into disjoint equivalence classes.*

Proof. Let an equivalence relation R be defined in a set S. Let $a \in$ S and T be a subset of S consisting of all those elements which are equivalent to a *i.e.*,

$$\text{T} = \{x : x \in \text{S and } x\text{R}a\}.$$

Then $a \in$ T, for $a\text{R}a$ (R is reflexive). Any two elements of T are equivalent to each other, for if $x, y \in$ T, then $x\text{R}a$ and $y\text{R}a$.

Again $\qquad x\text{R}a, y\text{R}a \Rightarrow x\text{R}a, a\text{R}y$ (R is symmetric)

$\Rightarrow \qquad x\text{R}y$. (R is transitive)

Thus T is an equivalence class.

Let T_1 be another equivalence class *i.e.*, $T_1 = \{x : x \in \text{S and } x\text{R}b\}$, where b is not equivalent to a. Then the classes T and T_1 must be disjoint. For if they have a common elements, $s\text{R}a$ and $s\text{R}b$, so that $b\text{R}a$ which is contrary to our hypothesis

The set S can now be decomposed into equivalence classes T, T_1, T_2, ... such that every element of S belongs to one of these classes. Since these classes are mutually disjoint, we obtain the required partition of S.

Theorem 2. *If R is an equivalence relation in a non-empty set X, then the quotient set X / R is a partition of X.*

Proof. Each $x \in X$ must belong to some equivalence class. Also the equivalence classes are pairwise disjoint, for if $z \in X|R \cap Y|R$ then xRz, yRz. Since $yRz \Rightarrow zRy$, xRz and $zRy \Rightarrow xRy$, it follows that X|R and Y|R must be identical. Hence two equivalence classes are either disjoint or identical. The set of equivalence classes is therefore a partition. Further, if x, y be any two members of the same set of this partition, they stand in a relation R to each other, showing that the partition induces the relation R.

Converse. If C be partition of X, then the induced relation is an equivalence relation whose set of equivalence classes is X | C.

30. PRODUCT OF EQUIVALENCE RELATIONS

Let A, B be two sets and R, S be equivalence relations in A and B respectively. The relation R × S in A × B, defined by (x, y) R × S(z, u) $\Leftrightarrow$ xRz and ySu, is an equivalence relation in A × B.

R × S is called the product of the relations R *and* S. Every equivalence class of A × B is of the form P × Q where P is an equivalence class of X mod. R and Q is an equivalence class of Y mod S.

Partial Order Relations. A relation R in a set S is called a partial order relation *if and only if it satisfies the following three conditions :*

(1) $aRa \ \forall \ a \in S$, **(reflexivity)**

(2) aRb and $bRa \Rightarrow a = b$, **(anti-symmetry)**

(3) aRb and $bRc \Rightarrow aRc$. **(transitivity)**

Example 1. *Show that in the set N of all natural numbers, the relation R defined by aRb if a divides b is a partial order relation.*

Solution. (1) We know $\forall \ a \in N$, a is a divisor of a *i.e.*, aRa. Therefore, R is reflexive.

(2) Again if a is a divisor of b then b cannot be a divisor of a unless $a = b$. Thus, aRb and $bRa \Rightarrow a = b$. Therefore, R is anti-symmetric.

(3) Finally a is a divisor of b and b is a divisor of c implies a is a divisor of c. Therefore, R is transitive.

Example 2. *If A be a family of sets, and R be the relation in A defined by "X is a subset of Y". Then show that R is a partial order relation.*

Solution. Since every set is a subset of itself, therefore $\forall \ X \in A$ we have XRX *i.e.*, R is reflexive.

Again R is anti-symmetric since

$$A \subset B \text{ and } B \subset A \Rightarrow A = B.$$

Finally, $A \subset B$ and $B \subset C \Rightarrow A \subset C$. Therefore R is transitive.

Since R is reflexive, anti-symmetric and transitive, therefore, R is a partial order relation.

EXERCISE 1(E)

1. Find all the partitions of $\{a, b, c\}$.
2. If Y = {1, 2} and C (a partition of X) = {{1}, {2}}, find what equivalence relation is induced by C.
3. If X = the set of real numbers and C = {{0}, {x : x is a positive real number}, {x : x is a negative real number}}. Find what equivalence relation is induced by C.
4. Let A = set of all living Indians. Prove that the relation '~' defined on A as follows $a \sim b \Leftrightarrow a$ and b belong to the same state, is an equivalence relation. Determine the equivalence classes.

Answer

1. $\{a, b, c\}$, $\{\{a\}, \{b, c\}\}$, $\{\{b\}, \{a, c\}\}$, $\{\{c\}, \{a, b\}\}$, $\{\{a\}, \{b\}, \{c\}\}$.

31. MAPPING OR FUNCTION

The concept of mapping of one set into another is of great importance in mathematics. It is not new concept to any of us as we have been considering mapping from the beginning of our mathematical training. For example, plotting of the relation $y = x^3$ is nothing but to study the particular mapping which takes every real number onto its cube. The following discussion will make the concept of mapping clear.

Suppose A and B are any two non-empty sets. Let A = $\{a, b, c, d\}$, B = $\{x, y, z\}$. Suppose by some rule or other, we assign to each element of A, a unique element of B. Suppose a is associated to x, b is associated to y, c is associated to x and d is associated to x. *The set of such assignments is called a 'function' or 'mapping'* from A to B. If we denote this set by f then we write.

$$f : A \to B$$

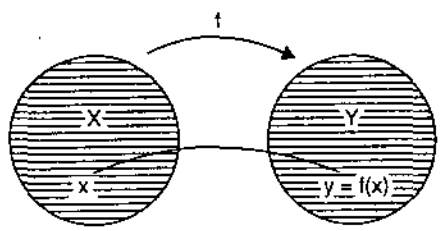
which is read as "f is function of A to B" or "f is a mapping from A to B".

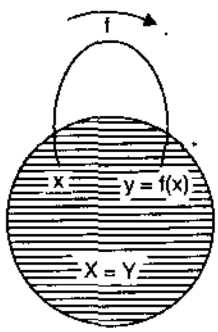
Loosely speaking, a mapping from one set into another may be defined as given below :

Definition. *Let A and B be two given sets. Suppose there exists a rule denoted by f, which associates to each member of A, a unique member of B. Then f is called a function or mapping of A to B.*

The mapping f of A to B is denoted by

$$f : A \to B \quad \text{or by} \quad A \xrightarrow{f} B$$





NOTES

Further, if $a \in$ A, then the element in B which is assigned to a is called the f-image of a or the value of the function f for a and is denoted by $f(a)$.

Above diagrams will help in understanding the idea of a function.

Now we think of a definition which serves to make the concept of a mapping precise and thus formally, we define mapping as follows :

Definition. *If A and B are non-empty sets then a mapping from A to B is a subset C of A × B such that for every $a \in$ A there is a unique $b \in$ B such that the ordered pair (a, b) is in C.*

For all practical purposes the following notion of mapping will be used hereafter and thus we arrive at another form of definition.

Definition. *A mapping $f : A \to B$, is a rule which associates any element $a \in$ A with some element $b \in$ B, the rule being to associate (or map) $a \in$ A with $b \in$ B iff (a, b) $\in$ C, C being the subset of A × B.*

Note. The rule f should possess the characteristic that there may be some elements of the set B which are not associated to any element of the set A but each element of the set A must be associated to one and only one element of the set B. Two or more elements of the set may be associated to the same element of the set B but association of one element of A to more than one element in B is not permissible.

Domain, Co-domain and Range of Functions. Let f be a mapping of A into B. Then A is called the 'domain' of the function f and B the 'co-domain' of the function f. It is evident from the definition that each element of B need not appear as the image of an element in A. We define the 'range' of f to consist of all those elements in B which appear as f-image of atleast one element in A. There can be more than one element of A which have the same image in B. The image set f [A] is called the range of f.

32. FUNCTIONS DEFINED AS SETS OF ORDERED PAIRS

Let A and B be any two non-empty sets, then a mapping f of A to B is a subset f of A × B satisfying the following conditions :

(*i*) for each $a \in$ A, $(a, b) \in f$ for some $b \in$ B ;

(*ii*) if $(a, b) \in f$ and $(a, b') \in f$, then $b = b'$.

The first condition ensures that we have a rule that assigns to each element $a \in$ A some element $b \in$ B. Thus each element in A will have f-image. The second condition guarantees that the image is unique. Accordingly f is a function from A to B.

Note. If f : A → B, it is important to distinguish between a function f and the value $f(x)$ of f for any element x. While f is a subset of A × B, $f(x)$ is an element of the set B.

33. OPERATOR

If the domain and co-domain of a function f are both the same set say

$$f : A \to A$$

then f is called an operator or transformation on A.

Following examples will make the notion of a function more clear.

Example 1. Let A = {1, 2, 3, 4} and B = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12}.

Now let *f assign to each number in* A *its square in* B. Then f is not a mapping from A to B since no number of B is assigned to the element 4 ∈ A.

Example 2. Suppose X = {*a*, *b*, *c*}, Y = {1, 2, 3} and mapping *f* is as follows :

$$f(a) = 3, f(b) = 1, f(c) = 2,$$

then *f*-image of X is {3, 1, 2} or in other words *f*-image of X is Y.

Example 3. Let *f* be a mapping of N into N such that

$$f(1) = 3 \; ; f(2) = 5 \; ; f(3) = 7 \; ; \ldots\ldots$$

We can express this mapping by the functional notation as

$$f : \mathrm{N} \rightarrow \mathrm{N} \; ; \text{ defined by } f(x) = 2x + 1, \forall\, x \in \mathrm{N}.$$

Example 4. Let X be the set of countries in the world and Y be the set of capital cities then (*i*) every country has a capital assigned to it (*ii*), no country will have two *capitals. Thus f is a function from the set* X *to set* Y *and the image of India under f is* Delhi *i.e.*, *f*(India) = Delhi.

Here domain of *f* is the list of countries in the world and the co-domain is the list of capital cities of these countries.

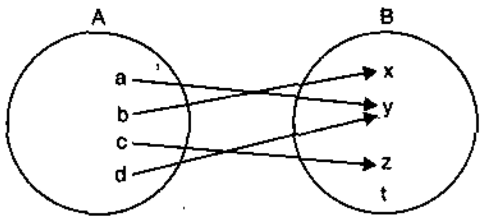
34. DIAGRAMMATIC REPRESENTATION OF FUNCTION

Sometimes a function may be represented by a diagram as will be obvious from the following example :

Let A = {*a*, *b*, *c*, *d*} and B = {*t*, *x*, *y*, *z*}.

Let *f* : A → B be defined by the correspondence

$$f(a) = y, f(b) = x, f(c) = z \quad \text{and} \quad f(d) = y.$$



We represent the two sets A and B by the points inside the two circles. The mapping *f* : A → B is represented by means of arrows joining the points which represent the elements of A to points representing the corresponding elements of B. By the definition of function, it is obvious that

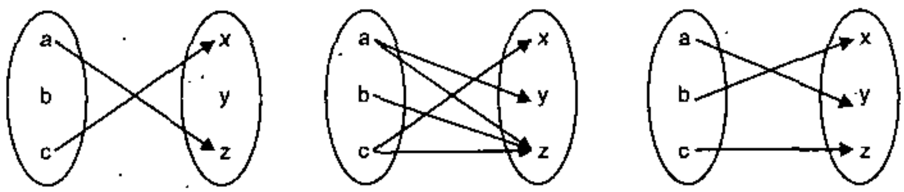
(*i*) every point in A is joined to some point in B by an arrow,

(*ii*) a point in A cannot be joined to two or more distinct points in B,

(*iii*) two or more points in A may be joined to the same point in B,

(*iv*) there may be some points in B which are not joined to any point in A.

Example. State with reasons whether or not each of the diagrams define a function of A = {*a*, *b*, *c*} into X = (*x*, *y*, *z*).



(i) (ii) (iii)

(*i*) Since there is no image of $b \in$ A it does not define a mapping,

(*ii*) Since $c \in$ A has two images $x \in$ X and $z \in$ X, it does not define a mapping,

(*iii*) It defines a mapping because every element of A has a unique image in B.

35. CONSTANT FUNCTION

The function $f : X \to Y$ s.t. $f(x) = c \ \forall \ x \in X, c \in Y$ is called a *constant function*.

In other words, $f : X \to Y$ is a constant function if the range of f consists of only one element, the adjoining figure, exhibits a constant mapping from a set $\{a, b, c\}$ into another set $\{1, 2, 3\}$.

Example. If $A = \{\pi, 2\pi, 3\pi\}$ and f is defined by

$$f(x) = \sin x, x \in A, \text{ then } f \text{ is constant function}$$

[since $\sin x = 0, \forall \ x \in A$].

36. VARIOUS KINDS OF MAPPING

(*i*) **One-one Mapping.** A mapping f defined from the set X to the set Y is said to be one-one if two different elements of X have two different f images in Y.

Thus $f : X \to Y$, is one-one mapping if,

$$f(x_1) = f(x_2) \quad \Rightarrow \quad x_1 = x_2, x_1, x_2 \in X$$

or equivalently, $$x_1 \neq x_2 \quad \Rightarrow \quad f(x_1) \neq f(x_2).$$

Note. No two elements of X can have the same image under this type of mapping.

Example. Let $X = \{1, 2, 3\}$, $Y = \{1, 4, 9, 16\}$ and $f : X \to Y$ s.t. $f(x) = x^2 \ \forall \ x \in N$, then f is one-one mapping of X into Y, as no two distinct elements of X have the same f-image in Y.

(*ii*) **Many-one Mapping.** A mapping $f : X \to Y$ is said to be many-one if $f(x_1) = f(x_2)$ even if $x_1 \neq x_2, x_1, x_2 \in X, f(x_1), f(x_2) \in Y$.

In other words, if a given element of Y may have more than one pre-image under the mapping f (but no element of X can have more than one f-image), then mapping is said to be many-one.

Example. Let $X = \{a, b, c, d\}$, $Y = \{3, 4, 5\}$, and $f(a) = 3, f(b) = 4, f(c) = 3, f(d) = 5$.

Then it is many-one mapping or many one function of X into Y, as two elements a and c of X have the same image 3 in Y.

(*iii*) **Into Mapping.** If the mapping $f : X \to Y$, is such that at least one element of Y is not the f-image of any element of X, we say that f is an into mapping.

In that case $f(X) \subset Y$ and $f(x) \neq Y$.

Note. A mapping which is both one-one and into is said to be *injective*.

Example. Let $f : R \to R$ be defined by

$$f(x) = x^2, \forall \ x \in R,$$

then f is an into function, because the negative numbers do not appear in the range of f, *i.e.*, no negative number is the square of a real number.

(*iv*) **Onto (or Surjective) Mapping.** If the mapping $f : X \to Y$ is such that each element of Y is the f-image of at least one element of X, then we say that f is a mapping of X onto Y. Thus if f is mapping of X onto Y and $y \in Y$, then there necessarily exists an element $x \in X$

such that $f(x) = y$.

In this case $f(X) = Y$.

i.e., image set of f = co-domain of f.

Example. Let X = {1, 2, 3, 4} and Y = {1, 4, 9, 16} and let $f(x) = x^2$, $\forall\ x \in X$ then $f : X \to Y$ is an onto function as every element of Y has a pre-image in X under f.

NOTES

(*v*) **One-one Into Mapping.** Any mapping which is one-one as well as into called *one-one into mapping*.

Example. Let I be the set of integers and Y the set of all even integers then the mapping $f : X \to Y$, s.t. $f(x) = 2x$, $x \in X$ is an into mapping which is also one-one.

(*vi*) **One-one Onto Mapping.** A mapping which is one-one as well as onto is called *one-one onto mapping*.

This mapping is also known as *bijection*. In other words a mapping $f : X \to Y$ is called one-one, onto if the following conditions are satisfied.

(*i*) $f(x_1) = f(x_2) \Rightarrow x_1 = x_2, x_1, x_2 \in X$

(*ii*) Given any element $y \in Y$; there exists an element $x \in X$ s.t. $y = f(x)$, *i.e.*, every element of Y has a pre-image.

Example. Let A be the set of even integers and B be the set of odd integers then the mapping $f : A \to B$ given by

$$f(x) = x + 1, \forall\ x \in A \quad \text{is one-one onto.}$$

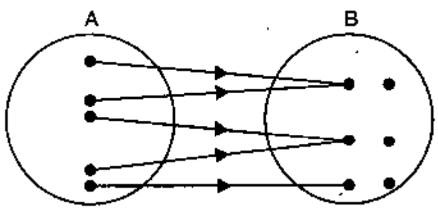
(*vii*) **Many-one Into Mapping.** A function which is many-one as well as into is called *many-one into mapping*.

Example. If $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2\}$ and if the function $f : X \to Y$, is defined as $f(x_1) = y_1$, $f(x_2) = y_1$, $f(x_3) = y_1$, then it is a many-one into function.

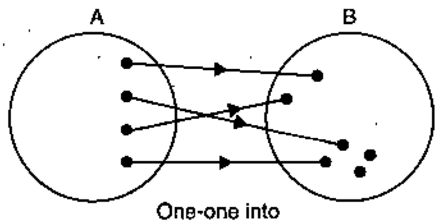
(*viii*) **Many-one Onto Mapping.** A function which is many-one as well as onto is called *many-one onto mapping*.

Example. Let X = {*a*, *b*, *c*, *d*}, Y = {3, 4, 5} and $f(a) = 3$, $f(b) = 4$, $f(c) = 4$, $f(d) = 5$, then it is many-one onto mapping.

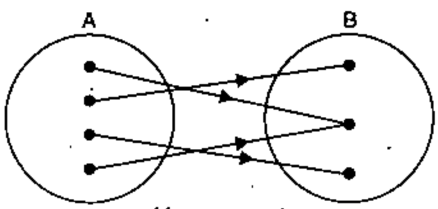
37. DIAGRAMMATIC REPRESENTATION OF DIFFERENT TYPE OF MAPPINGS



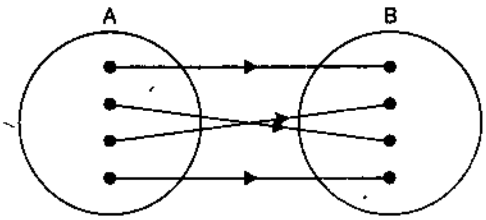
Many-one into



One-one into



Many-one onto



One-one onto

NOTES

Explanation from a daily life example. Let A be the set of students sitting on chairs in a class-room and let B be the set of chairs in the class-room. Let f be the correspondence which associates to each student the chair on which he sits.

Since every student has some chair to sit on (of course two or more than two students might sit on one chair) and no student can sit on two or more than two chairs, therefore, $f : A \to B$.

The following cases arise :

(*i*) If every student gets a separate chair and no chair is left vacant, then this is a case of one-one onto mapping.

(*ii*) If every student gets a separate chair and still some chairs lie vacant it is a one-one into mapping.

(*iii*) If every student does not sit on a separate chair and no chair is left vacant then this is the case of many-one onto mapping.

(*iv*) If every student does not sit on a separate chair and some chairs are left vacant, then this is a case of many-one into mapping.

ILLUSTRATIVE EXAMPLES

Example 1. *Decide whether or not the following are functions from* $A \to B$ *where*

$$A = \{1, 2, 3, 4, 5\}, B = \{a, b, c, d, e\}.$$

If they are functions, give the range of each. If they are not tell why ?

(*a*) $f = \{(1, a), (2, b), (3, b), (5, e)\}$.

(*b*) $g = \{(1, e), (5, d), (3, a), (2, b), (1, d), (4, a)\}$.

(*c*) $h = \{(5, a), (1, e), (4, b), (3, c), (2, d)\}$.

Solution. (*a*) Since the element $4 \in A$ is not associated to any element $\in B$ therefore, f is not a function from $A \to B$.

(*b*) The element $1 \in A$ is associated to two different elements e and $d \in B$. Therefore, g is not a function from $A \to B$.

(*c*) Each element A is associated to a unique element of B. Therefore, h is a function from $A \to B$. The range of h is the set of the h-images of all elements of A. Thus

$$\text{range of } h = h(A) = \{a, e, b, c, d\} = B.$$

Example 2. *Let* $A = \{-2, -1, 0, 1, 2\}$. *Let the function* $f : A \to R$ *be defined by*

$$f(x) = x^2 + 1.$$

Find the range of f.

Solution. The range of f consists of those elements of R which appear as f-images of different elements of A. So we calculate the f-image of each element of A.

$$f(-2) = (-2)^2 + 1 = 4 + 1 = 5$$

$$f(-1) = (-1)^2 + 1 = 1 + 1 = 2$$

$$f(0) = (0)^2 + 1 = 0 + 1 = 1$$

$$f(1) = (1)^2 + 1 = 1 + 1 = 2$$

$$f(2) = (2)^2 + 1 = 4 + 1 = 5.$$

The range of f is the set $\{5, 2, 1, 2, 5\}$ *i.e.*, the set $\{5, 2, 1\}$.

Example 3. *Show that the mapping* $f : I \to I$ *defined by* $f(x) = x^2$, $x \in I$ *where I is the set of positive integers, in one-one into.*

Solution. $f(x) = x^2$ means that the function f is such that f-image of x is x^2.

Domain of that the function is {1, 2, 3, ...} and the range is {1, 4, 9, ...}. Thus f-image is the subset of its domain *i.e.*, $\{f(x)\} \subset$ I. It is a mapping of I into I. Here two different elements of domain necessarily correspond to different elements of the range so that it is one-one mapping. *Hence it is a one-one into mapping.*

Example 4. *State whether or not each of the following functions is one-one.*

(*a*) *To each person on earth assign the number which corresponds to his age.*

(*b*) *To each country in the world assign the number of people who live in the country.*

(*c*) *To each book written by only one author assign the author.*

(*d*) *To each country in the world assign its capital city.*

Solution. (*a*) Here the domain is persons on earth and range is age of persons. As many people can be of the same age the mapping is not one-one. It is many-one.

(*b*) The domain consists of countries of the world and range is people in a country. As any two countries do not have the same number of people as shown by statistics, the different countries will be mapped to different number of people. Hence the mapping is one-one.

(*c*) Here domain of mapping is 'books' and range is 'authors'. Therefore, it is possible that two or more books may have the same author. So the mapping is not one-one.

(*d*) Since different countries in the world have different capital cities, the mapping is one-one.

Example 5. *If R is the set of real numbers, discuss the mapping* $f : R \to R$ *where*

$$f(x) = x^2,\ x \in R.$$

Solution. If x is any real number than x^2 is also a real number and it will be unique. Thus each element $x \in$ R has a unique f-image in R. Therefore, f : R $\to$ R.

Since there is no real number whose square is negative, therefore, any negative number in R is not the f-image of any element in R. Consequently f is mapping of R into R.

Also we see that $(-2)^2 = 4$ and $(2)^2 = 4$. Thus the elements 2 and -2 in R have the same *f-image 4 in R. Hence f is many-one into mapping of* R $\to$ R.

Example 6. *Let A = {0, 1} and let N be the set of natural numbers. Show that* $f : N \to A$ *defined by* $f(2x) = 0$, $f(2x + 1) = 1$, $(x \in N)$ *is a many-one onto mapping of N to A.*

Solution. Here, we see that each element in A is the f-image of at least one element in N. For example 0 is the f-image of 4 $\in$ N and 1 is the f-image of 9 $\in$ N. Therefore, f is a mapping of N onto A. Also we see that the element 1 $\in$ A is the f-image of every odd natural number $\in$ N. Hence f is a many-one onto mapping of N to A.

EXERCISE 1(F)

1. Each of the following formulae defines a function from R to R.
   Find the range of each function.
   (*a*) $f(x) = x^2$ (*b*) $g(x) = \sin x$, (*c*) $h(x) = x^2 + 1$.
2. Let A = {0, 1, 2, 3}. Decide whether or not the following are functions from A $\to$ A. If not, tell why ?
   (*a*) f = {(0, 1), (1, 2), (2, 3), (3, 0)} (*b*) f = {(1, 0), (2, 1), (3, 2), (0, 3)}
   (*c*) f = {(0, 1), (1, 0), (2, 1), (3, 0)}, (*d*) f = {(1, 0), (0, 1), (1, 2), (0, 3)}.

3. Examine the mapping $X = \{1, 2, 5\}, \rightarrow Y = \{x, y, z\}$ and $f(1) = x, f(2) = y, f(5) = z$.
4. Let $f : R^* \rightarrow R^*$ be given by $f(x) = 1/x$ when $R^* = R - \{0\} : x \in R^*$.
   Show that f is one-one onto.
5. Let $f : Q \rightarrow Q$ be given by $f(x) = 2x + 3, \forall x \in Q$
   Show that f is one-one onto.
6. Let P be the set of all triangles and R^+ be the set of positive real numbers. Prove that the function $f : P \rightarrow R^+$ given by
   $$f(\Delta) = \text{area of } \Delta, \Delta \in P$$
   is a many-one onto mapping.
7. Show that the mapping $f : R \rightarrow R$ defined by
   $$f(x) = \cos x, \forall x \in R$$
   is neither one-one nor onto. Modify the domain and co-domain of this mapping so that it may be both one-one and onto.
8. Let $f(x) = \begin{cases} 1 \text{ if } x \text{ is rational number} \\ 0 \text{ if } x \text{ is irrational number} \end{cases}$

   be a function from $R \rightarrow R$. Find $f(1/3)$, $f(\sqrt{7})$, also check if f is
   (*a*) one-one, (*b*) onto.
9. Prove that the mapping $f : I \rightarrow R$ define by $f(x) = x^2$, where is the set of all integers and R is the set of real numbers, is many-one and into.
10. Can a constant function be (*a*) one-one and (*b*) onto ? If so when ?

Answers

1. (*a*) f is an onto mapping and range if f is R.
   (*b*) g is an into mapping and range of g is $\{-1, 1\}$.
   (*c*) range of h is $\{1, a\}$.
2. (*a*) yes (*b*) yes (*c*) yes
   (*d*) no, since images of 0 and 1 are not unique, also images of 2 and 3 are missing.
3. f is a one-one into mapping.
7. $X = \left\{x : x \in R \text{ and } -\frac{\pi}{2} \le x \le \frac{\pi}{2}\right\}$. (domain)
   $Y = \{y : y \in R \text{ and } -1 \le y \le 1\}$. (Co-domain)
10. yes, (*a*) if domain of a function is a singleton
    (*b*) if co-domain of a function is a singleton.

38. INCLUSION MAP

If $X \subset Y$ the function $f : X \rightarrow Y$, defined by $f(x) = x$ for each $x \in X$ is called the *inclusion map*.

Example. Let $X = \{-3, -2, -1, 0, 1, 2, 3\}$ and $Y = \{...-4, -3, -2, -1, 0, 1, 2, 3, 4, ...\}$ and $f(-3) = -3, f(-2) = -2, f(-1) = -1, f(0) = 0, f(1) = 1, f(2) = 2, f(3) = 3$, then it is an inclusion map of X to Y because $X \subset Y$ and $f(x) = x, \forall x \in X$.

39. IDENTITY MAP OR IDENTITY FUNCTION

Let X be any set and the function $f : X \rightarrow X$ be defined by the formula $f(x) = x \ \forall x \in X$, *i.e.*, each element of X is mapped on itself, then f is called the *identity map* or *identity function* (transformation) on X. We denote this function by I_x usually. Thus if I_x denotes the identity mapping on a set X, we have

$$I_x(x) = x, \forall x \in X.$$

Example. Let A = {*a*, *b*, *c*, *d*}. Then $f = \{(a, a), (b, b), (c, c), (d, d), (e, e)\}$ is an identity mapping of A.

Identity mapping is always one-one onto.

40. EQUALITY OF FUNCTIONS

Two functions f and g are said to be equal if (*i*) they are defined on the same domain X and (*ii*) $f(x) = g(x)$; $\forall x \in X$.

Example. Let $f : R \to R$ and $g : R \to R$. Let f be defined as $f(y) = y^2, \forall y \in R$ and $g(x) = x^2, \forall x \in R$. Then the function f and g are equal, *i.e.*, $f = g$.

It is worth noting that x and y are merely dummy variables in the formula defining the functions.

41. CARDINALLY EQUIVALENT SETS

Let there be a mapping $f : X \to Y$ which is one-one and onto then the two sets X and Y are said to be *cardinally equivalent* or *equi-numerous*. This fact is denoted by writing X ~ Y.

Note. If any set X is equivalent to N, the set of natural numbers then X is said to be denumerable set.

Example 1. Let X = {1, 2, 3}, Y = {1, 4, 9} then $f : X \to Y$ defined by $f\{x\} = x^2, \forall x \in X$ is one-one and onto mapping, therefore, the set X and Y are cardinally equivalent *i.e.*, X ~ Y.

Example 2. *The set* $A = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\ldots, \frac{1}{n}, \ldots\ldots\right\}$ *is denumerable set as it can be* put in one-to-one correspondence with the set of natural numbers.

42. INVERSE IMAGE OF AN ELEMENT

Let f be a function defined from the set X to the set Y then the inverse image of an element $b \in Y$ under f is denoted by $f^{-1}(b)$ to be read as f inverse b and

$$f^{-1}(b) = \{x : x \in X \text{ and } f(x) = b\}$$

i.e., $f^{-1}(b)$ is the set of those elements in X which have b as their f image.

Example. Let $f : R \to R$, where $f(x) = x^2$ and R is the set of real numbers, then $f^{-1}(9) = \{3, -3\}$, for 9 is the f-image of both 3 and -3.

43. INVERSE IMAGE OF A SUBSET

Let f be function defined from the set X to the set Y and B be a subset of Y *i.e.*, $B \subset Y$ then the inverse of B under f is given by

NOTES

$$f^{-1}(B) = \{x : x \in X \text{ and } f(x) \in B\}$$

Obviously, $f^{-1}(B) \subset X$ and $f^{-1}[f(X)] = X$.

Note. If $b \in Y$, then $\{b\} \subset Y$ and $f^{-1}(b) = f^{-1}\{(b)\}$.

Example. Let $f : X \to Y$, where $X = \{1, 2, 3, 4, 5\}$ and $Y = \{1, 2, 4, 6, 9, 15, 16, 25, 30\}$.

and let $f(x) = x^2 \ \forall\, x \in X$; let $B_1 = \{2, 4, 9, 15, 16\}$ and $B_2 = \{2, 6\}$ than

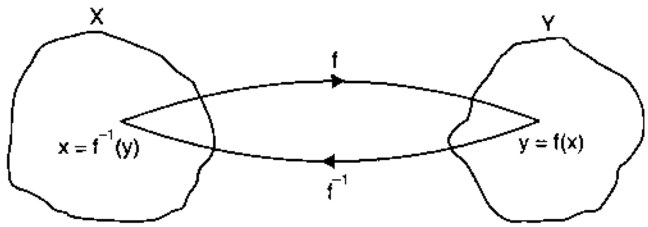
$$f^{-1}(B_1) = (2, 3, 4) \text{ and } f^{-1}(B_2) = \phi.$$

Note. It should be clear in the mind of the reader that we have not talked of inverse function so far.

44. INVERSE MAPPING (OR FUNCTION)

Let $f : X \to Y$ be one-one and onto mapping. Then the mapping $f^{-1} : Y \to X$ which associates to each element $y \in Y$, the element $x\ (= f^{-1}(y)) \in X$, whose f-image was $y \in Y$, is called the inverse of the map $f : X \to Y$.

The following diagram may be useful to understand the idea.



Alternately we can define inverse function as given below :

Let f be a function defined from the set $X \to Y$ and g be a function defined from the set $Y \to X$, then the function g is said to be inverse of f iff

$$g\{f(x)\} = x,\ \forall\, x \in X.$$

and the function g is denoted by f^{-1}.

Note. A function which possesses an inverse is called invertible.

Example. Let $f : R \to R$ where $f(x) = 2x$ and $g : R \to R$ be such that $g(y) = y/2$; then g is the inverse of f and vice-versa.

Remark. A function is invertible iff it is both one-one and onto. Identity mapping is always invertible for it is one-one onto.

Theorem 1. *If f is one-one onto function then f^{-1} is also one-one onto.*

Proof. Let X and Y be two sets and $f : X \to Y$ be a one-one onto function. Let $y_1, y_2 \in Y$. Since the function is onto, there exist $x_1, x_2 \in X$ such that

$$f(x_1) = y_1,\ f(x_2) = y_2.$$

Now $\quad f^{-1}(y_1) = x_1 \Leftrightarrow y_1 = f(x_1)$ and $f^{-1}(y_2) = x_2 \Leftrightarrow y_2 = f(x_2)$.

Now since f is one-one

$$x_1 \neq x_2 \Leftrightarrow f(x_1) \neq f(x_2)$$

i.e., $\quad f^{-1}(y_1) \neq f^{-1}(y_2) \Leftrightarrow y_1 \neq y_2$

Hence f^{-1} is an one-one mapping. Also the function f^{-1} is onto for any element $x \in X$ is the inverse image (*i.e.*, pre-image) of element $y \in Y$ where $y = f(x) \in Y$.

Hence the mapping f^{-1} is one-one onto if f is one-one onto.

Theorem 2. *If $f : X \to Y$ be one-one onto then the inverse map of f is unique.*

Proof. Let $g : Y \to X$ and $h : Y \to X$ be two inverse mappings of f. We have to prove that $g = h$.

Let $y \in Y, g(y) = x_1$ and $h(y) = x_2$. Since g and h are inverse mappings of f, therefore,

$$g(y) = x_1 \Rightarrow f(x_1) = y$$

and

$$h(y) = x_2 \Rightarrow f(x_2) = y.$$

But f is an one-one mapping.

$\therefore$ $f(x_1) = y$ and $f(x_2) = y \Rightarrow x_1 = x_2 \Rightarrow g(y) = h(y)$

Hence $g = h$.

ILLUSTRATIVE EXAMPLES

Example 1. *Let $R \to R$ defined by $f(x) = ax + b$, where $a, b, x \in R$ and $a \neq 0$. Prove that f is invertible.*

Solution. f is one-one :

For $x_1, x_2 \in R, f(x_1) = f(x_2)$

$\Rightarrow$ $ax_1 + b = ax_2 + b \Rightarrow ax_1 = ax_2$

$\Rightarrow$ $x_1 = x_2$

f is onto :

Let $y \in R$ such that

$$y = f(x) \Rightarrow y = ax + b$$

$\Rightarrow$ $ax = y - b$ and $a \neq 0 \in R$

$\Rightarrow$ $x = 1/a\ (y - b) \in R.$

$\therefore$ Given $y \in R, \exists$ some $x = 1/a\ (y - b) \in R$, s.t. $f(x) = y$

$\therefore$ $f : R \to R$ is both one-one and onto.

Hence f is invertible.

Example 2. *Let f be a function defined from the set X to the set Y and let A, B be the subset of Y, then*

(a) $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$. *(b) $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$.*

Solution. (*a*) Let x be an element of $f^{-1}(A \cup B)$, then

$$x \in f^{-1}(A \cup B) \Rightarrow f(x) \in A \cup B$$

$\Rightarrow$ $f(x) \in A$ or $f(x) \in B$

$\Rightarrow$ $x \in f^{-1}(A)$ or $x \in f^{-1}(B)$

$\Rightarrow$ $x \in \{f^{-1}(A)\} \cup \{f^{-1}(B)\}$

Therefore, $f^{-1}(A \cup B) \subset \{f^{-1}(A)\} \cup \{f^{-1}(B)\}$ ...(1)

Again let be an element of $f^{-1}(A) \cup f^{-1}(B)$ then $y \in f^{-1}(A) \cup f^{-1}(B)$

$\Rightarrow$ $y \in f^{-1}(A)$ or $y \in f^{-1}(B)$

$\Rightarrow$ $f(y) \in A$ or $f(y) \in B$

$\Rightarrow$ $f(y) \in A \cup B$

$\Rightarrow$ $y \in f^{-1}(A \cup B)$.

NOTES

Therefore, $f^{-1}(A) \cup f^{-1}(B) \subset f^{-1}(A \cup B)$. ...(2)

(1) and (2), imply, $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$.

Similarly part (*b*) can be proved.

Example 3. *If the mapping $f : I \to I$ be defined by $f(x) = x^2$ where I is the set of integers, evaluate $f^{-1}(16)$ and $f^{-1}(-3)$. Does the inverse mapping exist ? If so, find it.*

Solution. Let $f^{-1}(16) = x$ where $x \in I$

$$16 = f(x) = x^2$$

$$\therefore \quad x = 4 \quad \text{or} \quad -4$$

Hence $f^{-1}(16) = \{-4, 4\}$.

Also since there is no integer whose square is -3, if follows that $f^{-1}(-3) = \phi$, the empty set.

Second Part. Under the given mapping, the elements -4 and 4 have the same image 16. Accordingly, the given mapping is not one-one. Hence the inverse mapping does not exist.

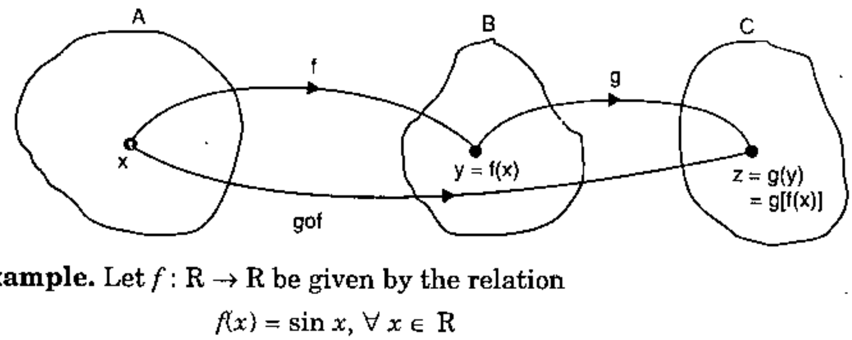
45. PRODUCT OF MAPPING OF COMPOSITE FUNCTIONS

The idea of a function of a function in mathematics has an analogue in abstract mathematics and it is known as a composite function. Let A, B, C be three sets and f be a function defined from A to B and g be a function defined from B to C.

i.e., $f : A \to B$ and $g : B \to C$.

By $f : A \to B$ we mean that to every element $x \in A$, there corresponds a unique element $f(x) \in B$. Since the domain of g is B, so by the function $g : B \to C$ we mean that to every element $f(x) \in B$ there corresponds a unique element $g[f(x)] \in C$. Thus we notice that to every element $x \in A$ there corresponds a unique element $g[f(x)] \in C$ under the mapping f and g. This implies that there exists a mapping from A to C. This mapping is called the composite mapping or the product mapping of f and g and is denoted by *gof* or *gf*.

Definition. *Let $f : A \to B$ and $g : B \to C$; then the composite of the function f and g denoted by gof or gf is mapping gof : $A \to C$ s.t. $(gof)(x) = g[f(x)]$, $\forall x \in A$.*



Example. Let $f : R \to R$ be given by the relation

$$f(x) = \sin x, \forall x \in R$$

and the mapping $g : R \to R$ be given by the relation

$$g(x) = x^2, \forall x \in R,$$

then the composite function $(gof) : R \to R$ is given by the relation

$$(gof)(x) = g[f(x)] = g[\sin x] = (\sin x)^2 = \sin^2 x$$

also $(fog)(x) = f[g(x)] = f[x^2] = \sin x^2 \ \forall x \in R.$

Note. If $f : A \to A$ and $g : A \to A$, then both the composite of functions gof and fog are defined. But in general $gof \neq fog$ for in the above example $(gof)(x) = \sin^2 x$, where as $(fog)(x) = \sin x^2$.

Theorem. *Associativity of composite functions. Let $f : A \to B$, $g : B \to C$ and $h : C \to D$; then $(hog)of = ho(gof)$.*

Proof. It is obvious that both $(hog)of$ and $ho(gof)$ are mappings from $A \to D$. These two mappings will be equal iff they assign the same image for every element $x \in A$. In other words,

$$(hog)of = ho(gof) \text{ if}$$
$$[(hog)of]\,(x) = [ho(gof)](x),\ \forall\, x \in A.$$

Now $[(hog)of](x) = (hog)[f(x)],\ \forall\, x \in A$ by the definition of composition

$$= h\,[gf(x)]$$
$$= h\,[(gof)(x)]$$
$$= [ho(gof)](x),\ \forall\, x \in A$$

Therefore, $(hog)of = ho(gof)$.

ILLUSTRATIVE EXAMPLES

Example 1. *If the map $f : R \to R$ be given by $f(x) = 4x - 1$ and the map $g : R \to R$ be given by $g(x) = x^3 + 2$, find $(gof)\,x$ and $(fog)\,x$, R being the set of real numbers.*

Solution. We have

$$(gof)x = g[f(x)] = g(4x - 1) \text{ as } f(x) = 4x - 1$$
$$= (4x - 1)^3 + 2 \text{ as } g(x) = x^3 + 2$$
$$= 64x^3 - 48x^2 + 12x + 1$$

Also $(fog)x = f[(g(x)] = f(x^3 + 2) = 4(x^3 + 2) - 1 = 4x^3 + 7.$

Example 2. *If $f : X \to Y$ be one-one and onto, prove that*

$$fof^{-1} = I_y, \text{ and } f^{-1} of = I_x,$$

where I_y = the identity map of Y onto itself and I_x = the identity map of X onto itself.

Solution. Let $f : X \to Y$ be given by $f(x) = y$, $(x \in X, y \in Y)$. Then $x = f^{-1}(y)$ *i.e.*, x is the inverse image of y under f.

i.e., $f^{-1} : Y \to X$ is defined by $f^{-1}(y) = x$. ...(*i*)

Now $(f^{-1} of\ )x = f^{-1}[f(x)],\ \forall\, x \in X$

$$= f^{-1}(y) = x \text{ from } (i).$$

Thus the map $(f^{-1}of)$ maps the element x onto itself. Hence $f^{-1}of = I_x$ the identity map of X onto itself. Again

$$(fof^{-1})y = f[f^{-1}(y)] = f(x) = y, \text{ as } f^{-1}(y) = x$$

Hence $fof^{-1} = I_y$, the identity map of Y onto itself.

EXERCISE 1(G)

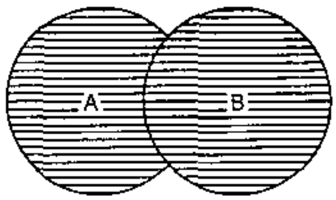
1. Define an invertible mapping. Prove that the inverse of an invertible mapping is invertible.
2. If the function $f : R \to R$ be defined by $f(x) = x^2 + 5x + 9$, where R is the set of real numbers, find $f^{-1}(3)$, $f^{-1}(9)$.

NOTES

3. Let $f : A \rightarrow B$ be given, prove the following :
(a) for each subset $X \subset A$, $X \subset f^{-1}[f(x)]$ (b) for each subset $Y \subset B$, $Y \supset [f^{-1}(Y)]$
(c) If $f : A \rightarrow R$ is one-one then $X \subset A \Rightarrow f^{-1}f(X) = X$,
(d) If $f : A \rightarrow B$ is onto then $ff^{-1}(Y) = Y$ for $Y \subset B$.
4. If $f : R \rightarrow R$ be defined by $f(x) = 3x - 4$, where R is the set real numbers, find a formula that defines the inverse map f^{-1}.
5. Prove that
(a) $f^{-1}(B') = [f^{-1}(B)]'$, (b) $f^{-1}(A - B) = f^{-1}(A) - f^{-1}(B)$.
6. *Show that the inverse mapping f^{-1} of a mapping f exists iff f is one-one onto.*
7. If $f : R \rightarrow R$ and $g : R \rightarrow R$ be defined by $f(x) = x^3 - 2$, $g(x) = x + 7 \ \forall x \in R$. Compute $(gof)x$ and $(fog)x$.
8. Let f be a one-one mapping of A onto B, and g be a one-one mapping of B onto C. Then gof is also invertible and $(gof)^{-1} = f^{-1}og^{-1}$.
9. If $f : R \rightarrow R$ be defined by $f(x) = | x |$ then prove that $fof = f$.
10. Prove that the product of any function and the identity function is the function itself.
11. Let $X = \left\{X : x \in R \text{ and } -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}\right\}$
and $Y = \{y : y \in R \text{ and } -1 \leq y \leq 1\}$.
Show that the function $f(x) = \sin x, x \in X$ in one-one onto. Also give the inverse mapping.

SUMMARY

1. A set or a collection is, therefore, a group of objects having some well-defined and common property or properties. The individual object of the collection or set is called an element or a member of the set.
2. **Roster Method (Tabular form).** In this method a set is represented by listing all its *elements* within braces { }. For example, the set of vowels in the English alphabets would be written as $\{a, e, i, o, u\}$.
3. **Rule Method.** In this method, a set is represented by describing its elements in terms of one or several characteristic properties which enable us to decide whether a given object is an element of the set under consideration or not.
4. **Null Set.** A set which contains no element is known as Null, Empty or Void set and is denoted by symbol F or { }.
5. A set which has only one member is known as a Singleton Set or Simply Singlet.
7. Let A and B be two sets.
If $B \subset A$ and $B \neq A$, then B is said to be proper subset of A.
8. ***Union of Sets.*** Let A *and* B *be two given sets.* The *set* which contains every element contained in A or B or both A and B is called the union (or join) of A and B. In a bit different language it is described that the union is as 'either' 'or' idea. The symbol $\cup$ is used to denote the union of sets. Thus $A \cup B$ is read as 'A union B' or 'A join B' or 'A cup B'. Symbolically $A \cup B = \{x : x \in A$ or/and $x \in B.\}$ or simply



A ∪ B (Shaded)

$$A \cup B = \{x : x \in A \text{ or } x \in B\}.$$

9. Let A and B be two sets. A relation from A to B is a subset of $A \times B$. Symbolically, R is a relation from A to B iff $R \subset A \times B$.
10. **Reflexive.** A relation R in a set A is a said to be reflexive, if $(x, x) \in R \ \forall x \in A$. In other words a relation is reflexive if every element of A related to itself in the sense of R
i.e., $xRx \ \forall x \in A$
11. **Symmetric.** A relation R on a set A is said to be symmetric,
if $x R y \Rightarrow yRx$ *i.e.*, $(x, y) \in R \Rightarrow (y, x) \in R$, or if x is related to y then y is related to x for $x, y \in A$.

NOTES

12. **Transitive.** A relation R in a set A is said to be transitive if for $x, y, z \in A$,

$$xRy \text{ and } yRz \Rightarrow xRz$$

i.e., $(x, y) \in R$ and $(y, z) \in R \Rightarrow (x, z) \in R$.

In other words, R is said to be transitive if x related to y and y related to z in the sense of R imply that x is related to z in the sense of R.

TEST YOURSELF

If A, B, C are the subsets of S, prove that

1. $A - (A - B) = A \cap B$.
2. $B \subset A \Rightarrow A - (A - B) = B$.
3. $(A \cap B)' \cup (A' \cap B) = A \cup B \Leftrightarrow A \cap B = \phi$.
4. $A \Delta B = (A \cup B) - (A \cap B)$.
5. $(A - B) - C = A - (B \cup C)$.
6. $X - (A \cap B) = (X - A) \cup (X - B)$.
7. If $A = \{x : x$ is a factor of $20.\}$
   $B = \{x : x$ is a multiple of 5 and $x \le 20\}$, find $A - B$ and $B - A$.
8. If A, B are two sets, show that $A - B \neq B - A$.
9. Show that
   (*i*) $A \cup B = A \cap B \Rightarrow A = B$ (*ii*) $B \subseteq A \cup B$.
10. If set $A = \{(x, y) : x^2 + y^2 = 1\}$ and $B = \{(x, y) : x + y = 1\}$, then, find $A \cap B$.
11. If R and S are two equivalence relations, than check $R \cup S$ for
    (*a*) reflexivity (*b*) transitivity (*c*) symmetry.
12. S is the set of real numbers and aRb, if $a = \pm b$, determine whether R is an equivalent relation.
13. Define an 'equivalence relation' on a set. Prove that the relation 'equal to' on the set of real numbers is an equivalence relation.
14. Prove that if a relation R is transitive then its inverse relation R^{-1} is also transitive.
15. Show that for the set of all points in a plane, the relation "at the same distance from the origin" is an equivalence relation.
16. If I is the set of integers and a relation R in I is defined as $a^b = b^a, ab \in I$, then show that R is not an equivalence relation.
17. Prove that the mapping $f : [0, \pi] \to [-1, 1]$ defined by
    $$f(x) = \cos x, x \in [0, \pi]$$
    is one-one and onto.
18. Explain that every function is a relation but every relation is not a function with the help of illustrative example.
19. Show that the mapping $f : R \to R$ defined by $f(x) = -\sin x, x \in R$ is neither one-one nor onto.
20. If $X = \{-1, 1\}$ and $f(x) = x^2$ and $f : X \to X$, prove that it is one-one onto mapping.
21. Examine the nature of the mapping $f : X \to Y$ if $X = \{1, 2, 3, 4\}$ and $Y = \{5, 7, 9, 11\}$ so that $1 \to 7, 2 \to 11, 2 \to 5, 3 \to 7, 4 \to 9$.
22. Show that the mapping $f : Q \to Q$ defined by $f(x) = 5x + 4, x \in Q$ is one-one and onto, Q being the set of rational numbers.
23. If f and g are two mappings of a set X into itself, show that the maps *gof* and *fog* are also mappings of X into itself.
24. If $f : R \to R, g : R \to R$ be defined by $f(x) = x^3, x \in R, g(x) = \sin x, x \in R$, prove that $fog \neq fog$.
25. If $f : R \to R$ is given by $f(x) = e^x$ and $g : R \to R$ is given by $g(y) = \sin y$, find the composite mappings *fog* and *gof*.

Answer

10. $\{(0, 1), (1, 0)\}$. **21.** Many one onto. **25.** $e^{\sin x}$, $\sin e^x$.

NOTES

CHAPTER

2

QUANTIFICATION THEORY

LEARNING OBJECTIVES

- Quantifiers
- Equivalence in the Meaning of the Quantifiers
- First Order Theories
- Free and Bound Variables
- Significance of Quantifiers
- Purifying Quantifiers
- Inference Rules for Quantified Statements
- The Generalization Principle
- The Predicate Calculus
- The Statement Function, Variables and Quantifiers
- Predicate Formulae
- Free and Bound Variables
- The universe of Discourse
- Inference Theory of the Predicate Logic
- Valid Formulae and Equivalence
- Some Valid Formulae Over Finite Universe
- Special Valid Formulae Involving Quantifiers
- Theory of Inference in Predicate Calculus
- Formula Involving More Than One Quantifier

1. QUANTIFIERS

We know that sets are defined by specifying a property $P(x)$ that elements of the set have in common. Thus an element of $\{x : P(x)\}$ is an object t for which the statement $P(t)$ is true. This type of a sentence $P(x)$ is called a predicate. $P(x)$ is also said to be a *prepositional function*, because each choice of x produces a preposition $P(x)$ that is either true or false. For example let $A = \{x : x \text{ is an integer less than } 6\}$.

Here $P(x)$ is the sentence "x is an integer less than 6".

The common property is "1 is an integer less than 6".

Since $P(1)$ is true, $1 \in A$.

The *universal quantification* of a predicate $P(x)$ the statement "For all values of x, $P(x)$ is true". We take it granted here that only values of x that make sense in $P(x)$

are considered. If some restriction is to be imposed, we write for example $\forall\, x \geq 0$ or $\forall\, n \in Z$. The universal quantification of $P(x)$ is denoted by $\forall\, x\, P(x)$. The symbol $\forall$ (for all) is called the universal quantifier.

NOTES

In fact there are so many forms of logical inferences which cannot be verified by prepositional calculus. For example

(1) Any friend of Ram is friend of Shyam

Mohan is not friend of Shyam

Therefore Mohan is not friend of Ram.

(2) All men are immortal

Hari is a man.

Therefore Hari is immortal.

(3) All men are animal.

Hence the head of a man is the head of an animal.

(4) If x is a real number and x is greater than zero, then for every x, for some y, $y = x^2$ exists, where y is a real number.

In the above conclusions the truth value does not depend only on the truth functional relation in the statement, rather it also depends on internal construction of these statements and the meaning of expression such as "for all", "some", "for every x", "for some y" etc. Modifying expression "for every x" or the **universal quantifier** and the expression "for any x" (or its biquistic equivalent) is said to be **existential quantifier.**

The quantifiers have an important role to play in mathematical logic. To use these quantifiers, we need some rules for which some notations are felt convenient. We use $\forall\, x$ for "for every x", $\exists x$ for "there exists x such that". Thus the symbol $\forall\, x$ is for universal quantifier and $\exists x$ for existential quantifier.

In discussion to follow we will accept the convention that the scope of every quantifier is the first statement variable of the statement or statement formula on the right hand side of the quantifier. Some times we write the scope of the quantifier by putting big bracket [pair of brackets]. Thus we write $[\exists x\, P]\, VQ$ in place of $\exists x\, PVQ$. If we write to use the quantifier for the complete disjunction then we write as $\exists x\, [PVQ]$.

In the formula $\forall y\, \exists x\, P$, $\exists x\, P$ is the statement just after $\forall y$, therefore the explanation of the whole formula should be made in the form $\forall y\, [\exists x\, P]$. The tilde sign used before quantifier expresses the negation of the quantified statement. Thus we write

$$\sim \forall x\, P \vee Q \text{ or } [\sim (\forall x\, P)] \vee Q \text{ and } \sim \exists x\, \forall y P \;\Rightarrow\; Q \text{ for } [\sim [\exists x[\forall y P]]] \;\Rightarrow\; Q\text{ ; etc.}$$

This must be clear that $\exists x P$ asserts that there is at least one x for which the statement P is true, there may be more than one value of such x. With linguistic equivalence point of view "for some x", "for at least one x", "there is an x such that", "there exists an x such that" all give the same meaning. $\forall x$ implies "for all x" or "for any x" etc.

Remark. If in any of these methods of verbalization appears to express "there is an x", the quantifier should not be taken as sub statement.

NOTES

2. EQUIVALENCE IN THE MEANING OF THE QUANTIFIERS

While talking about a shirt if we say "Every shirt here costs at least Rs two hundred fifty" then this statement carries the same meaning as the statement "There is no shirt here that costs less than Rs two hundred fifty". Similarly if we talk about cats then the statement "In the dark, all cats are gray" has the same meaning as the statement "In the dark, there is no cat that is not gray". If same statement P is true for every x then it implies that there is no x for which the statement P is false. Thus $\forall x$ P can be interchanged by $\sim \exists x \sim$ P and we can take the last statement as the definition of the first one where $\exists x$ is sole basic quantifier.

Definition. $\forall x$ P for $\sim \exists x \sim$ P.

For the strings of the quantifiers one more abbreviation is convenient.

Definition. $\forall x\, y$P for $\forall x\; \forall y$ P.

$\forall x\, yz$P for $\forall x\; \forall y\; \forall z$ P etc.

$\exists x\, y$P for $\exists x\; \exists y$ P

$\exists x\, yz$P for $\exists x\; \exists y\; \exists z$ P etc.

Remark. The existential quantifier should not be interchanged with the universal quantifier because a statement P for which $\forall x\; \exists y$ P is true but $\exists y\; \forall x$ P is not true, can be easily visualized. For example, let P : $x > y$ then we can find at least one y for each x is real number but there is no y for which every x is greater. Rather, contrary to it, we can prove that when the statement of the type $\exists x\; \forall\, y$ P is true then $\forall y\; \exists x$P is also true.

ILLUSTRATIVE EXAMPLES

Example 1. *Write down the symbolic statement $\exists x\; \forall\, y$(x is well wisher of y) in linguistic form.*

Solution. First form : There exists an x such that for every y, x is well wisher of y.

Second form : There exists some body who is well wisher of every body.

Third form : Some body who is well wisher of every body.

Example 2. *Explain the following statement with reference to theory of numbers.*

$$\forall x\; \forall y\; \exists z\; (x + y = z) \qquad (x, y, z \in R)$$

Solution. (*i*) For every x and every y there exists a z such that $x + y = z$ where, x, y, z are real numbers.

In other words (*ii*) For every x and y there exists a z such that $x + y = z$ where, x, y, $z \in$ R

or (*iii*) If two real numbers x, y are given, then there exists a real number z such that $x + y = z$

or (*iv*) The sum of the real numbers is a real number.

Example 3. *Let* $R(x) \equiv x$ *be a real number*

$Q(x) \equiv x$ *be a rational number*

$E(x, y) \equiv$ '$x = y$'

$G(x, y) \equiv$ '$x > y$'

Then write down the following sentences in the symbolic form

(a) Every rational number is a real number.

(b) Some real numbers are rational.

(c) The square of every real number is non-negative.

(d) There exists a real number between two different real numbers.

Solution. (*a*) $\forall x\ [Q(x) \Rightarrow R(x)]$

(*b*) $\exists x\ [R(x) \wedge Q(x)]$

(*c*) $\forall x\ [R(x) \Rightarrow \sim G(0, x^2)]$

(*d*) $\forall xy\ [R(x) \wedge R(y) \wedge \sim E(x, y) \Rightarrow \exists z\{Q(z) \wedge G(x, z) \wedge G(z, y) \vee G(y, z) \wedge G(z, x)]$

Example 4. *Write down the negation of the following*

(*a*) $\exists x\ (x^2 < 1)$

(*b*) $\forall a\ (x \neq 0) \Rightarrow (x^2 > 0)$

(*c*) *All Indians are honest.*

(*d*) *If there is a will, there is a way.*

Solution. (*a*) $\forall x\ (x^2 \geq 0)$

(*b*) $\exists x\ (x \neq 0) \wedge (x^2 < 0)$

(*c*) Some Indians are dishonest.

(*d*) There is a will but there is no way.

Example 5. *Using the quantifiers and symbol < for "less than", write down the following statements in the symbolic form.*

(a) A number x is less than 4 and greater than 1.

(b) For given number x there exists a smaller number y.

(c) There is no greatest number.

(d) For two numbers x and y, which are greater than 1, the sum of x and y is less than the product of x and y.

(e) The number x, y and z are such that the difference of x and y is less than the product of x and z.

(f) For the two given numbers x and y, there is a number z such that the difference of x and y is less than the product of x and z.

Solution. (*a*) $\exists x\ [(x < 4) \wedge (1 < x)]$

(*b*) $\forall x\ (\exists y)\ (y < x)$

(*c*) $\sim (\exists y)\ [(\forall x)\ (x < y)]$

(*d*) $(\forall x \parallel < x)\ (\forall x \parallel < y)\ (x + y < xy)$

(*e*) $(\exists x)\ (\exists y)\ (|\ x - y\ | < xy)$

(*f*) $(\forall x)\ (\forall y)\ (\forall z)\ (|\ x - y\ | < xy)$

Example 6. *Show that $(\exists x)\ (\sim P(x))$ is the negation of $(\forall x)\ P(x)$.*

Solution. Case 1. The value of $(\forall x)\ P(x)$ is T, then $P(x)$ is satisfied for all x, therefore

$\sim P(x)$ is not satisfied for any x.

Case 2. The value of $(\forall x)\ P(x)$ is F, then $P(x)$ is not satisfied for some x, therefore

$\sim P(x)$ is satisfied for some x.

Hence the value of $(\exists x)\ (\sim P(x))$ is T.

Example 7. *If $R(x) \equiv x$ is a number,*

$I(x) \equiv x$ is an integer,

NOTES

$$G(x, y) \equiv \text{`}x > y\text{'}$$
$$E(x, y) \equiv \text{`}x = y\text{'}$$
$$R \equiv \textit{number set,}$$

then write down (a) Symbolic form (b) Negation in the simple language (c) Negation in symbolic language of the following sentences.

(i) There is a number between any two different numbers.

(ii) The square of every negative number is positive.

(iii) The squares of different integers are different.

(iv) For two number x and y, where $x < y$, a number z is such that $x > z > y$.

Solution. (*i*) (*a*) $\forall\, xy\, (R(x) \wedge R(y) \wedge E(x, y))$

$\Rightarrow\ \exists z \in R\, (G(x, z) \wedge G(z, y) \wedge G(y, z) \wedge G(z, x))$

(*b*) There exist two different numbers that there is no number between them.

(*c*) $\exists x, y\, [R(x) \wedge R(y), \wedge \sim E(x, y) \wedge \forall z \in R \sim \{G(x, z) \wedge G(x, y) \vee G(y, z) \vee G(z, x)\}$

(*ii*) (*a*) $\forall x \in R\, [G(0, x) \wedge \sim G(x^2, 0)]$

(*b*) A negative number, whose square is not positive.

(*c*) $\exists x \in R[G(0, x) \wedge \sim G(x^2, 0)]$

(*iii*) (*a*) $\forall x, y[I(x) \wedge I(y) \sim E(x, y) \ \Rightarrow\ -E(x^2, y^2)]$

(*b*) There exist two different integers whose squares are equal.

(*c*) $\exists x, y[I(x) \wedge I(y) \wedge \sim E(x, y) \wedge E(x^2, y^2)]$

(*iv*) (*a*) $\forall x, y[R(x) \wedge R(y) \wedge G(x, y) \ \Rightarrow\ \exists z\{R(z)\, G(x, y) \wedge G(z, y)\}$

(*b*) There are some real numbers x, y for which $x > y$, $x > y > z$ is false for every z.

(*c*) $\exists x, y[R(x) \wedge R(y), \wedge G(x, y) \wedge \sim \forall z \in R\{G(x, z) \wedge G(z, y)\}]$

EXERCISE 2(A)

1. Applying scope, conventions and short forms, write down the following formulae using minimum number of brackets.

   (*i*) $((P \to R) \wedge (\sim Q) \to R)) \to ((P \vee Q) \to R)$ (*ii*) $(\sim P) \to ((\sim P) \vee Q)$
2. Write down the symbolic statement $\forall y\, \exists x$ (x is well wisher of y) in linguistic form.
3. Write the statement $\exists x\, \exists y$ (x is well wisher of y) in linguistic form.
4. Explain the following symbolic statement with reference to theory of numbers :

   $\forall x\, \forall y\, \forall z\, (x + y) + z = x + (y + z),\ x, y, z \in R$
5. Write down the following statements in linguistic form clearly.

   (*i*) $\exists y(x < y)$ (*ii*) $\exists z\, \forall y(z = y \vee y = x)$

   (*iii*) $(\forall x(x > 0)) \wedge (\exists y(y = x))$ (*iv*) $(\exists z(x + y = z) \vee ((\forall x(x > z)) \to (\exists y(y = z)))$.
6. Write down the three relations of equivalence in symbols.
7. Write down the Peano's axioms in symbols.
8. Write down the negation of the following sentences in symbolic form.

   (*i*) The function f tends to limit l near a if $\forall \varepsilon > 0$, $\exists \delta > 0$ such that

   $$\forall x,\ 0 < |\, x - a\, | < \delta \ \Rightarrow\ |\, f(x) - l\, | < \varepsilon$$

   (*ii*) No teacher is unwise (*iii*) $\forall x(x \in A \ \Rightarrow\ x \in B)$
9. Write down the negation of the sentence $\exists x \in I[x \geq 0 \ \Rightarrow\ x^2 > 0]$.

Answers

2. For every y, there exists an x, such that x is well wisher of y.
3. There exists an x such that there is y for whom x is well wisher.

NOTES

3. FIRST ORDER THEORIES

In first order theories we take a set and all the statements of the theory concerning the elements of the set. The set is said to be domain and elements of it are said to be **individuals.**

4. FREE AND BOUND VARIABLES

In any formula of the type "$(\exists x)$ F(x)" or "(x) F(x)" is called the x-bound portion of the formula. It is to be noticed that as per definition of *quantifier* the statement function F(x) is the scope of the quantifier. In the x-bound part the occurrence of "x" is called the bound occurrence. That occurrence of "x" which is not bound is said to be free occurrence.

Consider the following statement : $\sum_{k=1}^{k=5} k^2 = 55$ ...(1)

In this statement k is a dummy variable. When we write the statement (1) as follows, k does not occur at all.

$$1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 55$$

We could have taken any variable in place of k.

Similarly in $\int_0^1 \cos x \, dx = \sin 1,$ ...(2)

x is a dummy variable.

The meaning of the statement $\int_0^1 \cos t \, dt = \sin 1,$ ...(3)

is same as that of (2). In logic the dummy variable is said to be bound.

In the formula $\exists x \, (x < y)$ ...(4)

x is bound variable but y is not bound variable as when we write (4) in linguistic form :

"There exists a number which is less than y",

x does not occur while y exists.

If we take y as some fixed numbers then statement (4) gives information about y. In this statement y is a free variable. Similarly in the formula

$$\sum_{k=1}^{k=n} k = \frac{n(n+1)}{2}$$

k is bound variable and n is free variable.

Likewise in statement $\int_0^x \cos t \, dt = \sin x,$

t is bound variable and x is free variable.

A variable may be bound in different ways but we wish to talk about those variables also which are bound by quantifiers.

In the formula P(x), the occurrence of x is bound iff the occurrence is explicit for the quantifiers $\forall x$ or $\exists x$ *i.e.*, if it is in the scope of the quantifier $\forall x$ or $\exists x$. The occurrence of a variable is free only when it is not bound. In the above cited statement (4) the occurrence of x is bound and that of y is free.

Remark. The free and bound variables depend on the nature of the formula.

In the following example the same variable is free at one place and bound at the other.

In formula $(\exists x\,(x < 7)) \wedge (x + z) = 8$ ...(5)

Considering form left to right, the occurrence of first two x is bound but the third occurrence of x is free because it does not lie in the scope of the quantifier $\exists x$. So far as meaning is concerned there is nothing to do with the free and bound occurrence of a variable.

In the formula

$$\int_0^x \cos x \, dx = \sin x \quad ...(6)$$

the first and last occurrence of x are free while the two are bound. Writing (6) as follows will be better and free from this doubt

$$\int_0^x \cos t \, dt = \sin x \quad ...(7)$$

In formula $(\exists x\,(x < y)) \wedge (\forall x\,(x > 0))$ ...(8)

x is free everywhere but occurrence of x at first two places has nothing to do with the occurrence of x at last two places because they are bound by different quantifiers.

Statement (8) can be written as

$$(\exists x\,(x < y)) \wedge (\forall z\,(z > 0)) \quad ...(9)$$

Now we provide the definition of statement and predicate in the formula form.

A *statement* is the formula which does not contain any free variable.

A *predicate* is formula which contains one or more free variables.

For example in theory of numbers

$$\exists x\ \forall y\ (xy = x) \quad ...(10)$$

is a statement in which there is no free variable while the formula

$$\exists x\ (x + y = y) \quad ...(11)$$

is a *predicate where* y is one (and only one) free variable.

5. SIGNIFICANCE OF QUANTIFIERS

As has already been said mathematical logic plays an important role in Computer Science and quantifiers play an important role in logic. To realize this let us consider the following example.

NOTES

A student wrote a rule of Algebra as follows :

$$(x + 3)^2 = x^2 + 3^2$$

But this is wrong. We will like to prove by logic that this statement is wrong and for it we will prove that its negative statement is true.

The negative statement of the above statement is

$$(x + 3)^2 \neq x^2 + 3^2$$

But this is also not the universal truth because for $x = 0$ this statement is wrong. The statement which supports our point of view, should be

$$\exists x\ [(x + 3)^2 \neq x^2 + 3^2]$$

which is just the negation of the statement

$$\forall x\ [(x + 3)^2 = x^2 + 3^2]$$

Here we see that use of quantifier is essential here.

6. PURIFYING QUANTIFIERS

Let us consider the following statement :

"There exists a x greater than zero such that for every real number y, $xy = y$".

In this statement the part of the clause "there exists a x greater than zero" is not a pure quantifier because it is loaded by defining the property of x. Similarly in terms "for every real number y" there is description which is not the part of the universal quantifier and therefore to bring it to the universal quantifier form, we will have to purify it. With pure quantifier the first expression of the statement shall start as :

"There exists a x such that ...".

To maintain the original meaning we will add then

x is greater than x and ...

Generally the statement of the type

"There exists a x with property p ..."

is replaced by the statement

"There exists a x having property p and ..."

The rest of the statement under consideration will start as follows (by using pure quantifier) :

"For every y ..."

Now to include the information about the nature of y after this, we will write as :

"If y is a real number then ..."

In general, the statements of the type :

"For every x with property p ..."

is transformed in the form

"for every x, if x possesses the property p, ...".

A modified universal quantifier is changed into universal quantifier but after purification one conditional is loaded.

Similarly a modified existential quantifier is changed into existential quantifier but after purification a conjunction is added after it.

Now consider the following example :

For three different non-linear points, there exist two different planes containing all these points. We will start by $\forall xyz$ for using the quantifiers but question arises as to how the fact that x, y and z are non-linear be expressed.

For this complete description we can choose a sentence (variable).

P : x, y and z are different non-linear points.

Similarly the other statement can be taken as

Q : m and n are different planes which contain x, y and z.

Then our statement in symbolic form shall be

$$\forall xyz\ [P \Rightarrow \exists mn\ Q]$$

This statement can be further simplified as :

P_1 : x, y and z are points.

P_2 : x, y and z are different.

P_3 : x, y and z are non-linear.

(In the statement variables we prefer the positive sentences).

This can be further simplified as :

$P_{1,1}$: x is a point.

$P_{1,2}$: y is a point.

$P_{1,3}$: z is a point.

Similarly for in place of P_2, we write

$P_{2,1} : x \neq y$

$P_{2,2} : y \neq z$

$P_{2,3} : z \neq x$

and we let P_3 remain the same. We will not define collinearity, rather we accept it as undefined. But following nine substatements will be defined as :

$Q_{1,1}$: m is a plane

$Q_{1,2}$: n is a plane

$Q_2 : m = n$

$Q_{3,1} : x \in m$

$Q_{3,2} : y \in m$

$Q_{3,3} : z \in m$

$Q_{4,1} : x \in n$

$Q_{4,2} : y \in n$

$Q_{4,3} : z \in n$

Now formula becomes

$$\forall xyz\ [P_{1,1} \wedge P_{1,2} \wedge P_{1,3} \wedge \sim P_{2,1} \wedge \sim P_{2,2} \wedge P_{2,3} \wedge P_3$$

$$\Rightarrow \quad - \exists mn\ [Q_{1,1} \wedge Q_{1,2} \wedge \sim Q_2 \wedge Q_{3,1} \wedge Q_{3,2} \wedge Q_{3,3} \wedge Q_{4,1} \wedge Q_{4,2} \wedge Q_{4,3}]$$

7. INFERENCE RULES FOR QUANTIFIED STATEMENTS

We need rules for the following inferences :

1. Statement function from quantified statements.
2. Statement function from statement functions.

3. Quantified statement from statement functions.
4. Singular statements from quantified statements and vice-versa.

There is no necessity of adding any rules for quantified statements in statement calculus if the same are tried as different units other than logic. Otherwise no inference can be drawn without rules.

Rules. For some statement function "F(x)"

$$\frac{xF(x)}{F(y)}$$

Here y is used for some name or object in the universe. This rule is known as IU rule. This is an inference of statement function from universal statement.

The meaning of this rule is $(x)\ F(x) \rightarrow F(y)$

For example the symbolic translation of,

All men are mortal

Socrates is a man

Therefore Socrates is a mortal.

$(x)\ H(x) \rightarrow M(x)$

$H(s)$

$M(s)$

8. THE GENERALIZATION PRINCIPLE

Consider the following example. We state the theorem of isosceles triangles as follows :

"If there is an isosceles triangle, then angles opposite to equal sides are also equal."

This is a statement for all isosceles triangles. Symbolically

(x) [(x is a triangle) (x is isosceles) $\rightarrow$ ...]

To express this theorem the mathematicians do not follow the sequence of quantified statements. They directly start with an arbitrary triangle ABC in which AC = BC and prove that $\angle A = \angle B$. In the proof 'ΔABC' is taken unknown but fixed. Using the deduction law,

Axiom : ABC is a triangle $\mapsto$ (In ABC, AC = BC)

$\rightarrow$ ($\angle A = \angle B$ in ΔABC)

[Here symbol $\mapsto$ implies : such that]

Again by deduction

Axiom $\mapsto$ (ABC is a Δ) $\rightarrow$ [(AC = BC in ABC) $\rightarrow$ ($\angle A = \angle B$ in ABC)]

We can write this as follows too :

Axiom $\mapsto$ (ABC is a Δ) $\rightarrow$ [(In ABC, AC = BC) $\rightarrow$ ($\angle A = \angle B$ in ABC)] ...(I)

A mathematician writes without adding any thing.

Axiom $\mapsto$ (ABC is a Δ) $\rightarrow$ [(In ABC, AC = BC) $\rightarrow$ ($\angle A = \angle B$ in ABC)] ...(II)

Here ABC is taken a fixed Δ but in (I) the same statement is for any ABC.

NOTES

Therefore (II) from (I) is said to be generalization principle for universal statement. We call it PGU (Principle of Generalization to a Universal statement).

Thus, PGU : For any statement function,

if $A_1, A_2 ..., A_m \mapsto F(x)$ then $A_1, A_2, ..., A_m \mapsto (x) F(x)$ where each A is any statement-function but no A contains free variable.

ILLUSTRATIVE EXAMPLES

Example 1. *Decide the free and bound variables.*

(*i*) $A(x_1, x_2)$ (*ii*) $A(x_1, x_2) > (x_1) A(x_2)$

Solution. (*i*) x_1 is free, x_2 is also free variable

(*ii*) x_1 occurring first is free variable but occurring second time or third time is bound variable.

Example 2. *Prove that*

$$(x)\ [H(x) \to M(x)],\ H(y) \vdash M(Y) \text{ where } y \text{ is unknown.}$$

Solution. 1. $(x)\ [H(x) \to M(x)]$ (given)

2. $H(y)$ (given)

3. $H(y) \to M(y)$ IU

4. $M(y)$

EXERCISE 2(B)

Write the expressions after removing the brackets.

1. $(P \wedge Q) \to R$ **2.** $(P \vee Q) \leftrightarrow R$

3. $(\sim Q) \to (\sim P)$ **4.** $((\sim P) \to Q) \leftrightarrow R$

5. Prove that $(x)\ [I(x) \to R(x)],\ (x)\ [R(x) \to C(x)] \vdash (x)\ [I(x) \to C(x)]$

Answers

1. $P \wedge Q \to R$ **2.** $P \vee Q \leftrightarrow R$

3. $\sim Q \to \sim P$ **4.** $(-P) \to Q \leftrightarrow R$

Rules for Implications

$P \wedge Q \Rightarrow P,$	I_1	Simplification
$P \wedge Q \Rightarrow Q,$	I_2	
$P \Rightarrow P \vee Q,$	I_3	Addition
$Q \Rightarrow P \vee Q,$	I_4	
$\sim P \Rightarrow P \to Q,$	I_5	
$Q \Rightarrow P \to Q,$	I_6	
$\sim (P \to Q) \Rightarrow P,$	I_7	
$\sim (P \to Q) \Rightarrow \sim Q,$	I_8	
$P, Q \Rightarrow P \wedge Q,$	I_9	
$\sim P, P \vee Q \Rightarrow Q,$	I_{10}	disjunctive syllogism

NOTES

$P, P \to Q = Q,$	I_{11} modus ponens
$\sim Q, P \to Q \Rightarrow \sim P,$	I_{12} modus tollens
$P \to Q, Q \to R \Rightarrow P \to R,$	I_{13} hypothetical syllogisms
$P \vee Q, P \to R, Q \to R \Rightarrow R,$	I_{14} dilemma

Rules for Equivalence

$\sim\sim P \Leftrightarrow P,$	E_1	double negation
$P \wedge Q \Leftrightarrow Q \wedge P,$	E_2	Commutativity
$P \vee Q \Leftrightarrow Q \vee P,$	E_3	
$(P \wedge Q) \wedge R \Leftrightarrow P \wedge (Q \wedge R),$	E_4	Associativity
$(P \vee Q) \vee R \Leftrightarrow P \vee (Q \vee R),$	E_5	
$P \wedge (Q \vee R) \Leftrightarrow (P \wedge Q) \vee (P \wedge R),$	E_6	Distributivity
$P \vee (Q \wedge R) \Leftrightarrow (P \vee Q) \wedge (P \vee R),$	E_7	
$\sim (P \wedge Q) \Leftrightarrow \sim P \vee \sim Q,$	E_8	De Morgan' laws
$\sim (P \vee Q) \Leftrightarrow \sim P \wedge \sim Q,$	E_9	
$P \vee P \Leftrightarrow P,$	E_{10}	
$P \wedge P \Leftrightarrow P,$	E_{11}	
$R \vee (P \wedge \sim P) \Leftrightarrow R,$	E_{12}	
$R \wedge (P \vee \sim P) \Leftrightarrow R$	E_{13}	
$R \vee (P \vee \sim P) \Leftrightarrow T,$	E_{14}	
$R \wedge (P \wedge \sim P) \Leftrightarrow F,$	E_{15}	
$P \to Q \Leftrightarrow \sim P \vee Q,$	E_{16}	
$\sim (P \to Q) \Leftrightarrow P \wedge \sim Q,$	E_{17}	
$P \to Q \Leftrightarrow \sim Q \to \sim P,$	E_{18}	
$P \to (Q \to R) \Leftrightarrow (P \wedge Q) \to R,$	E_{19}	
$\sim (P \rightleftarrows Q) \Leftrightarrow (P \rightleftarrows \sim Q,$	E_{20}	
$P \rightleftarrows Q \Leftrightarrow (P \to Q) \wedge (Q \to P),$	E_{21}	
$(P \rightleftarrows Q) \Leftrightarrow (P \wedge Q) \vee (\sim P \wedge \sim Q),$	E_{22}	

9. THE PREDICATE CALCULUS

The symbolic logic has been limited to the consideration of statements and statement formulae. The inference theory was also restricted in the sense that the premises and conclusions are all statements. The symbol P, Q, R, ..., P_1, Q_1, R_1, ... are used for statements or statement variables. The statements are taken as basic units of statement calculus and nô analysis of any atomic statement is admitted. Only compound formulae are analysed and this analysis is done by studying the forms of compound formulae *i.e.*, the connection between the constituent atomic statements. It is not possible to express the fact that any two atomic statements have some features

NOTES

in common. In order to investigate questions of this nature, we have already introduced the concept of a predicate in an atomic statement. The logic based upon the analysis of the predicates in any statement is called predicate logic.

Predicates

Let us consider the two statements

Ramesh is a student.

Suresh is a student.

If we express these statements by symbols. We need two different symbols to denote them. Such symbols do not reveal the common features of these two statements *i.e.,* both are statements about two different individuals who are students. If we introduce some symbols to denote "is a student" and a method to join it with symbols denoting the names of individuals, then we will have a symbolic form to denote statements about any individual's being a student. The part "is a student" is called the predicate.

Another consideration which leads to some similar method for the representation of any feature is suggested by the following argument.

All army men are tall.

Sudhir is an army man.

Therefore Sudhir is tall.

Such a conclusion seems intuitively true. However, it does not form the inference theory of the statement calculus developed earlier. The reason for this deficiency is the fact that the statement "All army men are tall" cannot be analysed to say anything about an individual. If we could separate the part "are tall" from the part "All army men" then it might be possible to consider any particular army man.

We shall symbolize a predicate by a capital letter and the names of individuals or objects in general by small letters. We see that using capital letters to symbolize statements as well as predicates will not lead to any confusion. Every predicate describes some thing about one or more objects. Therefore a statement could be written symbolically in terms of predicate letter followed by the name or names of the objects to which the predicate is applied.

We again consider the statements.

Ramesh is a student. $\rightarrow$ (1)

Suresh is a student. $\rightarrow$ (2)

Denote the predicate "is a student" by the predicate letter B, "Ramesh" by r and Suresh by s. Then above statements can be written as B(r) and B(s) respectively. In general any statement of the type "P is Q" where Q is a predicate and P is the subject, can be denoted by Q(P).

A statement which is expressed by using a predicate letter must have at least one name of an object associated with the predicate. When an appropriate number of names are associated with a predicate, then we get a statement. Using a capital letter to denote a predicate may not indicate the appropriate number of names associated with it. Normally, this number is clear from the context or from the notation being used. This numbering can also be accomplished by attaching a superscript to a predicate letter indicating the number of names that are to be appended to the letter. A predicate requiring $m(m > 0)$ names is called an m-place predicate. For example "L : is less than" is a 2-place predicate. In order to extend our definition to $m = 0$, we shall call a statement a 0-place predicate because no names are associated with a statement.

Let R denote the predicate "is red" and let P denote "this flower". Then the statement

"This flower is red" $\rightarrow (3)$

can be symbolized by R(P). Further the connectives describe earlier can now be used to form compound statements such as "Ramesh is a student and this flower is red", which can be written as $B(r) \wedge R(P)$.

Other connectives can also be used to form statements such as

$$B(r) \rightarrow R(P) - R(P)\ B(r) \vee R(P) \text{ etc.}$$

Now consider the statements involving the names of two objects such as

Rahul is taller than Ram. $\rightarrow (4)$

Agra is in the north of India. $\rightarrow (5)$

The predicates "is taller than" and "is in the north of" are two place predicates because names of two objects are needed to complete a statement involving these predicates. If the letter G symbolizes "is taller than" r_1 denotes "Rahul" and r_2 denotes "Ram". Then the statement (4) can be translated as $G(r_1, r_2)$. The order in which the names appear in the statement as well as in the predicate is important. Similarly, if N denotes the predicate "is in the north of", a : Agra and s : India. Then (5) is symbolized as $N(a, s)$. Obviously $N(a, s)$ is the statement " Agra is in the north of India".

In general, an n-place predicate requires n names of objects to be inserted in fixed positions in order to obtain a statement. The position of these names is important. If S is n-place predicate letter and $a_1, a_2, ..., a_n$ are the names of objects then $S(a_1, a_2, ..., a_n)$ is a statement. If we use this convention, every predicate symbol is followed by an appropriate number of letters, which are the names of object, enclosed in Parentheses and separated by commas. Sometimes the Parentheses and the commas are dropped. The definition does not require that the names be chosen from any fixed set. For example if B denotes the predicate "is a brilliant student" and t denotes "Rakhee", then $B(t)$ symbolizes "Rakhee is a brilliant student".

10. THE STATEMENT FUNCTION, VARIABLES AND QUANTIFIERS

Let B be the predicate "is a brilliant" b the name "Rakhee" c "Agra" and s "A shirt". Then $B(b)$, $B(c)$ and $B(s)$ all denote statements. Infact these statements have a common form. If we write $B(x)$ for "is a brilliant". Then $B(b)$, $B(c)$, $B(s)$ and other having the same form can be obtained, from $B(x)$ by replacing x by an appropriate name. $B(x)$ is not a statement, but it converts in a statement when x is replaced by an appropriate name, the letter x used here is a place holder only.

A simple statement function of one variable is defined to be as expression consisting of a predicate symbol and an individual variable. Such a statement function becomes a statement when the variable is replaced by the name of any object. The statement resulting by a replacement is called a substitution instance of the statement function and is a formula of statement calculus.

The word "simple" in the above definition is to distinguish the simple statement function from those statement functions which result combining one or more simple statement functions and the logical connectives. For example if we assume $M(x)$ to be "x is a girl" and $H(x)$ to be "x is a mortal", then we can form compound statement function such as

$$M(x) \wedge H(x) \quad M(x) \rightarrow H(x) \quad - H(x) \quad M(x) \vee - H(x) \text{ etc.}$$

Consider the statement function of two variables

$B(x, y)$: x is better than y.

If both x and y are replaced by the names of objects, we have a statement. If m represents Mahavir and k represents Kunwarpal then we have

$G(m, k)$: Mahavir is better than Kunwarpal.

and $G(k, m)$: Kunwarpal is better than Mahavir.

It is possible to form statement functions of two variables by using statement functions of one variable. For example, if

$M(x)$: x is a teacher

$H(y)$: y is a punctual

Then we may write

$M(x) \wedge H(y)$: x is a teacher and y is a punctual.

However, it is not possible, to write every statement function of two variables with the use of statement function of one variable. Statements can be obtained from any statement function by replacing the variables by the names of the objects. There is another way in which statements can be obtained but to understand this method, we first consider some familiar equations of elementary algebra.

$$x + 5 = 7 \qquad ...(1)$$

$$x^2 + 1 = 0 \qquad ...(2)$$

$$(x - 1)(x + 1) = 0 \qquad ...(3)$$

$$x^2 - 1 = (x - 1)(x + 1) \qquad ...(4)$$

In algebra, the variable x is replaced by numbers such as real, complex, rational, integer etc. In the algebraic equations we would not normally consider substituting for x the name of a person or object instead of numbers. We may explain this idea by saying that the universe of the variable x is the set of real numbers or complex numbers or integers etc.. The restriction on x depends upon the problem under consideration. For example we may be interested in only the real solution or the positive solution in a particular case. In statement (1) if x is replaced by a real number, we get a statement. The resulting statement is true when 2 is substituted for x, while for every other substitution, the resulting statement is not true. In (2) there is no real number which, when substituted for x gives a true statement. In (3), if the universe of x is assumed to be integers then there is only one number which produces a true statement when substituted. The situation is slightly different in (4) in the sense that if any number is substituted for x, such integer is 1, then the resulting statement is true as this is this an identity.

In this case we may write

$$\forall x, x^2 - 1 \equiv (x - 1)(x + 1)$$

which is a statement and not the statement function.

Now let us first consider the following statements. Each one is a statement about objects belonging to a certain set.

1. All men are well educated
2. Every flower is white
3. Every integer is either positive or negative.

Let us paraphrase these in the following manner :

1(a). For all x, if x is a man, then x is well educated

2(a). For all x, if x is a flower, then x is white

3(a). For all x, if x is an integer, then x is either positive or negative.

It has already been shown that how statement function such as "x is a man" "x is a flower" or "x is white" can be written by using predicate symbols.

If we introduce a symbol to denote the phrase "for all x" then it would be possible to symbolise statements 1(a), 2(a), 3(a).

We symbolise "for all x" by the symbol "$(\forall x)$" or by "(x)" with the convention that this symbol be placed before the statement function to which this phrase is applied. If

M(x) : x is a man H(x) : x is well educated

B(x) : x is a flower R(x) : x is white

N(x) : x is an integer P(x) : x is either positive or negative,

then we write 1(a), 2(a), 3(a) as

$(x)\ (M(x) \rightarrow H(x))$

$(x)\ (B(x) \rightarrow R(x))$

$(x)\ (N(x) \rightarrow P(x))$

Sometimes $(x)\ (M(x) \rightarrow H(x))$ is also written as $(\forall x)\ (M(x) \rightarrow H(x))$. The symbols (x) or $(\forall x)$ are called universal quantifiers. The quantification symbol is "( )" or "$(\forall)$", and it contains the variable which is to be quantified. It is now possible for us to quantify any statement function of one variable to get statement. Thus (x) M(x) is a statement which can be written as

4. For all x, x is a man

4(a). For every x, x is a man

4(b). Everything is a man.

In order to determine the truth values of any one of these statements involving a universal quantifier, we may consider the truth value of the statement function which is quantified. This is not possible for two reasons. Firstly, statement functions do not have truth values. Infact when the variables are replaced by the number of objects, we obtain statements which have a truth value. Secondly in most cases there is an infinite number of statements that can be produced by such situation.

The particular variable appearing in the statements involving a quantifier is not important because the statements remain unchanged if x is replaced by y throught. Thus the statements

$$(x)\ (M(x) \rightarrow H(x)) \text{ and } (y)\ (M(y) \rightarrow H(y)) \text{ are equivalent.}$$

Sometimes it is necessary to use more than one universal quantifier in a statement. For example

G(x, y) : x is heavier than y.

We can say that "for any x and any y, if x is heavier than y, y is not heavier than x" or "for any x and y, if x is heavier than y, then it is not true that y is heavier than x". This statement can be symbolised as

$$(x)\,(y)\,(G(x, y) \rightarrow {\sim}G(x, y)).$$

The universal quantifier is used to translate expression such as "for all", "every".

We now introduce another quantifier to symbolise expressions such as "for some", "there is at least one", or "there exists some".

Now consider the following statements :

1. There exists a flower.

2. Some flowers are red.

3. Some real numbers are rational.

The first statement can be expressed in various ways, two such ways being

1(a). There exists an x such that x is a flower.

1(b). There is at least one x such that x is a flower.

Similarly statements 2 and 3 can be written as :

2(a). There exists an x such that x is a flower and x is red.

2(b). There exists at least one x such that x is a flower and x is red.

and

3(a). There exists an x such that x is a real number and x is rational.

3(b). There exists at least one x such that x is a real number and x is rational.

This type of rephrasing allows us to introduce the symbol "$(\exists x)$" called the existential quantifier, which symbolizes expressions such as "there is at least one x such that" or "there exists an x such that" or "for some x". If we write

$M(x)$: x is a flower.

$C(x)$: x is red.

$R_1(x)$: x is a real number.

$R_2(x)$: x is rational.

Then using the existential quantifier, we can write the above statements as :

$(\exists x)\,(M(x))$

$(\exists x)\,(M(x) \wedge C(x))$

$(\exists x)\,(R_1(x) \wedge R_2(x)).$

11. PREDICATE FORMULAE

We know that capital letters are used to denote some definite statements. Also these are used as place holders for the statements and in this sense these are called statement variables. These statement variables are also considered as special cases of statement formulae.

So far the capital letters have been introduced as definite predicates. It has also been said that a superscript n be used along with the capital letters in order to indicate that the capital letter is used as an n-place predicate. However, this notation is not necessary because an n-place predicate symbol must be followed by n object variables.

NOTES

Such variables are called object or individual variables and are denoted by lower case letters. The capital letter is used as an n-place predicate, followed by n individual variables which are enclosed in Parentheses and are separated by commas. For example $P(x_1, x_2, x_3, ..., x_n)$ denotes an n-place predicate formula in which the letter P is an n-place predicate and $x_1, x_2, x_3, ..., x_n$ are n individual variables. Generally speaking $P(x_1, x_2, x_3, ..., x_n)$ is called an atomic formula of predicate calculus. It may be noted that our symbolism includes the atomic formulae of the statement calculus as special cases ($n = 0$). The following are some examples of atomic formulae :

$$R \quad P(s) \quad Q(x, y) \quad A(x, y, z) \quad N(a, y) \quad \text{and} \quad B(x, a, z)$$

A well-formed formula of predicate calculus is obtained by using the following rules.

1. An atomic formula is a well-formed formula.
2. If A is well-formed formula, then–A is a well-formed formula.
3. If A and B are well-formed formulae, then $(A \wedge B)$, $(A \vee B)$, $(A \to B)$ and $(A \Leftrightarrow B)$ are also well-formed formulae.
4. If A is a well-formed formula and x is any variable, then (x) A and $(\exists x)$ A are well-formed formulae.

Remark. Only those formulae obtained by using rules (1) – (4) are well-formed formulae. We use simple word formula for "well-formed formula" for the sake of brevity".

12. FREE AND BOUND VARIABLES

We have explained the terms free and bound variables earlier. However it appears to be relevant to make some remarks about these variables.

It should be noted that in the bound occurrence of a variable, the letter which is used to represent the variable is not important, In fact, any other letter can be used to represent the variable without affecting the formula, provided that the new letter is not used elsewhere in the formula. Thus the formulae

x A (x, y) and (z) A(z, y) are the same.

Further the bound occurrence of a variable cannot be substituted by a constant. Only a free occurrence of a variable can be done. For example (x) A$(x) \wedge$ B(a) can be written in place of (x) A$(x) \wedge$ B(y) because (x) A$(x) \wedge$ B(a) can be expressed in English as "Every x has a car A, and a has a car B". A change of variables in the bound occurrence is not a substitution instance. Sometimes it is useful to change the variables in order to avoid confusion. It is better to write (y) A$(y) \wedge$ B(x) in place of (x) A$(x) \wedge$ B(x) so as to separate the free and bound occurrences of variables. Sometimes one may come across a formula of the type (x) A(y) in which the occurrence of y is free and the scope of (x) does not contain a x ; in such a case, we have a vacuous use of (x). In conclusion, it may be mentioned that in a statement every occurrence of a variable must be bound and no variable should have a free occurrence ; whenever a free variable occurs in a formula, we have a statement function.

Example 1. *A(x) : x is a man*

B(x, z) : x is the father of z

C(z, y) : z is mother of y.

Write the predicate "x is the father of the mother of y"

Solution. In order to symbolize the predicate, we name a person called z as the mother of y. Clearly we intend to say that x is the father of z and z is the mother of y. It is assumed that such person z exists. We symbolize the predicate as $(\exists z)\,(A(z) \wedge B(x, z) \wedge C(z, y))$.

Example 2. *Symbolize the expression "Every body likes the honest person".*

Solution. The quotation really means that every body like the honest person.

Now let $A(x)$: x is a person

$B(x)$: x is a honest person

$C(x, y)$: x likes y

The required expression is symbolized as

$$(x)\,A(x) \rightarrow (y)\,(A(y) \wedge B(y) \rightarrow C(x, y)).$$

13. THE UNIVERSE OF DISCOURSE

The process of symbolizing a statement in predicate calculus can be quite complicated. However, some simplification can be introduced by limiting the class of individuals or objects under consideration. This limitation means that the variables which are quantified stand for only those objects which are members of a particular set. Such a restricted set is called the universe of discourse or the domain or individuals or universal set or simply the universe. For example if the discussion refers to algebra of number theory, the universe of discourse could be numbers (real, complex, rational etc.).

ILLUSTRATIVE EXAMPLES

Example 1. *Symbolize the statement "All flowers are pretty".*

Solution. Using $A(x)$: x is pretty

$B(x)$: x is a flower.

The given statement can be symbolized as $(x)\,(B(x) \rightarrow A(x))$. However if we restrict the variable x to the universe which is the class of flowers. Then statement is $(x)\,A(x)$.

Example 2. *Consider the statement "Given any even number. There is a greater even number". Symbolize this statement with and without using set of even numbers as the universe of discourse.*

Solution. Let the variables x and y be restricted to the set of even numbers then the above statement can be paraphrased as follows : for all x, there exists a y such that y is greater than x. If $A(x, y)$ is "x is greater than y", then given statement is $(x)\,(\exists y)\,A(y, x)$. If we do not impose the restriction on the universe of discourse and if we write $B(x)$ for "x is a even number" then we can symbolize the given statement as $(x)\,(B(x) \rightarrow (\exists y)\,(B(y) \wedge A(y, x)))$.

Note. The universe of discourse, if any must be explicitly stated, because the truth value of a statement depends upon it. For example consider the predicate

$C(x)$: x is less than 8

and the statement $(x)\,C(x)$ and $(\exists x)\,C(x)$. If the universe of discourse is given by the sets

1. {– 1, 0, 1, 2, 4, 6, 8}

2. (– 2, 4, – 6, 13, 16}

3. {12, 15, 18}

NOTES

then (x) C(x) is true for the universe of discourse (1) and false for (2) and (3). The statement $(\exists x)$ C(x) is true for both (1) and (2) but false for (3).

It may be stated that there are two ways of obtaining a 0-place predicate from an n-place predicate. The first way is two substitute names of objects from the universe of discourse for the variables ; the second method is to quantify in such a way that all occurrences of individual variables are bound. *Evidently, the 0 – place predicates are* statements in statement calculus.

We know that in symbolizing expressions of the type "all X are Y" the correct connective that should be used is the conditional. But for symbolizing expressions of the type "Some X and Y" the correct connective is the conjunction. Below we now give examples to show that the meaning changes if the correct connectives are not used.

Consider the statement.

1. All persons are social animals.

This is true for any universe of discourse. In particular, let the universe of discourse be A = {Mahavir, Rakhee, 0, 1}, where the first two elements are the names of persons. Clearly statement (1) is true over A. Now let us consider the statements (x) (B$(x) \rightarrow$ C(x)) and (x) (B$(x) \wedge$ C(x)), where B(x) : x is a person, C(x) : x is social animal. In B$(x) \rightarrow$ C(x), if x is replaced by any element of A, then we get a true statement ; therefore (x) B$(x) \rightarrow$ C(x) is true over A. At the same time (x) (B$(x) \wedge$ C(x) is false over A because B(x) $\wedge$ C(x) assumes the value "false" when x is replaced by 0 or 1. This means that statement (1) cannot be symbolized as (x) (B$(x) \wedge$ C(x)).

Now consider the statement

2. Some cities are crowded.

Let A be {Delhi, Mumbai, 0, 1}

B(x) : x is crowded.

C(x) : x is a city.

In this case there is no uncrowded city in the universe A and (2) is false. The statement $(\exists x)$ (B$(x) \wedge$ C(x)) is also false over A because there is no crowded city in A. At the same time $(\exists x)$ (C$(x) \rightarrow$ B(x)) is true because C$(x) \rightarrow$ B(x) is true when x is replaced by 0 or 1.

Thus we have shown that the conditional is not the correct connective to use in this case.

EXERCISE 2(C)

1. Which of the following are statements ?

   (*a*) (x) (A$(x) \wedge$ B$(x) \wedge$ C(x)) (*b*) (x) M$(x) \vee$ N$(x) \wedge$ O

   (*c*) (x) P$(x) \wedge$ Q$(x) \wedge (\exists x)$ R(x).

2. If the universe of discourse is the set $\{a, b, c\}$, eliminate the quantifiers in the following formulae :

   (*i*) (x) – A$(x) \vee (x)$ A(x) (*ii*) x (A$(x) \rightarrow$ B(x))

   (*iii*) (x) P$(x) \wedge (\exists x)$ Q(x) (*iv*) (x) A$(x) \wedge (x)$ B(x)

   (*v*) (x) A(x).

3. Indicate the variables that are free and bound. Also write the scope of the quantifiers.

   (*i*) $(x)(A(x) \Leftrightarrow B(x) \wedge (\exists x) C(x)) \wedge D(x)$ (*ii*) $(x)(A(x) \wedge B(x) \rightarrow (x) A(x) \wedge C(x)$

   (*iii*) $(x)(P(x) \wedge (\exists x) Q(x)) \vee ((x) P(x) \rightarrow Q(x)$.

4. Show that $(\exists z)(B(z) \wedge C(z)$ is not implied by the formula $(\exists x)(A(x) \wedge C(x))$ and $(\exists y)(A(y) \wedge C(y))$, by assuming a universe of discourse which has two elements.

Answer

1. (*a*), (*c*).

14. INFERENCE THEORY OF THE PREDICATE LOGIC

We now first generalize the concept of equivalence and implication to give birth to the predicate calculus. We shall use the same terminology and symbolism as that used for the statement calculus. Here the discussion will include statement calculus as a special case. After defining the concept of validity involving predicate formulae, several valid formulae which are useful in the inference theory of predicate logic will be established.

15. VALID FORMULAE AND EQUIVALENCE

The formulae of the predicate calculus are assumed to contain statement variables, predicates and object variables. The object variables are assumed to belong to a set called the universe of discourse or the domain of the object variable. Such a universe may be finite or infinite. The term "variable" includes constants as a special case. In a predicate formula when all the object variables are replaced by definite names of objects and the statement variables by statements, we obtain a statement which has a truth value T or F always.

Formulae of predicate calculus do not contain predicate variables. These contain predicates *i.e.*, every predicate letter is intended to be a definite predicate, and therefore is not valid for substitution. In such cases predicate letters are interpreted as definite predicates, we use formula without specifying the definite predicate used.

If A and B are any two predicate formulae defined over a common universe denoted by symbol U and if for every assignment of object names from the universe of discourse U to each of the variables appearing in A and B the resulting statements have the same truth values, then the predicate formulae A and B are said to be equivalent to each other over U. This concept is symbolised by writing $A \Leftrightarrow B$ over U. If U is arbitrary, then we say that A and B are equivalent, that is $A \Leftrightarrow B$. The definition of implication can be extended in the same manner. It is to be noted that the same object names are assigned to the same variables through a reference in A and B. Likewise a formula A is said to be valid in U and written A in U, if for every assignment of object names from U to the corresponding variable in A and for every assignment of statements to statement variables, the resulting statements have the truth value T. If a formula is valid for arbitrary U, then it is expressed by writing kA.

$A \Leftrightarrow B$ requires that the equivalence of A and B should be examined over all

NOTES

universes, and same is true for kA, since these statements are made for any arbitrary universe. It is possible to determine by truth tables whether a formula is valid in U, where U is a finite universe of discourse. This method may not be practical when the number of elements in U is large rather it is not applicable when the number of elements in U is infinite. The verification of validity of a formula by truth table would involve examination for all possible universes which is impossible. In fact, methods of derivation are still available. In all such formulae of the predicate calculus that involve quantifiers and no free variables are also formulae of the statement calculus. Therefore, substitution examples of all the tautologies by these formulae yield any number of special tautologies. For example consider the tautologies of the statement calculus given by

$$P \vee - P, P \rightarrow Q \Leftrightarrow - P \vee Q$$

and substitute the formulae $(x)\, R(x)$ and $(\exists x)\, S(x)$ for P and Q respectively. It is assumed that $(x)\, R(x)$ and $(\exists x)\, S(x)$ do not contain any free variables.

Then the following tautologies are obtained

$$((x)\, R(x) \vee - ((x)\, R\,(x))$$

$$((x)\, R(x) \rightarrow ((\exists x)\, S(x)) \Leftrightarrow - ((x)\, R(x)) \vee ((\exists x)\, S(x))).$$

These tautologies form only a very special class of valid formulae of the predicate calculus. Another class of valid formulae that follows from the tautologies of the statement calculus is given below.

We know that any substitution instance of a tautology is also a tautology in the statement calculus. A substitution instance is one in which any variable in a formula is considered replaced by any other formula thought. A similar situation does exist for predicate formulae. A predicate formula is said to be a prime formula if no sentential connectives appear in it. A tautology of the statement calculus remains a valid formula of the predicate calculus when prime formulae are substituted for statement variables throughout the formula. In this way all the implications and equivalences of the statement calculus can be considered as implications and equivalences of the predicate calculus in which the statement variables are replaced by prime predicate formulae. As an example suppose $A(x)$, $B(x)$ and $C(x, y)$ denote any prime formula of the predicate calculus. Then the following are valid formulae of the predicate calculus

$$- - A(x) \Leftrightarrow A(x)$$

$$C(x, y) \wedge B(x) \Leftrightarrow B(x) \wedge C(x, y)$$

$$A(x) \rightarrow B(x) \Leftrightarrow - A(x) \vee B(x)$$

The valid formula obtained in this manner do not exhaust all possible valid formulae. There are several other valid formulae, particularly those involving quantifiers, which are useful. Such valid formulae are obtained by using the inference theory of predicate logic.

16. SOME VALID FORMULAE OVER FINITE UNIVERSE

If $A(x)$, $A(x, y)$, $B(y)$ and $C(x, y, z)$ are predicate formulae then some clarification is necessary in this regard. In $A(x)$, we convey that A is predicate formula in which x is

NOTES

one of the free variables. This variables x is of interest to us and we intend to emphasize the dependence of A on it. For example, we may write $B(x)$ for $(y)\ P(y) \vee Q(x)$.

In a formula $A(x)$ we replace each free occurrence of the variables x by another variable y, and we say that y is substituted for x in the formula and the resulting formula is denoted by $A(y)$. For such a substitution, the formula $A(x)$ must be free for y. A formula $A(x)$ is said to be free for y if no free occurrence of x is in the scope of the quantifiers (y) or $(\exists y)$. If $A(x)$ is not free for y, then it is essential to change the variable y, appearing as a bound variable to another variable before substituting y for x. If y is to be substituted then it is usually better to make all-the bound variables different from y. The following examples illustrate the equivalence of $A(y)$ for a given $A(x)$.

A(x)			A(y)
$P(x, y) \wedge (\exists y)\ Q(y)$	$P(yy) \wedge (\exists y)\ Q(y)$	or	$P(yy) \wedge (\exists z)\ Q(z)$
$(S(x) \wedge S(y)) \vee (x)\ R(x)$	$(S(y) \wedge S(y)) \vee (x)\ R(x)$	or	$(S(y) \wedge S(y)) \vee (z)\ R(z)$

- The following formulae are not free for y

$$P(x, y) \wedge (y)\ Q(x, y) \qquad (y)\ (S(y) \rightarrow S(x))$$

In order to substitute y in place of the variable x in these formulae it is essential to first make them free for y in the following manner.

A(x)	A(y)
$P(x, y) \wedge (z)\ Q(x, z)$	$P(yy) \wedge (z)\ Q(y, z)$
$(z)\ (S(z) \rightarrow S(x))$	$(z)\ (S(z) \rightarrow S(y))$

If the universe of discourse is a finite set, then all possible substitutions of the object variables can be counted. However it is not possible to enumerate all possible substitutions if the universe of discourse is infinite. We now illustrate some equivalences which hold for a finite universe. Later we will show that these equivalence also hold for an arbitrary universe.

Let the universe of discourse be denoted by a finite set S given by

$$S = \{a_1, a_2, a_3, \ldots, a_n\}$$

From the definition of the quantifiers and by simple counting of all the objects in S, we see that

$$(x)\ A(x) \Leftrightarrow A(a_1) \wedge A(a_2) \wedge \ldots \wedge A(a_n) \quad \ldots(1)$$

$$(\exists x)\ A(x) \Leftrightarrow A(a_1) \vee A(a_2) \vee \ldots \vee A(a_n) \quad \ldots(2)$$

Equivalences (1) and (2) over S exhibit that the quantifiers can be dispensed with if the universe of discourse is finite.

Two equivalence that can be derived with the help of (1), (2) and De Morgan's laws are :

$$-((x)\ A(x)) \Leftrightarrow (\exists x) - A(x) \quad \ldots(3)$$

$$-((\exists x)\ A(x)) \Leftrightarrow (x) - A(x) \quad \ldots(4)$$

The proof of (3)

$$-((x)\ A(x)) \Leftrightarrow -(A(a_1) \wedge A(a_2) \wedge \ldots \wedge A(a_n))$$

$$\Leftrightarrow -A(a_1) \vee -A(a_2) \vee \ldots \vee -A(a_n))$$

$$\Leftrightarrow (\exists x) - A(x).$$

Similarly (4) can be proved.

NOTES

If we assume that the negation appearing before a quantifier negates not the quantifier but the whole quantified statement, then we can drop certain parentheses used in (3) and (4) and rewrite the equivalences as :

$$-(x)\ A(x) \Leftrightarrow (\exists x) - A(x)$$

$$-(\exists x)\ A(x) \Leftrightarrow (x) - A(x).$$

If the universe and existential quantifiers are called duals of each other then the above equivalences can be summarized by saying that the negation of a quantified formula is equivalent to a formula in which the quantifier is replaced by its dual and the scope of quantifier is replaced by its negation.

Example 1. *Write the negation the following statements :*

(*a*) *Agra is a big city* (*b*) *Every city of U.P. is clean.*

Solution. Some possible negations are as follows :

(*a*) It is not the case that Agra is a small city

Agra is not a small city.

(*b*) It is not a case that every city of U.P. is clean

Some cities of U.P. are not clean.

Not every city in U.P. is clean.

Remark. 1. The difference between the negations of quantified and non-quantified statements. It is incorrect to negate (*b*) as "Every city of U.P. is not clean".

2. The above proofs are not possible if the universe is infinite.

17. SPECIAL VALID FORMULAE INVOLVING QUANTIFIERS

We do not wish to now put any restriction on the universe of discourse, and assume it to be arbitrary, it may be finite or infinite. In the beginning, we give four implications universe only. These will be used in the theory of inference of the predicate calculus and will allow us to either remove or add quantifiers during the course of a derivation.

Let A(x) be a predicate formula where x is a particular object variable under consideration. Then

$$x\ A(x) \Rightarrow A(y) \quad \text{...(1)}$$

where y is substituted for x to obtain A(y).

In order to prove (1), we assume that (x) A(x) is true. Clearly A(y) is also true. Therefore the implication (1) holds. In case (x) A(x) is false nothing is to be proved. This implication can be written in another convenient form as

$$x\ A(x) \Rightarrow A(x). \quad \text{(rule of universal specification)} \quad \text{...(2)}$$

If B is any formula free of any free occurrance of x, and A(x) any formula then

$$B \rightarrow A(x)) \ \Rightarrow \ (B \rightarrow (x)\ A(x)) \quad \text{...(3)}$$

Implication (3) states that if A(x) follows logically from B, then we can conclude (x) A(x) from B. For the truth of (3), if B → A(x) is assumed to be true for any variable x, then changing x does not change the truth value of B. Therefore B → (x) A(x) is also true. It is to be noted that variable x in A(x) is arbitrary in the sense that nothing that appears in B affects the arbitrariness of x. Certain restrictions are there which guarantee this statement. As a special case of (3), let us replace B by P ∨ – P, where P is any statement variable. Since P ∨ – P is tautology, from

$$(P \lor - P) \to A(x) \Rightarrow (P \lor - P) \to (x)\, A(x)$$

we obtain the implication

$$A(x) \Rightarrow (x)\, A(x) \qquad ...(4)$$

NOTES

Implication (4) permits us to conclude $(x)\, A(x)$ from $A(x)$ *i.e.*, it permits us to add the universal quantifier in the conclusion during the course of derivation. That is why this rule is called the rule of universal generalization and is denoted by writing UG.

According to the rules cited in (2) and (4) which permit the universal quantifier to be removed or added during the course of a derivation, there are two more rules which permit us to remove or add the existential quantifier during the course of a derivation.

The following implications, derivable under assumptions similar to those used for (2) and (4) establish these rules

$$(\exists x)\, A(x) \Rightarrow A(y) \qquad ...(5)$$

$$A(y) \Rightarrow (\exists x)\, A(x) \qquad ...(6)$$

Implication (5) is known as existential specification expressed as ES, while (6) is known as existential generalization and is denoted by EG. Some case is necessary in the interpretation of these rules and will be detailed in forth coming discussion. In the table (1) and (2) some other important implications and equivalences involving the quantifiers are given. These can be proved by using the definition of the quantifiers and arguments similar to the one given above. In some cases it is more convenient to prove them by using the method of derivation given in the coming section and $(\exists x)\, A(x) \Rightarrow A$ and the implications I_{15} and I_{16} become E_{27} and E_{28} as given in tables below. Using E_{11} we can easily prove

$$(\exists x)\, (A(x) \to B(x)) \Leftrightarrow (x)\, A(x) \to (\exists x)\, B(x) \qquad E_{33}$$

Similarly, from I_{15} and I_{16} we can prove

$$(\exists x)\, (A(x) \to (x)\, B(x)) \Leftrightarrow (x)\, A(x) \to B(x) \qquad E_{34}$$

We shall frequently use some of the equivalences and implications for the derivation of a conclusion from a set of premises.

Table 1

$(\exists x)\, (A(x) \lor B(x)) \Leftrightarrow (\exists x)\, (A(x) \lor (\exists x)\, B(x)$	E_{23}
$(x)\, (A(x) \land B(x)) \Leftrightarrow (x)\, A(x) \land (x)\, B(x)$	E_{24}
$-(\exists x)\, A(x) \Leftrightarrow (x) - A(x)$	E_{25}
$-(x)\, A(x) \Leftrightarrow (\exists x) - A(x)$	E_{26}
$(x)\, A(x) \lor B(x) \Rightarrow (x)\, (A(x) \lor B(x))$	I_{15}
$(\exists x)\, ((A(x) \land B(x)) \Rightarrow (\exists x)\, A(x) \land (\exists x)\, B(x)$	I_{16}

Table 2

$(x)\, (A \lor B(x)) \Leftrightarrow A \lor (x)\, B(x)$	E_{27}
$(\exists x)\, (A \lor B(x)) \Leftrightarrow A \land (\exists x)\, B(x)$	E_{28}
$(x)\, A(x) \to B \Leftrightarrow (\exists x)\, (A(x) \to B)$	E_{29}
$(\exists x)\, A(x) \to B \Leftrightarrow (x)\, (A(x) \to B)$	E_{30}
$A \to (x)\, B(x) \Leftrightarrow (x)\, (A \to B(x)$	E_{31}
$A \to (\exists x)\, B(x) \Leftrightarrow (\exists x)\, (A \to B(x)$	E_{32}

18. THEORY OF INFERENCE IN PREDICATE CALCULUS

NOTES

The method of derivation involving predicate formulae used the rules of inference given for the statement calculus and also certain additional rules which are required to deal with the formulae involving quantifiers. The rules P and T, regarding the introduction of a premise at any stage of derivation and the introduction of any formula which follows logically from the formulae already introduce remain the same. If the conclusion is given in the form of a conditional, we shall also use the rule of conditional proof abbreviated as CP. Some times, we may use the indirect method of proof in introducing the negation of the conclusion as an additional premise in order to reach at a contradiction.

The equivalences and implications of the statement calculus can be used in the process of derivation as before, except that the formulae involved are generalized to predicates. But these formulae do not have any quantifiers in them while some of the premise or conclusion may be quantified. In order to use the equivalences and implications, some rules are needed which help in how to eliminate quantifier during the course of derivation. This elimination is done by rules of specification called rules US and ES. Once the quantifiers are eliminated, the derivation proceeds as in the case of statement calculus and the conclusion is arrived. It may happen that the desired conclusion is quantified. In this case, the rules of generalization called rules UG and EG are used to attach a quantifier.

The rules of generalization and specification are as given below. We use $A(x)$ to denote a formula with a free occurrence of x. $A(y)$ denotes a formula obtained by the substitution of y for x in $A(x)$ for such a substitution $A(x)$ must be free for y.

Rule US (Universal Specification) : From $(x)\ A(x)$ one can conclude $A(y)$.

Rule ES (Existential Specification) : From $(\exists x)\ A(x)$ we can conclude $A(y)$ provided that y is not free in any given premise and also not free in any prior step of the derivation. These requirements can easily be meant by choosing new variable each time is used.

Rule EG (Existential Generalization) : From $A(x)$ we can conclude $(y)\ A(y)$ provided that x is not free in any of given premises and provided that if x is free in a prior step which resulted from the use of ES. Then no variables introduced by that use of ES appear free in $A(x)$.

We now show how an invalid conclusion could be arrived at if the second restriction on rule UG were not imposed. The other restrictions on ES and UG are easy to understand.

Let $D(u, v)$: u is divisible by v. Assume that the universe of discourse is {5, 7, 10, 11}, so that the statement $(\exists u)\ D(u, 5)$ is true because both $D(5, 5)$ and $D(10, 5)$ are true. On the other hand $(y)\ D(y, 5)$ is false because $D(7, 5)$ and $D(11, 5)$ are false. Consider now the following derivation

{1}	:	(1) $(\exists u)\ D(u, 5)$	P
{1}	:	(2) $D(x, 5)$	ES, (1)
{1}	:	(3) $(y)\ D(y, 5)$	UG, (2), neglecting second restriction.

In step 3 we have obtained from $D(x, 5)$ the conclusion $(y)\ D(y, 5)$. Evidently x is not free in premise and so that first restriction is satisfied but x is free in step 2 which resulted by use of ES, and that x has been introduced by use of ES and appears free in D(x, 5). Therefore it cannot be generalized. This is the reason why we obtain a false conclusion from a true premises.

NOTES

ILLUSTRATIVE EXAMPLES

Example 1. *Show that (x) (H(x)) → M(x) ∧ H(s) ⇒ M(s). This problem is a symbolic translation of*

All men are Indian

Tagore is a man

Therefore Tagore is an Indian.

If we denote H(x) : x is a man M(x) : x is an Indian.

Solution.

{1}	(1)	$(x)\,(H(x) \to M(x))$	P
{1}	(2)	$H(s) \to M(s)$	US, (1)
{3}	(3)	$H(s)$	P
{1, 3}	(4)	$M(s)$	T, (2), (3), I_{11}

It is to be observed that in step 2 first we have removed the universal quantifier.

Example 2. *Show that (x) (P(x) → Q(x) ∧ (x) (Q(x) → R(x)) ⇒ (x) (P(x) → R(x)).*

Solution.

{1}	(1)	$(x)\,(P(x) \to Q(x))$	P
{1}	(2)	$P(y) \to Q(y)$	US, (1)
{3}	(3)	$(x)\,(Q(x) \to R(x))$	P
{3}	(4)	$Q(y) \to R(y)$	US, (3)
{1, 3}	(5)	$P(y) \to R(y)$	T, (2), (4), I_{13}
{1, 3}	(6)	$(x)\,(P(x) \to R(x))$	UG, (5).

Example 3. *Show that (∃x) M(x) follows logically from the premises (x) (H(x) → M(x)) and (∃x) H(x).*

Solution.

{1}	(1)	$(\exists x)\, H(x)$	P
{1}	(2)	$H(y)$	ES, (1)
{3}	(3)	$(x)\,(H(x) \to M(x))$	P
{3}	(4)	$H(y) \to M(y)$	US, (3)
{1, 3}	(5)	$M(y)$	T, (2), (4), I_{11}
{1, 3}	(6)	$(\exists x)\, M(x)$	EG, (5).

It is to be observed that in step 2 variable y is introduced by ES. Therefore a conclusion such as (x) M(x) could not follow from step 5, because it would violate the rules given for UG.

Example 4. *Prove that (∃x) (P(x) ∧ Q(x)) ⇒ (∃x) P(x) ∧ (∃x) Q(x).*

Solution.

{1}	(1)	$(\exists x)\,(P(x) \wedge Q(x))$	P
{1}	(2)	$P(y) \wedge Q(y)$	ES, (1), y fixed
{1}	(3)	$P(y)$	T, (2), I_1
{1}	(4)	$Q(y)$	T, (2), I_2
{1}	(5)	$(\exists x)\, P(x)$	EG, (3)

{1}	(6)	$(\exists x)\, Q(x)$	EG, (4)
{1}	(7)	$(\exists x)\, P(x) \wedge (\exists x)\, Q(x)$	T, (4), (5), I_9

One may try to prove the converse which does not hold. The derivation is

(1)	$(\exists x)\, P(x) \wedge (\exists x)\, Q(x)$	P
(2)	$(\exists x)\, P(x)$	T, (1), I_1
(3)	$(\exists x)\, Q(x)$	T, (1), I_2
(4)	$P(y)$	ES, (2)
(5)	$Q(z)$	ES, (3)

It is to be observed that in step 4, y is fixed, and it is no longer possible to use that variable again in step 5.

Example 5. *Show that from*

(a) $(\exists x)\, (F(x) \wedge S(x)) \Rightarrow (y)\, (M(y) \to W(y))$

(b) $(\exists y)\, (M(y) \wedge \neg W(y))$

the conclusion $(x)\, (F(x) \to \neg S(x))$ *follows.*

Solution.

{1}	(1)	$(\exists y)\, (M(y) \wedge \neg W(y))$	P
{1}	(2)	$M(z) \wedge \neg W(z)$	ES, (1)
{1}	(3)	$\neg (M(z) \to W(z))$	T, (2), E_{17}
{1}	(4)	$(\exists y) \neg (M(y) \to W(y))$	EG, (3)
{1}	(5)	$\neg (y)\, (M(y) \to W(y))$	E_{26}, (4)
{6}	(6)	$(\exists x)\, (F(x) \wedge S(x) \to (y)\, (M(y) \to W(y))$	P
{1, 6}	(7)	$\neg (\exists x)\, (F(x) \wedge S(x))$	T, (5), (6), I_{12}
{1, 6}	(8)	$(x) \neg (F(x) \wedge S(x))$	T, (7), E_{25}
{1, 6}	(9)	$\neg (F(x) \wedge S(x))$	US, (8)
{1, 6}	(10)	$(F(x) \to \neg S(x))$	T, (9), E_9, I_{16}
{1, 6}	(11)	$(x)\, (F(x) \to \neg S(x))$	UG, (10).

Example 6. *Show that* $(x)\, (P(x) \vee Q(x)) \Rightarrow (x)\, P(x) \vee (\exists x)\, Q(x)$.

Solution. We shall use the indirect method of proof by assuming $\neg ((x)\, (P(x) \vee (\exists x)\, Q(x))$ as an additional premise

{1}	(1)	$\neg ((x)\, P(x) \vee (\exists x)\, Q(x))$	P (assumed)
{1}	(2)	$\neg (x)\, P(x) \wedge \neg (\exists x)\, Q(x)$	T, (1), E_9
{1}	(3)	$\neg (x)\, P(x)$	T, (2), I_1
{1}	(4)	$(\exists x) \neg P(x)$	T, (3), E_{26}
{1}	(5)	$\neg (\exists x)\, Q(x)$	T, (2), I_2
{1}	(6)	$(x) \neg Q(x)$	T, (5), E_{25}
{1}	(7)	$\neg P(y)$	ES, (4)
{1}	(8)	$\neg Q(y)$	US, (6)
{1}	(9)	$\neg P(y) \wedge \neg Q(y)$	T, (7), (8), I_9
{1}	(10)	$\neg P(y) \vee Q(y)$	T, (9), E_9
{11}	(11)	$(x)\, (P(x) \vee Q(x))$	P

{11}	(12)	$(P(y) \vee Q(y))$	US, (11)
{1,11}	(13)	$\neg (P(y) \vee Q(y)) \wedge (P(y) \vee Q(y))$	T, (10), (12), I_9 contradiction

19. FORMULA INVOLVING MORE THAN ONE QUANTIFIER

We have considered only those formulae in which the universal and existential quantifiers appear singly. We shall now consider cases in which the quantifiers occur in combinations. These combinations are possible even in the case of 1–place predicates and they become particularly important in the case of n–place predicates ($n \geq 2$). For example, if $P(x, y)$ is a 2–place predicate formula then the following possibilities exist.

$$(x)\,(y)\,P(x, y) \qquad (x)\,(\exists y)\,P(x, y) \qquad (\exists x)\,(y)\,P(x, y)$$

$$(\exists x)\,(\exists y)\,P(x, y) \qquad (y)\,(x)\,P(x, y) \qquad (\exists y)\,(x)\,P(x, y)$$

$$(y)\,(\exists x)\,P(x, y) \qquad (\exists y)\,(\exists x)\,P(x, y)$$

It is taken for granted that $(x)\,(y)\,P(x, y)$ stands for $(x)\,((y)\,P(x, y))$ and $(\exists x)\,(y)\,P(x, y)$ for $(\exists x)\,((y)\,P(x, y))$. The brackets are not used because even without them there is no possibility of any confusion. From the meaning of the quantifiers the following formulae can be obtained.

$$(x)\,(y)\,P(x, y) \Leftrightarrow (y)\,(x)\,P(x, y) \qquad \text{...(1)}$$

$$(x)\,(y)\,P(x, y) \Rightarrow (\exists y)\,(x)\,P(x, y) \qquad \text{...(2)}$$

$$(y)\,(x)\,P(x, y) \Rightarrow (\exists x)\,(y)\,P(x, y) \qquad \text{...(3)}$$

$$(\exists y)\,(x)\,P(x, y) \Rightarrow (x)\,(\exists y)\,P(x, y) \qquad \text{...(4)}$$

$$(\exists x)\,(y)\,P(x, y) \Rightarrow (y)\,(\exists x)\,P(x, y) \qquad \text{...(5)}$$

$$(x)\,(\exists y)\,P(x, y) \Rightarrow (\exists y)\,(\exists x)\,P(x, y) \qquad \text{...(6)}$$

$$(y)\,(\exists x)\,P(x, y) \Rightarrow (\exists x)\,(\exists y).P(x, y) \qquad \text{...(7)}$$

$$(\exists x)\,(\exists y)\,P(x, y) \Leftrightarrow (\exists y)\,(\exists x)\,P(x, y) \qquad \text{...(8)}$$

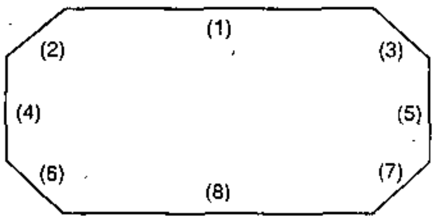


Fig. 1

Figure 1 shows implication (2)—(7) and equivalences (1) and (8). One can also prove these implications and equivalences using the method of derivation given in the previous section. The negation of any of the above formulae can be obtained by repeated application of equivalences E_{25} and E_{26}. For example

$$\neg (\exists y)\,(x)\,P(x, y) \Leftrightarrow (y)\,(\neg (x)\,P(x, y) \Leftrightarrow (y)\,((\exists x) \neg P(x, y)$$

The negation of any of the above formulae can be obtained by similar manner.

The inference rules and the method of derivation also apply to n-place predicate formulae. Obviously, some special is needed in the use of the rules UG, EG, US and ES. In the case of US and ES, the specific variable should be chosen in such a way that it is different from the bound variable used elsewhere. To illustrate this discussion, we consider the formula $(x)\,(\exists y)\,P(x, y)$. Using US, we can write any of the formulae $(\exists y)\,P(x, y)\,(\exists y)\,P(z, y)$ but we should not write $(\exists y)\,P(y, y)$ because the variable y is used as a bound variable; that is $(\exists y)\,P(x, y)$ is not free for y. Similarly, in using EG, one should be careful. For example from $(x)\,P(x, y)$, we can generalize $(\exists y)\,(x)\,P(x, y)$ or $(\exists z)\,(x)\,P(x, z)$ but not $(\exists x)\,(x)\,P(x, x)$. Similar case is required in the use of UG and ES.

NOTES

Example. *Show that* – *P(a, b) follows logically from (x) (y) (P(x, y) → W(x, y)) and* – *W(a, b).*

Solution.

{1}	(1)	$(x)\,(y)\,(P(x, y) \to W(x, y))$	P
{1}	(2)	$(y)\,(P(a, y) \to W(a, y)$	US, (1)
{1}	(3)	$P(a, b) \to W(a, b)$	US, (2)
{1}	(4)	$- W(a, b)$	P
{1, 4}	(5)	$- P(a, b)$	T, (3), (4), I_{12}.

EXERCISE 2(D)

1. Prove that $(x)\,[I(x) \to R(x)],\ (x)\,[R(x) \to C(x)] \to (x)\,[I(x) \to C(x)]$
2. Show that $P(x) \wedge (x)\,Q(x) \Rightarrow (\exists x)\,(P(x) \wedge Q(x))$
3. What is wrong in the following steps of derivation ?

(a)	(1)	$P(x) \to Q(x)$	P
	(2)	$(\exists x)\,P(x) \to Q(x)$	(1), EG
(b)	(1)	$P(a) \to Q(b)$	P
	(2)	$(\exists x)\,(P(x) \to Q(x))$	(1), EG
(c)	(1)	$P(a) \wedge (\exists x)\,(P(a) \wedge Q(x))$	P
	(2)	$(\exists x))\,(P(x) \wedge (\exists x)\,(P(x) \wedge Q(x)))$	(1), EG

4. Explain why the following steps in the derivations are not correct.

(a)	(1)	$(x)\,P(x) \to Q(x)$	
	(2)	$P(x) \to Q(x)$	(1), US
(b)	(1)	$(x)\,P(x) \to Q(x)$	
	(2)	$P(y) \to Q(x)$	(1), US
(c)	(1)	$(x)\,(P(x) \vee Q(x))$	
	(2)	$P(a) \vee Q(b)$	(1), US
(d)	(1)	$(x)\,(P(x) \vee (\exists x)\,(Q(x) \wedge R(x)))$	
	(2)	$P(a) \vee (\exists x)\,(Q(x) \wedge R(a))$	(1), US

5. Given a premise $(x)\,(\exists y)\,P(x, y)$, find the mistake in the following derivations.

{1}	(1)	$(x)\,(\exists y)\,P(x, y)$	P
{1}	(2)	$(\exists y)\,P(z, y)$	US, (1)
{1}	(3)	$P(z, w)$	ES, (2)
{1}	(4)	$(x)\,P(x, w)$	UG, (3)
{1}	(5)	$(\exists y)\,(x)\,P(x, y)$	EG, (4)

6. Construct the derivation of the following equivalences.
   (a) $(\exists x)\,P(x) \to (x)\,Q(x) \Leftrightarrow (x)\,(P(x) \to Q(x))$ – without using E_{34}
   (b) $P \to (\exists x)\,Q(x) \Leftrightarrow (\exists x)\,(P \to Q(x))$ – without using E_{32}.
7. Demonstrate the following implications.
   (a) $\sim ((\exists x)\,P(x) \wedge Q(a)) \Rightarrow (\exists x)\,P(x) \to \sim Q(a)$
   (b) $(x)\,(\sim P(x) \to Q(x)),\ (x) \sim Q(x) \Rightarrow P(a)$
   (c) $(x)\,(P(x) \to Q(x)),\ (x)\,(Q(x) \to R(x)) \Rightarrow P(x) \to R(x)$

(d) $(x)\,(P(x) \vee Q(x)),\ (x) \sim P(x) \Rightarrow (\exists x)\,Q(x)$

(e) $(x)\,(P(x) \vee Q(x)),\ (x) \sim P(x) \Rightarrow (x)\,Q(x)$

(f) $\sim (x)\,(P(x) \wedge Q(x)),\ (x)\,P(x) \Rightarrow \sim (x)\,Q(x)$

NOTES

SUMMARY

1. We know that sets are defined by specifying a property $P(x)$ that elements of the set have in common. Thus an element of $\{x : P(x)\}$ is an object t for which the statement $P(t)$ is true. This type of a sentence $P(x)$ is called a predicate. $P(x)$ is also said to be a *prepositional function*, because each choice of x produces a preposition $P(x)$ that is either true or false. For example let $A = \{x : x$ is an integer less than $6\}$.
2. In first order theories we take a set and all the statements of the theory concerning the elements of the set. The set is said to be domain and elements of it are said to be **individuals.**
3. In any formula of the type "$(\exists x)\,F(x)$" or "$(x)\,F(x)$" is called the x-bound portion of the formula. It is to be noticed that as per definition of *quantifier* the statement function $F(x)$ is the scope of the quantifier. In the x-bound part the occurrence of "x" is called the bound occurrence. That occurrence of "x" which is not bound is said to be free occurrence.
4. As has already been said mathematical logic plays an important role in Computer Science and quantifiers play an important role in logic. To realize this let us consider the following example.

   A student wrote a rule of Algebra as follows :

   $$(x + 3)^2 = x^2 + 3^2$$

   But this is wrong. We will like to prove by logic that this statement is wrong and for it we will prove that its negative statement is true.

   The negative statement of the above statement is

   $$(x + 3)^2 \neq x^2 + 3^2$$

   But this is also not the universal truth because for $x = 0$ this statement is wrong. The statement which supports our point of view, should be

   $$\exists x\ [(x + 3)^2 \neq x^2 + 3^2]$$

   which is just the negation of the statement

   $$\forall x\ [(x + 3)^2 = x^2 + 3^2]$$

   Here we see that use of quantifier is essential here.

TEST YOURSELF

1. Write down the negation of the following sentences in symbolic form.

   (i) The function f tends to limit l near a if $\forall \varepsilon > 0, \exists \delta > 0$ such that

   $$\forall x,\ 0 < |\,x - a\,| < \delta \ \Rightarrow\ |\,f(x) - l\,| < \varepsilon$$

   (ii) No teacher is unwise (iii) $\forall x (x \in A \ \Rightarrow\ x \in B)$
2. Write down the negation of the sentence $\exists x \in I[x \geq 0 \ \Rightarrow\ x^2 > 0]$.
3. Find the values of

   (i) $(\exists x)\,(A(x) \rightarrow B(x) \wedge T$ where $A(x) : x > 2$ $B(x) : x = 0$ and T is any tautology with the universe of discourse as $\{1\}$.

   (ii) $x\,(R(x) \vee S(x))$, where $R(x) : x = 1$, $S(x) : x = 2$ and the universe of discourse is $\{1, 2\}$.

   (iii) $(x)\,(P(x) \rightarrow Q(x) \vee R(a)$ where $P : 2 > 1$, $Q(x) : x < = 3$, $R(x) : x > 5$ and $a : 5$ with the universe being $\{-2, 3, 6\}$.
4. Are the following conclusions validly derivable from the premises given ?

   (a) $(x)\,(P(x) \rightarrow Q(x)),\ (\exists y)\,P(y)$ C : $(\exists z)\,Q(z)$

   (b) $(\exists x)\,(P(x) \wedge Q(x))$ C : $(x)\,P(x)$

(c) $(\exists x)\ P(x), (\exists x)\ Q(x)$ C : $(\exists x)\ P(x) \wedge Q(x))$

(d) $(x)\ (P(x) \to Q(x)), \sim Q(a)$ C : $(x) \sim P(x)$

NOTES

5. Show the following by constructing derivations.

(a) $(\exists x)\ P(x) \to (x)\ ((P(x) \vee Q(x)) \to R(x)), (\exists x)\ P(x),$
$(\exists x)\ P(x) \Rightarrow (\exists x)\ (\exists y)\ (R(x) \wedge R(y))$

(b) $(x)\ (P(x) \to (Q(y) \wedge R(x))), (\exists x)\ P(x) \Rightarrow Q(y) \wedge (\exists x)\ P(x) \wedge R(x))$

(c) $(x)\ (H(x) \to A(x)) \Rightarrow (x)\ ((\exists y), (H(y) \wedge N(x, y) \to (\exists y)\ (A(y) \wedge N(x, y)))$

6. Using **CP** or otherwise, show the following equivalences.

(a) $(\exists x)\ P(x) \to (x)\ Q(x) \Rightarrow (x)\ (P(x) \to Q(x))$

(b) $(x)\ (P(x) \to Q(x)), (x)\ (R(x) \to \sim Q(x)) \Rightarrow (x)\ (R(x) \to \sim P(x))$

(c) $(x)\ (P(x) \to Q(x)) \Rightarrow (x)\ P(x) \to (x)\ Q(x)$

7. There is a mistake in the following derivation. Find it. Is the conclusion valid ? If so, obtain a correct derivation.

{1}	(1)	$(x)\ (P(x) \to Q(x))$	P
{1}	(2)	$P(y) \to Q(y)$	US, (1)
{3}	(3)	$(\exists x)\ P(x)$	P
{3}	(4)	$P(y)$	ES, (3)
{1, 3}	(5)	$Q(y)$	T, (2), (4), I_{11}
{1, 3}	(6)	$(\exists x)\ Q(x)$	EG, (5)

SECTION B

3. Posets and Lattices
4. Boolean Algebra and Boolean Function with Their Applications

C H A P T E R

3

POSETS AND LATTICES

NOTES

LEARNING OBJECTIVES

- Partial Ordered Sets (Posets)
- Hasse Diagrams
- Topological Sorting
- Isomorphism
- Extremal Elements of Partially Ordered Sets
- Lattices
- Some Theorems
- Lattices as Algebraic Systems
- Existence of Partial Ordering Relation on L
- Sublattices, Direct Product and Homomorphism

1. PARTIAL ORDERED SETS (POSETS)

Definition. A relation R on a set A is called a **partial order** if R is reflexive, antisymmetric and transitive. The set A together with the partial order R is called a partially ordered set or simply **poset.** This algebraic structure is denoted by (A, R) or (A, $\leq$).

Remark. If there are no chances of confusion about the partial order, we refer to the poset as A only. We provide some illustrations for the poset.

Illustration 1. Let N be the set of natural numbers. The usual relation $\leq$ (less than or equal to) is a partial order on N and also $\geq$ (greater than equal to) is a partial order relation.

Illustration 2. Let A be a collection of subsets of a set S. The relation $\subseteq$ of set inclusion is a partial order on A *i.e.*, (A, $\subseteq$) is a poset.

Illustration 3. The relation of divisibility *i.e.*, a R b iff $a \mid b$ is a partial order on N.

Illustration 4. *Let R be the set of all equivalence relations on a set A. R is a* partially ordered set under the partial order of containment because it consists of subsets of A × A. If R and S are equivalence relations on A *i.e.*, R $\subseteq$ S if and only if x R $y \Rightarrow x$ S y $\forall$ $x, y \in$ A,

then (R, $\subseteq$) is a poset.

NOTES

Illustration 5. The relation < on N is not partial order as it is not reflexive and therefore (N, <) is not a poset.

ILLUSTRATIVE EXAMPLES

Example 1. *Let R be a partial order on a set A and let R^{-1} be the inverse relation of R. Then prove that R^{-1} is also a partial order.*

Solution. Since R is a partial order on a set A, therefore it is reflexive, antisymmetric and transitive *i.e.*, $\Delta \subseteq R$, $R \cap R^{-1} \subseteq \Delta$ and $R^2 \subseteq R$.

By taking inverses, $\Delta = \Delta^{-1} \subseteq R^{-1}$, $R^{-1} \cap (R^{-1})^{-1} = R^{-1} \cap R \subseteq \Delta$ and $(R^{-1})^2 \subseteq R^{-1}$.

Hence R^{-1} is reflexive, antisymmetric and transitive. Thus R^{-1} is also a partial order relation on A. The poset (A, R^{-1}) is said to be the **dual** of the poset (A, R) and the partial order R^{-1} is said to be **dual** of the partial order R.

The most familiar partial order are the relations ≤ and ≥ on N and R. That is why, when we talk about partial order on a set A in general, we use the symbols ≤ and ≥ for partial order.

Note 1. Whenever (A, ≤) is a poset, we will use ≥ for the inverse of the partial order ≤.

Note 2. If (A, ≤) is a poset, the elements a and b of A are said to be **comparable** if $a \leq b$ or $b \leq a$. It is to be taken carefully that in a partially ordered set every pair of elements need not be comparable. Whenever every pair of elements in a poset A is comparable then A is said to be a linearly ordered set and the partial order is called a linear order. We also say then that A is a **chain.**

Theorem 1. *If (A, ≤) and (B, ≤) are posets, then (A × B, ≤) is a poset, with partial order ≤ defined by $(a, b) \leq (a', b')$ if $a \leq a'$ in A and $b \leq b'$ in B.*

Here the symbol ≤ is being used to denote three distinct partial orders. One should find it easy to determine which of the three is meant at any time.

Proof. 1. Reflexivity. If $(a, b) \in A \times B$, then $(a, b) \leq (a', b')$ since $a \leq a$ in A and $b \leq b$ in B, therefore ≤ satisfies the reflexive property in A × B.

2. Antisymmetry. Now suppose that $(a, b) \leq (a', b')$ and $(a', b') \leq (a, b)$, where a and $a' \in A$ and b and $b' \in B$.

Then $a \leq a'$ and $a' \leq a$ in A

and $b \leq b'$ and $b' \leq b$ in B.

Since A and B are posets, the antisymmetry of the partial orders in A and B implies that

$$a = a' \quad \text{and} \quad b = b'$$

Hence ≤ satisfies the antisymmetric property in A × B.

3. Transitivity. Finally, suppose that

$$(a, b) \leq (a', b') \quad \text{and} \quad (a', b') \leq (a'', b''),$$

where $a, a', a'' \in A$ and $b, b', b'' \in B$. Then

$$a \leq a' \quad \text{and} \quad a' \leq a'',$$

therefore, $a \leq a''$, by the transitive property of the partial order in A.

Similarly, $b \leq b'$ and $b' \leq b''$,

therefore, $b \leq b''$, by the transitive property of the partial order in B. Hence

$$(a, b) \leq (a'', b'').$$

Consequently, the transitive property holds for the partial order in A × B. Thus we prove that A × B is poset.

Definition 1. The partial order $\leq$ defined on the Cartesian product A × B as above is called *product partial order.*

Definition 2. If (A, $\leq$) is a poset, we say that $a < b$ if $a \leq b$, but $a \neq b$. Suppose now that (A, $\leq$) and (B, $\leq$) are posets. In theorem given above we have defined the product partial order on A × B. Another useful partial order on A × B, denoted by <, is defined as follows :

$(a, b) < (a', b')$ if $a < a'$ or if $a = a'$ and $b \leq b'$.

This ordering is called **lexicographic**, or "*dictionary*" order.

Note. The ordering of the elements in the first co-ordinate dominates, except in case of **"ties"**. In that case attention passes on to the second co-ordinate. If (A, $\leq$) and (B, $\leq$) are linearly ordered sets, then the lexicographic order < on A × B is also a linear order.

Example 2. Let A = R, with the usual ordering $\leq$. Then the plane $R^2 = R \times R$ may be given lexicographic order. This is shown in figure given below. We see that the plane is linearly ordered by lexicographic order. Each vertical line has the usual order and points on one line are less than any point on a line farther to the right.

Thus in the following figure, $p_1 < p_2, p_1 < p_3$ and $p_2 < p_3$.

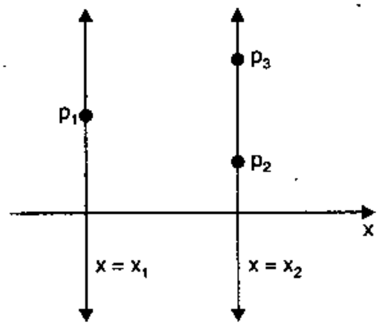


Fig. 1

Lexicographic ordering is easily extended to Cartesian products $A_1 \times A_2 \times .... \times A_n$ in the following way:

$(a_1, a_2, ... , a_n) < (a_1', a_2', ..., a_n')$ if and only if

$a_1 < a_1'$ or $a_1 = a_1'$ and $a_2 \leq a_2'$

or $a_1 = a_1', a_2 = a_2'$ and $a_3 \leq a_3'$ or ....

$a_1 = a_1', a_2 = a_2', a_3 = a_3' \ldots\ldots a_{n-1} = a_{n-1}'$ and $a_n \leq a_n'$

Thus the first co-ordinate dominates except for equality and in that case we consider the second coordinate. If equality holds again, we focus our attention to the next coordinate, and so on.

Example 3. Let S = $(a \leq b, b \leq c, \ldots , y \leq z)$ be the set of ordinary alphabets, linearly ordered in the usual way $(a \leq b, b \leq c, ..., y \leq z)$. Then $S^n = S \times S \times .... \times S$ (n factors) can be identified with the set of all words having length n. Lexicographic order on S^n has the property that if $w_1 < w_2$ $(w_1, w_2 \in S^n)$, then w_1 would precede w_2 in a dictionary listing. This fact is responsible for the name of the ordering.

Thus park < part, help < hind, jump < mump. The third is true since $j < m$; the second, since $h = h, e < i$ and the first is true since $p = p, a = a, r = r, k < t$.

If S is a poset, we can extend lexicographic order to S^* (the set consisting of all finite sequences of elements of A) in the following way.

If $x = a_1 a_2 \ldots\ldots a_n$ and $y = b_1 b_2 \ldots\ldots b_k$ are in S^* with $n \le k$, we say that $x < y$ if , $(a_1 a_2 \ldots\ldots a_n) < (b_1 b_2 \ldots\ldots b_n)$ in S^n under lexicographic ordering of S^n. In other words, we chop off to the length of the shortest word and then compare.

In the above discussion, we use the fact that the n-tuple $(a_1, a_2, \ldots, a_n) \in S^n$ and the string $a_1, a_2, \ldots., a_n \in S^*$ are really the same sequence of length n, written in two different notations. Though the notations differ for historical reasons, we will use them interchangeably depending on the context in a certain reference.

Example 4. Let S be $\{a, b, \ldots, z\}$, ordered as usual. Then S^* is the set of all possible "words" of any length, whether such words are meaningful or not.

Thus we have *help* < *helping*

in S^* since

help < *helpi*

in S^4.

Similarly, we have

helper < *helping*

since

helper < *helpin*

in S^6.

As the example *help* < *helping*

shows, this order includes *prefix order* ; that is any word is greater than all of its prefixes (beginning parts). This is also the way that words occur in the dictionary.

Thus we have dictionary ordering again, of course this time for any finite length.

Since a partial order is a relation, we can look at the digraph (see chapter 12) of any partial order on a finite set. We shall notice that the digraphs of partial orders can be represented in a similar manner as those of general relations. The following theorem provides the first result in this connection. (This theorem may be read after reading digraph).

Theorem 2. *The digraph of a partial order has no cycle of length greater than 1.*

Proof. Suppose that the digraph of a partial order $\le$ on the set A contains a cycle of length $n \ge 2$. Then there exist distinct elements $a_1, a_2, \ldots, a_n \in$ A such that

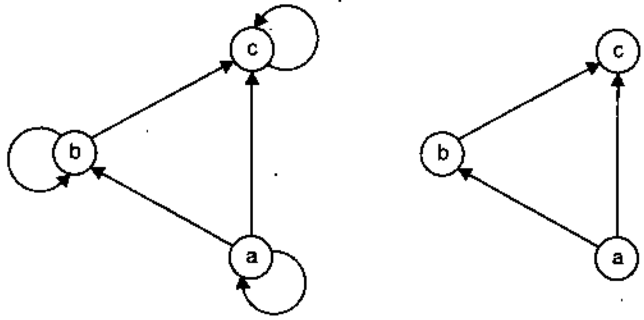
$$a_1 \le a_2, a_2 \le a_3, \ldots\ldots a_{n-1} \le a_n, a_n \le a_1.$$

By the transitivity property of the partial order, used $n - 1$ times, $a_1 \le a_n$. By antisymmetry, $a_n \le a_1$ and $a_1 \le a_n$ imply that $a_n = a_1$, a contradiction to the assumptions that $a_1, a_2, \ldots, a_n$ are distinct.

2. HASSE DIAGRAMS

Let A be a finite set. In the above Theorem, we have seen that the digraph of a partial order on A has only cycles of length 1. Infact, since a partial order is reflexive, every vertex in the digraph of the partial order is contained in a cycle of length 1. To simplify things, we delete all such cycles from the digraph. Thus the digraph shown in figure below on left would be drawn as shown in figure below on right.

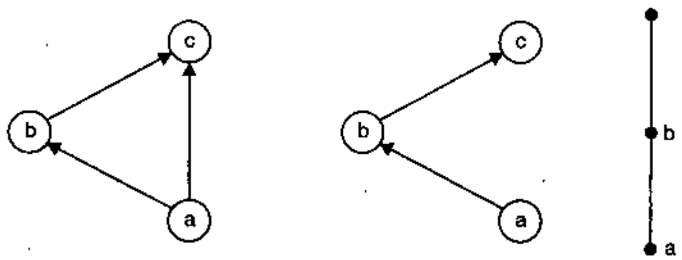
NOTES



(a) (b)

Fig. 2

We also eliminate all edges that are implied by the transitive property. Thus, if $a \leq b$ and $b \leq c$, it follows that $a \leq c$. In this case, we omit the edge from a to c; however, we certainly draw the edges from a to b and from b to c. For example, the digraph shown in following figure (a) would be drawn as shown in figure (b) below. We also draw the digraph of a partial order with all edges pointing upward, so that arrows may be omitted from the edges. Finally, we replace the circles representing the vertices by dots. Thus the diagram shown in figure on the extreme right gives the final form of the digraph shown in figure (a) above. The resulting diagram of a partial order, which is much simpler than its digraph, is said to be **Hasse diagram** of the partial order of the poset. Since Hasse diagram completely describes the associated partial order, we find it to be a very useful tool to analyse a problem involving partial ordering. **Hasse diagrams** should not be confused with graphs, both are simplified ways of representing different types of digraphs.



(a) (b)

Fig. 3 **Fig. 4**

ILLUSTRATIVE EXAMPLES

Example 1. *Let A = {1, 2, 3, 4, 12}. Consider the partial order of divisibility on A. That is, if a and b ∈ A, a < b if and only if a | b. Draw the Hasse diagram of the poset (A, ≤).*

Solution. The Hasse diagram is shown in left upper side of the following figures 9.5. To highlight the simplicity of the Hasse diagram it shown in right upper side of the following figures, the diagraph of the poset in figure (a).

Example 2. *Let S = {a, b, c} and A = P(S). Draw the Hasse diagram of the poset with the partial order ⊆ (set inclusion).*

Solution. We first determine A, obtaining

$$A = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}.$$

The Hasse diagram can then be drawn as shown in following figure 9.5(c).

Note. The Hasse diagram of a finite linearly ordered set is always of the form shown in figure 9.5 (*d*) below.

Remark 1. It is easily seen that if (A, ≤) is a poset and (A, ≥) is a dual poset, the Hasse diagram of (A, ≥) is just the Hasse diagram of (A, ≤) turned upside down.

Remark 2. Figure 9.6 (*a*) given below shows the Hasse diagram of the poset (A, ≤), where A = {*a*, *b*, *c*, *d*, *e*, *f*}. Figure (*b*) shows the Hasse diagram of the dual poset (A, ≥). It is observed that, as mentioned above, each of these diagrams can be constructed by turning the other upside down.

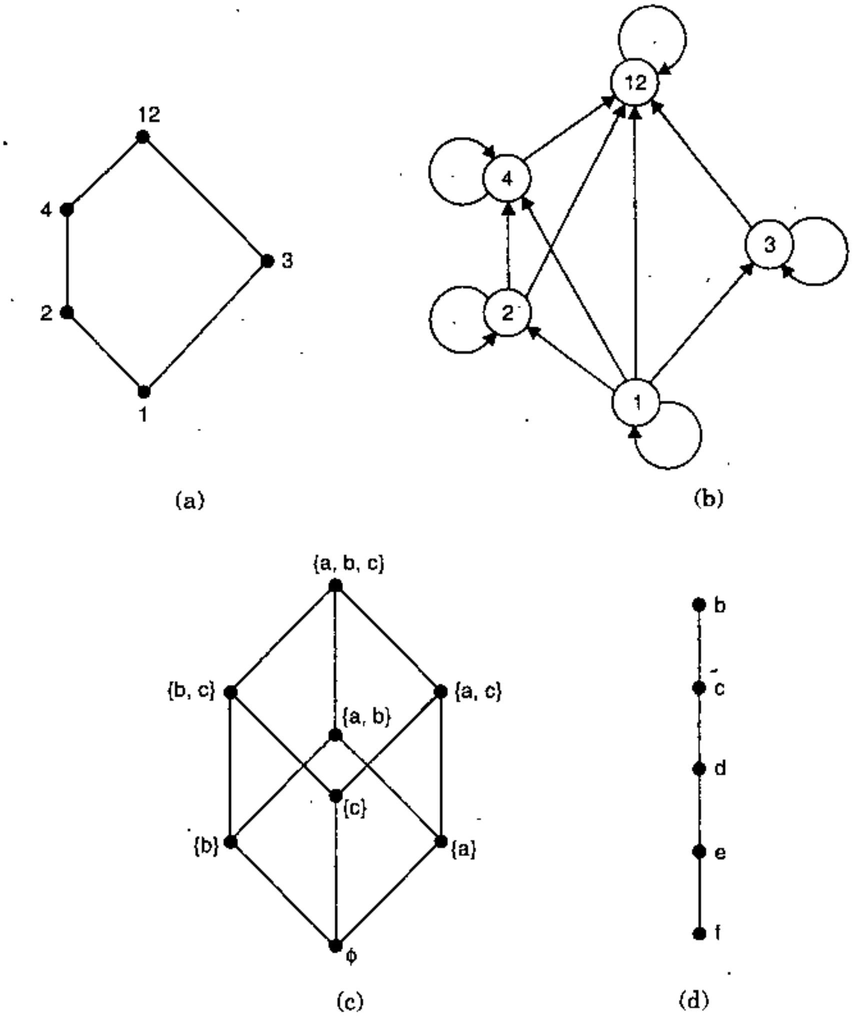


Fig. 5

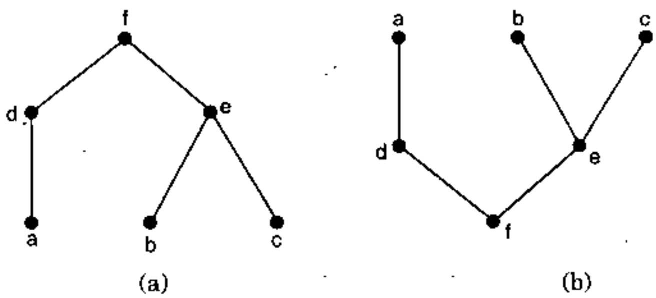


Fig. 6

3. TOPOLOGICAL SORTING

NOTES

Definition. If A is a poset with partial order ≤, we some times need to find a linear order < for the set A that will merely be an extension of the given partial order in the sense that if $a \leq b$, then $a < b$. The process of constructing a linear order such as < is said to be **topological sorting**. This problem might arise when we have to enter a finite poset A into a computer. The elements of A must be entered in some order and we might require them to be entered so that the partial order is preserved. That is, if $a \leq b$, then a is entered before b. A topological sorting < will give an order of entry of the elements that meets this condition.

ILLUSTRATIVE EXAMPLE

Example. *Give a topological sorting for the poset whose Hasse diagram is shown in figure given below on left.*

Solution. The partial order < whose Hasse diagram is shown in figure 8 (a) is clearly a linear order. It is easy to see that every pair in ≤ is also in the order <, so < is a topological sorting of the partial order <. Figures 8(b) and (c) below show other representations to this problem for topological sorting.

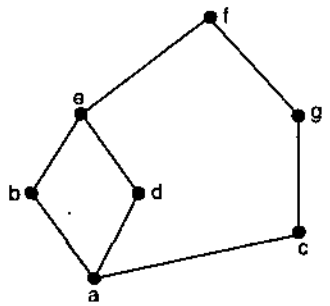
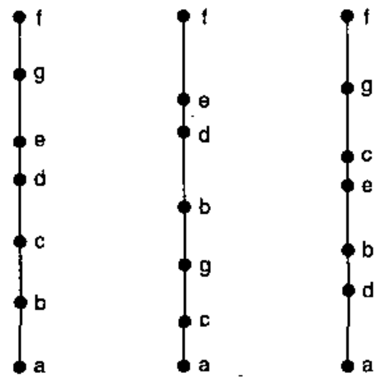


Fig .7



(a) (b) (c)

Fig .8

As it has been seen above, there are many ways of topologically sorting a given poset. The algorithm for generating topological sortings is given as under :

Input : A digraph G = (V, E), with n vertices

Output : A topological enumeration $S_n = < s_1, \ldots, s_n >$ of V with respect to E, provided E* is a partial ordering on V.

Procedure :

1. Let $U_0 = V$, $S_0 = < >$ (the sequence of length zero) and $T_0(x) = \{u : (u, x) \in E$ and $u \neq v\}$

2. Repeat the following for $i = 1, \ldots, n$;

   (a) Choose s_i from U_{i-1} s.t. $T_{i-1}/s_i) = \phi$, provided such an s_i exists ; otherwise, halt and output a massage that E* is not antisymmetric.

   (b) Let $U_i = U_{i-1} - \{s_i\}$, $S_i = S_{i-1} \cdot < s_i >$, and $T_i(v) = T_{i-1}(x) - \{s_i\}$ for all $v \in V$.

3. If not already halted, output s_n.

NOTES

4. ISOMORPHISM

Let (A, ≤) and (A′, ≤′) be posets and let f: A → A′ be a one-to-one correspondence between A and A′. The function f is called an isomorphism from (A, ≤) to (A′, ≤′) if for any a and b in A,

$$a \leq b \text{ if and only if } f(a) \leq' f(b).$$

If f : A → A′ is an isomorphism, we say that (A, ≤) and (A′, ≤′) are **isomorphic** posets.

ILLUSTRATIVE EXAMPLES

Example 1. *Let A be a set Z^+ of positive integers and let ≤ be a usual partial order on A. Let A′ be the set of positive even integers and let ≤′ be the usual partial order on A′. Prove that the function f : A → A′ given by f(a) = 2 a is an isomorphism from (A, ≤) to (A′, ≤′).*

Solution. f is one-to-one since $f(a) = f(b) \Rightarrow 2a = 2b \Rightarrow a = b$. Now, Dom ($f$) = A, therefore f is defined everywhere. Finally, $c \in$ A′, then $c = 2a$ for some $a \in Z^+$; therefore $c = f(a)$. This shows that f is onto, so we see that f is a one-to-one onto correspondence. Thus, if a and b are elements of A, then it is clear that $a \leq b$ if and only if $2a < 2b$. Hence f is isomorphism.

Suppose that f : A → A′ is an isomorphism from a poset (A, ≤) to a poset (A′, ≤′). Suppose also that B is a subset of A and B′ = f(B) is a corresponding subset of A′. Then we see from the definition of isomorphism that the following general principle must hold.

Theorem 3. (Principle of Correspondence). If the elements of B have any property relating to one another or to other elements of A and if this property can be defined entirely in terms of the relation ≤, then the elements of B′ must posses exactly the same property, defined in terms of ≤′.

Example 2. Let (A, ≤) be the poset whose Hasse diagram is shown in figure below, and suppose that f is an isomorphism from (A, ≤) to some other poset (A′, ≤′). Let $d \leq x$ for any x in A (later we will call an element such as d, a "least element" of A), then the corresponding element $f(d)$ in A′ must satisfy the property $f(d) \leq' y$ for all $y \in$ A′. As another example, if $a \neq b$ and $b \neq a$. Such a pair is called **incomparable** in A. It then follows from the principle of correspondence that $f(a)$ and $f(b)$ must be incomparable in A′.

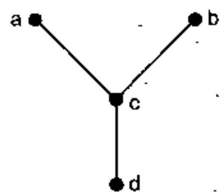


Fig. 9

For a finite poset, one of the objects that is defined entirely in terms of the partial order is its Hasse diagram. It follows from the principle of correspondence that two finite isomorphic posets must have the same Hasse diagrams.

To be precise, let (A, ≤) and (A′, ≤′) be finite posets, let f : A → A′ be a one-to-one correspondence and let H be any Hasse diagram of (A, ≤). Then

1. If f is an isomorphism and each label a of H is replaced by $f(a)$, then H will become a Hasse diagram for (A′, ≤′).

Conversely,

2. If H becomes a Hasse diagram for (A′, ≤′), whenever each label a is replaced by $f(a)$, then f is an isomorphism.

NOTES

Example 3. Let A = {1, 2, 3, 6} and let $\leq$ be the relation | (divides). Figure (*a*) below shows a Hasse diagram for (A, $\leq$). Let A′ = P ({*a*, *b*}) = {ϕ, {*a*}, {*b*}, {*a*, *b*}} and let $\leq'$ be the set containment, $\subseteq$. If $f : A \to A'$ is defined by

$$f(1) = \phi, f(2) = \{a\}, f(3) = \{b\}, f(6) = \{a, b\},$$

then it is easily seen that *f* is a one-to-one correspondence. If each label $a \in$ A of the Hasse diagram is replaced by *f* (*a*), the result is shown in figure (*b*). Since this is clearly a Hasse diagram for (A′, $\leq'$), the function *f* is an isomorphism.

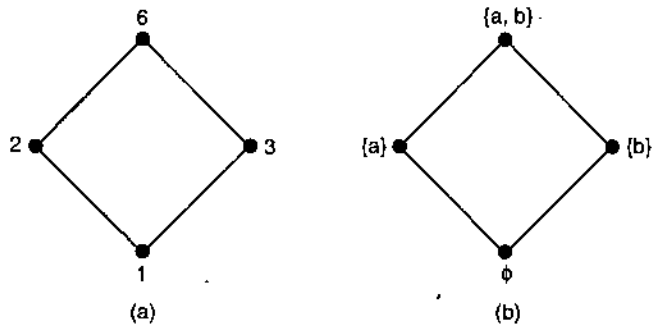


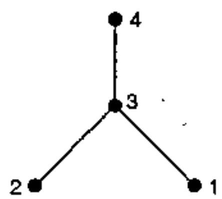
Fig. 10

EXERCISE 3 (A)

1. What can you say about the relation R on a set A if R is partial order and an equivalence relation ?
2. Determine whether the relation R is a linear order on the set A.
   (*a*) A = R, and *a* R *b* if and only if $a \leq b$.
   (*b*) A = R, and *a* R *b* if and only if $a \geq b$.
   (*c*) A = P(S), where S is a set. The relation R is a set inclusion.
   (*d*) A = R × R, and (*a*, *b*) R (*a*′, *b*′) if and only if $a \leq a'$ and $b \leq b'$, where $\leq$ is the usual partial order on R.
3. Determine whether the relation R is a partial order on the set A.
   (*a*) A = Z, and *a* R *b* if and only if $a = 2b$.
   (*b*) A = Z, and *a* R *b* if and only if $b^2 \mid a$.
   (*c*) A = Z, and *a* R *b* if and only if $a = b^k$ for some $k \in Z^+$. Note that *k* depends on *a* and *b*.
   (*d*) A = R, and *a* R *b* if and only if $a \leq b$.
4. Find all partial orders $\leq$ on the set A = {*a*, *b*, *c*} in which $a \leq b$.

   Make the Hasse diagram of the relation R in the following.
5. A = {1, 2, 3, 4}, R = {(1, 1), (1, 2), (2, 2), (2, 4), (1, 3), (3, 3), (3, 4), (1, 4), (4, 4)}.
6. A = {*a*, *b*, *c*, *d*, *e*}, R = {(*a*, *a*), (*b*, *b*), (*c*, *c*), (*a*, *c*), (*c*, *d*), (*c*, *e*), (*a*, *d*), (*d*, *d*), (*a*, *e*), (*b*, *c*), (*b*, *d*), (*b*, *e*), (*e*, *e*)}.
7. Describe the ordered pairs in the relation determined by the Hasse diagram on the set A in the following figures.

(*a*) A = {1, 2, 3, 4}

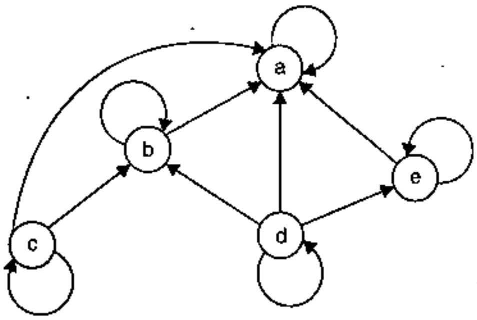


(*b*) A = {1, 2, 3, 4}

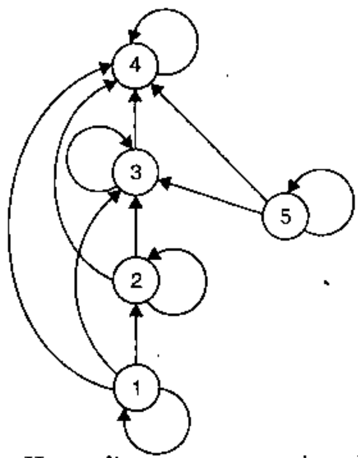
NOTES

In the following questions, determine the Hasse diagram of the partial order having the following digraphs.

8.

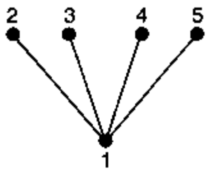


9.

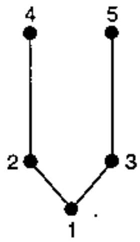


10. Determine the matrix of the partial order whose Hasse diagrams are as given below :

(*a*)



(*b*)



11. Determine the Hasse diagram of the relation on A = {1, 2, 3, 4, 5} whose matrix is shown below :

(*a*) $\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ (*b*) $\begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$.

12. Let $A = Z^+ \times Z^+$ have lexicographic order. Mark each of the following as true or false.

(*a*) (2, 12) < (5, 3) (*b*) (3, 6) < (3, 24)

(*c*) (4, 8) < (4, 6) (*d*) (15, 93) < (12, 3).

In the following questions, consider the partial order of divisibility on the set A. Draw the Hasse diagram of the poset and determine which posets are linearly ordered.

13. (*a*) A = {1, 2, 3, 5, 6, 10, 15, 30} (*b*) A = {2, 4, 8, 16, 32}

14. (*a*) A = {3, 6, 12, 36, 72} (*b*) A = {1, 2, 3, 4, 5, 6, 10, 12, 15, 30, 60}.

In the following questions, draw the Hasse diagram of a topological sorting of the given poset

15. Let A = {A, B, C, E, O, M, P, S} have the usual alphabetical order, where □ represents a "blank" character and □ ≤ *a* for all *x* ∈ A.

Arrange the following in lexicographic order (as elements of A × A × A × A).

(*a*) MOP □ (*b*) MOPE (*c*) CAP □

(*d*) MAP □ (*e*) BASE (*f*) ACE □

(*g*) MACE (*h*) CAPE.

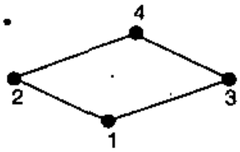
NOTES

Answers

3. (*a*) No (*b*) No (*c*) yes
(*d*) yes

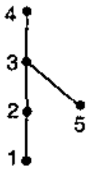
4. $\{(a, a), (b, b), (c, c), (a, b)\}$, $\{(a, a), (b, b), (c, c), (a, b), (a, c)\}$, $\{(a, a), (b, b), (c, c), (a, b), (c, b)\}$
$\{(a, a), (b, b), (c, c), (a, b), (b, c), (a, c)\}$, $\{(a, a), (b, b), (c, c), (a, b), (c, b), (c, a)\}$

5.



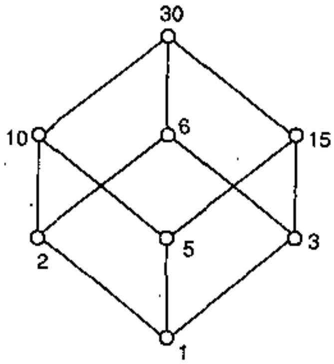
7. (*a*) $\{(1, 1), (2, 2), (3, 3), (4, 4), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\}$.
(*b*) $\{(1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\}$.

9.



10. (*a*) $\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ (*b*) $\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

13. (*a*)



(*b*)

Linear

15. ACE, BASE, CAP, CAPE, MACE, MAP, MOP, MOPE.

5. EXTREMAL ELEMENTS OF PARTIALLY ORDERED SETS

Certain elements in a poset are of special importance for many of the properties and applications of posets. Let us consider a poset (A, ≤) with partial order ≤.

An element $a \in$ A is called a *maximal element* of A if there is no element $c \in$ A such that $a < c$. An element $b \in$ A is called a *minimal element* of A if there is no element $c \in$ A such that $c < b$.

NOTES

It follows immediately that, if (A, ≤) is a poset and (A, ≥) is its dual poset, an element $a \in$ A is a maximal element of (A, ≥) if and only if a is a minimal element of (A, ≤). Also, a is a minimal element of (A, ≥) if and only if a is a maximal element of (A, ≤).

ILLUSTRATIVE EXAMPLES

Example 1. Consider the poset A whose Hasse diagram is shown in figure given below. The elements a_1, a_2 and a_3 are maximal elements of A and the elements b_1, b_2 and b_3 are the minimal elements. We see that, since there is no line between b_2 and b_3 therefore, neither $b_3 \le b_2$ nor $b_2 \le b_3$.

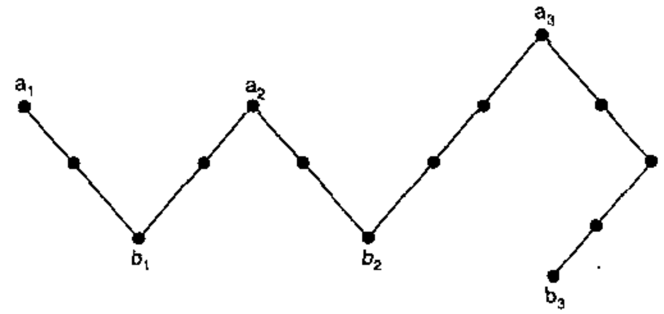


Fig. 11

Example 2. Let A be a poset of non-negative real numbers with the usual partial order ≤. Then we have 0 as a minimal element of A and there are no maximal elements of A.

Example 3. The poset Z with the usual partial order ≤ has no maximal elements and has no minimal elements.

Theorem 1. *Let A be a finite nonempty poset with partial order ≤. Then A has at least one maximal element and at least one minimal element.*

Proof. Let a be any element of A. If a is not maximal, we can find an element $a_1 \in$ A such that $a < a_1$. If a_1 is not maximal, we can find an element $a_2 \in$ A such that $a_1 < a_2$. This argument cannot be continued indefinitely, since A is a finite set. Thus we finally obtain the finite chain

$$a < a_1 < a_2 < \ldots\ldots < a_{k-1} < a_k,$$

which cannot be extended any more. Therefore we cannot have $a_k < b$ for any $b \in$ A, hence a_k is a maximal element of (A, ≤).]

The same argument implies that the dual poset (A, ≥) has a maximal element, hence (A, ≤) has a minimal element.

By using the concept of a minimal element, we can give an algorithm for finding a topological sorting of a given finite poset (A, ≤). We know that if $a \in$ A and B = A – {a}, then B is also a poset under the restriction of ≤ to B × B. We then have the following algorithm, which produces a linear array named SORT. We assume that SORT is ordered by increasing index, that is, SORT [1] < SORT [2] < ..... The relation < on A defined in this way is a topological sorting of (A, ≤).

Algorithm for finding a topological sorting of a finite poset (A, ≤) is as given below.

Step 1. *Choose a minimal element a of* A.

Step 2. Make a the next entry of SORT and replace A with A – {a}.

NOTES

Step 3. Repeat steps 1 and 2 until A = { }.

End of Algorithm.

An element $a \in$ A is called a greatest element of A if $x \leq a$ for all $x \in$ A. An element $a \in$ A is called a least element of A if $a \leq x$ for all $x \in$ A.

As before, an element a of (A, ≤) is greatest (or least) element if and only if it is a least (or greatest) element of (A, ≥).

Example 4. Consider the poset (A, <) where A is the set of non-negative real numbers. Then 0 is a least element ; there is no greatest element.

Example 5. Let S = {a, b, c} and consider the poset (A = P(S), C). The empty set is a least element of A, and the set S is a greatest element of A.

Example 6. The poset Z with the usual partial order has neither a least nor a greatest element.

Example 7. Let A = {a, b, c, d, e} and let the Hasse diagram of a partial order ≤ on A be as shown in figure (a) given below. A minimal element of this poset is the vertex

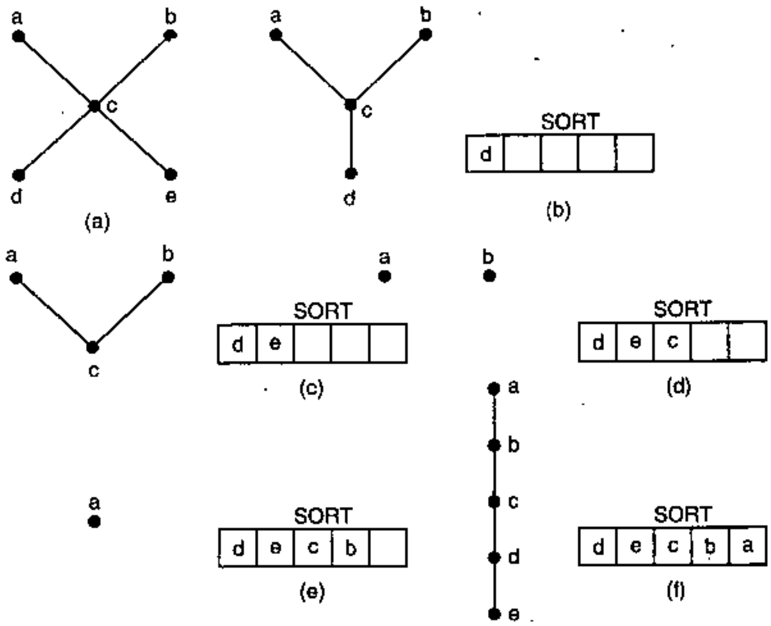


Fig. 12

labeled d (we could also have chosen e). We put d in SORT[1] and in figure (b) is the Hasse diagram of A – {d}. The minimal element of the new A is e, therefore e becomes SORT[2] and A – {e} is shown in figure (c). This process continues until A is exhausted and SORT is filed. Figure (f) shows the completed array SORT and the Hasse diagram of the poset corresponding to SORT. This is a topological sorting of (A, ≤).

Theorem 2. *A poset has at most one greatest element and at most one least element.*

Proof. Suppose that a and b are greatest elements of a poset A. Then, since b is greatest element, we have $a \leq b$. Similarly, since a is a greatest element, we have $b \leq a$. Hence $a = b$ by the antisymmetric property. Thus, if the poset has a greatest element, it only has one such element. Since this fact is true for all posets, the dual poset (A, ≥) has at most one greatest element, therefore (A, ≤) also has at most one least element. The greatest element of a poset, if it exists, is denoted by 1 and is often called the unit element. Similarly, the least element of a poset, if it exists, is denoted by 0 and is often called the zero element.

Definition. Consider a poset A and a subset B of A. An element $a \in$ A is called an **upper bound** of B if $b \leq a$ for all $b \leq$ B. An element $a \in$ A is called a **lower bound** of B if $a \leq b$ for all $b \in$ B.

ILLUSTRATIVE EXAMPLES

Example 1. *Consider the poset A = {a, b, c, d, e, f, g, h}, whose Hasse diagram is shown in the following figure. Find all upper and lower bounds of the following subsets of A ;*

(*i*) $B_1 = \{a, b\}$; (*ii*)$B_2 = \{c, d, e\}$ *and lower and upper bounds.*

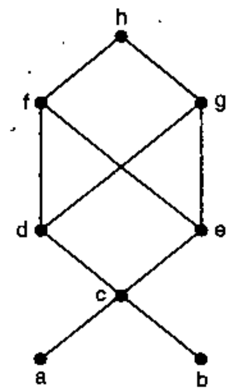


Fig. 13

Solution. (*i*) B_1 has no lower bounds; its upper bounds are c, d, e, f, g and h.

(*ii*) The upper bounds of B_2 are f, g and h ; its lower bounds are c, a and b.

Remark. This example shows that, a subset B of a poset may or may not have upper or lower bounds (in A). Moreover, an upper or lower bound of B may or may not belong to B itself.

Definition. Least Upper and Greatest Lower Bounds

Let be a poset and B a subset of A. An element $a \in$ A is called a **least upper bound** ($l.u.b$) of B if a is an upper bound of B and $a \leq a'$,whenever a' is an upper bound of B. Thus $a = l.\ u.b.$(B) if $b \leq a$ for all $b \leq$ B and if whenever $a' \in$ A is also an upper bound of B, then $a \leq a'$.

Similarly, and element $a \in$ A is called a **greatest lower bound** ($g.l.b$) of B if a is a lower bound of B and $a' \leq a$, whenever a' is a lower bound of B. Thus $a = g.l.b.$ (B) if $a \leq b$ for all $b \in$ B and if whenever $a' \in$ A is also a lower bound of B, then $a' \leq a$.

Note. It goes without saying that upper bounds in (A, $\leq$) corresponds to lower bounds in (A, $\geq$) (for the same set of elements) and lower bounds in (A, $\leq$) corresponds to upper bound (A, $\geq$). Similar statements hold for greatest lower bounds and least upper bounds too.

Example 2. *Let A be a poset considered in example 1 with subsets B_1 and B_2 as defined in that example. Find all least upper bounds and all greatest lower bounds of (i) B_1 and (ii) B_2.*

Solution. (*i*) Since B_1 has no lower bounds, it has no greatest lower bounds. However,

$$l.u.b\ (B_1) = c.$$

(*ii*) Since the lower bounds of B_2 are f, g and h. Since f and g are not comparable, we conclude that B_2 has no least upper bound.

Theorem 3. *Let (A, ≤) be a poset. Then a subset B of A has at most one l.u.b and at most one g.l.b.*

Proof. The proof is similar to the proof of theorem 2.

Remark. Finally we make some remarks about *l.u.b* and *g.l.b* in a finite poset A, as viewed from the Hasse diagram of A. Let $B = \{b_1, b_2, \ldots, b_r\}$. If $a = l.u.b$ (B), then a is the first vertex that can be reached from $b_1, b_2, \ldots, b_r$ by upward paths. Similarly, if $a = g.l.b$ (B), then a is the first vertex that can be reached from $b_1, b_2, \ldots, b_r$ by downward paths.

Example 3. *Let A = { 1, 2, 3, 4, 5, 6,...... , 1 1 } be the poset whose Hasse diagram is shown in figure given below. Find the lub and the glb of B = {6, 7, 10} , if they exist.*

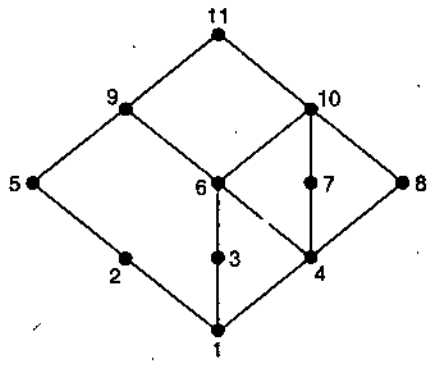


Fig. 14

Solution. Examining all upward paths from vertices 6, 7 and 10, we find that *lub* (B) = 10. Similarly, by considering all downward paths from 6, 7 and 10, we find that *glb* (B) = 4.

The following theorem follows immediately from the principle of correspondence.

Theorem 4. *Suppose that (A, ≤) and (A′, ≤′) are isomorphic posets under the isomorphism* $f : A \to A'$.

(*i*) If a is a maximal (minimal) element of (A, ≤), then $f(a)$ is a maximal (minimal) element of (A′, ≤′).

(*ii*) If a is a greatest (least) element of (A, ≤), then $f(a)$ is a greatest (least) element of (*A*′, ≤′).

(*iii*) If a is an upper bound (lower bound, least upper bound, greatest lower bound) of a subset B, then $f(a)$ is an upper bound (lower bound, least upper bound, greatest lower bound) for the subset f (B) A′.

(*iv*) If every subset of (A, ≤) has a *l.u.b* (*g.l.b*), then every subset of (A′, ≤′) has a *l.u.b* (*g.l.b*).

Example 4. *Show that the posets (A, ≤) and (A′, ≤′), whose Hasse diagrams are shown in the following figures (a) and (b), respectively, are not isomorphic.*

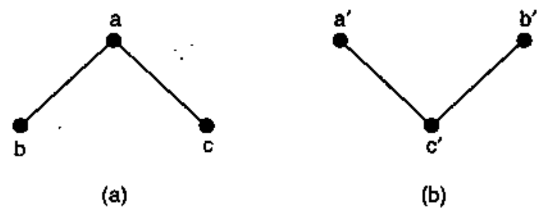


Fig. 15

Solution. The two posets are not isomorphic because (A, ≤) has a greatest element a, while (A′, ≤′) does not have a greatest element. We could also say that they are not isomorphic because (A, ≤) does not have a least element, while (A′, ≤′) does have a least element.

6. LATTICES

In this chapter, we introduce a more general algebraic system called a lattice. The difference between algebraic system presented earlier and in this section is the fact that the ordering relation plays an important role in this system. It will not be out of place to mention that Lattices have important applications in the theory and design of computers. We aim to first introduce lattice on a partially ordered set and then as an algebraic system.

Definition. A lattice is a partially ordered set $< L, \leq >$ in which any two elements $a, b \in L$ have a greatest lower bound (*g.l.b*) and a least upper bound (*l.u.b*), the *g.l.b.* of a and b being an element c such that $c \leq a, c \leq b$ and there is no d for which $c < d \leq a$ and $d \leq b$. The *l.u.b.* may be defined analogously.

The *g.l.b.* of a subset $\{a, b\} \subseteq L$ is denoted by $a \wedge b$ and the *l.u.b.* by $a \vee b$. $a \wedge b$ is said to be the meet or product of a and b and $a \vee b$ is called the join or sum of a and b. Other symbols such as* and ⊕ or . and + are also used to indicate the meet and join of two elements respectively. When we use . for meet, it is customary to write ab instead of $a.b$.

From the definition of a lattice it follows that both ∧ and ∨ are binary operations on L because of the uniqueness of the l.u.b. and g.l.b. of any subset of a partially ordered set. A totally ordered set is trivially a lattice, but not all partially ordered sets are lattices. Below we give some examples of lattices.

ILLUSTRATIVE EXAMPLES

Example 1. Let I^+ be the set of all positive integers and let D denote the relation of "division" in I^+ s.t. for all $a, b \in I^+$, a D b iff a divides b. Then $< I^+, D >$ is a lattice in which the join of a and b is LCM (Least common multiple) of a and b *i.e.*, $a \vee b$ = LCM of a and b and the meet of a and b is the greatest common divisor (GCD) of a and b *i.e.*, $a \wedge b$ = GCD of a and b.

Example 2. Let S be any set and p (S) be its power set. The partially ordered set $< p(S), \subseteq >$ is a lattice :

Here the meet and join are operations ∧ and ∨ respectively. In particular, when S is singleton, the corresponding lattice is a chain containing two elements when S has two and three elements, the diagrams are as given below, diagram (*b*), (*e*) respectively.

Example 3. Let n be a positive integer and S_n be the set of all divisor of n ; for example $n = 6$, $S_6 = \{1, 2, 3, 6\}$ and for $n = 24$, $S_{24} = \{1, 2, 3, 4, 6, 8, 12, 24\}$ etc. Let D denote the relation "division" in I^+ (set of positive integers). Then partially ordered sets $< S_6, D >$, $< S_{24}, D >$, $< S_8, D >$ and $< S_{30}, D >$ are lattices. [see diagrams (*b*), (*f*), (*a*), (*e*)]. Below we give Hasse diagrams as referred above.

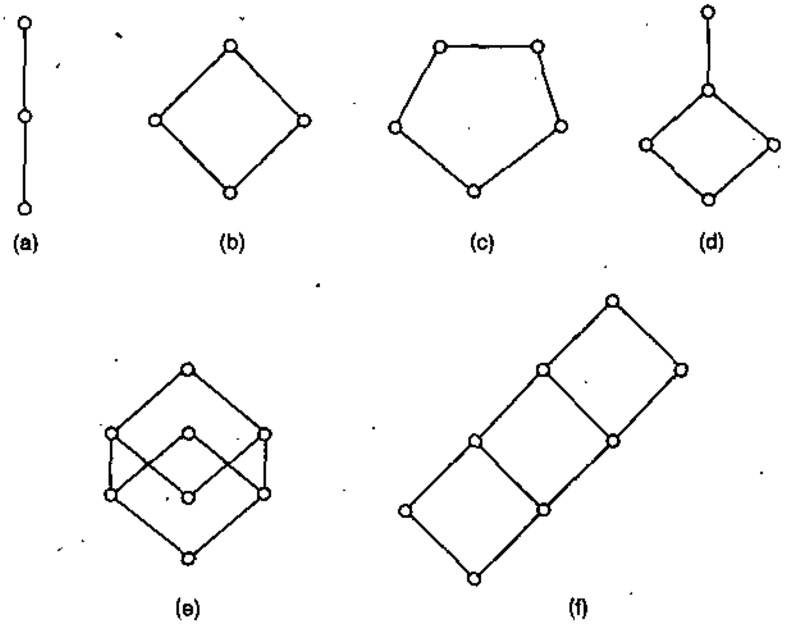


Fig. 16. *Lattices*

Example 4. Let S be a non-empty set and II (s) be the set of all partitions of S. If we define a corresponding ordering relation $\leq$ on II (s) s.t. for $II_1, II_2 \in II(S)$, $II_1 < II_2$ iff every block of II_1 is a subset of some block of II_2 then $< II(S), \leq >$ is a *lattice in which the operations* $\wedge$ and $\vee$ are required meet and join operations respectively.

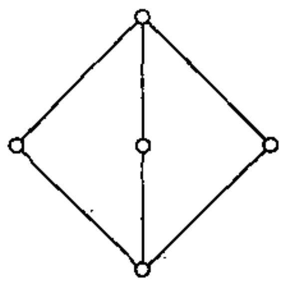


Fig. 17.

To be more specific, let

$$S = \{a, b, c\} \text{ then}$$

$$II(s) = \{II_1, II_2, II_3, II_4, II_5\}$$

where $\quad II_1 = \{a, b, c\}$, $II_2 = \{a, b, c\}$, $II_3 = \{a, c, b\}$,

$$II_4 = \{a, b, c\}, II_5 = \{a, b, c\}$$

The diagram of $< II(s), \leq >$ is given on right hand side.

Remark 1. Different lattices can be represented by the same diagram except that the nodes have different labels.

Remark 2. For any partial ordering relation $\leq$ on set S, the converse relation $>$ is also a partial ordering relation on S. Partially ordered sets $< S, \leq >$ and $< S, \geq >$ are called duals of each other.

Definition 1. A lattice is complete if every subset has *g.l.b.* and *l.u.b.*

Definition 2. A lattice is modular if it has the property that

$$x \geq z \quad \Rightarrow \quad x \wedge (y \vee z) = (x \wedge y) \vee z \ \forall\, y.$$

Definition 3. The lattices $< L, \leq >$, and $< L, \geq >$ are called duals of each other.

Properties of Lattices. Let $\wedge$ and $\vee$ be two binary operations of meet and join on a lattice $< L, \leq >$. For any $a, b, c \in L$, we have

(L – 1) $a \wedge a = a$ (L – 1)′ $a \vee a = a$ (Idempotent law)

(L – 2) $a \wedge b = b \wedge a$ (L – 2)′ $a \vee b = b \vee a$ (Commutative law)

(L – 3) $(a \wedge b) \wedge c = a \wedge (b \wedge c)$ (L – 3)′ $(a \vee b) \vee c = a \vee (b \vee c)$ (Associative law)

(L – 4) $a \wedge (a \vee b) = a$ (L – 4)′ $a \vee (a \wedge b) = a$ (Absorption law)

NOTES

The identities (L – 1) to (L – 4) can be easily proved by using the definition of the operators $\wedge$ and $\vee$. The identities (L – 1)′ to (L – 4)′ follow from the principle of duality. We can prove them directly by the definition of operators too. We present below the proof of two identities as illustrations for the reader.

Proof of (L – 1). For any $a \in L$, $a \wedge a \leq a$ by the definition of $\wedge$.

Also $a \leq a \wedge a$, therefore $a \wedge a = a$.

Proof of (L – 4). For any $a \in L$, $a \leq a \vee b$ by definition of $\vee$.

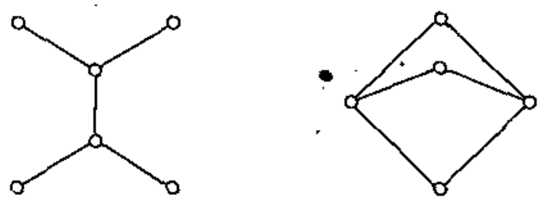
Hence $$a \leq a \wedge (a \vee b)$$

Also $a \wedge (a \vee b) \leq a$ by definition of $\wedge$.

$$\therefore \quad a \wedge (a \vee b) = a.$$

EXERCISE 3 (B)

1. Explain when the partially ordered sets given in the following figure are not lattices.



2. Show that the operations of meet and join in a lattice are commutative and associative.
3. Let $S = \{a, b, c\}$. Draw the diagram of $< \rho(S), \subseteq >$.
4. Let R be the set of real numbers in [0, 1] and ≤ be the usual operation of "less than or equal to" on R. Show that < R, ≤ > is a lattice. What are the operations of meet and join on this lattice ?

7. SOME THEOREMS

Theorem 1. *Let < L, ≤ > be a lattice in which $\wedge$ and $\vee$ denote the operations of meet and join respectively. Prove that for any a, b, ∈ L*

$$a \leq b \iff a \wedge b = a \iff a \vee b = b.$$

Proof. Let $a \leq b$. Also we know that $a \leq a$. Therefore $a \leq a \wedge b$. From the definition of $a \wedge b$, we have $a \wedge b \leq a$. Hence $a \wedge b = a$

$$\therefore \quad a \leq b \Rightarrow a \wedge b = a \quad \text{...(1)}$$

Again suppose $a \wedge b = a$; but it is only possible when

$$a \leq b \text{ i.e., } a \wedge b = a \Rightarrow a \leq b \quad \text{...(2)}$$

∴ from (1) and (2), we have

$$a \leq b \iff a \wedge b = a$$

Next $$a \wedge b = a \Rightarrow b \vee (a \wedge b) = b \vee a = a \vee b \quad \text{...(3)}$$

But $$b \vee (a \wedge b) = b \quad \text{...(4)}$$

∴ from (3) and (4), $a \vee b = b$

Similarly we can show that $a \wedge b = a$. It follows from

$$a \vee b = b$$

$\therefore \qquad a \wedge b = a \;\Rightarrow\; a \vee b = b$ and $a \vee b = b \;\Rightarrow\; a \wedge b = a$

so that finally, $\qquad a \le b \;\Leftrightarrow\; a \wedge b = a \;\Leftrightarrow\; a \vee b = b.$

Remark. The theorem provides a connection between the partial ordering relation $\le$ and the two binary operations $\wedge$ and $\vee$ on the meet and join in a lattice $< L, \le >$.

Theorem 2. *Let $< L, \le >$ be a lattice. For any a, b, c, $\in$ L, the following properties called* ***isotonicity*** *hold*

$$b \le c \;\Rightarrow\; \begin{cases} a \wedge b \le a \wedge c \\ a \vee b \le a \vee c \end{cases}$$

Proof. From the theorem 1, we know

$$b \le c \;\Leftrightarrow\; b \wedge c = b \qquad ...(1)$$

To show that $a \wedge b \le b \wedge c$, we will prove

$$(a \wedge b) \wedge (a \wedge c) = a \wedge b$$

Now $\quad (a \wedge b) \wedge (a \wedge c) = (a \wedge a) \wedge (b \wedge c) = a \wedge (b \wedge c) = a \wedge b \qquad ...(2)$

Therefore, in view of (1), we have

$$(a \wedge b) \wedge (a \wedge c) = a \wedge b \;\Rightarrow\; a \wedge b \le a \wedge c$$

Further, $(a \vee b) \vee (a \vee c) = (a \vee a) \vee (b \vee c)$

$$= a \vee (b \vee c) = a \vee b \qquad \text{(Using theorem 1)}$$

$\therefore \quad (a \vee b) \vee (a \vee c) = a \vee b \Rightarrow a \vee b \le a \vee c.$

Note. Below we enlist some multiplications which hold for any $a, b, c \in$ L where $< L, \le >$ is a lattice. These are direct outcomes of the definitions of operators $\wedge$ and $\vee$ on L.

Alternatively, the same can be proved by using the properties of isotonicity.

$$a \le b \vee a \le c \;\Rightarrow\; a \le b \vee c \qquad ...(1)$$

$$a \le b \wedge a \le c \;\Rightarrow\; a \le b \wedge c \qquad ...(2)$$

As duals of (1) and (2), following are also true

$$a \ge b \wedge a \ge c \;\Rightarrow\; a \ge b \wedge c \qquad ...(3)$$

$$a \ge b \vee a \ge c \;\Rightarrow\; a \ge b \vee c \qquad ...(4)$$

Theorem 3. *Let $< L, \le >$ be a lattice. For any a, b, c $\in$ L, the following inequalities, called distributive inequalities, hold* :

$$a \vee (b \wedge c) \le (a \vee b) \wedge (a \vee c)$$

$$a \wedge (b \vee c) \ge (a \wedge b) \vee (a \wedge c).$$

Proof. $a \le a \vee b$ and $a \le a \vee c$, implies

$$a \le (a \vee b) \wedge (a \vee c) \qquad ...(5)$$

Also $\qquad b \wedge c \le b \le a \vee b$

and $\qquad b \wedge c \le c \le a \vee c \qquad$ [On using (2)]

$\therefore \qquad b \wedge c \le (a \vee b) \wedge (a \vee c) \qquad$ [On using (5)] $\quad ...(6)$

From (5) and (6) using (4), we have

$$a \vee (b \wedge c) \le (a \vee b) \wedge (a \vee c)$$

Similarly the other distributive inequally can be proved or by using the principle of duality we have $a \wedge (b \vee c) \ge (a \wedge b) \vee (a \wedge c)$.

Theorem 4. *Let $< L, \le >$ be a lattice. For any a, b, c $\in$ L, the following holds :*

$$a \le c \;\Leftrightarrow\; a \vee (b \wedge c) \le (a \vee b) \wedge c \qquad ...(7)$$

NOTES

Proof. From theorem 1, $a \le c \Leftrightarrow a \vee c = c$. Hence replacing $a \vee c$ by c in the first distributive inequality, we get

$$a \vee (b \wedge c) \le (a \vee b) \wedge c$$

Therefore, $a \le c \quad \Leftrightarrow \quad a \vee (b \wedge c) < (a \vee b) \wedge c.$

Remarks. This inequality is called the modular inequality. The modular inequalities are also expressed as follows :

$$(a \wedge b) \vee (a \wedge c) \le a \wedge [b \vee (a \wedge c)] \quad \text{...(8)}$$

$$(a \vee b) \wedge (a \vee c) \le a \vee [b \wedge (a \vee c)] \quad \text{...(9)}$$

EXERCISE 3 (C)

1. Show that in a lattice if $a \le b$ and $c \le d$, then $a \wedge c \le b \wedge d$.
2. Show that in a lattice if $a \le b \le c$, then $a \vee b = b \wedge c$ and $(a \wedge b) \vee (a \wedge c) = b = (a \vee b) \wedge (a \vee c)$.
3. Prove that every finite subset of a lattice has a LUB and GLB. What can be said about a finite lattice ?
4. Show that the identities (L – 1) and (L –1)′ follow from the identities (L – 2) to (L – 4) and their duals.
5. In a lattice show that
   (*i*) $(a \wedge b) \vee (c \vee d) \le (a \vee c) \wedge (b \vee d)$
   (*ii*) $(a \wedge b) \vee (b \wedge c) \vee (c \wedge a) \le (a \vee b) \wedge (b \vee c) \wedge (c \vee a)$.

8. LATTICES AS ALGEBRAIC SYSTEMS

Now we define a lattice as an algebraic system wherein it is possible to define a partial ordering relation. The advantage in doing so is that many concepts which are associated with algebraic systems can be applied to lattices also.

Definition. A lattice is an algebraic system $< L, \wedge, \vee >$ with two binary operations $\wedge$ and $\vee$ on L which are both (1) commutative and *(2) associative and (3) satisfy the* absorption laws.

Alternatively we say that the operations $\wedge$ and $\vee$ on L, satisfy the identities (L –2) to (L – 4) and (L – 2)′ to (L – 4)′.

Remark 1. The absence of the identities (L – 1) and (L – 1)′ in the definition is due to the fact that (L – 1) and its dual imply the identities (L – 1) and (L – 1)′ . This can be shown as given below :

Let $a \in L$, then

$$a \wedge a = a \wedge [a \vee (a \wedge a)] = a$$

where replacing the second a in $a \wedge a$ by $a \vee (a \wedge a)$ and using (L – 4)′ we get a in the second step. The identity $a \vee a = a$ can be proved in a similar manner or by the principle of duality.

Remark 2. Definition given above does not assume the existence of any partial ordering on L. We now show that a partial ordering relation on L follows as a result of the properties of the operations $\wedge$ and $\vee$.

9. EXISTENCE OF PARTIAL ORDERING RELATION ON L

Let there be a relation R on L such that for

$$a, b \in L, a \, R \, b \Leftrightarrow a \wedge b = a$$

Evidently, for any $a \in L$, $a \wedge a = a$, hence $a \mathrel{R} a$, *i.e.*, the relation R is reflexive.

Again for some $a, b \in L$, let us assume that $a \mathrel{R} b$ and $b \mathrel{R} a$, so that

$$a \wedge b = a \quad \text{and} \quad b \wedge a = b$$

But we know that $a \wedge b = b \wedge a$ (Commutative law)

$$\therefore \quad a \wedge b = a,\ b \wedge a = b \Rightarrow a = b$$

$\therefore$ $a \mathrel{R} b$ and $b \mathrel{R} a \Rightarrow a = b$ *i.e.*, the relation is anti symmetric. Finally, let us assume that for some $a, b, c \in L$,

$$a \mathrel{R} b \quad \text{and} \quad b \mathrel{R} c \quad \textit{i.e.,} \quad a \wedge b = a,\ b \wedge c = b$$

Thus $$a \wedge c = (a \wedge b) \wedge c = a \wedge (b \wedge c) = a \wedge b = a \text{ or } a \mathrel{R} c$$

Thus $a \mathrel{R} b$ and $b \mathrel{R} c \Rightarrow a \mathrel{R} c$ *i.e.*, the relation is transitive. Thus R is a partial ordering. It can be easily shown that $a \wedge b = a \Leftrightarrow a \vee b = b$.

Hence the same partial ordering relation R on L could have been defined as $a \mathrel{R} b \Leftrightarrow a \vee b = b$ for any $a, b \in L$.

Now we show that for any two elements $a, b \in L$, the greast lower bound and the least upper bound of $\{a, b\} \subseteq L$ with respect to the partial ordering R are $a \wedge b$ and $a \vee b$ respectively.

Using the absorption laws $a \wedge (a \vee b) = a$ and $b \wedge (a \vee b) = b$, we get $a \mathrel{R} (a \vee b)$ and $b \mathrel{R} (a \vee b)$. We now assume that there exists an element $c \in L$ such that $a \mathrel{R} c$ and $b \mathrel{R} c$. This implies that

$$a \vee c = c \quad \text{and} \quad b \vee c = c$$

or $$(a \vee c) \vee (b \vee c) = (a \vee b) \vee c = c \vee c = c.$$

meaning there by that $(a \vee b) \mathrel{R} c$. Finally $a \vee b$ is the least upper bound of a and b.

Similarly it can be shown that $a \wedge b$ is the greatest lower bound of $\{a, b\}$ with respect to the partial ordering relation R. Summarizing by on a lattice $< L, \wedge, \vee >$ it is possible to define a partial ordering relation R such that for any $a, b \in L$.

$$a \mathrel{R} b \quad \Leftrightarrow \quad a \wedge b = a \quad \Leftrightarrow \quad a \vee b = b$$

and that $l.u.b\ \{a, b\} = a \vee b$ and $g.l.b\ \{a, b\} = a \wedge b$ with respect to the relation R on L.

Also it has already been shown that in a lattice $< L, \wedge, \vee >$ defined as a partially ordered set, it is possible to define two binary operations $\wedge$ and $\vee$ such that for any a, $b \in L$.

$$a \wedge b = g.l.b\ \{a, b\} \quad \text{and} \quad a \vee b = l.u.b\ \{a, b\}$$

and $$a \leq b \Leftrightarrow a \wedge b = a \quad \Leftrightarrow \quad a \vee b = b,$$

where the operations $\wedge$ and $\vee$ are both commutative and associative and satisfy the absorption laws. Then the equivalence of the two definitions is established where R plays the same role as $\leq$ on L.

10. SUBLATTICES, DIRECT PRODUCT AND HOMOMORPHISM

By defining a lattice as an algebraic system, we are in the position to introduce the concept of sublattices in a natural way.

Sublattice. Let $< L, \wedge, \vee >$ be a lattice and let $S \subseteq L$ be a subset of L. The algebra $< S, \wedge, \vee >$ is a sublattice of $< L, \wedge, \vee >$ iff S is closed under both operations $\wedge$ and $\vee$.

Note. 1. Evidently definition given above shows that a sublattice itself is a lattice. But any subset of L, which is a lattice need not be a sublattice.

NOTES

2. Every set of a partially ordered set is also a partially ordered set under the same partial ordering relationship. Thus if $< P, \wedge, \vee >$ is a partially ordered set and $Q \subseteq P$, the $< Q, \wedge, \vee >$ is also partially ordered set.

3. For a lattice $< L, \wedge, \vee >$ and for any two elements $a, b \in L$ s. t. $a \leq b$ the closed interval $[a, b]$ consisting of all elements $x \in L$ such that $a \leq x \leq b$ is a sublattice of L.

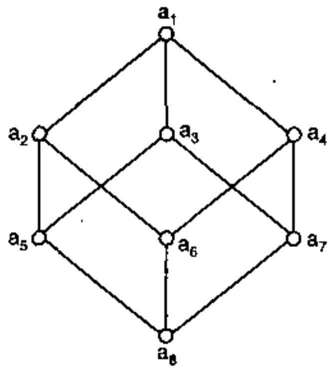
ILLUSTRATIVE EXAMPLES

Example 1. *Let $< L, \wedge, \vee >$ be a lattice in which $L = \{a_1, a_2, ..., a_n\}$ and set S_1, S_2 and S_3 be the subsets of L given by*

$S_1 = \{a_1, a_2, a_4, a_6\}$, $S_2 = \{a_3, a_5, a_7, a_8\}$ and $S_3 = \{a_1, a_2, a_4, a_8\}$. Which one of $< S_1, \wedge, \vee >$, $< S_2, \wedge, \vee >$ and $< S_3, \wedge, \vee >$ is a lattice ?

Solution. The diagram of $< L, \wedge, \vee >$ is given below.

Clearly $< S_1, \leq >$ and $< S_2, \leq >$ are sublattices of $< L, \leq >$ but $< S_3, \leq >$ is not sublattice because for $a_2, a_4 \in S_3$, $a_2 \wedge a_4 = a_6 \notin S_3$. At the same time $< L, \leq >$ is a lattice.



Example 2. *Let n be a positive integer and S_n be the set of all divisors of n. Let I^+ be the set of all positive integers and let D denote the relation of "division" in I^+. Prove that the lattice*

$< S_n, D >$ is a sub lattice of $< I^+, D >$.

Solution. Clearly $S_n \subseteq I^+$. Also every pair of elements $a, b \in S_n$ has a greatest lower bound and a least upper bound. Hence $< S_n, D>$ is a sublattice of $< I^+, D >$.

Direct Product. Let $< L, \wedge, \vee >$ and $< S, *, \oplus >$ be two lattices. The algebraic system $< L \times S, ., + >$ in which the binary operations + and . on $L \times S$ are such that for any $< a_1, b_1 >$ and $< a_2, b_2 >$ in $L \times S$,

$$< a_1, b_1 > \odot < a_2, b_2 > = < a_1 \wedge a_2, b_1 * b_2 >$$
$$< a_1, b_1 > + < a_2, b_2 > = < a_1 \vee a_2, b_1 \oplus b_2 >$$

is called the direct product of the lattices $< L, \wedge, \vee >$ and $< S, *, \oplus >$.

Remark. The operation + and . on $L \times S$ are commutative and associative and satisfy the absorption laws because they are defined in terms of the operations $\wedge, \vee, *$ and $\oplus$. Therefore, the direct product is itself a lattice. As $< L \times S, ., + >$ is a lattice, one can form a direct product of this lattice with another lattice and so on.

Note. The order of the lattice formed by the direct product of two lattices is equal to the product of the orders of the lattices appearing in the direct product. Further it is worth mentioning that all lattices can be expressed as a direct product of other lattices.

Lattice Homomorphism. Let $< L, \wedge, \vee >$ and $< S, *, \oplus >$ be two lattices. A mapping $f : L \rightarrow S$ is called a lattice homomorphism from the lattice

$< L, \wedge, \vee >$ to $< S, *, \oplus >$ if for any $a, b \in L$

$$f(a \wedge b) = f(a) * f(b) \quad \text{and} \quad f(a \vee b) = f(a) \oplus f(b).$$

Remark. Here both the operations of meet and join are preserved. There may be mappings which preserve only one of the two operations and such mappings are not lattice homomorphisms.

NOTES

Let $< L, \wedge, \vee >$ and $< S, *, \oplus >$ be two lattices and the partial ordering relations on L and S corresponding to the operations of meet and join be $\leq$ and $\leq'$, respectively. If $f : L \to S$ is a homomorphism, then f preserves the ordering relations also *i.e.*, for any $a, b \in L$ such that

$$a \leq b, \qquad f(a) \leq' f(b).$$

From $\qquad a \leq b \iff a \wedge b = a,$

we have $\qquad f(a \wedge b) = f(a) * f(b) = f(a) \Leftrightarrow f(a) \leq' f(b)$

Hence $a \leq b \Rightarrow f(a) \leq' f(b)$ if f is homomorphism.

If a homomorphism $f : L \to S$ of two lattices $< L, \wedge, \vee >$ and $< S, *, \oplus >$ is one-one onto then f is called isomorphism. If there exists an isomorphism between two lattices, then the lattices are said to be isomorphic.

If the lattices $< L, \wedge, \vee >$ and $< S, *, \oplus >$ are isomorphic and f denotes an isomorphism, then f preserve the ordering relation *i.e.*, for any $a, b \in L$,

$$a \leq b \Rightarrow f(a) \leq' f(b).$$

Also $f(a) \leq' f(b) \Rightarrow a \leq b$. Hence the two lattices which are isomorphic, can be presented by the same diagram in which the nodes are replaced by the images.

ILLUSTRATIVE EXAMPLES

Example 1. Let L = {0, 1} and the lattice $< L, \leq >$ is as shown below :

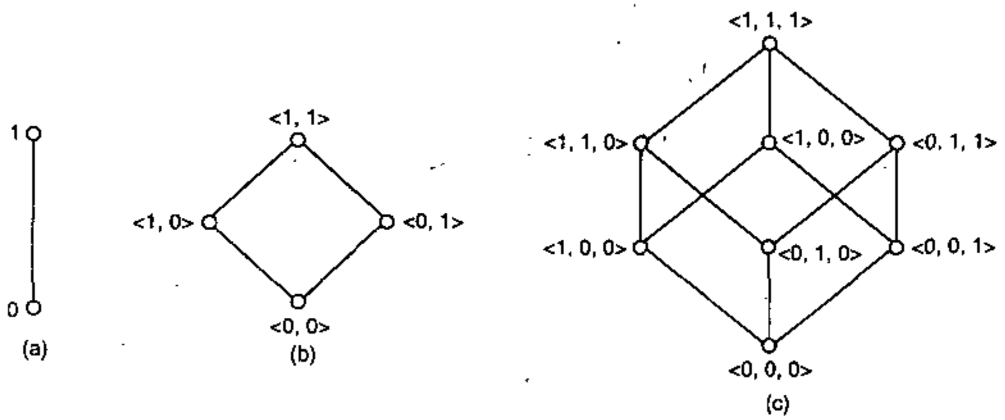


Fig. 18

The lattices $< L^2, \leq_2 >$, $< L^3, \leq_3 >$ are shown in figure given above.

In general, the diagram showing $< L^n, \leq_n >$ can be an n cube.

Example 2. Let S be any set containing n elements and ρ (S) be its power set. The lattice $< \rho(S), \cap, \cup >$ or $< \rho(S), \subseteq >$ is isomorphic to the lattice $< L^n, \leq n >$.

Remark. A homomorphism $f : L \to L$ where $< L, \leq >$ is a lattice is called an *endomorphism*. If $f : L \to L$ is an isomorphism, then the image set of f is a sublattice of L.

Though the concepts of homomorphism and isomorphism are associated with any algebraic system, following definitions exhibit as to how these concepts can be applied to partially ordered sets also.

Definition 1. Let < P, ≤ > and < Q, ≤′ > be two partially ordered sets. A mapping $f : P \to Q$ is said to be order-preserving relative to the ordering ≤ in P and ≤′ in Q iff for any $a, b \in P$ such that $a \le b$, $f(a) \le' f(b)$ in Q.

If < P, ≤ > and < Q, ≤′ > are lattices and $g : P \to Q$ is a lattice homomorphism, then g is order preserving.

Definition 2. Two partially ordered sets < P, ≤ > and < Q, ≤′ > are called order isomorphic if there exists a mapping $f : P \to Q$ which is bijective and if both f and f^{-1} are order-preserving.

Note. It is just possible that a mapping $f : P \to Q$ is bijective and order-preserving, but that f^{-1} is not order-preserving. In such a situation P and Q are not order-isomorphic.

For lattices < L, ≤ > and < S, ≤′ >, an order isomorphic is equivalent to lattice isomorphism. Hence lattices which are order-isomorphic as partially ordered sets are isomorphic. The significance of order-isomorphism is that two partially ordered sets which are order-isomorphic can be represented by the same diagram.

Example. Consider the lattice < S_n, D > for $n = 12$ *i.e.*, the lattice of divisors of 12. Consider another lattice < S_n, ≤ > in which ≤ denotes the ordering relation "less than or equal to". A mapping $f : S_n \to S_n$ given by $f(x) = x$ is order-preserving and bijective, but f^{-1} is not order preserving. Hence < S_n, D > and < S_n, ≤ > are neither order-isomorphic nor isomorphic.

Some Definitions

Definition 1. A lattice is called complete if each of its non-empty subsets has a least upper bound and a greatest lower bound.

Definition 2. In a bounded lattice < L, ∧, ∨, 0, 1 >, an element $b \in L$ is called a complement of an element $a \in L$ if $a \wedge b = 0$ and $a \vee b = 1$.

Definition 3. A lattice < L, ∧, ∨, 0, 1 > is said to be complemental lattice if every element of L has at least one complement.

Definition 4. A lattice < L, ∧, ∨ > is called a distributive lattice if for any $a, b, c \in L$,

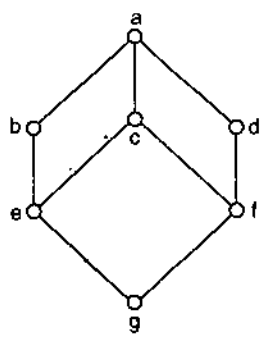
$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \qquad \ldots(1)$$

and

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) \qquad \ldots(2)$$

EXERCISE 3 (D)

1. Show that the following diagram is a lattice and is not a sublattice of diagram 9.15 (*e*).



2. Show that every interval of a lattice is sublattice.
3. Find all sublattices of the lattice < S_n, D > for $n = 12$.
4. Show that the lattice < S_n, D > for $n = 216$ is isomorphic to the direct product of lattices for $n = 8$ and $n = 27$.

NOTES

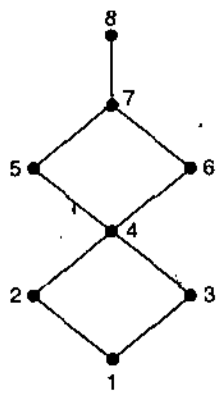
SUMMARY

1. A relation R on a set A is called a **partial order** if R is reflexive, antisymmetric and transitive. The set A together with the partial order R is called a partially ordered set or simply **poset.** This algebraic structure is denoted by (A, R) or (A, ≤).
2. The partial order ≤ defined on the Cartesian product A × B as above is called *product partial order.*
3. If (A, ≤) is a poset, we say that $a < b$ if $a \le b$, but $a \ne b$. Suppose now that (A, ≤) and (B, ≤) are posets. In theorem given above we have defined the product partial order on A × B. Another useful partial order on A × B, denoted by <, is defined as follows :
   $(a, b) < (a', b')$ if $a < a'$ or if $a = a'$ and $b \le b'$.
   This ordering is called **lexicographic**, or "*dictionary*" order.
4. A lattice *is a partially ordered set* < L, ≤ > *in which any two elements* $a, b \in$ L have a greatest lower bound (*g.l.b*) and a least upper bound (*l.u.b*), the *g.l.b.* of a and b being an element c such that $c \le a$, $c \le b$ and there is no d for which $c < d \le a$ and $d \le b$. The *l.u.b.* *may be defined analogously.*
   The *g.l.b.* of a subset $\{a, b\} \subseteq$ L is denoted by $a \wedge b$ and the *l.u.b.* by $a \vee b$. $a \wedge b$ is said to be the meet or product of a and b and $a \vee b$ is called the join or sum of a and b. Other *symbols such as** *and* ⊕ *or* . *and* + *are also* used to indicate the meet and join of two elements respectively. When we use . for meet, it is customary to write ab instead of $a.b$.
5. If < L, ≤ > is a lattice in which ∧ and ∨ denote the operations of meet and join respectively, *then for any* $a, b, \in$ L
   $$a \le b \iff a \wedge b = a \iff a \vee b = b.$$
6. If < L, ≤ > is a lattice, for any a, b, c, ∈ L, the following properties called **isotonicity** hold
   $$b \le c \Rightarrow \begin{cases} a \wedge b \le a \wedge c \\ a \vee b \le a \vee c \end{cases}$$
7. If < L, ≤ > is a lattice, for any $a, b, c \in$ L, the following inequalities, called distributive inequalities, hold :
   $$a \vee (b \wedge c) \le (a \vee b) \wedge (a \vee c)$$
   $$a \wedge (b \vee c) \ge (a \wedge b) \vee (a \wedge c)$$
8. If < L, ≤ > is a lattice, for any $a, b, c \in$ L, the following holds ;
   $$a \le c \iff a \vee (b \wedge c) \le (a \vee b) \wedge c \qquad ...(7)$$
9. Let < P, ≤ > and < Q, ≤′ > be two partially ordered sets. A mapping f : P → Q is said to be order-preserving relative to the ordering ≤ in P and ≤′ in Q iff for any $a, b \in$ P such that $a \le b$, $f(a) \le' f(b)$ in Q.
   If < P, ≤ > and < Q, ≤′ > *are lattices and* g : P → Q is a lattice homomorphism, then g is order preserving.
10. Two partially ordered sets < P, ≤ > and < Q, ≤′ > are called order isomorphic if there exists a mapping f : P → Q which is bijective and if both f and f^{-1} are order-preserving.

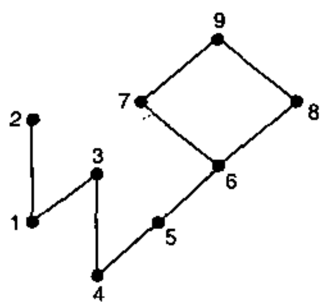
TEST YOURSELF

In the following questions, draw the Hasse diagram of a topological sorting of the given poset.

1.



2.



3. Show that if R is a linear order on the set A, then R^{-1} is also a linear order on A.
4. If (A, ≤) is a poset and A′ is a subset of A, show that (A′, ≤′) is also a poset, where ≤′ is the relation of ≤ to A′.
5. A relation R on a set a is called quasi-order if it is transitive and reflexive. Let A = P(S) be the power set of a set S and consider the following relation R on A : URT if and only if $U \subsetneq T$ (proper containment). Show that R is a quasi-order.
6. Let A = {x | x is a real number and $-5 \le x \le 20$}. Show that the usual relation < is a quasi order on A.
7. If R is quasi-order on A show that R^{-1} is also a quasi-order.
8. Let B = {2, 3, 6, 9, 12, 18, 24} and let A = B × B. Define the following relation on A : $(a, b) < (a', b')$ if and only if $a \mid a'$ and $b \le b'$, where ≤ is the usual partial order. Show that < is the partial order.
9. Let A = {1, 2, 4, 8} and let ≤ be the partial order of divisibility on A. Let A′ = {0, 1, 2, 3} and let ≤′ be the usual relation "less than or equal to" on integers. Show that (A, ≤) and (A′, ≤′) are isomorphic posets.
10. Let A = {1, 2, 3, 5, 6, 10, 15, 30} and consider the partial order ≤ of divisibility on A. That is, define $a \le b$ to mean that $a \mid b$. Let A′ = P(S), where S = {e, f, g}, be the poset with partial order ⊆. Show that (A, ≤) and (A, ⊆) are isomorphic.
11. Show that there are 15 partitions of a set of four elements.
12. Draw the diagram of a lattice and draw a diagram of another lattice of which the previous one is not sublattice.

Answer

2.

NOTES

CHAPTER

4

BOOLEAN ALGEBRA AND BOOLEAN FUNCTION WITH THEIR APPLICATIONS

LEARNING OBJECTIVES

- Introduction
- Boolean Algebra
- Properties of Boolean Algebra
- Principle of Duality in Boolean Algebra
- Algebra of Classes
- Subalgebras
- Isomorphism
- Partial Orders
- Some Definitions
- Application of Boolean Algebra
- Application to Switching Circuits
- Boolean Switching Circuit
- Conversion of Operations
- Equivalence of Two Circuits
- Simplification of Circuits
- Boolean Polynomial
- Boolean Expression and Function
- Definition
- Fundamental Forms of A Boolean Function
- Some Definitions and Theorems
- Bool's Theorem
- Normal Forms
- Disjunctive Normal Form or Canonical Form
- Complete Disjunctive Normal Form
- Complement Function of a Boolean Function
- Some Theorems
- Conjunctive Normal Form
- Boolean Matrix Operations

NOTES

1. INTRODUCTION

Boolean algebra was first introduced by British mathematician George Boole. It is the generalization of the algebraic structure in the set theory and in symbolic logic, which are similar in nature. Boolean algebra provides the arithmetic operations in binary numbers 1 and 0. The computer science as a whole, is based on these two numbers.

There are three fundamental laws which govern the structure of Boole's System.

1. Concept of the operation of 'election' and of elective symbols.
2. The laws of thought expressible as rules of operation upon these symbols.
3. The fact that these rules are the same as would hold in the algebra of 0 and 1.

The operation of 'electing' x, y, z as symbols of classes in the universe can be treated as analogous to algebraic multiplication. The result of two operations x and y selected in succession represented by x, y will be class of things both x's and y's. Order does not matter obviously *i.e.*, $xy = yx$.

Also if $x = y$ then $zx = zy$; *i.e.*, **both z and x will be same as both z and y.**

Repetition of same operations yields to the same thing *i.e.*, the result is not altered :

$$x \cdot x = x \quad \text{or} \quad x^2 = x.$$

This is one fundamental law which is different from the usual law of algebra.

Boole uses '+' to indicate the operation if 'aggregation'. In other words it serves for "either ...... or". The classes which are either x's or y's but not both are represented by $x + y$. This operation obeys the law of commutativity *i.e.*,

$$x + y = y + x$$

The operation of multiplication (election) is associative (distributive) over aggregation (addition) *i.e.*,

$$z(x + y) = zx + zy.$$

The operation of 'exception' is denoted by 'subtraction'.

Thus if $\quad x$ = women, y = Indians

Then $\quad x - y$ = all women except Indians

= all women but not Indians.

Multiplication is associative with respect to subtraction *i.e.*,

$$z(x - y) = zx - zy$$

Boole accepts the general law of algebra namely $-y + x = x - y$.

In respect to it there is a convention that these two are equivalent in meaning. In this system '1' represents universe (or everything) while '0' denotes the null class (or nothing)

$1 \cdot x = x$ (selecting x's from universe gives the class x)

$0 \cdot x = 0$ (selecting x's from the null class gives null class)

The negative of any class x is $1 - x$ *i.e.*, it represents not x's or everything excepts x's.

Also $\quad x(1 - y) = x - xy.$

In other words what is x but not y is the something as x except what is both x and y.

The sum of any class and its negative constitutes everything *i.e.*,

$$x + (1 - x) = x + 1 - x = 1.$$

Thus everything is either x or not x.

The multiplication of a class and its negative gives zero (nothing) *i.e.*,

$$x \,.\, (1 - x) = x - x^2 = x - x = 0.$$

Thus nothing is both x and not x.

An important consequence from this principle is derived as

$$z = z \,.\, 1 = z \,.\, (x + 1 - x)$$
$$= zx + z(1 - x)$$

i.e., class z = either z and x or both z and not x. Here it is to be looked that this law permits any class x to be present in any expression even if it does not exist there originally.

In $x \,.\, x = x$, 1 = Universe and 0 = Null class, Boole establishes the analogy of the basic logical principle with those of his algebra.

Following three obvious difficulties are worth noting.

1. What does $x + x$ mean ?
2. What is the meaning of $1 + x$?
3. What is the concept if inverse with respect to multiplication ?

Boole overcomed these difficulties by restricting the elective symbol to binary numbers 0 and 1 only. The distinctive laws such as $x \,.\, x = x$ hold for 0 and 1 only and for no other classes.

However it is to be mentioned that in original form Boole's algebra is an entirely workable calculus. All those features which were not logically true were finally eliminated by his successors.

English Mathematician W.S. Jevens (1835–1882), B. Peirie (1809–1880) from U.S.A. and another English Mathematician A. De Morgan (1806–1871) developed Boolean algebra with five modifications.

(*i*) $a + b$ implies either a or b or both

(*ii*) $a + a = a$ ($\neq 2a$), (elimination of numerical coefficients).

(*iii*) De Morgan's Theorems of systematic connection between sum and product,

$$a + a = a \text{ and } a \,.\, a = a$$

(*iv*) Elimination of operations of subtraction and division.

(*v*) Introduction of 'inclusion' notation '$\subset$' namely $a \subset$ B, [a is contained in B].

The modification (*i*), (*ii*) and (*iv*) resulted in disappearance of all expressions not logically interpretable.

Thus is developed an abstract system B composed of any set with two binary operations (+) and (.) and one binary operation of complementation along with identity elements 0 and 1.

2. BOOLEAN ALGEBRA

A set B with two binary operations on B denoted by (+) and (.) along with a binary operation on B denoted by (′) and two specific elements 0 and 1 of B, is called Boolean Algebra if the following axioms are satisfied :

$\mathbf{B_1}$ $a + b = b + a$, for all $a, b \in B$
$\mathbf{B_2}$ $a \,.\, b = b \,.\, a$, for all $a, b \in B$ (Commutative laws).

$\mathbf{B_3}$ $a \,.\, (b + c) = (a \,.\, b) + (a \,.\, c)$, for $a, b, c \in B$
$\mathbf{B_4}$ $a + (b \,.\, c) = (a + b) \,.\, (a + c)$, for $a, b, c \in B$

Distributive laws *i.e.*, both the operations are distributive one on the other.

$\mathbf{B_5}$ $a + 0 = a \;\forall\; a \in B$

$\mathbf{B_6}$ $a \,.\, 1 = a \;\forall\; a \in B$

$\mathbf{B_7}$ There exists a' in B such that

$$a + a' = 1 \;\forall\; a \in B$$

$\mathbf{B_8}$ There exists a' in B such that

$$a \,.\, a' = 0 \;\forall\; a \in B.$$

Note. The element 0 is called the zero element, element 1 is called the unit element as usual. a' is the complement of a.

3. PROPERTIES OF BOOLEAN ALGEBRA

Let (B, +, .) be Boolean algebra with '′' as sign of complementation.

Theorem 1. *For $a \in B$, (i) $a + a = a$ and (ii) $a \,.\, a = a$.*

Proof. (*i*) We know from definition,

$$\begin{aligned} a &= a + 0 \\ &= a + (a \,.\, a') && (\because\ a \,.\, a' = 0) \\ &= (a + a) \,.\, (a + a') && (\because\ a + (b \,.\, c) = (a + b) \,.\, (a + c)) \\ &= (a + a) \,.\, 1 && (\because\ a + a' = 1) \\ &= a + a && (\because\ a \,.\, 1 = a) \end{aligned}$$

(*ii*) Also,

$$\begin{aligned} a &= a \,.\, 1 \\ &= a \,.\, (a + a') && (\because\ a + a' = 1) \\ &= (a \,.\, a) + (a \,.\, a') && \text{(Distributive law)} \\ &= (a \,.\, a) + 0 && (\because\ a \,.\, a' = 0) \\ &= a \,.\, a \end{aligned}$$

Theorem 2. *The complement of an element is unique.*

Proof. If complement of an element a is not unique let there be two complements a_1' and a_2' then we have,

$$a + a_1' = 1,\ a \,.\, a_1' = 0 \quad \text{and} \quad a + a_2' = 1,\ a \,.\, a_2' = 0$$

Now,

$$\begin{aligned} a_1' &= a_1' \,.\, 1 && (\text{By } B_6) \\ &= a_1' \,.\, (a + a_2') && \text{(By hypothesis)} \\ &= (a_1' \,.\, a) + (a_1' \,.\, a_2') && (\text{By } B_3) \\ &= (a \,.\, a_1') + (a_1' \,.\, a_2') && (\text{By } B_2) \\ &= 0 + (a_1' \,.\, a_2') && \text{(By hypothesis)} \\ &= a \,.\, a_2' + (a_1' \,.\, a_2') && \text{(By hypothesis)} \\ &= a_2' \,.\, a + a_2' \,.\, a_1' && (\text{By } B_2) \end{aligned}$$

$= a_2' . (a + a_1')$ (By B_3)
$= a_2' . 1$ (By hypothesis)
$= a_2'$.

Hence the result.

NOTES

Theorem 3. *If $a \in B$, then (i) $a + 1 = 1$, (ii) $a . 0 = 0$*

Proof. (*i*) $a + 1 = a + (a + a')$ (By B_7)
$= (a + a) + a'$ (Associativity)
$= a + a' = 1$ (By B_7)

(*ii*) $a . 0 = (a . 0) + 0$ (By B_5)
$= (a . 0) + (a . a')$ (By B_8)
$= a . (0 + a')$ (By B_3)
$= a . (a' + 0)$ (By B_1)
$= a . a' = 0$ (By B_5 and B_8)

Theorem 4. *If $a \in B$ then $(a')' = a$.* **(Involution law)**

Proof. $(a')' = (a')' + 0$ (By B_5)
$= 0 + (a')'$ (Commutativity)
$= (a . a') + (a')'$ (By B_8)
$= [a + (a')'] . [a' + (a')']$ (By B_4)

or $(a')' = [a + (a')'] . 0$ (By B_8)
$= 0 . [a + (a')']$
$= [a + a'] . [a + (a')']$ (By B_5)
$= a + [a' . (a')']$
$= a + 0 = a$.

4. PRINCIPLE OF DUALITY IN BOOLEAN ALGEBRA

The proposition obtained by putting + for '.', '.' For '+', 0 for 1, and 1 for 0 in a proposition concerning to a Boolean Algebra, is known as the dual of the proposition concerned. The converse of this is also true.

For example, the dual of the proposition

$a . (b + c) = (a . b) + (a . c)$ is

$a + (b . c) = (a + b) . (a + c)$.

Similarly, dual of the proposition

$a . a' = 0$ is $a + a' = 1$ and vice versa.

Theorem 5. *If $a, b \in B$, then*

(*i*) $a . (a + b) = a$, (*ii*) $a + (a . b)$ **(Absorption laws)**

Proof. (*i*) $a . (a + b) = (a + 0) . (a + b)$ (By B_5)
$= a + (0 . b)$ (By B_4)
$= a + (b . 0) = a + 0 = a$.

(*ii*) The dual of $a . (a + b) = a$ is

$a + (a . b) = a$.

Theorem 6. *If $a, b, c \in B$, then*

$b . a = c . a$ *and* $b . a' = c . a'$ $\Rightarrow$ $b = c$.

NOTES

Proof. $b \,.\, a = c \,.\, a$ and $b \,.\, a' = c \,.\, a'$

$\Rightarrow$ $b = b \,.\, 1 = b \,.\, (a + a') = b \,.\, a + b \,.\, a'$

$= c \,.\, a + c \,.\, a' = c \,.\, (a + a') = c \,.\, 1 = c.$

Theorem 7. *If $a, b, c \in B$, then*

(i) $a + (b + c) = (a + b) + c$

(ii) $a \,.\, (b \,.\, c) = (a \,.\, b) \,.\, c$ **(Associative laws)**

Proof. *(i)* We have shown in last theorem that

$$b \,.\, a = c \,.\, a \text{ and } b \,.\, a' = c \,.\, a' \Rightarrow b = c.$$

Therefore, if we write $a + (b + c)$ in place of b and $(a + b) + c$ in place of c then

$$[a + (b + c)] \,.\, a = [(a + b) + c] \,.\, a$$

and $[a + (b + c)] \,.\, a' = [(a + b) + c] \,.\, a'.$

$\Rightarrow$ $[a + (b + c)] = [(a + b) + c]$

Now it remains to prove the validity of the results on left of the implication sign. To do so, we have

$$[a + (b + c)] \,.\, a = a \,.\, [(a + b) + c]$$

$= a$ (By theorem 5)

and $[(a + b) + c] \,.\, a = a \,.\, [(a + b) + c]$

$= [a \,.\, (a + b)] + [a \,.\, c]$

$= a + (a \,.\, c)$ (By theorem 5)

$= a$ (By theorem 5)

$\therefore$ $[a + (b + c)] \,.\, a = [(a + b) + c] \,.\, a$

To prove the other result, we have

$$[(a + b) + c] \,.\, a' = a' \,.\, [(a + b) + c]$$

$= [a' \,.\, (a + b)] + [a' \,.\, c]$

$= [(a' \,.\, a) + (a' \,.\, b)] + [a' \,.\, c]$

$= [0 + (a' \,.\, b)] + [a' \,.\, c]$

$= (a' \,.\, b) + (a' \,.\, c) = a' \,.\, (b + c)$

and $[a + (b + c)] \,.\, a' = a' \,.\, [a + (b + c)]$

$= a' \,.\, a + a' \,.\, (b + c)$

$= 0 + a' \,.\, (b + c) = a' \,.\, (b + c)$

$\therefore$ $[(a + b) + c] \,.\, a' = [a + (b + c)] \,.\, a'$

Thus we complete the proof.

(ii) It is true by the principle of duality for *(i)*.

Theorem 8. *If $a, b \in B$, then*

(i) $(a + b)' = a' \,.\, b'$, *(ii)* $(a \,.\, b)' = a' + b'$. **(De Morgan's laws)**

Proof. *(i)* To establish this result we shall show

$$(a + b) + (a' \,.\, b') = 1 \text{ and } (a + b) \,.\, (a' \,.\, b') = 0$$

Now $(a + b) + (a' \,.\, b') = [(a + b) + a'] \,.\, [(a + b) + b']$

$= [b + (a + a')] \,.\, [a + (b + b')]$

$$= (b + 1)(a + 1) = 1 \,.\, 1 = 1$$

and $$(a + b) \,.\, (a' \,.\, b') = a \,.\, (a' \,.\, b') + b \,.\, (a' \,.\, b')$$
$$= (a \,.\, a') \,.\, b' + b \,.\, (b' \,.\, a')$$
$$= (a \,.\, a') \,.\, b' + (b \,.\, b') \,.\, a'$$
$$= 0 \,.\, b' + 0 \,.\, a' = 0 + 0 = 0$$

Hence $$(a + b)' = a' \,.\, b'$$

(*ii*) It is true by the principle of duality.

Theorem 9. *The identity elements are compliments of each other i.e.,*

(*i*) $1' = 0$, (*ii*) $0' = 1$

Proof. (*i*) $1' = 1' \,.\, 1 = 1 \,.\, 1' = 0$

(*ii*) By the principle of duality for above result, we have

$$0' = 1.$$

Below we shall give some other important theorems.

Theorem 10. *If $a, b \in B$ (Boolean algebra), then*

(*i*) $a + a'b = a + b$, (*ii*) $a \,.\, (a' + b) = a \,.\, b$, **(Redundancy laws)**

Proof. (*i*) $a + a'b = (a + a') \,.\, (a + b)$

$$= 1 \,.\, (a + b) = (a + b) \,.\, 1 = a + b$$

(*ii*) This result is obtained from (*i*) by using principle of duality.

Theorem 11. *If $a, b, c \in B$ (Boolean algebra), then*

(*i*) $ab + a'c + bc = ab + a'c$

(*ii*) $(a + b) \,.\, (a' + c) \,.\, (b + c) = (a + b) \,.\, (a' + c)$ **(Consensus laws)**

Proof. (*i*) $ab + a'c + bc = ab + a'c + bc(a + a')$

$$= ab(1 + c) + a'c(1 + b)$$
$$= (ab)(c + 1) + a'c\,(b + 1)$$
$$= ab \,.\, 1 + a'c \,.\, 1 = ab + a'c.$$

(*ii*) This result is obtained by using duality principle in result (*i*).

Theorem 12. *In Boolean algebra B the elements 0 and 1 are unique.*

Proof. Suppose 0 and 0_1 are two identity elements with respect to operation + then,

$$0 + 0_1 = 0_1 + 0 = 0_1 \text{ with 0 as identity element}$$
$$0 + 0_1 = 0_1 + 0 = 0 \text{ with } 0_1 \text{ as identity element}$$

Therefore, $$0 = 0_1.$$

i.e., 0 is unique element

Now suppose there are two identity elements 1 and 1_1 in B with respect to operation. ;

then, $$1 \,.\, 1_1 = 1_1 \,.\, 1 = 1 \text{ when } 1_1 \text{ is an identity element}$$

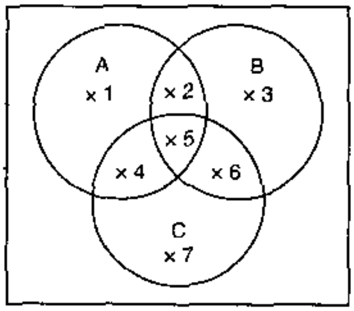
and $$1_1 \,.\, 1 = 1 \,.\, 1_1 = 1_1 \text{ when 1 is an identity element}$$

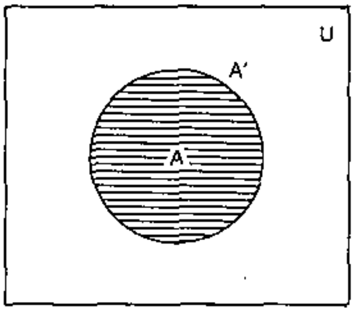
$\therefore$ $$1 = 1_1$$

Therefore, 1 is unique element.

5. ALGEBRA OF CLASSES

Let us consider the following :





Here operation ∪ denotes : either in one or the other or in both

Operation ∩ denotes : common to both

$A \cap A' = \phi$ (null class)

$A \cup A' = U = A' \cup A$

(A' is for complement of A)

U = universal class

ϕ = null class

$$A = \{1, 2, 4, 5\},\ B = \{2, 3, 5, 6\},\ C = \{4, 5, 6, 7\}$$
$$A \cup B = \{1, 2, 3, 4, 5, 6\},\ A \cap B = \{2, 5\}$$
$$B \cup C = \{2, 3, 4, 5, 6, 7\},\ B \cap C = \{5, 6\}$$
$$C \cup A = \{1, 2, 4, 5, 6, 7\},\ C \cap A = \{4, 5\}$$
$$A \cup (B \cap C) = \{1, 2, 4, 5, 6\} = (A \cup B) \cap (A \cup C)$$
$$A \cap (B \cup C) = \{2, 4, 5\} = (A \cap B) \cup (A \cap C)$$

It can be observed that associative, commutative and other laws are satisfied.

Additive identity is ϕ since $A \cup \phi = A$.

Multiplicative identity is U since $A \cap U = A$.

The system considered is isomorphic with that of Boolean Algebra.

6. SUBALGEBRAS

Let (B, +, .,') be a Boolean Algebra. If A is a subset of B and also {A, +, .,'} is a Boolean Algebra then A is said to be subalgebra of B.

7. ISOMORPHISM

A function *f* is said an isomorphism from a Boolean algebra $B = \{B, +_B, \cdot_B, '\}$ into a Boolean Algebra $C = \{C, +_C, \cdot_C, '\}$ iff

(*i*) *f* is a one-one function from B into C,

(*ii*) for any *a*, *b* in B,

$$f(a \cdot_B b) = f(a) \cdot_C f(c)$$

$$f(a \cdot_B b) = f(a) +_C f(b)$$
$$f(a'_B) = f(a)' \cdot_C$$

We say that B is isomorphic with C iff there is an isomorphism from B onto C.

8. PARTIAL ORDERS

In a Boolean Algebra **B**, we define a binary relation $\leq$ on **B** by saying that $a \leq b$ iff $a \,.\, b = a$

Theorem 1. *$a \leq b$ iff $a + b = b$.*

Proof. Let $a \leq b$, then

$$a + b = (a \,.\, b) + b = b.$$

Conversely if $a + b = b$, then

$$a \,.\, b = a \,.\, (a + b) = a$$

Theorem 2. (*i*) *$a \leq a$ (reflexivity)*

(*ii*) *$(a \leq b$ and $b \leq c) \Rightarrow a \leq c$ (Transitivity)*

(*iii*) *$(a \leq b$ and $b \leq a) \Rightarrow a = b$ (Anti-symmetry)*

Proof. (*i*) $a \,.\, a = a$

(*ii*) Assume $a \,.\, b = a$ and $b \,.\, c = b$, then

$$a \,.\, c = (a \,.\, b) \,.\, c = a \,.\, (b \,.\, c) = a \,.\, b = a$$

(*iii*) Assume $a \,.\, b = a$ and $b \,.\, a = b$, then

$$a = a \,.\, b = b \,.\, a = b.$$

Partial Order Relation

In Boolean Algebra the partial order relation $\leq$ is defined as follows :

For all elements a and b of Boolean algebra B,

$$a \leq b \Leftrightarrow ab' = 0.$$

Theorem 1. *For all arbitrary different elements x, y and z of Boolean algebra B following four relations are valid.*

(*i*) $x \leq y$ *and* $y \leq z \Rightarrow x \leq z$

(*ii*) $x \leq y$ *and* $y \leq z \Rightarrow x \leq yz$

(*iii*) $x \leq y \Rightarrow x \leq y + z$

(*iv*) $x \leq y \Leftrightarrow y' \leq x'$

Proof. (*i*) $x \leq yx \Rightarrow y' = 0$

And $\quad y \leq z \Rightarrow yz' = 0$

Now $\quad xz' = xz' \,.\, 1$ (1 is identity element)

$= xz'(y + y')$ (law of complementation)

$= (xz')\, y + (xz')\, y'$ (distributive law)

$= x(z'y) + x(z'y')$ (associative law)

$= x(yz') + x(y'z')$ (commutative law)

$= x\,(yz') + (xy')\, z'$ (associative law)

$= x \,.\, 0 + 0 \,.\, z'$ (by hypothesis)

$= 0 + 0$ (property of zero)

$= 0$

But $\qquad xz' = 0 \quad \Rightarrow \quad x \le z$

Therefore, $x \le y$ and $y \le z \quad \Rightarrow \quad x \le z$

(*ii*) $x \le y$ and $x \le z \quad \Rightarrow \quad xy' = 0$ and $xz' = 0$

$\Rightarrow \qquad xy' + xz' = 0 + 0 = 0 \quad \Rightarrow \quad x(y' + z') = 0$

$\Rightarrow \qquad x(yz)' = 0 \qquad [\because \; y' + z' = (yz)']$

$\Rightarrow \qquad x \le yz.$

(*iii*) $x \le y \quad \Rightarrow \quad xy' = 0$

Now $\qquad x(y + z)' = x(y'z') = (xy')\, z' = 0 \qquad$ (From 1)

i.e., $\qquad x(y + z)' = 0$ so that $x \le y + z$

(*iv*) Assume that $x \le y$ then,

$$x \le y \quad \Rightarrow \quad xy' = 0 \quad \Rightarrow \quad 0\, xy' = (x')'y' = y'(x')'$$

$$y'(x')' = 0 \quad \Rightarrow \quad y' \le (x')'$$

conversely if $y' \le x'$ then

$$(x')' \le (y')' \quad \text{or} \quad x \le y.$$

9. SOME DEFINITIONS

Inverse order. If a relation R is a partially ordered then inverse relation R^{-1} is also partially ordered.

Remark. Two elements x and y in a partially ordered set are said to be 'not comparable' if x R̸ y.

Total order. The word 'partial' is used in defining a partial order in a set A because some elements in A need not be comparable. If, on the contrary, every two elements in a partially ordered set A are comparable, then the partial order in A is called a total order in A.

Lower bound and greatest lower bound. *Any element a of a partially ordered set A is said to be a lower bound of a subset B of A, if for every $x \in B$, a precedes x.*

If a lower bound of B dominates every other lower bound of B, then it is called the greatest lower bound or infimum of B.

Upper bound and least upper bound. An element $a \in$ A is said to be an upper bound of a subset B of a set A if for every $x \in$ B, a dominates x *i.e.*, $\forall x \in$ B, x precedes a.

If an upper bound of B precedes every other bound of B, it is said to be least upper bound or supremum of B.

ILLUSTRATIVE EXAMPLES

Example 1. *If B = [{1, 0}, +,·] be an algebraic structure where + and · are defined as given below :*

+	1	0
1	1	1
0	1	0

·	1	0
1	1	0
1	0	0

Then prove that this structure is a Boolean algebra.

Solution. The elements of both the tables are present in {1, 0}, therefore, the two operations are binary.

B_1, B_2 : In both of the tables elements equidistant from the principal diagonal are equal, therefore, the commutative law holds with respect to both the operations.

B_3, B_4 : Both the operations are distributive, one on the other.

B_5, B_6 : 0 is identity element for '+' because in the composition table for +, second row is identical to the horizontal border and the element to the left of this row in the vertical column is 0. Similarly in the second table for '·' first row is identical to the border row and the element in the left most column against it is 1. Therefore, 1 is identity element for '·'.

B_7, B_8 : here $1' = 0$, $0' = 1$ so that in B = {0, 1} there exist a' such that

$$a + a' = 1, \ \forall a \in B$$

And
$$a.a' = 0, \ \forall a \in B$$

Therefore, the given structure is Boolean algebra.

Example 2. *For any element A of Boolean algebra B prove that (i) $a \leq 1$ and (ii) $0 \leq a$.*

Solution. (*i*) here $a.\, 1' = a\, .\, 0$ since $1' = 0$

$$= 0$$

$\therefore$ $a\, .\, 1' = 0 \ \Rightarrow \ a \leq 1$

(*ii*) $0\, .\, a' = 0$, for $a \in B$ and $a' \in B$

$\therefore$ $0 \leq a$.

Therefore, we see that for any element $a \in B$

$$0 \leq a \leq 1.$$

Example 3. *If the set of divisors of 30 be B = {1, 2, 3, 5, 6, 10, 15, 30} and $a \cup b$ stands for l. c.m. (a, b), $a \cap b$ stands for g.c.f. (a, b) and $a = 30/a$ then prove that $(B, \cup, \cap, -)$ is a Boolean algebra.*

Solution.

∪	1	2	3	5	6	10	15	30
1	1	2	3	5	6	10	15	30
2	2	2	2	10	6	10	30	30
3	3	6	6	15	6	30	15	30
5	5	10	15	5	30	10	15	30
6	6	6	6	30	6	30	30	30
10	10	10	30	10	30	10	30	30
15	15	30	15	15	30	30	15	30
30	30	30	30	30	30	30	30	30

∩	1	2	3	5	6	10	15	30
1	1	1	1	1	1	1	1	1
2	1	2	1	1	2	2	1	2
3	1	1	3	1	3	1	3	3
5	1	1	1	5	1	5	5	5
6	1	2	3	1	6	2	3	6
10	1	2	1	5	2	10	5	10
15	1	1	3	5	3	5	15	15
30	1	2	3	5	6	10	15	30

NOTES

From the above tables it is clear that all elements of the tables are contained in B, therefore, B is closed for both the operations *i.e.*, the two operations are binary.

B_1, B_2 : Since in both the tables the elements equidistant from the principle diagonal are equal, the operations obey the commutative law.

B_3, B_4 : Each operation is distributive over the other.

B_5 : From the table for $\cup$ on the left it is evident that the first row is identical to the top border row, therefore, the element 1 opposite to this row in left border column is an identity element for $\cup$. Thus the zero elements is 1.

B_6 : Testing as above it is clear that 30 is an identity element for $\cap$ *i.e.*, the unit element is 30.

B_7 : For each $a \in$ B there exists a' (= 30/a) such that

$$a \cup a' = 1$$

B_8 : for each $a \in$ B there exists a' such that

$$a \cap a' = 30$$

Therefore, the given structure is Boolean algebra.

Example 4. *Prove that for a, b, c* $\in$ *B,*

$$b = c \iff a + b = a + c \text{ and } ab = ac.$$

Solution. Firstly we shall prove that

$$a + b = a + c \quad \text{and} \quad ab = ac \quad \Rightarrow \quad b = c.$$

$$b = b + b \,.\, a = b \,.\, b + b \,.\, a = b \,.\, (b + a)$$

$$= b \,.\, (c + a) \qquad [\because \quad a + b = a + c]$$

or

$$b = b \,.\, c + b \,.\, a \qquad \text{[From distributive law]}$$

$$= (c \,.\, b) + (c \,.\, a) \qquad [\because \quad a \,.\, b = a \,.\, c \quad \text{or} \quad b \,.\, a = c \,.\, a]$$

$$= c \,.\, (b + a)$$

$$= c \,.\, (c + a) \qquad [\because \quad b + a = c + a]$$

$$= c \,.\, c + c \,.\, a \qquad \text{[From distributive law]}$$

$$= c + c \,.\, a$$

$$= c$$

Similarly it can be proved that

$$b = c \quad \Rightarrow \quad a + b = a + c \text{ and } ab = ac$$

Note. The cancellation law is not true in Boolean algebra. Here one should remember that $a + b = b + c$ and $ab = ac \quad \Rightarrow \quad b = c$ while

$$a + b = a + c \neq b = c \text{ and also } ab = ac \neq b = c$$

i.e., when both the relations are given simultaneously the only $b = c$.

EXERCISE 4(A)

1. If B = [(x, y), + , .,′], where operations are defined as given below then prove that B is a Boolean algebra :

+	x	y
x	x	y
y	y	y

.	x	y
x	x	x
y	x	y

2. If B = {a, b, c, d} and Binary operations '+' and '.' are defined on B as follows, then prove that the triplet (B, +, ·} is a Boolean algebra.

+	a	c	b	d
a	a	c	b	d
b	b	d	b	d

.	a	c	b	d
a	a	a	a	a
b	a	a	b	b

3. Show that there cannot be three distinct elements in a Boolean algebra.
4. Prove that the set of all subsets of a given set is a Boolean algebra with binary operations $\cup$, $\cap$, and a unary operation is a Boolean algebra.
5. For all a, b in Boolean algebra B prove that

   (i) $(a + b) = (a' \,.\, b')$ (ii) $a \,.\, b = (a' + b')'$

   (iii) $(a \,.\, b) + [(a + b') \,.\, b]' = 1$

10. APPLICATION OF BOOLEAN ALGEBRA

The results of mathematical logic can be proved, without preparing truth tables, with the help of Boolean algebra. The operations $\wedge$, $\vee$ and ~ are replaced by '·', '+' and ''', respectively and tautology is replaced by 1 whereas contradiction by 0.

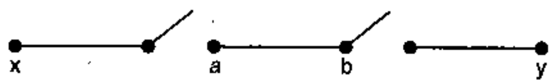
The conditional $\Rightarrow$ is expressed by its equivalent form, as $p \Rightarrow q$ by $\sim p \vee q$ and the biconditional as $p \Leftrightarrow q$ by $(p \Rightarrow q) \wedge (q \Rightarrow p)$. These results are written as $a + b$ and $ab + ab$ respectively. All the results of logic can be verified easily.

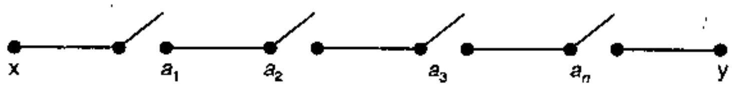
11. APPLICATION TO SWITCHING CIRCUITS

Switching Circuits. It is known that a switch is a device which is attached to a point in an electric circuit.

A switch may assume either of two states closed or open. In the closed state the switch allows current to flow through the point while in the open state it does not allow the flow of electricity. If a switch is denoted by the symbol $\underset{a}{—/—}$, a denotes a sentence such that the switch is closed when a is true and open if a is false. We say that two points are connected by a switching circuit if and only if they are connected by wires on which a finite number of switches are situated.

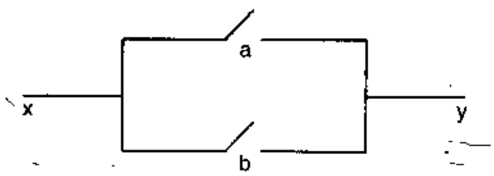
Switches in series. If the points x and y are connected by switching circuits as shown in the following figures they are said to be in series.



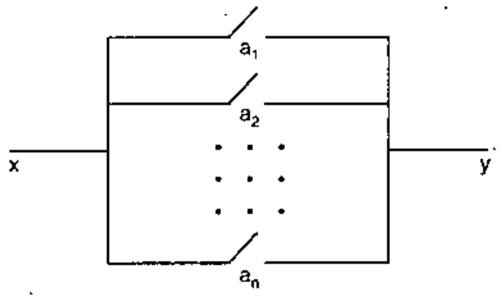


Clearly the current will flow if all the circuits are closed otherwise it will not flow.

Switches in parallel. If the points x and y are connected by a switching circuit as shown are said to be in parallel.



The most general case can be exhibited by the following figure :



Whenever in a circuit there are switches in parallel as well as in series as shown in the first figure of article 7.14, the arrangement of this type is known as series-parallel switching circuit.

12. BOOLEAN SWITCHING CIRCUIT

Nowadays binary system is used in all efficient type of digital computing machines.

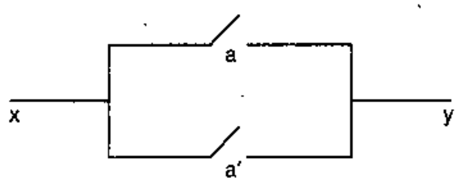
We know that the Boolean algebraic structure is based on elements 0 and 1. Also we use two numbers 0 and 1 in binary number system. With this view we will discuss the design of Boolean switching circuit. Evidently we will associate the binary mechanism of closed and open switches for electric current flow with the numbers 1 and 0. If the switch or circuit is closed (on state) it will be denoted by 1 and if it is open (off state), then by 0. Thus if a and a' be situated in a circuit then a will be 1 if and only a' is 0.

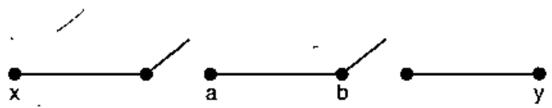
13. CONVERSION OF OPERATIONS

Combining switches in parallel is a binary operation which we denote by '+'. Combining switches in series is also a binary composition which we denote by '·'. Two switches are equal if they are open or closed simultaneously. A switch which is open when a is closed or closed when a is open is denoted by 'a''.

A switch which is always closed will be denoted by 1 and one which is always open by 0.

Remark. From the following figures, we see $a + a' = 1$ and $a.a' = 0$.





Below we give tables for circuits in series $x \cdot y$ and circuits in parallel $x + y$.

x	y	$x \cdot y$
1	1	1
1	0	0
0	1	0
0	0	0

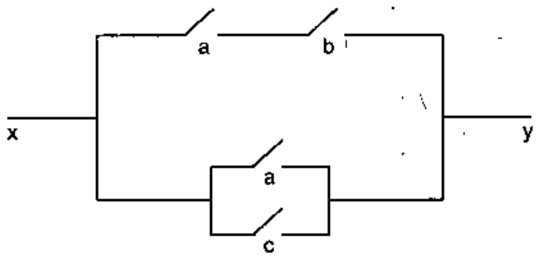
x	y	$x + y$
1	1	1
1	0	1
0	1	1
0	0	0

Following table exhibits the relation between switches a and a'.

a	a'
1	0
0	1

14. EQUIVALENCE OF TWO CIRCUITS

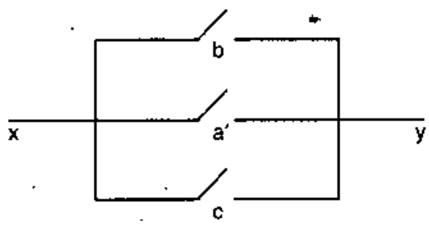
The two circuits in which one situated the switches a, b, c, ... are said to be equivalent when the closure conditions of the two circuits be same in all states. In other words for all states of switches in both circuits current flows (this will happen when switches are closed) or does not flow in any one of them (this will happen when some switch is open). In view of logic, in the switching circuit of the diagram given below, the current will flow if the sentence $(a \wedge b) \vee (a \vee c)$ is true.



The above statement is logically equivalent to $b \vee a' \vee c$ because

$$(a \wedge b) \vee (a' \vee c) = [(a \wedge b) \vee a'] \vee c = [(a \vee a') \wedge (b \vee a')] \vee c$$
$$= [U \wedge (b \vee a')] \vee c = b \vee a' \vee c.$$

Therefore, the above figures is equivalent to the figure given below :

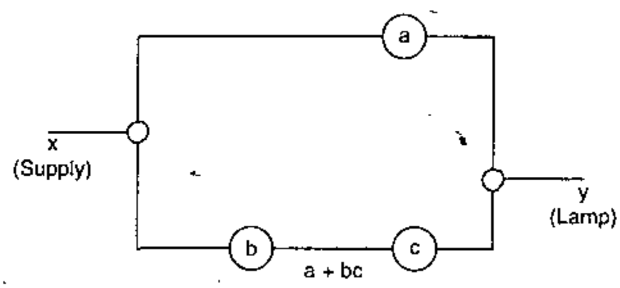


Theorem. *The algebra of Boolean switching circuits is a Boolean algebra.*

Proof. If there exist switches a, a', b, c in a circuit then the theorem is proved immediately with the help of the table given below.

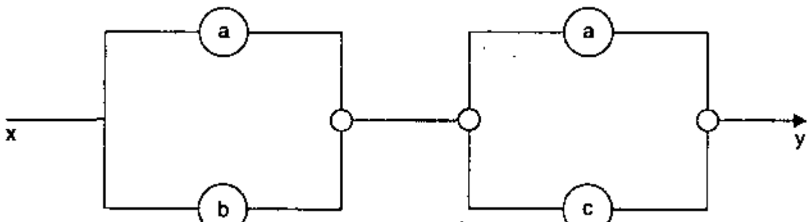
a	b	c	a'	$a+b$	$b+a$	ab	ba	$a+a'$	aa'	$a+0$	$a+bc$	$(a+b)(b+c)$	$a(b+c)$	$(ab+ac)$
1	1	1	0	1	1	1	1	1	0	1	1	1	1	1
1	1	0	0	1	1	1	1	1	0	1	1	1	1	1
1	0	1	0	1	1	0	0	1	0	1	1	1	1	1
0	1	1	1	1	1	0	0	1	0	0	1	1	0	0
1	0	0	0	1	1	0	0	1	0	1	1	0	0	0
0	1	0	1	1	1	0	0	1	0	0	0	0	0	0
0	0	1	1	0	0	0	0	1	0	0	0	0	0	0
0	0	0	1	0	0	0	0	1	0	0	0	0	0	0

From what has been said earlier it is evident that all axioms of Boolean algebra can be verified by switching circuits. That is why algebra of switching circuits is important.



a + bc

In the following diagram the verification of distributive law for Boolean algebra has been done with the help of switching circuits.



(a + b)(a + c)

It is clear from the figure that current will flow in on position of the circuit if switch a is in on position or b and c both are in on position ; the current will not flow in off position of the circuit if a and b or c are in off position. Therefore, the above two circuits are equivalent to each other *i.e.*, they verify the distributive law. The other axioms can be verified similarly.

Note. The laws of Boolean algebra are the same as the laws of algebra of sets or algebra of sentences if +, ., ′, 0, 1, are replaced by ∪, ∩, ′, ϕ, U and ∧, ∨, ~, F, T, respectively.

15. SIMPLIFICATION OF CIRCUITS

When some statement is logically equivalent to the statement form given, it is in the simplified form when it involves fewer switches.

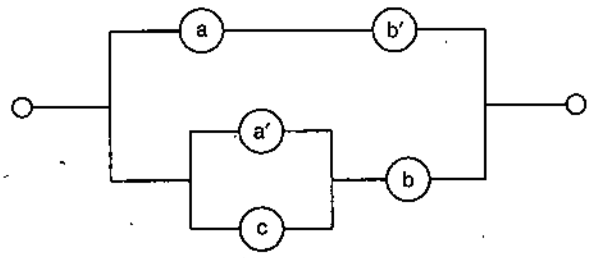
16. BOOLEAN POLYNOMIAL

A Boolean polynomial is an expression (or combination) obtained by the application of connectives '+', '.' (bivariate operations) and ''' (one variate operation) on the finite elements of a Boolean algebra.

Note. Variate operation (′) means a symbol which may express any element of the set B.

ILLUSTRATIVE EXAMPLES

Example 1. *Write down the Boolean polynomial corresponding to following electric circuit and by constructing the truth tables find in which positions current will flow ?*



Solution. In given circuit a, a', b, b' and c are switches where a and a' (b and b') means if one switch is on then the other is off. As has been already said 1 and 0 denote the on and off positions of a switch respectively.

Therefore, the Boolean polynomial representing the circuit is following :

$$(a \,.\, b') + [(a' + c) \,.\, b]$$

Now prepare a truth table for it :

a	b	c	a'	b'	$a \,.\, b'$	$a' + c$	$[(a' + c) \,.\, b]$	$(a \,.\, b') + [(a' + c) \,.\, b]$
1	1	1	0	0	0	1	1	1
1	1	0	0	0	0	0	0	0
1	0	1	0	1	1	1	0	1
1	0	0	0	1	1	0	0	1
0	1	1	1	0	0	1	1	1
0	1	0	1	0	0	1	1	1
0	0	1	1	1	0	1	0	0
0	0	0	1	1	0	1	0	0

From the table it is evident that current will flow in following cases :

(*i*) When switches a, b and c are in on positions.

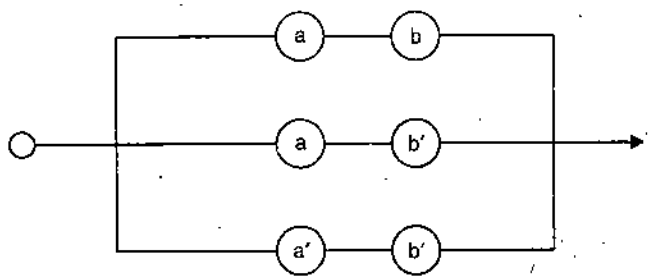
(*ii*) When switches a and c are in on positions.

(*iii*) When switch a is in on position and b is in off position.

(*iv*) When switches b and c are in on positions.

(*v*) When switch b is in on position and a is in off position.

Example 2. *Draw a circuit equivalent to the following circuit design.*



Solution. The Boolean polynomial for the given circuit is

$$(a \,.\, b) + (a \,.\, b') + (a' \,.\, b')$$

But $(a \,.\, b) + (a \,.\, b') + (a' \,.\, b') = [a \,.\, (b + b')] + (a' \,.\, b')$

$= [a \,.\, 1] + (a' \,.\, b')$ ($b + b' = 1$ by complementary law)

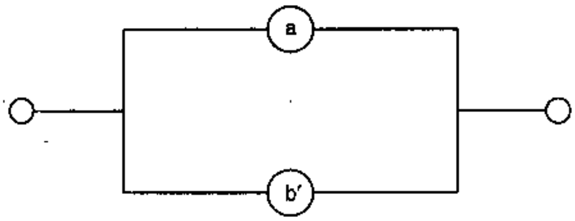
$= a + (a' \,.\, b')$ (property identity)

$= (a + a') \,.\, (a + b')$ (distributive law)

$= 1 \,.\, (a + b')$ (by complementry law)

$= a + b'$ (identity law)

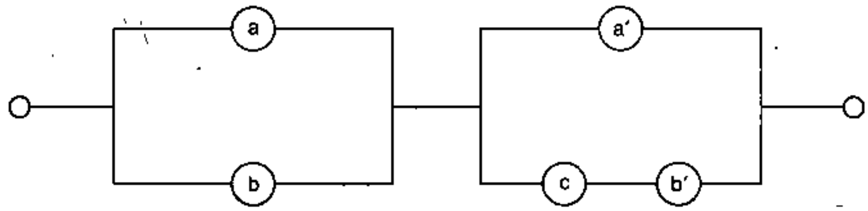
Therefore, the equivalent circuit is following :



Example 3. *Draw an electric circuit design for the Boolean polynomial*

$$(a + b) \,.\, [a' + (c \,.\, b')].$$

Solution. $(a + b)$ implies that switches a and b are parallel. Therefore, we draw the circuit for this expression as given in the following figure. Now the meaning of the second term $(c.\, b')$ in second expression is that the switches c and b' are in series and $[a' + (c \,.\, b')]$ means a' and $(c \,.\, b')$ are parallel. In view of it we prepare the right portion of the following circuit :



Thus the whole design exhibits the circuits for the above mentioned Boolean polynomial.

Example 4. *Comment on the following :*

"A teacher is good if and only if he imparts knowledge to his students."

Solution. Let p be the statement, 'A teacher is good 'and' q be the statement 'he imparts knowledge to his students' then above comment can be written as $p \Leftrightarrow q$.

In Boolean algebraic symbols this can be expressed as

$a'b' + ab$.

If $a = 1$, $b = 1$ (*i.e.*, p and q both are true) then

$$a' = 1' = 0,\, b' = 1' = 0$$

$$\therefore \qquad a'b' + ab = 0 \,.\, 0 + 1 \,.\, 1 = 1$$

which is a tautology. Hence statement is valid if p and q both are true.

Again, if $a = 1$, $b = 0$ (*i.e.*, p is true, q is false) then

$$a' = 1' = 0,\, b' = 0' = 1$$

$\therefore \quad a'b' + ab = 0.1 + 1.0 = 0$

which is contradiction. Hence the given statement is false if p is true and q is false.

Similarly there is contradiction if $a = 0, b = 1$ because then

$$a'b' + ab = 0.$$

Hence given statement is invalid if any one of p and q is false.

Example 5. *Using Boolean algebra test the validity of the following :*

"Either the candidate is not good at studies or paper is out of course. Paper is not out of course. Therefore, the candidate is not good at studies."

Solution. Let a stands for "The candidate is not good at studies" b for "Paper is not out of course".

Then $\quad [(a + b) . b'] + a = [(a + b)' + (b')] + a$

$$= [(a' . b') + b] + a$$

$$= [(a' + b) \ . (b' + b)] + a$$

$$= [(a' + b) . 1] + a$$

$$= (a' + b) + a \qquad [\because \quad b' + b = 1]$$

$$= (b + a') + a = b + (a' + a)$$

$$= b + 1 = 1 \qquad [\because \quad b + 1 = 1]$$

Hence this is a tautology *i.e.*, the argument is valid.

EXERCISE 4(B)

1. Simplify the following circuits :

   (*i*) $(a \wedge b) \vee (a \wedge b') \vee (a' \wedge b')$, (*ii*) $(a \wedge c) \vee (a' \wedge b)$

2. Simplify the following :

   (*i*) $(a + b) . (a + c) . (a' . b')$, (*ii*) $[a + \{b . (c + a')\}]$,

   (*iii*) $(a' . b)' + (a . b')$, (*iv*) $(a + b') . (a' + b) . (a' + b')$.

3. Using Boolean algebra, test the validity of the following :

   "If today is Friday, then yesterday was Thursday. Yesterday was Thursday therefore, today is Friday."

Answers

1. (*i*) $a \vee b'$ (*ii*) $(c \wedge a') \vee b$
2. (*i*) a (*ii*) $a' b'$ (*iii*) $a + b'$
   (*iv*) $a' b'$
3. Invalid.

BOOLEAN FUNCTION

In this chapter we shall use the technical term such as monomial, polynomial, term and factor in reference to an arbitrary Boolean algebra. The use of the word 'constant' is made for some specified single symbol in a Boolean algebra. For example the identity elements 0 and 1 are called constants. The symbol denoting an arbitrary or unspecified element of a Boolean algebra is said to be variable.

NOTES

17. BOOLEAN EXPRESSION AND FUNCTION

Any expression made up of variables $x, y, z, x_1, y_1, z_1, x_2, y_2, z_2, \ldots\ldots$ with the use of operations +, ., ′ a finite number of times is said to be a Boolean expression. Thus all variables are Boolean expressions and if p and q are Boolean expressions, then $(p + q)$, $(p \,.\, q)$ and (p') are also Boolean expressions.

The following are Boolean expressions :

$$((x + (y')) \,.\, (z_2')), ((y.z) + (x').\, y),$$

$$(y \,.\, (z + x)'), ((((y')') \,.\, z) \,.\, y), (((y')') \,.\, (z \,.\, y))$$

The convention of omitting the parenthesis will make writing convenient. Thus above expression can be written as

$$(x + y') \,.\, z_2', (y \,.\, z) + (x' \,.\, y), (y \,.\, (z + x))', y'' \,.\, z \,.\, y, y'' \,.\, (z \,.\, y)$$

Let $\mathbf{B}$ = (B, +, ., ′, 0, 1) be a Boolean algebra and let $f(u_1, \ldots\ldots u_k)$ be a Boolean *expression having its variables among* $u_1 \ldots\ldots u_k$. We can determine a corresponding Boolean function $f^B(u_1, \ldots\ldots u_k)$: for each k-tuple $(b_1, \ldots\ldots, b_k)$ of elements of B, f^B $(b_1, \ldots\ldots b_k)$, is the element of B obtained by assigning the values $b_1, \ldots\ldots, b_k$ to $u_1, \ldots\ldots u_k$ respectively, and interpreting the symbols ., +,′ to mean the corresponding operation in **B**.

Remark. In order to make the corresponding function unique, the variables $u_1, \ldots u_k$ are *always listed in the order in which they occur in the list* $x, y, z, x_1, y_1, z_1, x_2, y_2, z_2, \ldots\ldots$. Thus, $y + x'$ determines the function $f(x, y) = y + x'$ and therefore, $f(1, 0) = 0$ and $f(0, 1) = 1$. The Boolean expression $x + y'$ denotes the function $f(x, y)$ with respect to the two element Boolean algebra B_0, such that

$$f(0, 0) = 1, f(0, 1) = 0, f(1, 0) = 1, f(1, 1) = 1.$$

Note. If $b_1, \ldots, b_k$ are in {0,1} and $(u_1, \ldots u_k)$ is a Boolean expression, then $f^B(b_1, \ldots b_k)$ is also in {0, 1}, since {0, 1} is closed under + , . and ′. It is also to be noted that different Boolean expressions may determine the same Boolean function.

Theorem. *Given a Boolean expression f(u), which may contain other variables* $u_1, u_2, \ldots u_k$ *as well as u, then the equation* $f(u) = [f(0).\, u'] + [f(1) \,.\, u]$ *is derivable from the axioms for Boolean algebras.*

Proof. Induction on the number m of occurrences of +, ., ′ in f will be used. If m = 0 then f is either u or u_i for some i. If f is u then $f(0) = 0$ and $f(1) = 1$. Therefore,

$$f(u) = u = [0.u'] + [1.u] = [f(0).\, u'] + [f(1).\, u]$$

If f is u_i then $f(0) = f(1) = u_i$. Thus

$$f(u) = u_i = u_i \,.\, (u' + u) = (u_i \,.\, u') + (u_i \,.\, u) = [f(0) \,.\, u'] + [f(1) \,.\, u]$$

Again let $m > 0$ and suppose that the result is true for all expressions with less than m occurrences of +, .,′.

Case 1. $f(u) = [g(u)]'$. By induction. $g(u) = [g(1) \,.\, u'] + [g(1).\, u]$

Therefore,

$$\begin{aligned} f(u) &= (g(u))' = [g(0).u'] + [g(1).u'] \\ &= [g(0) \,.u']'.\, [g(1).u]' = [g(0)' + u''].\, [g(1)' + u'] \\ &= [f(0) + u] \,.\, [f(1) + u'] \\ &= [f(0) \,.\, f(1)] \,.\, [f(0).u'] + [f(1).u] + [u.u'] \\ &= [f(0) \,.\, f(1)] + [f(0).u'] + [f(1).u] \\ &= [f(0) \,.\, f(1) \,.\, (u + u')] + [f(0).u'] + [f(1) \,.\, u] \\ &= [f(0).f(1).u] + [f(0).f(1).u'] + [f(0).u'] + [f(1).u] \\ &= [f(0).f(1).u] + [f(1).u] + [f(0).f(1).u')] + [f(0).u')] \\ &= [f(1).u] + [f(0).u]. \end{aligned}$$

Case 2. $f(u) = g(u) + h(u)$. Then inductive hypothesis holds for g and h, therefore,

$$\begin{aligned} f(u) &= g(u) + h(u) \\ &= [(g(0).u') + (g(1).u)] + [(h(0).u') + (h(1).u)] \\ &= [(g(0).u') + (h(0).u')] + [(g(1).u) + (h(1).u)] \\ &= [g(0) + h(0).u'] + [g(1) + h(1).u] \\ &= [f(0).u'] + [f(1).u] \end{aligned}$$

Case 3. $f(u) = g(u) \,.\, h(u)$. This can be proved as in case 2.

18. DEFINITION

Alternately we can define Boolean function as under:

Boolean Function. A Boolean function is an *expression which is obtained* by the application of bivariate operations +,. and one variate operation ′ on the finite elements of a Boolean algebra. For example $(x + y') \,.\, z + x' \,.\, y$ is a Boolean function where x, y, z are elements of a Boolean algebra. Evidently every polynomial is a function. In brief a function of one or more Boolean variables is called a Boolean function.

Remark. Since $x^k = x + x + \ldots\ldots + x(k \text{ times}) = x$ and $x^k = x.x.x \ldots\ldots x(k \text{ factors}) = x$

Therefore, we conclude that there are no multiples and no powers of the elements of a Boolean function.

19. FUNDAMENTAL FORMS OF A BOOLEAN FUNCTION

The totality of distinct Boolean functions that can be formed from a given set of independent Boolean variables is known as the set of fundamental Boolean functions of that set of Boolean variables.

Note 1. If F denotes the positions of switches situated in series or parallel in a switch network then $f(a, b, c, \ldots)$, where $a, b, c, \ldots$ are switches is called a switching function.

2. Equation $x + x' = 1$ implies that the function $x + x'$ of variable x is equal to the constant 1.

3. There are only four Boolean functions in one variable:

$$a, a', a.a', a + a'$$

20. SOME DEFINITIONS AND THEOREMS

Below we give some necessary definitions and theorems which will be most useful in studying switching network.

Minimal Boolean Function. A minimal Boolean Function is the product of n letters $x_1, x_2, \ldots\ldots x_n$ in n variables $x_1, x_2, \ldots x_n$ where i^{th} letters is x_i or x_i'.

For example the minimal Boolean functions in two variables x_1 and x_2 are:

$$x_1 \,.\, x_2 \,,\, x_1' . x_2, x_1 . x_2' \quad \text{or} \quad x_1' . x_2'$$

In three variables x_1, x_2, x_3 the minimal Boolean functions are :

$$x_1 . x_2 . x_3, x_1 . x_2 . x_3', x_1 . x_2' . x_3, x_1' . x_2 \,.\, x_3,$$
$$x_1' \,.\, x_2' \,.\, x_3, x_1' \,.\, x_2 \,.\, x_3', x_1 \,.\, x_2' \,.\, x_3', x_1' \,.\, x_2' \,.\, x_3'$$

Example. *Prove that there are exactly sixteen fundamental Boolean Functions of two independent Boolean variables.*

Solution. Let x, x', y, y' forms distinct values of the pair (x, y).

(*i*) Number of functions having zero associated with each of the possible 4 distinct values of the pair = ${}^4c_4 = 1$.

(*ii*) Number of functions having zero associated with 3 values of the pair (x, y) and 1 associated with remaining one value of the pair $(x, y) = {}^4c_3 = 4/1 = 4$.

(*iii*) Number of functions having zero associated with 2 values of the pair (x, y) and 1 associated with remaining 2 values of the pair $(x, y) = {}^4c_2 = 4.3/1.2 = 6$.

(*iv*) Number of functions having zero associated with 1 value of the pair (x, y) and 1 associated with the remaining 3 values of the pair $(x, y) = {}^4c_1 = 4$.

(*v*) Number of functions remaining 1 associated with all 4 values of the pair

$$(x, y) = {}^4c_0 = 1.$$

$\therefore$ Total number of distinct functions = 1 + 4 + 6 + 4 + 1 = 16.

Theorem. *The number of fundamental functional form for a Boolean function of n-variable is* $(2)^{2^n}$.

Proof. Let $f = f(x_1, x_2, ... x_n)$, where x_1, x_2, x_n are n-independent binary Boolean variables each of which can take two values, namely x_i and x_i'. The total number of independent values that n-tuples $(x_1, x_2, ...x_n)$ can have is $2 \times 2 \times .... \times 2$ (n factors) = 2^n.

Now, number of functions having zero associated with each one of these 2^n independent values of the n-tuple $(x_1, x_2, ... x_n) = {}^{2^n}C_{2^n}$

....

Number of functions having a zero associated with any r of these 2^n independent values of the n-tuple $(x_1, x_2,.....x_n)$ and 1 associated with the remaining $(2^n - r)$ independent values of

the n-tuple $(r = 1, 2,......, n\text{-}1) = {}^{2^n}C_r$.

....

Number of functions having 1 associated with each one of the 2^n independent values of the n-tuple $(x_1, x_2...... x_n) = {}^{2^n}C_0$.

$\therefore$ The total number of distinct fundamental functions

$$= {}^{2^n}C_{2^n} + ... + {}^{2^n}C_r + ... + {}^{2^n}C_0$$

$$= \sum_{r=1}^{2^n} {}^{2^n}C_r = (1+1)^{2^n} = (2)^{2^n}$$

21. BOOL'S THEOREM

There are 2^n minimal Boolean functions in n-variables.

Proof. There are two ways of selecting the i^{th} variable in a minimal Boolean function namely x_i or x_i' therefore, there are 2 ways of selection for each variable.

Thus the number of minimal Boolean functions in n-variables

$$= 2.2.2 ...... 2(n\text{-factors})$$

$$= 2^n.$$

22. NORMAL FORMS

NOTES

We are familiar with the terms conjunction and disjunction.

A statement form P is said to be in *disjunctive normal form or dnf in brief if* either (*i*) P is a fundamental conjunction, or (*ii*) P is a disjunction of two or more fundamental conjunctions, of which none is included in another.

Now we present below a normal form theorem for Boolean Algebras which is a generalization of the disjunctive normal theorem for propositions in logic. Before the theorem is given it is in the fitness of the things that the symbols used be made clear in advance. We shall use, for any expression f,

$$f^{i} = \begin{cases} f, \text{ if } i = 1 \\ f' \text{ if } i = 0 \end{cases}.$$

The symbol Σ, with relevant indices will indicate repeated use of +. Thus $\sum_{\alpha=0}^{1} g(\alpha)$ stands for $g(0) + g(1)$ while $\sum_{a_1=0}^{1} \sum_{a_2=0}^{1} g(\alpha_1, \alpha_2)$ for $g(0, 0) + g(1, 0) + g(1, 1)$.

Theorem 1. *Disjunctive Normal Form.*

For any Boolean expression $f(u_1, u_2, \ldots\ldots, u_k)$, the equation $f(u_1, u_2, \ldots, u_k)$

$$= \sum_{\alpha_1=0}^{1} \sum_{\alpha_2=0}^{1} \ldots\ldots \sum_{\alpha_k=0}^{1} f(a_1, a_2, \ldots\ldots, a_k)$$

is derivable from axioms for Boolean algebra.

Proof. We shall use induction on k. For $k = 1$ we get the result from theorem 1. Now assume that the result holds for k and we shall prove it for an expressions $f(u_1, u_2, \ldots u_k)$.

We know $f(u_1, u_2, \ldots\ldots, u_{k+1}) = [f(0, u_2, \ldots\ldots, u_{k+1}) \,.\, u'_1] + [f(1, u_2, \ldots\ldots, u_{k+1}).\, u_1]$

But by induction

$$f(0, u_2, \ldots\ldots, u_{k+1}) = \sum_{\alpha_2=0}^{1} \ldots\ldots \sum_{\alpha_{k+1}=0}^{1} \left[f(0, \alpha_2, \ldots, \alpha_{k+1}) \,.\, u_2^{\alpha_2} \ldots\ldots u_{k+1}^{\alpha_{k+1}}\right]$$

and $$f(1, u_2, \ldots, u_{k+1}) = \sum_{\alpha_2=0}^{1} \ldots\ldots \sum_{\alpha_{k+1}=0}^{1} \left[f(1, \alpha_2, \ldots, \alpha_{k+1}) \,.\, u_2^{\alpha_2} \ldots\ldots u_{k+1}^{\alpha_{k+1}}\right]$$

Therefore, $f(u_1, u_2, \ldots\ldots, u_{k+1})$

$$= \left\{ \sum_{\alpha_2=0}^{1} \ldots\ldots \sum_{\alpha_{k+1}=0}^{1} \left[f(0, \alpha_2, \ldots, \alpha_{k+1}) \,.\, u_2^{\alpha_2} \ldots\ldots u_{k+1}^{\alpha_{k+1}}\right] . u_1' \right\}$$

$$\left\{ \left(\sum_{\alpha_2=0}^{1} \ldots\ldots \sum_{\alpha_{k+1}=0}^{1} f(1, \alpha_2, \ldots, \alpha_{k+1}) \,.\, u_2^{\alpha_2} \ldots\ldots u_{k+1}^{\alpha_{k+1}} \right) . u_1' \right\}$$

NOTES

$$= \left(\sum_{\alpha_2=0}^{1} \cdots\cdots \sum_{\alpha_{k+1}=0}^{1} \left[f(0, \alpha_2, \ldots, \alpha_{k+1}) \,.\, u_1' \, u_2^{\alpha_2} \ldots\ldots u_{k+1}^{\alpha_{k+1}} \right] \right) .$$

$$\left(\sum_{\alpha_2=0}^{1} \cdots\cdots \sum_{\alpha_{k+1}=0}^{1} \left[f(1, \alpha_2, \ldots, \alpha_{k+1}) \,.\, u_1 \, u_2^{\alpha_2} \ldots\ldots u_{k+1}^{\alpha_{k+1}} \right] \right)$$

$$= \sum_{\alpha_1=0}^{1} \sum_{\alpha_2=0}^{1} \cdots\cdots \sum_{\alpha_{k+1}=0}^{1} \left[f(\alpha_1, \alpha_2, \ldots, \alpha_{k+1}) \,.\, u_1^{\alpha_1} \,.\, u_2^{\alpha_2} \ldots\ldots u_{k+1}^{\alpha_{k+1}} \right]$$

Corollary. In view of above theorem the corresponding equation with f replaced by f^B, holds in any Boolean algebra B.

Before we present a theorem for conjunctive Normal Form, we explain the symbol II to indicate repeated application of '.' Thus $\prod_{\alpha=0}^{1} g(\alpha)$ stands for $g(0).\, g(1)$.

Theorem 2. ***Conjunctive Normal Form.***

Given a Boolean expression $f(u_1, u_2, \ldots u_n)$ having its variables among $u_1, u_2, \ldots\ldots, u_n$ the equation $f(u_1, u_2, \ldots\ldots, u_n)' = \sum_{\alpha_1=0}^{1} \cdots\cdots \sum_{\alpha_n=0}^{1} \left[f(a_1, \ldots, a_n)' \,.\, u_1^{a_1} \ldots\ldots u_n^{a_n} \right]$ is derivable from the axioms for Boolean algebra and therefore, holds in every Boolean algebra.

Proof. *$[f(u_1, u_2, \ldots\ldots, u_n)]'$ is a Boolean expression* and, therefore, by the disjunctive Normal Form Theorem, the equation

$$f(u_1, u_2, \ldots\ldots, u_n) = \sum_{\alpha_1=0}^{1} \cdots\cdots \sum_{\alpha_n=0}^{1} \left[f(\alpha_1, \ldots, \alpha_n)' \,.\, u_1^{\alpha_1} \ldots\ldots u_n^{\alpha_n} \right]$$

is derivable. Taking the complement of both sides of (1) and using De Morgan's Laws, we get the result.

Remark. When $n = 1$, we have

$$F(u) = (f(0) + u) \,.\, (f(1) + u')$$

When $n = 2$

$$F(u_1, u_2) = (f(0, 0) + u_1 + u_2) \,.\, (f(0, 1) + u_1 + u_2').$$
$$(f(1, 0) + u_1' + u_2).\, (f(1, 1) + u_1' + u_2')$$

Normal Form-Explanation. By taking an example we will consider the fact that a given Boolean expression can be expressed as a sum of minimal Boolean function or in the form of 0.

Let us consider a Boolean polynomial f given by

$$F = f(x, y, z) \;\; [(x' + y)'.\, z] + [x' \,.\, (x + z)]$$

The complementary signs in it can be removed with the help of De Morgan's laws. Double complementary sign can be removed with the help of relevant theorem. Then the result is an expression of complementary and non-complementary symbols connected by operations + and.

$$F = [((x')'.\, y').z] + [x' \,.\, (x + z)]$$
$$= [(x.y').z] + [x' \,.\, (x + z)]$$
$$= [(x.y').z] + (x'.x + x'z] \quad \text{(By distributive law)}$$
$$= (x.y').z + 0 + (x'z) \quad \text{(Since } x'x = 0\text{)}$$
$$= (x.y'.z) + (x'.z)$$

Here f is expressed in terms of a sum of the expressions which themselves are in the form of products of symbols and the complements. This can be further simplified when in each term each element or its complement is present. If in any term some element x or x'(say) is not present then the same can be included by multiplying with $(x + x')$ because $x + x' = 1$. Thus

$$F = (x.y'.z) + ((x'.z).1)$$

$$= (x.y'.z) + (x'.z).(y + y') \quad \text{(To include } y \text{ in second term)}$$

$$= (x.y'.z) + (x'z.y) + (x'.z.y') \quad \text{(By distributive law)}$$

$$= (x.y'.z) + (x'.y.z) + (x'.y'.z) \quad \text{(By commutative law)}$$

In the result obtained each group is the product of each element or its complement and the Boolean expression as a whole is sum of these terms. This method can be employed for other functions. Therefore, we express this fact in general by the following theorem.

Theorem 1. *Any Boolean function can be written as 0 or a sum of minimal polynomials.*

Some authors write this theorems as 'Bool's Expansion Theorem'.

Note. The example considered above is an example of Normal form.

23. DISJUNCTIVE NORMAL FORM OR CANONICAL FORM

Definition. When a Boolean polynomial is written as a sum of the minimal Boolean functions, it is called *disjunctive normal form or canonical form.*

For example

(*i*) $x + x'$, (*ii*) $x'.y$, (*iii*) $x.y.z' + x.y'.z + x'.y.z$

are disjunctive normal forms in one, two and three variables respectively.

Since these are 2^n minimal polynomials in n-variables, there can be at the most 2^n distinct terms in a disjunctive normal form.

24. COMPLETE DISJUNCTIVE NORMAL FORM

Definition. If a disjunctive normal form contains 2^n terms, it is called complete disjunctive normal form in n-variables.

Equal Boolean polynomials. Two Boolean polynomials are said to be equal if and only if their canonical forms are identical *i.e.*, in both the forms all the terms are exactly the same.

Definition 1. Two Boolean polynomials are said to be equal when their canonical forms are such that the terms contained in each one of them are exactly the same.

This is worth noting that *the complete Disjunctive normal forms are always identical to 1.*

Let $n = 1$, then $f(x) = x + x' = 1$.

Again Let $n = 2$ then the disjunctive normal form in 2 variables y, x is as given below:

$$f(x, y) = (x.y) + (x.y') + (x' . y) + (x'.y')$$
$$= x . (y + y') + x'.(y + y') \quad \text{(By distributive law)}$$
$$= (x + x') . (y + y') \quad \text{(By distributive law)}$$
$$= 1.1 = 1 \quad (\text{Since } x + x' = 1, y + y' = 1)$$

Similarly complete disjunctive form in three variables is

$$f(x, y, z) = (x.y.z) + (x.y.z') + (x.y'.z) + (x'.y.z) + (x.y'.z') + (x'.y'.z) + (x'.y.z') + (x'.y'.z')$$
$$= (x.y).(z + z') + (x.y').(z + z') + (x'.y).(z + z') + (x'.y').(z + z')$$
$$= \{(x.y) + (x.y') + (x'.y) + (x'.y')\}.(z + z')$$
$$= x.(y + y') + x'.(y + y') \quad (\because z + z' = 1)$$
$$= (x + x').(y + y') \quad (\because x + x' = 1, y + y' = 1)$$
$$= 1.1 = 1$$

we can verify the result for other functions having more variables.

Finally in general the result can be proved for a function of $n-$ variables. With the help of it we give below the definition of the complement function of a Boolean function.

25. COMPLEMENT FUNCTION OF A BOOLEAN FUNCTION

Definition 2. *Let f be a Boolean function expressed in canonical form. Then f', complement function of f is a Boolean function which is the sum of all those terms of complete canonical form which are not present in the canonical form of f.*

Example. *Find the complement function $f'(x, y)$ of the given function*

$$f(x, y) = (x.y) + (x'.y) + (x'.y').$$

Solution. The complete canonical form in two variables is

$$(x.y) + (x.y') + (x'.y) + (x'.y')$$

and the given function is

$$f(x, y) = (x.y) + (x'.y) + (x'.y')$$

Therefore, $f'(x, y) = (x . y')$ because the term $x . y'$ does not exist in Boolean function f.

This is the required complement function.

26. SOME THEOREMS

Theorem 1. *Every function without constants in Boolean algebra is equal to a function in disjunctive normal form.*

Proof. Let f be an arbitrary function without constants in n-variables $x_1, x_2 \ldots\ldots x_n$. Let f contain the expressions of the form $(A + B)'$ or $(AB)'$ in functions A and B. Then by De Morgan's law,

$$(A + B)' = A'B' \quad \text{and} \quad (AB)' = A' + B'$$

We shall use this fact till every prime expression of f represents a single variable x_i'.

Now for the operations '+' and '.' using distributive law f can be converted into a Boolean polynomial. Again suppose that for a variable x_i there does not exist x_i or x_i' in a term T of the function f then as explained earlier the existence of these can be ensured by multiplying that term by $x_i + x_i'(= 1)$. Continuing this process repeatedly for every missing variable in each term of f, we can get an equivalent function in whose each term will exists x_j or x_j' where $j = 1, 2, ..., n$. Finally by using the laws $a + a = a$, $a.a = a$ the duplicate term can be removed.

ILLUSTRATIVE EXAMPLE

Example. *Write the function $f = (xy' + xz)' + x'$ in the disjunctive normal form.*

Solution.

$$f = (xy' + xz)' + x'$$
$$= (xy')' (xz)' + x' \quad \text{(By De Morgan's law)}$$
$$= (x' + y)(x' + z') + x' \quad \text{(By De Morgan's law and } (y')' = y)$$
$$= x' + x'y + x'z' + yz' + x'$$
$$= x'(y + y')(z + z') + yz'(x + x') + x'y(z + z') + x'z'(y + y')$$
$$= x'yz + x'yz' + x'y'z + x'y'z' + xyz' + x'yz'$$
$$= x'yz + xyz' + x'yz' + x'y'z + x'y'z'.$$

Note. A function can be written in more than one normal forms by changing the number of variables. For example

$$f = xy = xy \cdot 1 = xy(z + z') = xyz + xyz' \qquad (\because \; z + z' = 1)$$

This expression is in normal form in variables x, y and z.

Similarly $F = x'yz + xyz + x'yz' + xyz'$, is in normal form in variables x, y and z and can be transformed in the form $F = x'y + xy$ by removing z.

Theorem 2. *If in the complete disjunctive form represented in n-variables the n-variables are assigned values 0 or 1 arbitrarily then in the complete disjunctive normal form in n-variables, the value of exactly one term will be 1 while the values of other terms will be 0.*

Proof. Let the complete disjunctive normal form of a function be expressed in n-variables x_i, $i = 1, 2, ..., n$.

Let the variables $x_1, x_2, ..., x_n$ be assigned values $a_1, a_2, ..., a_n$ respectively where a_i is 0 or 1. Now choose a term in the complete disjunctive normal form. If $a_i = 1$ then use x_i in this term and if $a_i = 0$ then use x_i'. Perform this process for each x_i, $i = 1, 2, ..., n$. Therefore, the selected term will be multiplication of 1 and consequently its value will be 1. In all other terms. (except this) at least one factor will be 0 and therefore their values will be 0.

EXERCISE 4(C)

1. Write the following Boolean functions in disjunctive normal forms:

   (*a*) $f(x, y, z) = [(x.y')' + z'] . [z + x']'$ (*b*) $f(x, y, z) = (x' + y)'.(x + z)' + (y . z)'$

2. Write down the following function in disjunctive normal forms in which the variables be used in least number.

   (*i*) $x + x'.y$ (*ii*) $x.y' + x.z + x.y$

   (*iii*) $x.y.z + (x + y)(x + z)$ (*iv*) $(x'.y + x.y.z' + x.y'.z + x'.y.z'.t + t')'$

(v) $(x + y')(y + z')(z + x')(x' + y')$

(vi) $x'.y.z + x.y'.z' + x'.y'.z + x'.y.z' + x.y'.z + x'.y'.z'$

3. Write down the following functions in disjunctive normal forms in three variables x, y, z:

(i) $x + y'$ (ii) $x'z + xz'$

(iii) $(x + y)(x' + y')$ (iv) x'

(v) $x' + y'$

Answers

1. (a) $f(x, y, z) = x.y.z'$

(b) $f(x, y, z) = (x.y.z') + (x.y'.z) + (x.y'.z') + (x'.y.z') + (x'.y'.z) + (x'.y'.z')$

2. (i) $(x.y) + (xy') + (x'.y)$ (ii) $xy'z + xy'z' + xyz + xyz'$

(iii) $xyz + xyz' + xy'z + x'yz + xy'z'$ (iv) $xyzt + x'y'zt + xy'z't$

(v) $x'y'z'$

3. (i) $xyz + xyz' + xy'z + xy'z' + x'y'z + x'y'z'$ (ii) $xyz' + x'yz + xy'z' + x'y'z$

(iii) $xy'z + x'yz + xy'z' + x'yz'$ (iv) $xyz + xyz' + xy'z + xy'z'$

(v) $xy'z + xy'z' + x'yz + x'y'z + x'yz' + x'y'z'$

27. CONJUNCTIVE NORMAL FORM

A Boolean function is said to be in conjunctive Normal form if it is a product of the factors, each of which is sum of the variables or their complements and in each factor the variable or its complement occurs only once. This is also called dual canonical form.

Definition When a Boolean polynomial is a product of such factors where each factor is sum of letters or their complements and in each factor letters or their complements occur only once, it is called conjunctive normal form or dual canonical form.

For example if we consider the Boolean function

$$f(x, y) = (x.y) + (x'.y) + (x'.y')$$

then we know that $(x.y) + (x.y') + (x'.y) + (x'.y')$

is a complete canonical form in two variables x, y', therefore, $f'(x.y) = (x.y')$ which does not exist in $f(x, y)$.

Now if we consider the following Boolean function then

$$\begin{aligned} f &= [(x + y')(xy'z)']' \\ &= (x + y')' + ((xy'z)')' \\ &= (x')(y')' + (xy'z) \\ &= x'y + xy'z \qquad \text{(Sum of products)} \end{aligned}$$

Also $f = [(x + y')(xy'z)]' = (x + y')' + ((xy'z)')'$

$$\begin{aligned} &= (x'y) + (xy'z) \\ &= (x' + xy'z)[y + (xy'z)] \\ &= (x' + x)(x' + y')(x' + z)(y + x)(y + y')(y + z). \\ &= 1.(x' + y')(x' + z)(x + y).1.(y + z) \\ &= (x' + y')(x' + z)(x + y)(y + z) \qquad \text{(Product of sums)} \end{aligned}$$

In each bracket of this expression all the three variable x, y, z are not present, therefore, we add 0 in each bracket to include these variables. Here in first bracket there is no z, therefore 0 which is not a variable will be replaced by $z.z'$. Similarly this method will be used for other brackets too.

Then $$f = (x + y + 0)(y + z + 0)(x' + z + 0)(x' + y' + 0)$$
$$= (x + y + z.z')(y + z + x.x')(x' + z + y.y')(x' + y' + z.z')$$
$$= (x + y + z)(x + y + z')(y + z + x)(y + z + x')$$
$$(x' + z + y)(x' + z + y')(x' + y' + z)(x' + y' + z')$$
$$= (x + y + z)(x + y + z')(x' + y + z)(x' + y' + z)(x' + y' + z') \quad ...(A)$$

because $(x + y + z)(x + y + z) = (x + y + z)$

$(x' + z + y)(x' + z + y) = (x' + z + y)$

$(x' + y' + z)(x' + y' + z) = (x' + y' + z)$

Above expression (A) is in conjunctive normal form.

Theorem 1. *Every function of Boolean algebra which is without constants is equal to a function in the conjunctive normal form.*

Proof. By using principle of duality the proof is immediate from theorem 1 of article 7.26.

Definition (Complete conjunctive normal form)

A conjunctive normal form in n-variables is said to be complete conjunctive normal form if there are 2^n distinct factors in it.

Theorem 2. *If the n-variables in the complete disjunctive normal form in n-variables are arbitrary but definitely 0 or 1 then in the complete disjunctive normal form exactly one term will have the value 0 while the value of rest of the terms shall be 1.*

Proof. The proof is immediate from Theorem 2 of article 7.26 on applying duality principle.

Procedure to convert a function in the conjunctive normal form.

First of all the parentheses are cleared from the prime terms and the function is factorized in linear factors. After this the additional terms like yy' are added according to necessity. Finally removing the repeated factors (same factors, so that they are not repeated) expansion in linear factors is done again.

ILLUSTRATIVE EXAMPLE

Example. *Write down the function $f = (xy' + xz)' + x'$ in conjunctive normal form.*

Solution.
$$f = (xy' + xz)' + x'$$
$$= (x' + y)(x' + z') + x'$$
$$= (x' + x' + y)(x' + x' + z')$$
$$= (x' + y)(x' + z')$$
$$= (x' + y + z.z')(x' + z' + y.y')$$
$$= (x' + y + z)(x' + y + z')(x' + z' + y)(x' + z' + y')$$
$$= (x' + y + z)(x' + y + z')(x' + y' + z')$$

NOTES

EXERCISE 4(D)

1. Express the following functions in conjunctive normal form in which minimum number of variables is used:

   (*i*) $x + x'y$ (*ii*) $xy' + xz + xy$

   (*iii*) $(x + y + z)(xy + x'z)'$ (*iv*) $xyz + (x + y)(x + z)$

   (*v*) $(x'y + xyz' + xy'z + x'y'z't + t')'$ (*vi*) $(x + y')(y + z')(z + x')(x' + y')$

   (*vii*) $(x + y)(x + y')(x' + z)$ (*viii*) $x'yz + xy'z' + x'y'z + x'yz' + xy'z + x'y'z'$.

2. Write down the following functions in conjunctive normal form in variables x, y, z :

   (*i*) x (*ii*) $x + y'$

   (*iii*) $x'z + xz'$ (*iv*) $(x + y)(x' + y')$

3. Convert the following functions in conjunctive normal form from disjunctive normal form:

   (*i*) $xy + x'y + x'y'$ (*ii*) $xyz + xy'z' + x'yz' + x'y'z + x'y'z'$

4. Convert the following functions in conjunctive normal form from disjunctive normal form :

   (*i*) $(u + v')(u' + v)(u' + v')$

   (*ii*) $(a + b + c)(a + b + c')(a + b' + c)(a' + b + c')(a' + b' + c)(a' + b' + c')$

Answers

1. (*i*) $x + y$

   (*ii*) $(x + y + z)(x + y + z')(x + y' + z)(x + y' + z')$

   (*iii*) $(x + y + z)(x + y + z')(x + y' + z')(x' + y' + z)(x' + y' + z')$

   (*iv*) $(x + y + z)(x + y + z')(x + y' + z)$

   (*v*) $(x + z)(x + z')(x' + z)$

2. (*i*) $(x + y + z)(x + y + z')(x + y' + z)(x + y' + z')$

   (*ii*) $(x + y' + z)(x + y' + z')$

   (*iii*) $(x + y' + z)(x + y' + z')(x' + y + z)(x' + y' + z)$

   (*iv*) $(x + y + z)(x + y + z')(x' + y' + z)(x' + y' + z')$

3. (*i*) $x' + y$

4. (*i*) $x'y'$

28. BOOLEAN MATRIX OPERATIONS

A matrix of m × n order is called a Boolean matrix if its entries are either zero or one. We shall define three operations on Boolean matrices which have useful application in computer science.

Three operations on Boolean Matrices.

(*i*) Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be m × n matrices. We define $A \vee B = C = [c_{ij}]$, the join of A and B by

$$c_{ij} = \begin{cases} 1 & if\ a_{ij} = 1\ or\ b_{ij} = 1 \\ 0 & if\ a_{ij}\ and\ b_{ij}\ are\ both\ zero \end{cases}$$

(*ii*) $A \wedge B = D = [d_{ij}]$, the meet of A and B is defined by

$$d_{ij} = \begin{cases} 1 & if\ a_{ij}\ and\ b_{ij}\ are\ both\ 1 \\ 0 & if\ a_{ij} = 0\ or\ b_{ij} = 0. \end{cases}$$

These operations are only possible when A and B have the same size, just as in the case of matrix addition. Instead of adding corresponding elements in A and B, to compute the entries of the result, we simply examine the corresponding elements for practical pattern.

ILLUSTRATIVE EXAMPLE

Example. *Let* $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ *and* $B = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

(a) Compute $A \vee B$ *(b) Compute* $A \wedge B$

Solution. (*a*) Let $A \vee B = [c_{ij}]$. Then, since a_{43} and b_{43} are both 0, we see that $c_{43} = 0$. In all other cases, either a_{ij} or b_{ij} is 1, and as such c_{ij} is also 1

$$A \vee B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

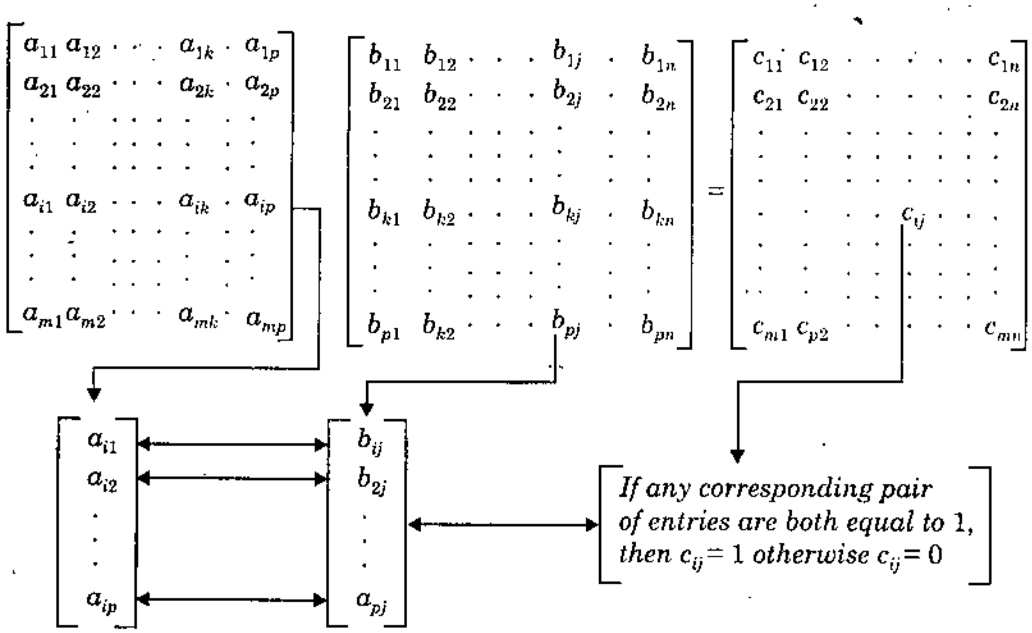
(*b*) Let $A \wedge B = [d_{ij}]$. Then since a_{11} and b_{11} both are 1, $d_{11} = 1$, and since a_{23} and b_{23} are both 1, $d_{23} = 1$. In all other cases either a_{ij} or b_{ij} is zero and hence $d_{ij} = 0$.

$$A \wedge B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

(*iii*) **Boolean Product:** Suppose that $A = [a_{ij}]$ is $m \times p$ Boolean matrix, $B = [b_{ij}]$ is a $p \times n$ Boolean matrix. Notice that the condition on the sizes of A and B is exactly the same condition as needed to form the matrix product AB. We now define another kind of product. The Boolean product of A and B, denoted by $A \odot B$, is the $m \times n$ Boolean matrix $C = [c_{ij}]$ defined by

$$c_{ij} = \begin{cases} 1 & \text{if } a_{ik} = 1 \text{ and } b_{kj} = 1 \text{ for some } k, 1 \le k \le p \\ 0 & \text{otherwise} \end{cases}$$

This multiplication is similar to ordinary matrix multiplication. The preceding formula states that for any i and j the element c_{ij} of $C = A \odot B$ can be computed in the following way



We can easily perform the indicated comparison and checks for each position of the Boolean product. Thus at least for students, the computation of elements in A ⊙ B is considerably easier than the computation of elements in AB.

ILLUSTRATIVE EXAMPLE

Example. *Let* $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ *and* $B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix}$,

Compute A ⊙ B.

Solution. Let A ⊙ B = $[e_{ij}]$. Then $e_{11} = 1$, since row 1 of A and column 1 of B each has a 1 as the first entry. Similarly, $e_{12} = 1$, since $a_{12} = 1$ and $b_{22} = 1$; that is, the first row of A and the second column of B have a 1 in the second position. In a similar way we see that $e_{13} = 1$. On the other hand $e_{14} = 0$, since row 1 of A and column 4 of B do not have common 1's in any position. Proceeding in this way, we obtain

$$A \odot B = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

Theorem 1. *If A, B and C are Boolean matrices of compatible sizes, then*

1. (*a*) A ∨ B = B ∨ A (*b*) A ∧ B = B ∧ A
2. (*a*) (A ∨ B) ∨ C = A ∨ (B ∨ C) (*b*) (A ∧ B) ∧ C = A ∧ (B ∧ C)
3. (*a*) A ∧ (B ∨ C) = (A ∧ B) ∨ (B ∧ C) (*b*) A ∨ (B ∧ C) = (A ∨ B) ∧ (B ∨ C)
4. (A ⊙ B) ⊙ C = A ⊙ (B ⊙ C)

NOTES

EXERCISE 4(E)

1. If A is an $m \times n$ matrix. Show that $I_m A = AI_n = A$
2. Let $A = \begin{bmatrix} 2 & 1 \\ 3 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 2 \\ 3 & 4 \end{bmatrix}$. Prove that $AB \neq BA$.
3. If $\begin{bmatrix} a+b & c+d \\ c-d & a-b \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 10 & 2 \end{bmatrix}$, find the values of a, b, c and d.
4. If $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, show that $A^2 = I_2$.
5. Prove that the j^{th} column of the matrix product AB is equal to the matrix product AB_j, where B_j is the j^{th} column of B.
6. Prove that $I_n^T = I_n$. (T stands for transpose).
7. If $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$.

   (a) Find A^3 (b) What is A^k.
8. If a and B are two symmetric matrices, show that

   (a) A + B is also symmetric. (b) Is AB also symmetric ?

Answer

3. $a = 3, b = 1, c = 8, d = -2$

SUMMARY

1. A set B with two binary operations on B denoted by (+) and (.) along with a binary operation on B denoted by (′) and two specific elements 0 and 1 of B, is called Boolean Algebra if the following axioms are satisfied :

   B_1 $a + b = b + a$, for all $a, b \in B$
   B_2 $a \,.\, b = b \,.\, a$, for all $a, b \in B$ } (Commutative laws).

   B_3 $a \,.\, (b + c) = (a \,.\, b) + (a \,.\, c)$, for $a, b, c \in B$
   B_4 $a + (b \,.\, c) = (a + b) \,.\, (a + c)$, for $a, b, c \in B$ }

   Distributive laws *i.e.*, both the operations are distributive one on the other.

   B_5 $a + 0 = a \ \forall \ a \in B$

   B_6 $a \,.\, 1 = a \ \forall \ a \in B$

   B_7 There exists a' in B such that
   $$a + a' = 1 \ \forall \ a \in B$$
   B_8 There exists a' in B such that
   $$a \,.\, a' = 0 \ \forall \ a \in B.$$
2. For all arbitrary different elements x, y and z of Boolean algebra B following four relations are valid.

   (i) $x \leq y$ *and* $y \leq z \Rightarrow x \leq z$

   (ii) $x \leq y$ *and* $y \leq z \Rightarrow x \leq yz$

   (iii) $x \leq y \Rightarrow x \leq y + z$

   (iv) $x \leq y \Leftrightarrow y' \leq x'$
3. **Inverse order.** If a relation R is a partially ordered then inverse relation R^{-1} is also partially ordered.
4. **Total order.** The word 'partial' is used in defining a partial order in a set A because some elements in A need not be comparable. If, on the contrary, every two elements in a partially ordered set A are comparable, then the partial order in A is called a total order in A.

NOTES

5. **Lower bound and greatest lower bound.** Any element a of a partially ordered set A is said to be a lower bound of a subset B of A, if for every $x \in$ B, a precedes x.

   If a lower bound of B dominates every other lower bound of B, then it is called the greatest lower bound or infimum of B.

6. **Upper bound and least upper bound.** An element $a \in$ A is said to be an upper bound of a subset B of a set A if for every $x \in$ B, a dominates x *i.e.*, $\forall x \in$ B, x precedes a.

   If an upper bound of B precedes every other bound of B, it is said to be least upper bound or supremum of B.

TEST YOURSELF

1. Prove that for all elements a and b of each Boolean algebra,
$$a + a'b = a + b.$$
2. Prove that for every three elements a, b, c of a Boolean algebra the identity
$$ab + bc + ca = (a + b)(b + c)(c + a)$$
   holds.
3. If $a + x = b + x$ and $a + x' = b + x'$, then prove that $a = b$.
4. If $a \cdot x = b \cdot x$ and $a \cdot x' = b \cdot x'$, then prove that $a = b$.
5. If elements a and b of a Boolean algebra B, satisfy the relation $a \leq b$ then prove that
$$a + bc = b(a + c), \forall c \in B.$$
   [This is called Modular law.]
6. Write down a function in three variables x, y, z which is 1 if either $x = y = 1$ and $z = 0$ or if $x = z = 1$ and $y = 0$, otherwise 0.
7. If A is a square matrix of order n, show that

   (*a*) AA^T and A^TA are symmetric (*b*) $A + A^T$ is symmetric

   where A^T is the transpose of matrix A.
8. Find two square matrices A and B of order 2, with $A \neq 0$ and $B \neq 0$, such that $AB = 0$, where 0 is zero matrix of order 2×2.

Answer

6. $xyz' + xy'z$

SECTION C

5. Matrices and Applications
6. Determinant
7. Permutations and Combinations
8. Probability

CHAPTER

5

MATRICES AND APPLICATIONS

NOTES

LEARNING OBJECTIVES

- Introduction
- Types of Matrices
- Matrix Addition and Subtraction
- Properties
- Matrix Multiplication
- Properties
- Identity Matrix
- Product of Matrices
- Matrix Transpose
- Inverse of a Matrix
- Solving Systems of Equations Using Matrices
- Eigen Values and Eigen Vectors

1. INTRODUCTION

Arthur Calyey was the first person to introduce the concept of matrices. Latter Eisenberg used matrices as a tool to explain his famous Quantum Principle. The whole world knew the importance of the application of it. Here we deal with some of the operations and properties of matrices.

A matrix is a collection of numbers ordered by rows and columns. It is customary to inclose the elements of a matrix in parenthesis, brackets, braces.

For example, $X = \begin{pmatrix} 5 & 8 & 2 \\ 7 & 1 & 5 \end{pmatrix}$

Hence each number within the array is called an element. The horizontal lines and the vertical lines formed by the elements are known as rows and columns respectively. If there are m rows and n columns in a matrix, then it is known as m by n matrix or a matrix of order $m \times n$. The above example has two rows and three columns. So it is known as 2×3 (read as 2 by 3) matrix.

Generally, the capital letters of English alphabets are assigned to denote matrix.

$$X = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$$

NOTES

The first subscript in a matrix refers to the row and the second subscript refers to the column. So the element a_{ij} is at the intersection of the i-th row and j-th column.

For example, construct a matrix of order 2×3 where $a_{ij} = i + 2j$.

Suppose the required matrix be

Now, $a_{11} = i + 2j = 1 + 2(1) = 3$ $\quad a_{12} = i + 2j = 1 + 2(2) = 5$

$a_{21} = i + 2j = 2 + 2(1) = 4$ $\quad a_{22} = i + 2j = 2 + 2(2) = 6$

$a_{31} = i + 2j = 3 + 2(1) = 5$ $\quad a_{32} = i + 2j = 3 + 2(2) = 7$

Hence the required matrix is

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix} = \begin{pmatrix} 3 & 5 \\ 4 & 6 \\ 5 & 7 \end{pmatrix}$$

Example. *Construct 3×3 matrix with elements a_{ij} such that $a_{ij} = i + 2j$.*

Solution. Let us consider a 3×3 matrix with elements a_{ij} as follows

$$x = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Here $a_{ij} = i + 2j$

Hence $a_{11} = 1 + 2.1 = 3; a_{12} = 1 + 2.2 = 5; a_{13} = 1 + 2.3 = 7$

$a_{21} = 2 + 2.1 = 4; a_{22} = 2 + 2.2 = 6; a_{23} = 2 + 2.3 = 8$

$a_{31} = 3 + 2.1 = 5; a_{32} = 3 + 2.2 = 7; a_{33} = 3 + 2.3 = 9$

Hence the matrix is

$$x = \begin{pmatrix} 3 & 5 & 7 \\ 4 & 6 & 8 \\ 5 & 7 & 9 \end{pmatrix}$$

2. TYPES OF MATRICES

Square Matrix. A matrix is called a square matrix of the number of rows is equal to number of columns.

For example, $A = \begin{pmatrix} 5 & 6 \\ 2 & 3 \end{pmatrix}$ $B = \begin{pmatrix} 1 & 2 & 3 \\ 6 & 7 & 8 \\ 4 & 8 & 7 \end{pmatrix}$

Row Matrix. *A* matrix of order $i \times m$ is called a row matrix.

For example, (5 6), (5 6 3) of order 1×2 and 1×3 respectively.

Column Matrix. A matrix of order $m \times 1$ is known as column matrix.

$\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ are column matrices of order 2×1 and 3×1 respectively.

Zero Matrix. If all the elements of a matrix are zero , then it is known as zero matrix denoted by (0). But, this matrix may be of any order.

NOTES

For example, $\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

Unit Matrix. The square matrix whose elements on its main diagonal (left top to right botom) are '1's and the rest of its elements are zeros is known as unit matrix.

For example, (1), $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Singular and Non-singular Matrices. A square matrix *A is called a singular* matrix *iff* its determinant is zero and is called non-singular (or regular) matrix if determinant is not equal to zero.

For example, $A = \begin{pmatrix} 2 & 3 \\ 4 & 6 \end{pmatrix} \Rightarrow \det A = \begin{vmatrix} 2 & 3 \\ 4 & 6 \end{vmatrix} = 12 - 12 = 0$

$\Rightarrow$ A is a singular matrix.

If $A = \begin{bmatrix} 1 & 5 \\ 2 & 3 \end{bmatrix} \Rightarrow \det A = \begin{vmatrix} 1 & 5 \\ 2 & 3 \end{vmatrix} = 3 - 10 = -7$

$\Rightarrow$ A is a non-singular matrix.

Symmetric Matrix. *A* symmetric matrix is a square matrix in which $X_{ij} = X_{ji}$ for all i and j.

For example, $A = \begin{pmatrix} 8 & 2 & 4 \\ 2 & 4 & 1 \\ 4 & 1 & 6 \end{pmatrix}$ $B = \begin{pmatrix} 8 & 2 & 4 \\ 1 & 4 & 1 \\ 4 & 2 & 6 \end{pmatrix}$

Matrix A is symmetric; Matrix B is not symmetric

Diagonal Matrix. *A* diagonal matrix is a symmetric matrix where all of diagonal elements are 0.

$$A = \begin{pmatrix} 7 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 9 \end{pmatrix}$$

Matrix A is diagonal.

Upper Triangular and Lower Triangular Matrix. A square matrix $A = [a_{ij}]$ is called upper triangular matrix if all the elements below the main diagonal are zero *i.e.,* if $a_{ij} = 0$ for all $i > j$

For example, $A = \begin{pmatrix} 1 & 5 & 1 \\ 0 & 4 & 2 \\ 0 & 0 & 3 \end{pmatrix}$

Similarly, a square matrix is called lower triangular matrix if all the elements above the main diagonal are zero *i.e.,* if $a_{ij} = 0$ for all $i < j$.

For example, $A = \begin{pmatrix} 5 & 0 & 0 \\ 3 & 1 & 0 \\ 6 & 1 & 2 \end{pmatrix}$

3. MATRIX ADDITION AND SUBTRACTION

NOTES

Definition : Two matrices A and B can be added or subtracted if and only if their dimensions are the same. (*i.e.,* both matrices have the identical amount of rows and columns.

Let us take

$$A = \begin{pmatrix} 1 & 2 & -3 \\ 3 & 1 & 4 \\ 5 & -2 & -1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 2 & -1 & 1 \\ -2 & 0 & -3 \\ -3 & 2 & -1 \end{pmatrix}$$

Addition. If A and B above are matrices of the same type, then the sum is found by adding the corresponding elements $a_{ij} + b_{ij}$.

Here is an example of adding A and B, together

$$A + B = \begin{pmatrix} 1 & 2 & -3 \\ 3 & 1 & 4 \\ 5 & -2 & -1 \end{pmatrix} + \begin{pmatrix} 2 & -1 & 1 \\ -2 & 0 & -3 \\ -3 & 2 & -1 \end{pmatrix} = \begin{pmatrix} 3 & 1 & -2 \\ 1 & 1 & 1 \\ 2 & 0 & 2 \end{pmatrix}$$

Subtraction. If A and B are matrices of the same type, then the subtraction is found by subtracting the corresponding elements $a_{ij} + b_{ij}$.

Here is an example of subtracting matrices ,

$$A - B = \begin{pmatrix} 1 & 2 & -3 \\ 3 & 1 & 4 \\ 5 & -2 & -1 \end{pmatrix} - \begin{pmatrix} 2 & -1 & 1 \\ -2 & 0 & -3 \\ -3 & 2 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 3 & -4 \\ 5 & 1 & 7 \\ 8 & 0 & 0 \end{pmatrix}$$

ILLUSTRATIVE EXAMPLES

Example 1. *If* $A = \begin{pmatrix} 1 & 2 & -3 \\ 3 & 1 & 4 \\ 5 & -2 & -1 \end{pmatrix}$ *and* $B = \begin{pmatrix} 2 & -1 & 1 \\ -2 & 0 & 3 \\ 3 & 2 & 2 \end{pmatrix}$

Then find out $A + B$ *and* $A - B$.

Solution. Given $A = \begin{pmatrix} 1 & 2 & -3 \\ 3 & 1 & 4 \\ 5 & -2 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & -1 & 1 \\ -2 & 0 & 3 \\ 3 & 2 & 2 \end{pmatrix}$

$$A + B = \begin{pmatrix} 1 & 2 & -3 \\ 3 & 1 & 4 \\ 5 & -2 & -1 \end{pmatrix} + \begin{pmatrix} 2 & -1 & 1 \\ -2 & 0 & 3 \\ 3 & 2 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1+2 & 2-1 & -3+1 \\ 3-2 & 1+0 & 4+3 \\ 5+3 & -2+2 & -1+2 \end{pmatrix} = \begin{pmatrix} 3 & 1 & -2 \\ 1 & 1 & 7 \\ 8 & 0 & 1 \end{pmatrix}$$

$$A - B = \begin{pmatrix} 1 & 2 & -3 \\ 3 & 1 & 4 \\ 5 & -2 & -1 \end{pmatrix} - \begin{pmatrix} 2 & -1 & 1 \\ -2 & 0 & 3 \\ 3 & 2 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1-2 & 2+1 & -3-1 \\ 3+2 & 1-0 & 4-3 \\ 5-3 & -2-2 & -1-2 \end{pmatrix} = \begin{pmatrix} -1 & 3 & -4 \\ 5 & 1 & 1 \\ 2 & -4 & -3 \end{pmatrix}.$$

NOTES

Example 2. *If* $A = \begin{pmatrix} 1 & -2 \\ 3 & 2 \end{pmatrix}, B = \begin{pmatrix} 2 & -1 \\ -2 & 3 \end{pmatrix}, C = \begin{pmatrix} 1 & 0 \\ -3 & -1 \end{pmatrix}$

Find (*i*) $A + B$ (*ii*) $A - 2B$ (*iii*) $A - 3B + 4C$

Solution. Given $A = \begin{pmatrix} 1 & -2 \\ 3 & 2 \end{pmatrix}, B = \begin{pmatrix} 2 & -1 \\ -2 & 3 \end{pmatrix}, C = \begin{pmatrix} 1 & 0 \\ -3 & -1 \end{pmatrix}$

(*i*) $$A + B = \begin{pmatrix} 1 & -2 \\ 3 & 2 \end{pmatrix} + \begin{pmatrix} 2 & -1 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} 3 & -3 \\ 1 & 5 \end{pmatrix}$$

(*ii*) $$A - 2B = \begin{pmatrix} 1 & -2 \\ 3 & 2 \end{pmatrix} - 2\begin{pmatrix} 2 & -1 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 3 & 2 \end{pmatrix} - \begin{pmatrix} 4 & -2 \\ -4 & 6 \end{pmatrix}$$

$$= \begin{pmatrix} 1-4 & -2+2 \\ 3+4 & 2-6 \end{pmatrix} = \begin{pmatrix} -3 & 0 \\ 7 & -4 \end{pmatrix}$$

(*iii*) $$A - 3B + 4C = \begin{pmatrix} 1 & -2 \\ 3 & 2 \end{pmatrix} - 3\begin{pmatrix} 2 & -1 \\ -2 & 3 \end{pmatrix} + 4\begin{pmatrix} 1 & 0 \\ -3 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -2 \\ 3 & 2 \end{pmatrix} - \begin{pmatrix} 6 & -3 \\ -6 & 9 \end{pmatrix} + \begin{pmatrix} 4 & 0 \\ -12 & -4 \end{pmatrix}$$

$$= \begin{pmatrix} -5 & 1 \\ 9 & -7 \end{pmatrix} + \begin{pmatrix} 4 & 0 \\ -12 & -4 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ -3 & -11 \end{pmatrix}.$$

Example 3. *If* $A = \begin{pmatrix} 1 & 0 & 2 \\ 3 & 0 & 5 \end{pmatrix}, B = \begin{pmatrix} 6 & 9 & 10 \\ 5 & 3 & 2 \end{pmatrix}$

Then find $A + B, A + 2B, A - B$

Solution. Given $A = \begin{pmatrix} 1 & 0 & 2 \\ 3 & 0 & 5 \end{pmatrix}, B = \begin{pmatrix} 6 & 9 & 10 \\ 5 & 3 & 2 \end{pmatrix}$

Then $$A + B = \begin{pmatrix} 1 & 0 & 2 \\ 3 & 0 & 5 \end{pmatrix} + \begin{pmatrix} 6 & 9 & 10 \\ 5 & 3 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1+6 & 0+9 & 2+10 \\ 3+5 & 0+3 & 5+2 \end{pmatrix}$$

$$= \begin{pmatrix} 7 & 9 & 12 \\ 8 & 3 & 7 \end{pmatrix}$$

$$A + 2B = \begin{pmatrix} 1 & 0 & 2 \\ 3 & 0 & 5 \end{pmatrix} + 2\begin{pmatrix} 6 & 9 & 10 \\ 5 & 3 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 2 \\ 3 & 0 & 5 \end{pmatrix} + \begin{pmatrix} 12 & 18 & 20 \\ 10 & 6 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 13 & 18 & 22 \\ 13 & 6 & 9 \end{pmatrix}$$

$$A - B = \begin{pmatrix} 1 & 0 & 2 \\ 3 & 0 & 5 \end{pmatrix} + \begin{pmatrix} 6 & 9 & 10 \\ 5 & 3 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1-6 & 0-9 & 2-10 \\ 3-5 & 0-3 & 5-2 \end{pmatrix} = \begin{pmatrix} -5 & -9 & -8 \\ -2 & -3 & 3 \end{pmatrix}.$$

4. PROPERTIES

Commutative. The addition of matrices is commutative, that is, if A and B are two matrices of same order, then $A + B = B + A$.

From the definition of addition of matrices, it follows that $A + B$ and $B + A$ are of same order. Further if $A = (a_{ij})$ and $B = (b_{ij})$, then $A + B = (a_{ij} + b_{ij})$

and $B + A = (b_{ij} + a_{ij})$

But, $(a_{ij} + b_{ij}) = (b_{ij} + a_{ij})$

i.e., each element of $(A + B)$ is equal to the corresponding element of $(B + A)$.

Hence the result.

Associative. The matrix addition is associative *i.e.*, if A, B and C are three matrices of same order, then $A + (B + C) = (A + B) + C$

Since A, B and C are of same order

$A + (B + C)$ and $(A + B) + C$ are of the same order,

Further, if $A = (a_{ij})$, $B = (b_{ij})$ and $C = (c_{ij})$,

Then $A + (B + C) = a_{ij} + (b_{ij} + c_{ij})$ and $(A + B) + C = (a_{ij} + b_{ij}) + c_{ij}$

But, $a_{ij} + (b_{ij} + c_{ij}) = (a_{ij} + b_{ij}) + c_{ij}$

i.e., each element of $A + (B + C)$ and $(A + B) + C$ are equal *i.e.*, the addition of matrices is associative.

Additive Identity. The identity matrix for addition is the zero matrix or null matrix denoted by '0'. Thus, if A is a matrix, then

$A + 0 = A$, provided the order of the zero matrix is same as that of A.

Thus, $\begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}$

and $(x \quad y) + (0 \quad 0) = (x \quad y)$

Additive Inverse. The matrix in which each element is the negative of the corresponding element of a given matrix, A is called the inverse of A and is denoted by $(-A)$.

Thus if $$A = \begin{pmatrix} 2 & 1 & 0 \\ -3 & 2 & 1 \\ 1 & 2 & 0 \end{pmatrix}$$

NOTES

Then $(-A) = \begin{pmatrix} -2 & -1 & 0 \\ 3 & -2 & -1 \\ -1 & -2 & 0 \end{pmatrix}$

Further, $A + (-A) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

and $A + (-A) = (-A) + A$

i.e., $A + (-A) = 0 = (-A) + A$

5. MATRIX MULTIPLICATION

Feasibility. *For matrices A and B, AB is possible only when number of columns* of A = number of rows of B and then the product AB has as many rows as A has and as many columns as B has.

Procedure. The $(i,j)^{th}$ element of the product AB is obtained by dot multiplication of i^{th} row of the first matrix with j^{th} column of the 2nd.

Here is an example of matrix multiplication for two 2×2 matrices.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}$$

Here is an example of matrix multiplication for a 3×3 matrices.

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\begin{pmatrix} j & k & l \\ m & n & o \\ p & q & r \end{pmatrix} = \begin{pmatrix} aj+bm+cp & ak+bn+cq & ai+bo+cr \\ dj+em+fp & dk+en+fq & di+eo+fr \\ gj+hm+ip & gk+hn+iq & gi+ho+ir \end{pmatrix}$$

Now, lets look at the $n \times n$ matrix case, where A has dimensions $m \times n$, B has dimensions $n \times p$. Then the product of A and B is the matrix C, which has dimensions $m \times p$. The ij^{th} element of matrix C is found by multiplying the entries of the i-th row of A with the corresponding entries in the j-th column of B and summing the n terms. The elements of C are,

$$C_{11} = a_{11}b_{11} + a_{12}b_{21} + \ldots\ldots\ldots\ldots + a_{1n}b_{n1} = a_{ij}b_{ji}$$

$$C_{12} = a_{11}b_{12} + a_{12}b_{22} + \ldots\ldots\ldots\ldots + a_{1n}b_{n2}$$

$$C_{mp} = a_{m1}\, b_{1p} + a_{m2}\, b_{2p} + \ldots\ldots\ldots\ldots + a_{mn}b_{np}$$

NOTE That $A \times B$ is not the same as $B \times A$.

6. PROPERTIES

Commutative. The multiplication of matrices is not commutative *i.e.*, if A and B are two matrices, then A may not be equal to BA.

From the defination of product of two matrices, it follows that if A and B are of different orders and the product AB is defined, then BA may not be defined. Similarly, whenever BA is defined, AB matrix be defined. Hence both the products AB and BA are defined when A and B are square matrices of same order.

NOTES

For example, let $A = \begin{pmatrix} 1 & 2 \\ -1 & -2 \end{pmatrix}, B = \begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}$.

Then
$$AB = \begin{pmatrix} 1 & 2 \\ -1 & -2 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}$$
$$= \begin{pmatrix} 1\times 2+2\times 1 & 1\times(-1)+2\times 3 \\ -1\times 2+(-2)\times 1 & (-1)\times(-1)+(-2)\times 3 \end{pmatrix}$$
$$= \begin{pmatrix} 4 & 5 \\ -4 & -5 \end{pmatrix}$$
$$BA = \begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ -1 & -2 \end{pmatrix}.$$
$$= \begin{pmatrix} 2\times 1+(-1)\times(-1) & 2\times 2+(-1)\times(-2) \\ 1\times 1+3\times(-1) & 1\times 2+3\times(-2) \end{pmatrix}$$
$$= \begin{pmatrix} 3 & 6 \\ -2 & -4 \end{pmatrix}$$

From the above, we get $AB \neq BA$.

But when $A = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}, B = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

Then
$$AB = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}$$
$$BA = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}$$
$$AB = BA.$$

Associative. The multiplication of matrices is associative *i.e.*, if A, B, C are three matrices, then $AB(C) = A(BC)$, provided the products are defined.

For example, let $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}, B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}, C = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$.

Then
$$AB = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}\begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} = \begin{pmatrix} a_1b_1+a_2b_3 & a_1b_2+a_2b_4 \\ a_3b_1+a_3b_3 & a_3b_2+a_4b_4 \end{pmatrix}$$
$$AB(C) = \begin{pmatrix} a_1b_1+a_2b_3 & a_1b_2+a_2b_4 \\ a_3b_1+a_3b_3 & a_3b_2+a_4b_4 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$
$$= \begin{pmatrix} a_1b_1c_1+a_2b_3c_1+a_1b_2c_2+a_2b_4c_2 \\ a_3b_1c_1+a_3b_3c_1+a_3b_2c_2+a_4b_4c_2 \end{pmatrix}$$
$$BC = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} b_1c_1+b_2c_2 \\ b_3c_1+b_3c_2 \end{pmatrix}$$
$$A(BC) = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}\begin{pmatrix} b_1c_1+b_2c_2 \\ b_3c_1+b_4c_2 \end{pmatrix}$$

$$= \begin{pmatrix} a_1b_1c_1 + a_1b_2c_2 + a_2b_3c_1 + a_2b_4c_2 \\ a_3b_1c_1 + a_3b_2c_2 + a_4b_3c_1 + a_4b_4c_2 \end{pmatrix}$$

From the above, we get

$$(AB)C = A(BC)$$

NOTES

Example. *If* $A = \begin{pmatrix} 1 & 0 & 2 \\ 2 & -1 & -2 \\ 1 & 2 & -3 \end{pmatrix}, B = \begin{pmatrix} -1 & 1 & 0 \\ -2 & 0 & 1 \\ 3 & -1 & -2 \end{pmatrix}, C = \begin{pmatrix} 1 & 0 \\ 2 & -1 \\ 0 & 1 \end{pmatrix}$

Then show that, $(AB)C = A(BC)$.

Solution.

$$AB = \begin{pmatrix} 1 & 0 & 2 \\ 2 & -1 & -2 \\ 1 & 2 & -3 \end{pmatrix}\begin{pmatrix} -1 & 1 & 0 \\ -2 & 0 & 1 \\ 3 & -1 & -2 \end{pmatrix} = \begin{pmatrix} 5 & -1 & -4 \\ -6 & 4 & 3 \\ -14 & 4 & 8 \end{pmatrix}$$

$$AB(C) = \begin{pmatrix} 5 & -1 & -4 \\ -6 & 4 & 3 \\ -14 & 4 & 8 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 2 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & -3 \\ 2 & -1 \\ 6 & 4 \end{pmatrix}$$

7. IDENTITY MATRIX

The identity matrix for multiplication for the set of all square matrices of a given order is the unit matrix of the same order.

For example, let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ a square matrix of order 2.

Taking $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, it can be easily shown that $AI = A = IA$.

$$AI = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a \times 1 + b \times 0 & a \times 0 + b \times 1 \\ c \times 1 + d \times 0 & c \times 0 + d \times 0 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$IA = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 \times a + 0 \times c & 1 \times b + 0 \times d \\ 0 \times a + 1 \times c & 0 \times b + 1 \times d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Similarly, if A is a square matrix of order 3,

Taking $I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, it can be proved that

$\therefore$ $AI = A = IA$.

Cancellation Law. The cancellation law does not hold for matrix multiplication *i.e.,* $CA = CB$ does not imply $A = B$, even if the products are defined.

For example, let $A = \begin{pmatrix} -2 & 3 \\ 1 & -5 \\ 4 & 1 \end{pmatrix}, B = \begin{pmatrix} 4 & 1 \\ -4 & -3 \\ 3 & 1 \end{pmatrix}, C = \begin{pmatrix} 1 & 1 & 1 \\ 3 & 3 & 3 \end{pmatrix}$

NOTES

Then, $$CA = \begin{pmatrix} 1 & 1 & 1 \\ 3 & 3 & 3 \end{pmatrix} \begin{pmatrix} -2 & 3 \\ 1 & -5 \\ 4 & 1 \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ 9 & -3 \end{pmatrix}$$

$$CB = \begin{pmatrix} 1 & 1 & 1 \\ 3 & 3 & 3 \end{pmatrix} \begin{pmatrix} 4 & 1 \\ -4 & -3 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ 9 & -3 \end{pmatrix}$$

$\therefore$ $CA = CB$, when $A \neq B$.

Distributive Law. The distributive laws hold for matrices *i.e.*, if A, B and C are three matrices, then

$$A(B + C) = AB + AC, \quad (A + B)\,C = AC + BC$$

Provided, the addition and multiplication in above equations are defined.

For example, let $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}, B = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, C = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$

Then $$B + C = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} + \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} b_1 + c_1 \\ b_2 + c_2 \end{pmatrix}$$

and $$A(B + C) = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \begin{pmatrix} b_1 + c_1 \\ b_2 + c_2 \end{pmatrix} = \begin{pmatrix} a_1b_1 + a_1c_1 + a_2b_2 + a_2c_2 \\ a_3b_1 + a_3c_1 + a_4b_2 + a_4c_2 \end{pmatrix}$$

$$AB = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} a_1b_1 + a_2b_2 \\ a_3b_1 + a_4b_2 \end{pmatrix}$$

$$AC = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} a_1c_1 + a_2c_2 \\ a_3c_1 + a_4c_2 \end{pmatrix}$$

Then, $$AB + AC = \begin{pmatrix} a_1b_1 + a_2b_2 \\ a_3b_1 + a_4b_2 \end{pmatrix} + \begin{pmatrix} a_1c_1 + a_2c_2 \\ a_3c_1 + a_4c_2 \end{pmatrix} = \begin{pmatrix} a_1b_1 + a_2b_2 + a_1c_1 + a_2c_2 \\ a_3b_1 + a_4b_2 + a_3c_1 + a_4c_2 \end{pmatrix}$$

$\therefore$ $A(B + C) = AB + AC$

Taking $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}, B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}$, and $C = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$

Similar way, we can prove that,

$$(A + B)C = AC + BC$$

8. PRODUCT OF MATRICES

Theorem 1. *If A and B are two square matrices of the same order, then*

$$det\,(AB) = (det\,A)\,(det\,B)$$

Proof. Let us consider two matrices A and B of order 2×2 as

$$A = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} \text{ and } B = \begin{pmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{pmatrix}$$

Now, $AB = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} \begin{pmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{pmatrix} = \begin{pmatrix} \alpha_{11}\beta_{11} + \alpha_{12}\beta_{21} & \alpha_{11}\beta_{12} + \alpha_{12}\beta_{22} \\ \alpha_{21}\beta_{11} + \alpha_{22}\beta_{21} & \alpha_{21}\beta_{12} + \alpha_{22}\beta_{22} \end{pmatrix}$

$\Rightarrow$ $\det(AB) = (\alpha_{11}\beta_{11} + \alpha_{12}\beta_{21})(\alpha_{21}\beta_{12} + \alpha_{22}\beta_{22}) - (\alpha_{11}\beta_{12} + \alpha_{12}\beta_{22})(\alpha_{21}\beta_{11} + \alpha_{22}\beta_{21})$

$= \alpha_{11}\beta_{11}\alpha_{22}\beta_{22} + \alpha_{12}\beta_{21}\alpha_{21}\beta_{21} - \alpha_{12}\beta_{22}\alpha_{21}\beta_{11} - \alpha_{11}\beta_{12}\alpha_{22}\beta_{21}$

R.H.S. $A = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix}, B = \begin{pmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{pmatrix}$

Then $\det A = (\alpha_{11}\alpha_{22} + \alpha_{12}\alpha_{21})$ and $\det B = (\beta_{11}\beta_{22} + \beta_{12}\beta_{21})$

$\Rightarrow (\det A)(\det B) = (\alpha_{11}\alpha_{22} + \alpha_{12}\alpha_{21})(\beta_{11}\beta_{22} + \beta_{12}\beta_{21})$

$= \alpha_{11}\alpha_{22}\alpha_{11}\alpha_{22} - \alpha_{11}\alpha_{22}\beta_{12}\beta_{21} - \alpha_{12}\alpha_{21}\beta_{11}\beta_{22} + \alpha_{12}\alpha_{21}\beta_{12}\beta_{21}$

$= \det(AB)$ **Proved.**

Theorem 2. *If A is a non-singular matrix,*

then $\det(A)^{-1} = (\det A)^{-1} = \dfrac{1}{\det A}$

Proof. Suppose A is a non-singular matrix,

$AA^{-1} = I$

$\Rightarrow$ $\det(AA^{-1}) = \det I = 1$

$\Rightarrow$ $(\det A)(\det A^{-1}) = 1$

Since, we know that if A and B are two square matrices of the same order, then

$\det(AB) = (\det A)(\det B)$

$\Rightarrow$ $(\det A^{-1}) = \dfrac{1}{\det A} = (\det A)^{-1}$ as $\det A \neq 0$ **Proved.**

Example. *If* $A = \begin{pmatrix} 2 & 1 & -2 \\ 1 & 2 & 3 \end{pmatrix}, B = \begin{pmatrix} -1 & -2 & -3 \\ 0 & -2 & 1 \\ 3 & 11 & 0 \end{pmatrix}$, *then find AB.*

Solution. Given $A = \begin{pmatrix} 2 & 1 & -2 \\ 1 & 2 & 3 \end{pmatrix}, B = \begin{pmatrix} -1 & -2 & -3 \\ 0 & -2 & 1 \\ 3 & 11 & 0 \end{pmatrix}$

$$AB = \begin{pmatrix} 2 & 1 & -2 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} -1 & -2 & -3 \\ 0 & -2 & 1 \\ 3 & 11 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 2\times(-1) + 1\times 0 + (-2)\times 3 & 2\times(-2) + 1\times(-2) + (-2)\times 11 & 2\times(-3) + 1\times 1 + (-2)\times 0 \\ 1\times(-1) + 2\times 0 + 3\times 3 & 1\times(-2) + 2\times(-2) + 3\times 11 & 1\times(-3) + 2\times 1 + 3\times 0 \end{pmatrix}$$

$$= \begin{pmatrix} -8 & -28 & -5 \\ 8 & 27 & -1 \end{pmatrix}$$

9. MATRIX TRANSPOSE

Transpose of an $m \times n$ matrix A is the matrix of order $n \times m$ obtained by interchanging the rows and columns of $A = (a_{ij})$. The transpose of a matrix is denoted by a prime (A′) or a superscript t or T ($A^t = A^T$).

For example. The transpose of a matrix would be,

If $$A = \begin{pmatrix} 2 & 1 & 2 & 0 \\ -2 & 1 & 3 & 1 \\ 1 & -3 & 1 & 2 \end{pmatrix}, A^T = \begin{pmatrix} 2 & -2 & 1 \\ 1 & 1 & -3 \\ 2 & 3 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

$$A = \begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix}, A^T = \begin{pmatrix} 2 & 4 \\ 3 & 1 \end{pmatrix}$$

$$A = \begin{pmatrix} 4 & 1 & 2 \\ 5 & 6 & 3 \\ 7 & 4 & 8 \end{pmatrix}, A^T = \begin{pmatrix} 4 & 5 & 7 \\ 1 & 6 & 4 \\ 2 & 3 & 8 \end{pmatrix}$$

Example. *If* $A = \begin{pmatrix} -1 & 3 \\ 2 & 4 \end{pmatrix}, B = \begin{pmatrix} -3 & -2 \\ 1 & 2 \end{pmatrix}$, *show that*

(*i*) $(A')' = A$ (*ii*) $(A + B)' = A' + B'$ (*iii*) $(AB)' = B'A'$

Solution. (*i*) If $A = \begin{pmatrix} -1 & 3 \\ 2 & 4 \end{pmatrix} \Rightarrow A' = \begin{pmatrix} -1 & 2 \\ 3 & 4 \end{pmatrix}$

$$(A')' = \begin{pmatrix} -1 & 3 \\ 2 & 4 \end{pmatrix} \Rightarrow (A')' = A$$

(*ii*) Here $A = \begin{pmatrix} -1 & 3 \\ 3 & 4 \end{pmatrix}, B = \begin{pmatrix} -3 & -2 \\ 1 & 2 \end{pmatrix}$

We have to show that,

$$(A + B)' = A' + B'$$

L.H.S. $$(A + B) = \begin{pmatrix} -1 & 3 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} -3 & -2 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} -4 & 1 \\ 3 & 6 \end{pmatrix}$$

$$(A + B)' = \begin{pmatrix} -4 & 3 \\ 1 & 6 \end{pmatrix}$$

R.H.S. $$A' = \begin{pmatrix} -1 & 2 \\ 3 & 4 \end{pmatrix}, B' = \begin{pmatrix} -3 & 1 \\ -2 & 2 \end{pmatrix}$$

$$A' + B' = \begin{pmatrix} -1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} -3 & 1 \\ -2 & 2 \end{pmatrix} = \begin{pmatrix} -4 & 3 \\ 1 & 6 \end{pmatrix}$$

From the above, we get $(A + B)' = A' + B'$

(*iii*) Here given that $A = \begin{pmatrix} -1 & 3 \\ 2 & 4 \end{pmatrix}, B = \begin{pmatrix} -3 & -2 \\ 1 & 2 \end{pmatrix}$

We have to show that, $(AB)' = B'A'$

L.H.S. $$AB = \begin{pmatrix} -1 & 3 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} -3 & -2 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 6 & 8 \\ -2 & 4 \end{pmatrix}$$

$$(AB)' = \begin{pmatrix} 6 & -2 \\ 8 & 4 \end{pmatrix}$$

L.H.S. $$B' = \begin{pmatrix} -3 & 1 \\ -2 & 2 \end{pmatrix}, A' = \begin{pmatrix} -1 & 2 \\ 3 & 4 \end{pmatrix}$$

$$B'A' = \begin{pmatrix} -3 & 1 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} -1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 6 & -2 \\ 8 & 4 \end{pmatrix}$$

Hence, $(AB)' = B'A'$

Now, it follows immediately from the definition of transpose that,

Theorem. *If A and B are two N × n matrices, then*

(*i*) $(A + B)^T = A^T + B^T$

(*ii*) $(\alpha A)^T = \alpha A^T$ *where* α *is any scalar*

(*iii*) $(A^T) = A$

The proof is left to the reader.

Theorem. *If A is a non-singular square matrix, then* A^T *is also non-singular, and* $(A^T)^{-1} = (A^{-1})^T$.

Proof. Suppose A is a non-singular square matrix. Let there exists a matrix B such that

$$AB = I = BA$$

$$\Rightarrow \quad (AB)^T = I^T = (BA)^T$$

$$\Rightarrow \quad A^T B^T = I = A^T B^T$$

$\Rightarrow A$ has the inverse *i.e.*, B^T

and $$(A^T)^{-1} = B^T = (A^{-1})^T \text{ as } B = A^{-1}$$ **Proved.**

10. INVERSE OF A MATRIX

Definition : Assuming, we have a square matrix A, which is non-singular (*i.e.*, det (A) does not equal zero), then there exists an $n \times n$ matrix A^{-1} which is called the inverse of A, such that this property holds :

$$AA^{-1} = A^{-1}A = I \text{ where } I \text{ is the identity matrix.}$$

Inverse of a 2 × 2 matrix

Take for example, a arbitrary 2 × 2 matrix A whose determinant $(ad - bc)$ is not equal to zero.

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

NOTES

where a, b, c, d are numbers. The inverse is :

$$A^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Inverse of a $n \times n$ matrix

The inverse of a general $n \times n$ matrix A can be found by using the following equation

$$A^{-1} = \frac{Adj\,(A)}{Det\,(A)}$$

where $Adj\,(A)$ denotes the adjoint (or adjugate) of a matrix. It can be calculated by the following method.

1. Given the $n \times n$ matrix A, define

$$B = (b_{ij})$$

to be the matrix whose co-efficients are found by taking the determinant of the $(n^{-1}) \times (n^{-1})$ matrix obtained by deleting the i^{th} row and j^{th} column of A. The terms of B (*i.e.*, $B = b_{ij}$) are known as the co-factors of A.

2. And define the matrix C, where

$$C_{ij} = (-1)^{i+j} b_{ij}.$$

3. The transpose of C (*i.e.*, C^T) is called the adjoint of matrix A.

   Lastly, to find the inverse of A, divide the matrix C^T by the determinant of A to give its inverse.

ILLUSTRATIVE EXAMPLES

Example 1. *Find the inverse of the following 2 × 2 matrix*

(*a*) $\begin{pmatrix} 1 & 4 \\ -1 & 0 \end{pmatrix}$ (*b*) $\begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}$ (*c*) $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

Solution. (*a*) Let $A = \begin{pmatrix} 1 & 4 \\ -1 & 0 \end{pmatrix}$

$\therefore$ $|A| = \text{det. } A = \begin{vmatrix} 1 & 4 \\ -1 & 0 \end{vmatrix} = 0 + 4 = 4 \neq 0.$

Hence A^{-1} exists.

(*b*) Let $A = \begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}$

$\therefore$ $|A| = \text{det. } A = \begin{vmatrix} 3 & 5 \\ 1 & 2 \end{vmatrix} = 6 - 5 = 1 \neq 0$

Hence A^{-1} exists.

(*c*) Let $A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

$$|A| = \det A = \begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix} = -1 - 0 \neq 0$$

Hence A^{-1} exists.

Example 2. *Find the inverse of the following 3 × 3 matrix.*

(a) $\begin{pmatrix} 1 & 2 & 5 \\ 2 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{pmatrix}$

NOTES

Solution. (a) Let $A = \begin{pmatrix} 1 & 2 & 5 \\ 2 & 3 & 1 \\ -1 & 1 & 1 \end{pmatrix}$

Then we find $|A| = \begin{vmatrix} 1 & 2 & 5 \\ 2 & 3 & 1 \\ -1 & 1 & 1 \end{vmatrix} = 1\begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix} - 2\begin{vmatrix} 2 & 1 \\ -1 & 1 \end{vmatrix} + 5\begin{vmatrix} 2 & 3 \\ -1 & 1 \end{vmatrix}$

$$= 1(3 - 1) - 2(2 + 1) + 5(2 + 3) = 2 - 6 + 25 = 21$$

Since $|A| \neq 0$, therefore A^{-1} exist.

We know that $A^{-1} = \dfrac{Adj\ A}{|A|}$

For this, we find co-factors of A.

$$A_{11} = (-1)^{1+1} M_{11} = (-1)^2 \begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix} = 2$$

$$A_{21} = (-1)^{1+2} M_{12} = (-1)^3 \begin{vmatrix} 2 & 5 \\ 1 & 1 \end{vmatrix} = 3$$

$$A_{31} = (-1)^{1+3} M_{31} = (-1)^4 \begin{vmatrix} 2 & 5 \\ 3 & 1 \end{vmatrix} = -13$$

$$A_{12} = (-1)^{2+1} M_{12} = (-1)^3 \begin{vmatrix} 2 & 1 \\ -1 & 1 \end{vmatrix} = -3$$

$$A_{22} = (-1)^{2+2} M_{22} = (-1)^4 \begin{vmatrix} 1 & 5 \\ -1 & 1 \end{vmatrix} = 6$$

$$A_{32} = (-1)^{2+3} M_{22} = (-1)^5 \begin{vmatrix} 1 & 5 \\ 2 & 1 \end{vmatrix} = -9$$

$$A_{31} = (-1)^{3+1} M_{31} = (-1)^4 \begin{vmatrix} 2 & 3 \\ -1 & 1 \end{vmatrix} = 5$$

$$A_{32} = (-1)^{3+2} M_{32} = (-1)^5 \begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix} = -3$$

$$A_{33} = (-1)^{3+3} M_{33} = (-1)^6 \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = -1$$

$$Adj\ A = \begin{vmatrix} 2 & 3 & -13 \\ -3 & 6 & -9 \\ 5 & 3 & -1 \end{vmatrix}$$

NOTES

Hence $$A^{-1} = \frac{Adj\ A}{|A|} = \frac{1}{21}\begin{vmatrix} 2 & 3 & -13 \\ -3 & 6 & -9 \\ 5 & -3 & -1 \end{vmatrix}$$

(*b*) Let $$A = \begin{pmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{pmatrix}$$

$$|A| = \begin{vmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{vmatrix} = 3\begin{vmatrix} -3 & 4 \\ -1 & 1 \end{vmatrix} + 3\begin{vmatrix} 2 & 4 \\ 0 & 1 \end{vmatrix} + 4\begin{vmatrix} 2 & -3 \\ 0 & -1 \end{vmatrix}$$

$$= 3(-3 + 4) + 3(2 - 0) + 4(-2 - 0)$$

$$= 3 + 6 - 8 = 1 \neq 0$$

Hence, A^{-1} exist.

We know that $A^{-1} = \dfrac{Adj\ A}{|A|}$

Now find $Adj\ A$. For this, we find co-factors of A.

$$A_{11} = (-1)^{1+1} M_{11} = (-1)^2 \begin{vmatrix} -3 & 4 \\ -1 & 1 \end{vmatrix} = 1$$

$$A_{21} = (-1)^{2+1} M_{21} = (-1)^3 \begin{vmatrix} -3 & 4 \\ -1 & 1 \end{vmatrix} = -1$$

$$A_{31} = (-1)^{3+1} M_{31} = (-1)^4 \begin{vmatrix} -3 & 4 \\ -3 & 4 \end{vmatrix} = 0$$

$$A_{12} = (-1)^{1+2} M_{12} = (-1)^3 \begin{vmatrix} 2 & 4 \\ 0 & 1 \end{vmatrix} = -2$$

$$A_{22} = (-1)^{2+2} M_{22} = (-1)^4 \begin{vmatrix} 3 & 4 \\ 0 & 1 \end{vmatrix} = 3$$

$$A_{32} = (-1)^{3+2} M_{32} = (-1)^5 \begin{vmatrix} 3 & 4 \\ 2 & 4 \end{vmatrix} = -4$$

$$A_{13} = (-1)^{1+3} M_{13} = (-1)^4 \begin{vmatrix} 2 & -3 \\ 0 & -1 \end{vmatrix} = -2$$

$$A_{23} = (-1)^{2+3} M_{23} = (-1)^5 \begin{vmatrix} 3 & -3 \\ 0 & -1 \end{vmatrix} = 3$$

$$A_{33} = (-1)^{3+3} M_{33} = (-1)^6 \begin{vmatrix} 3 & -3 \\ 2 & -3 \end{vmatrix} = -3$$

Now, $$Adj\ A = \begin{vmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{vmatrix}$$

$$A^{-1} = \frac{Adj\ A}{|A|} = \frac{1}{1}\begin{vmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{vmatrix}$$

11. SOLVING SYSTEMS OF EQUATIONS USING MATRICES

NOTES

Definition : A system of linear equations is a set of equations with n equations and n unknowns, is of the form of

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n &= b_n \end{aligned}$$

The unknown are denoted by $x_1, x_2, ..., x_n$ and the co-efficients (a's and b's above) are assumed to above can be written as :

A simplified way of writing the above is like this :

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

$$AX = b$$

$\Rightarrow$ Pre multiplying both sides by A^{-1}

We get, $A^{-1}(AX) = A^{-1}b$

$$\Rightarrow \qquad X = A^{-1}b$$

NOTE Inverse of A does not exist if A is a singular matrix *i.e.*, $|A| = 0$. In such case, there are two possibilities, either the system has no solution or it has an infinite number of solutions.

ILLUSTRATIVE EXAMPLES

Example 1. *Solve the following equation by matrix method*

$$2x - 3y = -1$$
$$3x + 5y = 5$$

Solution. The matrix representation of the system of equation is

$$AX = b \qquad ...(1)$$

where $\quad A = \begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix}, X = \begin{pmatrix} x \\ y \end{pmatrix}, b = \begin{pmatrix} -1 \\ 5 \end{pmatrix}$

Since $\quad |A| = \begin{vmatrix} 2 & -3 \\ 3 & 2 \end{vmatrix} = 4 + 9 = 13 \neq 0$

Hence A^{-1} exists.

We know that $\quad A^{-1} = \dfrac{1}{|A|}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ where $|A| = \det A$

$$= \frac{1}{13}\begin{pmatrix} 2 & 3 \\ -3 & 2 \end{pmatrix}$$

Now multiplying both sides of (1) by A^{-1},

$$A^{-1}(AX) = A^{-1}b$$

$$\Rightarrow \quad X = A^{-1}b$$

$$\Rightarrow \quad \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{13}\begin{pmatrix} 2 & 3 \\ -3 & 2 \end{pmatrix}\begin{pmatrix} -1 \\ 5 \end{pmatrix} = \frac{1}{13}\begin{pmatrix} 13 \\ 13 \end{pmatrix} = \begin{pmatrix} \frac{13}{13} \\ \frac{13}{13} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\therefore \quad x = 1, y = 1.$$

Example 2. *Solve the following equation by matrix method.*

$$3x - 2y + 2 = 1$$
$$2x + y - 5z = 2$$
$$x + y - 2z = 3$$

Solution. The matrix representation of the system of equation is

$$AX = b \qquad \text{....(1)}$$

where

$$A = \begin{pmatrix} 3 & -2 & 1 \\ 2 & 1 & -5 \\ 1 & 1 & -5 \end{pmatrix}, X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, b = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

Since

$$|A| = \begin{vmatrix} 3 & -2 & 1 \\ 2 & 1 & -5 \\ 1 & 1 & -5 \end{vmatrix}$$

$$= 3(-2 + 5) + 2(-4 + 5) + 1(2 - 1)$$
$$= 9 + 2 + 1 = 12 \neq 0$$

Hence A^{-1} exists.

Now we find co-factors of A.

$$A_{11} = (-1)^{1+1} M_{11} = (-1)^2 \begin{vmatrix} 1 & -5 \\ 1 & -2 \end{vmatrix} = 3$$

$$A_{21} = (-1)^{2+1} M_{21} = (-1)^3 \begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix} = -3$$

$$A_{31} = (-1)^{3+1} M_{31} = (-1)^4 \begin{vmatrix} -2 & 1 \\ 1 & -5 \end{vmatrix} = 9$$

$$A_{12} = (-1)^{1+2} M_{12} = (-1)^3 \begin{vmatrix} 2 & -5 \\ 1 & -2 \end{vmatrix} = -1$$

$$A_{22} = (-1)^{2+2} M_{22} = (-1)^4 \begin{vmatrix} 3 & 1 \\ 1 & -2 \end{vmatrix} = -7$$

$$A_{32} = (-1)^{3+2} M_{32} = (-1)^5 \begin{vmatrix} 3 & -2 \\ 2 & 1 \end{vmatrix} = -7$$

$$A_{13} = (-1)^{1+3} M_{13} = (-1)^4 \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = -1$$

$$A_{23} = (-1)^{2+3} M_{23} = (-1)^5 \begin{vmatrix} 3 & -2 \\ 1 & 1 \end{vmatrix} = -5$$

$$A_{33} = (-1)^{3+3} M_{33} = (-1)^6 \begin{vmatrix} 3 & -2 \\ 2 & 1 \end{vmatrix} = 7$$

NOTES

Now, $$\text{Adj } A = \begin{pmatrix} 3 & -3 & 9 \\ -1 & -7 & -7 \\ 1 & -5 & 7 \end{pmatrix}$$

$$A^{-1} = \frac{Adj\ A}{|A|} = \frac{1}{12}\begin{pmatrix} 3 & -3 & 9 \\ -1 & -7 & -7 \\ 1 & -5 & 7 \end{pmatrix}$$

Now, pre multiplying both sides of (1) by A^{-1}

$$A^{-1}(AX) = A^{-1}b$$

$$\Rightarrow \quad X = A^{-1}b$$

$$\Rightarrow \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{12}\begin{pmatrix} 3 & -3 & 9 \\ -1 & -7 & -7 \\ 1 & -5 & 7 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$= \frac{1}{12}\begin{pmatrix} 24 \\ -36 \\ 12 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$$

$$\Rightarrow \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$$

$$\therefore \quad x = 3, y = 3, z = 1$$

Example 3. *Solve the following simultaneous equations using inverse method.*

$$x - y + z = 2$$
$$2x + y - 3z = 5$$
$$3x - 2y - z = 4$$

Solution. The matrix representation of the system of equation is

$$AX = b \qquad \ldots(1)$$

where $$A = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 3 & -2 & -1 \end{pmatrix}, x = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, b = \begin{pmatrix} 2 \\ 5 \\ 4 \end{pmatrix}$$

Since $$|A| = \begin{vmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 3 & -2 & -1 \end{vmatrix} = 1(-1-6) + 1(-2+9) + 1(-4-3)$$

$$= -7 + 7 - 7 = -7 \neq 0$$

Hence, A^{-1} exists.

NOTES

Now we find co-factors of A.

$$A_{11} = (-1)^{1+1} M_{11} = (-1)^2 \begin{vmatrix} 1 & -3 \\ -2 & -1 \end{vmatrix} = -7$$

$$A_{21} = (-1)^{2+1} M_{21} = (-1)^3 \begin{vmatrix} -1 & 1 \\ -2 & -1 \end{vmatrix} = -3$$

$$A_{31} = (-1)^{3+1} M_{31} = (-1)^4 \begin{vmatrix} -1 & 1 \\ 1 & -3 \end{vmatrix} = 2$$

$$A_{12} = (-1)^{1+2} M_{12} = (-1)^3 \begin{vmatrix} 2 & -3 \\ 3 & -1 \end{vmatrix} = -7$$

$$A_{22} = (-1)^{2+2} M_{22} = (-1)^4 \begin{vmatrix} 1 & 1 \\ 3 & -1 \end{vmatrix} = -4$$

$$A_{32} = (-1)^{3+2} M_{32} = (-1)^5 \begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix} = -5$$

$$A_{13} = (-1)^{1+3} M_{13} = (-1)^4 \begin{vmatrix} 2 & 1 \\ 3 & -2 \end{vmatrix} = -7$$

$$A_{23} = (-1)^{2+3} M_{23} = (-1)^5 \begin{vmatrix} 1 & -1 \\ 3 & -2 \end{vmatrix} = -1$$

$$A_{33} = (-1)^{3+3} M_{33} = (-1)^6 \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} = 3$$

Now $$\text{Adj } A = \begin{pmatrix} -7 & -3 & 2 \\ -7 & -4 & 5 \\ -7 & -1 & 3 \end{pmatrix}$$

$$A^{-1} = \frac{Adj\ A}{|A|} = \frac{1}{-7}\begin{pmatrix} -7 & -3 & 2 \\ -7 & -4 & 5 \\ -7 & -1 & 3 \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 7 & 3 & -2 \\ 7 & 4 & -5 \\ 7 & 1 & -3 \end{pmatrix}$$

Now pre multiplying both sides of (1) by A^{-1},

$$A^{-1}(Ax) = A^{-1}b$$

$$\Rightarrow \quad x = A^{-1}b$$

$$\Rightarrow \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 7 & 3 & -2 \\ 7 & 4 & -5 \\ 7 & 1 & -3 \end{pmatrix}\begin{pmatrix} 2 \\ 5 \\ 4 \end{pmatrix}$$

$$= \frac{1}{7}\begin{pmatrix} 14+15-8 \\ 14+20-20 \\ 14+5-12 \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 21 \\ 14 \\ 7 \end{pmatrix}$$

$$\Rightarrow \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$$

$$\therefore \quad x = 3, y = 2, z = 1$$

NOTES

12. EIGEN VALUES AND EIGEN VECTORS

Suppose A is a square matrix of order n containing elements a_{ij} ($i = 1, 2, 3, \ldots n$). Then to find a vector x and a scalar λ such that the equation

$$Ax = x\lambda \quad \ldots(1)$$

has non-trivial solutions, then is called eigen value and x is an eigen vector.

Hence, equation (1) can be rewritten as

$$(A - \lambda I)x = 0 \quad \ldots(2)$$

$$\Rightarrow \quad \begin{pmatrix} \alpha_{11} - \lambda & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} - \lambda & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} - \lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \quad \ldots(3)$$

where $$A = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix}, \text{ and } x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

such equation (3) has non-trivial solution, if det $(A - \lambda I) = 0$

i.e., $$\begin{pmatrix} \alpha_{11} - \lambda & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} - \lambda & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} - \lambda \end{pmatrix} = 0 \quad \ldots(4)$$

Then the values of λ are known as Eigen values or characteristic values or proper values of A. By denoting the set of these non-trivial solutions of equation (1) for a given λ by $E(\lambda)$. And each vector, belonging to $E(\lambda)$ is known as eigen vector or characteristic vector or proper vector, corresponding to the eigen value.

Characteristic Equation

The equation $\det(A - \lambda I)_{ij}$ is known as characteristic equation of A.

Characteristic Polynomial

The determinant $|A - \lambda I|$ when expanded will give a polynomial of degree n is λ, which is known as characteristic polynomial.

Example 4. *Determine the eigen values and corresponding eigen spaces of the matrix :*

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Solution. Here A is a matrix of order 2×2. The eigen values of A is

$$|A - \lambda I| = 0$$

NOTES

or $$\det\left\{\begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}-\lambda\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\right\}=0$$

or $$\left|\begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}-\begin{pmatrix}2 & 0\\0 & 2\end{pmatrix}\right|=0$$

or $$\begin{vmatrix}-\lambda & -1\\1 & -\lambda\end{vmatrix}=0$$

or $$\lambda^2+1=0 \quad \text{or,} \quad \lambda^2=-1$$

So, $$\lambda=\pm i$$

So, $\lambda_1 = i$ and $\lambda_2 = -i$ are the eigen values.

Now, to find $E(\lambda_1) = E(i)$, we write

$$\begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}=i\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}ix_1\\ix_2\end{pmatrix}$$

or $$\begin{pmatrix}-x_1\\x_2\end{pmatrix}=\begin{pmatrix}ix_1\\ix_2\end{pmatrix} \quad \text{or,} \quad \begin{pmatrix}ix_1-x_2\\x_1-ix_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

So, $$-ix_1-x_2=0 \text{ and } x_1-ix_2=0$$

$\Rightarrow$ $$x_2=-ix_1,\ x_1\neq 0$$

So, the eigen vectors corresponding to eigen i are of the form $(x_1, -ix_1)$, $x_1 \neq 0$

So, $$E(i) = [(1, -i)]/\{0\}$$

Next to find $E(\lambda_2) = E(-i)$, we write

$$\begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}=-i\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}-ix_1\\-ix_2\end{pmatrix}$$

or $$\begin{pmatrix}-x_2\\x_1\end{pmatrix}+\begin{pmatrix}ix_1\\ix_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

or $$ix_1-ix_2=0 \text{ and } x_1+ix_2=0$$

$\Rightarrow$ $$x_2=ix_1,\ x_1\neq 0$$

So the eigen vectors corresponding to the eigen value $-i$ are of the form (x_1, ix_1), $x_1 \neq 0$.

So $$E(-i) = [(1, i)]/\{0\}.$$

Example 5. *Determine the eigen values and the corresponding eigen spaces for the following matrix.*

$$A=\begin{pmatrix}3 & 1\\6 & 2\end{pmatrix}$$

Solution. Given $$A=\begin{pmatrix}3 & 1\\6 & 2\end{pmatrix}$$

Then the eigen values λ are the roots of $\det(A - \lambda I) = 0$.

or $$\begin{vmatrix}3 & 1\\6 & 2\end{vmatrix}-\lambda\begin{vmatrix}0 & 1\\1 & 0\end{vmatrix}=0$$

or $$\begin{vmatrix}3-\lambda & 1\\6 & 2-\lambda\end{vmatrix}=0$$

or $$(3-\lambda)(2-\lambda)-6=0$$

or $$6+\lambda^2-5\lambda-6=0$$

or $$\lambda^2-5\lambda=0$$

or $$\lambda(\lambda-5)=0 \quad \text{or,} \quad \lambda=0,\ \lambda=5$$

So the eigen values of A are $\lambda_1 = 0$, $\lambda_2 = 5$

Now to find $E(\lambda_1) = E(0)$, we write

$$\begin{pmatrix} 3 & 1 \\ 6 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad \begin{pmatrix} 3x_1 + x_2 \\ 6x_1 + 2x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

So, $$3x_1 + x_2 = 0 \text{ and } 6x_1 + 2x_2 = 0$$
$$x_2 = -3x_1$$

The eigen vectors corresponding to the eigen value 0 are therefore of the form $(x_1, -3x_1)$ where $x_1 \neq 0$

So, $$E(0) = [(1, -3)]/\{0\}$$

Next to find $E(\lambda_2) = E(5)$, we write

$$\begin{pmatrix} 3 & 1 \\ 6 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 5\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 5x_1 \\ 5x_2 \end{pmatrix}$$

or, $$\begin{pmatrix} 3x_1 + x_2 \\ 6x_1 + 2x_2 \end{pmatrix} - \begin{pmatrix} 5x_1 \\ 5x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

or, $$\begin{pmatrix} -2x_1 + x_2 \\ 6x_1 - 3x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

So, $$-2x_1 + x_2 = 0 \text{ and } 6x_1 - 3x_2 = 0$$

Hence $$x_2 = 2x_1 \quad \text{where } x_1 \neq 0$$

So the eigen vectors corresponding to the eigen value 5 are therefore of the form $(x_1, 2x_1)$ where $x_1 \neq 0$

So; $$E(5) = [(1, 2)]/\{0\}.$$

Example 6. *Determine the eigen values and the corresponding eigen spaces of the matrix*

$$A = \begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix}$$

Solution. Given $$A = \begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix}$$

Then the eigen values λ are the roots of the equation $\det(A - \lambda I) = 0$

$$\det\left\{\begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right\} = 0$$

or $$\begin{vmatrix} 3-\lambda & 2 & 4 \\ 2 & -\lambda & 2 \\ 4 & 2 & 3-\lambda \end{vmatrix} = 0$$

or $$(3-\lambda)[-3\lambda + \lambda^2 - 4] - 2[6 - 2\lambda - 8] + 4[4 + 4\lambda] = 0$$

or $$-9\lambda + 3\lambda^2 - 12 + 3\lambda^2 - \lambda^3 + 4\lambda - 12 + 4\lambda + 16 + 16 + 16\lambda = 0$$

or $$-\lambda^3 + 6\lambda^2 + 15\lambda + 8 = 0$$

or $$\lambda^3 - 8\lambda^2 + 2\lambda^2 - 15\lambda - 8 = 0$$

or $$(\lambda + 1)^2 (\lambda - 8) = 0$$

So, the eigen values are $\lambda_1 = -1, \lambda_2 = -1, \lambda_3 = 8$

Now to find the eigen vectors corresponding to the eigen value $\lambda_1 = -1$, we write

$$\begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = -1 \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

or $$\begin{pmatrix} 3x_1 + 2x_2 + 4x_3 \\ 2x_1 + 0 + 2x_3 \\ 4x_1 + 2x_2 + 3x_3 \end{pmatrix} - \begin{pmatrix} 8x_1 \\ 8x_2 \\ 8x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

So, $$-5x_1 + 2x_2 + 4x_3 = 0$$
$$2x_1 - 8x_2 + 2x_3 = 0$$
$$4x_1 + 2x_2 - 5x_3 = 0$$

So, $$x_1 = 2, x_2 = 1, x_3 = 2$$

Hence $$E_3 = [(2, 1, 2)]/\{0\}$$

Example 7. *Determine the eigen values and corresponding eigen spaces for the following matrix :*

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 0 \end{pmatrix}$$

Solution. Let $$A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 0 \end{pmatrix}$$

Then the eigen values are the roots of the equation $\det(A - \lambda I) = 0$

or $$\det \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 0 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\} = 0$$

or $$\begin{vmatrix} 1-\lambda & 0 & 0 \\ 2 & 1-\lambda & 0 \\ 3 & 2 & -\lambda \end{vmatrix} = 0$$

or $$(1-\lambda)(1-\lambda)(-\lambda) = 0$$

So, $$\lambda_1 = 0, \lambda_2 = \lambda_3 = 1$$

Now, to find $E(\lambda_1) = E(0)$, we write

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

or $$\begin{pmatrix} x_1 \\ 2x_1 + x_2 \\ 3x_1 + 2x_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

So, $x_1 = 0,\ 2x_2 + x_2 = 0,\ 3x_1 + 2x_2 = 0$

Hence, $x_1 = x_2 = 0$ and take $x_3 = 1$

So, $E(0) = [(0, 0, 1)]$

Next to find $E(\lambda_2) = E(1)$, we have

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 1\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

or $$\begin{pmatrix} x_1 \\ 2x_1 + x_2 \\ 3x_1 + 2x_1 \end{pmatrix} - \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

So, $2x_1 = 0,\ 3x_1 + 2x_2 - x_3 = 0$

$x_1 = 0,\ x_3 = 2x_2,\ x_2 \neq 0$

Hence the eigen vectors are of the form $(0, x_2, 2x_2)$

So, $E(I) = [(0, 1, 2)]/\{0\}$

Example 8. *Determine the eigen values and the corresponding eigen spaces of the matrix*

$$\begin{pmatrix} 0 & i & i \\ i & 0 & i \\ i & i & 0 \end{pmatrix}$$

Solution. Let $$A = \begin{pmatrix} 0 & i & i \\ i & 0 & i \\ i & i & 0 \end{pmatrix}$$

Then the eigen vectors λ are the roots of the equation $\det(A - \lambda I) = 0$

or $$\det\left\{\begin{pmatrix} 0 & i & i \\ i & 0 & i \\ i & i & 0 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right\} = 0$$

or $$\begin{vmatrix} -\lambda & i & i \\ i & -\lambda & i \\ i & i & -\lambda \end{vmatrix} = 0$$

or $$-\lambda(\lambda^2 - i^2) - i(-i\lambda - i^2) + i(i^2 + i\lambda) = 0$$

NOTES

or $$-\lambda(\lambda^2 + 1) - i(1 - i\lambda) + i(i\lambda - 1) = 0$$

or $$-\lambda^3 - \lambda - i - \lambda - \lambda - i = 0$$

or $$\lambda^3 + 3\lambda + 2i = 0$$

So, $\lambda_1 = \lambda_2 = -i$, $\lambda_3 = 2$ which are the eigen values.

Now to find $E(\lambda_1) = E(-i)$, we write $Ax = \lambda_1 x$

i.e., $$\begin{pmatrix} 0 & i & i \\ i & 0 & i \\ i & i & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = -i\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

or, $$\begin{pmatrix} ix_2 + ix_3 \\ ix_1 + ix_3 \\ ix_1 + ix_2 \end{pmatrix} + \begin{pmatrix} ix_1 \\ ix_2 \\ ix_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

So, $$ix_1 + ix_2 + ix_3 = 0$$

Now putting $x_1 = 0$, $x_3 = -x_2$

putting $x_3 = 0$, $x_2 = -x_1$

Hence, the eigen vectors are one of the form (0, 1, –1) and (1, –1, 0)

So, $E(-i) = [(0, 1, -1), (1, -1, 0)]/\{0\}$

Next to find $E(\lambda_3) = E(2)$, we write

$$\begin{pmatrix} 0 & i & i \\ i & 0 & i \\ i & i & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 2i\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

or $$\begin{pmatrix} ix_2 + ix_3 \\ ix_1 + ix_3 \\ ix_1 + ix_2 \end{pmatrix} + \begin{pmatrix} 2ix_1 \\ 2ix_2 \\ 2ix_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

So, $$-2ix_1 + ix_2 + ix_3 = 0$$
$$ix_1 - 2ix_2 + ix_3 = 0$$
$$ix_1 + ix_2 - 2ix_3 = 0$$
$$x_1 = x_2 = x_3 = 1$$

Hence, $$E(i) = [(1, 1, 1)]/\{0\}.$$

Example 9. *Determine the eigen values and the corresponding eigen spaces for the following matrix*

$$\begin{pmatrix} 3 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

Solution. Then the eigen vectors λ are the roots of the equation $\det(A - \lambda I) = 0$

or, $$\begin{vmatrix} 3-\lambda & 2 & 1 & 0 \\ 0 & 1-\lambda & 0 & 1 \\ 0 & 2 & -1-\lambda & 0 \\ 0 & 0 & 0 & \frac{1}{2}-\lambda \end{vmatrix} = 0$$

NOTES

or, $$\begin{vmatrix} 1-\lambda & 0 \\ 2 & -1-\lambda \\ 0 & 0 \end{vmatrix} = 0$$

or, $$(3-\lambda)(\tfrac{1}{2}-\lambda)(1-\lambda)(-1-\lambda) = 0$$

$\therefore \lambda_1 = 3, \lambda_2 = \frac{1}{2}, \lambda_3 = 1, \lambda_4 = -1$ which are the eigen values.

Now to find $E(\lambda_1) = E(3)$, we write

$$\begin{pmatrix} 3 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = 3\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

$$\begin{pmatrix} 3x_1 + 2x_2 + x_3 \\ x_2 + x_4 \\ 2x_3 - x_3 \\ \frac{1}{2}x_4 \end{pmatrix} - \begin{pmatrix} 3x_1 \\ 3x_2 \\ 3x_3 \\ 3x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

So, $$2x_2 + x_3 = 0, -2x_2 + x_4 = 0$$

$$2x_2 - 4x_3 = 0, \tfrac{5}{2}x_4 = 0$$

So, $x_2 = x_3 = x_4 = 0$. Take $x_1 = 1$

Hence, $E(3) = [(1, 0, 0, 0)]/\{0\}$

In a similar manner, we find

$$E(-1) = [(1, 0, -4, 0)]$$

$$E(1) = [(-3, 2, 2, 0)]$$

and $$E\left(\frac{1}{2}\right) = [(-8, 6, 8, -3)].$$

EXERCISE 5(A)

1. If $A = \begin{pmatrix} 1 & 2 \\ 3 & -4 \end{pmatrix}, B = \begin{pmatrix} 5 & 6 \\ 2 & 3 \end{pmatrix}, C = \begin{pmatrix} 9 & 7 \\ 2 & -1 \end{pmatrix}$ and $D = \begin{pmatrix} 6 & -3 \\ -5 & 8 \end{pmatrix}$

then find out,

(a) $A + 2B$ (b) $A + 3B + 2C$ (c) $A + B + 4D$

(d) $A - 2B$ (e) $A - 3B + 2C$ (f) A^T

(g) $A^T + 2B + C$ (h) $A^T + B^T + C^T + D^T$ (i) $A + BC$

(j) $B^2 + AD$ (k) $D^3 + C$

2. Find x, y, if

(i) $\begin{pmatrix} 2 & 3 \\ 4 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 9 \end{pmatrix}$ (ii) $\begin{pmatrix} 8 & 3 \\ 5 & x \end{pmatrix}\begin{pmatrix} -1 & 3 \\ -9 & y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

3. If $A = \begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ 3 & 1 \end{pmatrix}$

   Prove that $(AB)^{-1} = B^{-1} A^{-1}$.

4. If $A = \begin{pmatrix} 2 & 3 \\ 5 & -7 \end{pmatrix}$, then prove that $(A^2)^T = (A^T)^2$.

5. If $A = \begin{pmatrix} 1 & 2 \\ 5 & -3 \end{pmatrix}$, then find $f(A)$, where $f(n) = n^3 - 2n^2 - 5$.

6. Find the inverse of each of the following matrix

   (*i*) $\begin{pmatrix} -2 & 5 \\ 1 & -1 \end{pmatrix}$ (*ii*) $\begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}$

   (*iii*) $\begin{pmatrix} -1 & 1 & 0 \\ -2 & -3 & 2 \\ 2 & 0 & -1 \end{pmatrix}$ (*iv*) $\begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 3 & -2 & -1 \end{pmatrix}$

Answers

6. (*i*) $\begin{pmatrix} -4/23 & -5/23 \\ -3/23 & 2/23 \end{pmatrix}$ (*ii*) $\begin{pmatrix} 2 & -1 \\ -5 & 3 \end{pmatrix}$

   (*iii*) $\begin{pmatrix} 3/2 & 1/2 & 1 \\ 1 & 1/2 & 1 \\ 3 & 1 & 5/2 \end{pmatrix}$ (*iv*) $\begin{pmatrix} 1 & 3/7 & -2/7 \\ 1 & 4/7 & -5/7 \\ 1 & 1/7 & -3/7 \end{pmatrix}$

SUMMARY

1. **Square Matrix.** A matrix is called a square matrix of the number of rows is equal to number of columns.
2. **Unit Matrix.** The square matrix whose elements on its main diagonal (left top to right bottom) are '1's and the rest of its elements are zeros is known as unit matrix.
3. **Singular and Non-singular Matrices.** A square matrix A is called a singular matrix *iff* its determinant is zero and is called non-singular (or regular) matrix if determinant is not equal to zero.
4. **Symmetric Matrix.** A symmetric matrix is a square matrix in which $X_{ij} = X_{ji}$ for all i and j.
5. **Diagonal Matrix.** *A* diagonal matrix is a symmetric matrix where all of diagonal elements are 0.
6. **Upper Triangular and Lower Triangular Matrix.** A square matrix $A = [a_{ij}]$ is called upper triangular matrix if all the elements below the main diagonal are zero *i.e.*, if $a_{ij} = 0$ for all $i > j$
7. Two matrices A and B can be added or subtracted if and only if their dimensions are the same. (*i.e.*, both matrices have the identical amount of rows and columns.
8. **Commutative.** The addition of matrices is commutative, that is, if A and B are two matrices of same order, then $A + B = B + A$.
9. **Associative.** The matrix addition is associative *i.e.*, if A, B and C are three matrices of same order, then $A + (B + C) = (A + B) + C$
10. **Additive Identity.** The identity matrix for addition is the zero matrix or null matrix denoted by '0'. Thus, if A is a matrix, then

    $A + 0 = A$, provided the order of the zero matrix is same as that of A.
11. **Distributive Law.** The distributive laws hold for matrices *i.e.*, if A, B and C are three matrices, then

    $$A(B + C) = AB + AC, \quad (A + B)C = AC + BC$$

TEST YOURSELF

NOTES

1. Find the eigen values and eigen vectors of the following matrix

(i) $\begin{pmatrix} 3 & 4 \\ 4 & 3 \end{pmatrix}$ (ii) $\begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix}$

(iii) $\begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix}$ (iv) $\begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix}$

2. Solve the following equations, by matrix method

(i) $x + y + z = 3$
$2x - y + z = 2$
$x - 2y + 3z = 2$

(ii) $x + y + z = 3$
$x + 2y + 3z = 4$
$x + 4y + 9z = 6$

(iii) $x - y + z = 4$
$2x + y - 3z = 0$
$x + y + z = 2$

(iv) $x + 2y - 3z = 4$
$2x + 3y - 5z = 12$
$3x - y + z = 3$

(v) $x + 2y + 3z = 14$
$2x - y + 5z = 15$
$2y + 4z - 3x = 13$

Answers

1. (i) ± 5, $\left[-1/\sqrt{5}, 2/\sqrt{5}\right]5$

(ii) (1, 6)

(iii) 1, $1 \pm \sqrt{2}$, [−4, 2, 1], [$\sqrt{2}$, 0, 1], [$-\sqrt{2}$, 0, 1]

(iv) [−1, −1, 8], [1, −2, 0], [0, −2, 1], [2, 1, 2].

CHAPTER

6

DETERMINANT

LEARNING OBJECTIVES

- Introduction
- Determinant of Order 3
- Working Rule of the Expression of a Determinant
- Determinant of n-th Order
- Minor
- Solutions of Simultaneous Linear Equations

1. INTRODUCTION

This chapter brings an introduction and an application of determinant. A determinant is a symbolical compact form representation in the study of the theory of linear equations.

Let us consider the following system of linear equations

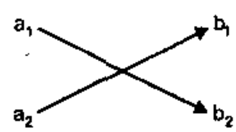
$$a_1x + b_1y = 0 \quad ...(i)$$

$$a_2x + b_2y = 0 \quad ...(ii)$$

By eliminating x and y from (i) and (ii), we get $a_1b_2 - a_2b_1 = 0$

This expression $a_1b_2 - a_2b_1$ is symbolically written as $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ which is known as determinant of 2nd order. Here there are two rows and two columns a_1 and b_2 are in the first row, a_2 and b_2 are in the second row, a_2 and b_2 are in the first column, b_1 and b_2 are in the second column. As it contains 2 rows and 2 columns, it is known as determinant of 2nd order. Each element a_1, a_2, b_1 and b_2 are known as elements (on constituents) of the determinant. *Solving procedure of determinant of 2nd order.*

Step 1. Write down the elements in the following manner in two rows



Step 2. Multiply the elements as shown by the arrows.

Arrows going downward *i.e.*, clockwise direction as +ve sign and the arrows going upward *i.e.*, anticlockwise direction as –ve sign.

Step 3. Add the two products as

$$a_1b_2 + (-a_2b_1) = a_1b_2 - a_2b_1 \text{ the require value of determinant.}$$

NOTES

2. DETERMINANT OF ORDER 3

Let us consider the following equations

$$a_1x + b_1y + c_1 z z o \ldots \qquad \ldots(i)$$
$$a_2x + b_2y + c_2 z z o \ldots \qquad \ldots(ii)$$
$$a_3x + b_3y + c_3 z z o \ldots \qquad \ldots(iii)$$

Eliminating x, y and z from these equation, we get

$$a_1(b_2c_3 - b_3c_2) + b_1(c_2a_3 - c_3a_2) + c_1(a_2b_3 - a_3b_2) = 0$$

This expression is symbolically written as a compact form

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

which is known as a determinant of 3rd order.

Here

(*i*) It is a function of 3^2 elements arranged in three horizontal lines known as rows and three vertical lines known as columns in the form of a solid square.

(*ii*) It has 3! = 6, elements, half of them are positive and half of them are negative.

(*iii*) Each term in the expression has one and only one element from each row and from each column.

3. WORKING RULE OF THE EXPRESSION OF A DETERMINANT

Multiply each constituent of the first row by the determinant obtained by deleting the row and the column in which the element is lying, the signs of the product being alternatively positive and negative.

4. DETERMINANT OF *n*-TH ORDER

The above expression enables us to generalise the definition of the determinant of the 3rd order to n-th order.

A determinant of n-th order is a function of n^2 elements arranged in n-rows and n-columns in the form of a solid square contains $n!$ terms where half of them are positive and half of them are negative and each term in the expression has one and only one element from each row and from each column.

For example, find the value of $\begin{vmatrix} 1 & 2 & 3 \\ 1 & 0 & 4 \\ 2 & 1 & 2 \end{vmatrix}$

NOTES

$$= 1\begin{vmatrix} 0 & 4 \\ 1 & 2 \end{vmatrix} - 2\begin{vmatrix} 1 & 4 \\ 2 & 2 \end{vmatrix} + 3\begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix}$$

$$= 1(0-4) - 2(2-8) + 3(1-0)$$

$$= -4 + 12 + 3 = 11$$

5. MINOR

The determinant obtained by suppressing the row and the column in which a particular element occurs is called the minor of that element.

In the determinant $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

The minor of c_3 is $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$

The minor of b_2 is $\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix}$.

The minor of b_3 is $\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$

The minors of $a_1b_1c_1, a_2b_2c_2, a_3b_3c_3$ are denoted by $A_1, B_1, C_1, A_2, B_2, C_2, A_3, B_3, C_3$ respectively.

Hence $A_1 = \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$ $A_2 = \begin{vmatrix} b_1 & c_2 \\ b_3 & c_3 \end{vmatrix}$ $A_3 = \begin{vmatrix} b_1 & c_2 \\ b_2 & c_2 \end{vmatrix}$

$B_1 = \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}$ $B_2 = \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix}$ $B_3 = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$

$C_1 = \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$ $C_2 = \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}$ $C_3 = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$

If Δ stands for the determinant, then

$$\Delta = a_1A_1 - b_1B_1 + c_1C_1 = a_1A_1 - a_2A_2 + a_3A_3$$

NOTE

1. The value of the above determinant of third order is equal to each of the following expressions.

   $a_1A_1 - b_1B_1 + c_1C_1$ $\quad$ $a_1A_1 - a_2A_2 + a_3A_3$

   $-a_2A_2 + b_2B_2 - c_2C_2$ $\quad$ $-b_1B_1 + b_2B_2 - b_3B_3$

   $a_3A_3 - b_3B_3 + c_3C_3$ $\quad$ $c_1C_1 - c_2C_2 + c_3C_3$

2. In the expression of the determinant of 3rd order the signs with which the elements are multiplied may be remembered by the following rule.

   $$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

   Before analysing the problems of determinants let us know the following theorems (the properties of determinants)

Theorem 1. *The value of the determinant is unchanged by changing its rows into columns and columns into rows.*

Proof. Let
$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

be a given determinant.

Then,
$$\Delta = a_1(b_2c_3 - c_2b_3) - b_1(a_2c_3 - c_2a_3) + c_1(a_2b_3 - b_2a_3)$$
$$= a_1b_2c_3 - a_1c_2b_3 - b_1a_2c_3 + b_1c_2a_3 + c_1a_2b_3 - c_1b_2a_3$$

Let be the determinant formed from by changing its rows into columns and columns into rows.

i.e.,
$$\Delta' = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$
$$= a_1(b_2c_3 - c_2b_3) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$$
$$= a_1b_2c_3 - a_1c_2b_3 - a_2b_1c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1$$

This shows $\Delta = \Delta'$.

Theorem 2. *If two adjacent rows or columns of a determinant are interchanged, then the numerical value of the determinant is unchanged, but its sign is reversed.*

Proof. Let
$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

be a given determinant.

Then,
$$\Delta = a_1b_2c_3 - a_1c_2b_3 - b_1a_2c_3 + b_1c_2a_3 + c_1a_2b_3 - c_1b_2a_3$$

Let
$$\Delta' = \begin{vmatrix} b_1 & a_1 & c_1 \\ b_2 & a_2 & c_2 \\ b_3 & a_3 & c_3 \end{vmatrix} \quad [c_1 \to c_2 \text{ and } c_2 \to c_1]$$

$\Rightarrow$
$$\Delta' = b_1(a_2c_3 - c_2a_3) - a_1(b_2c_3 - c_2b_3) + c_1(b_2a_3 - a_2b_3)$$
$$= b_1a_2c_3 - b_1c_2a_3 - a_1b_2c_3 + a_1c_2b_3 + c_1b_2a_3 - a_2c_1b_3$$
$$= -(b_1c_2a_3 + a_1b_2c_3 + a_2c_1b_3 - b_1a_2c_3 - a_1c_2b_3 - c_1b_2a_3)$$

$\Rightarrow$ $\Delta = \Delta'$

This above property remains true if we interchange any two adjacent rows or any two adjacent columns. It may be clarified here that for determinants of 2nd and 3rd order, any two rows or any two columns are adjacent.

Theorem 3. *If two rows or columns of a determinant are identical, then the determinant vanishes.*

Proof. Let
$$\Delta = \begin{vmatrix} a_1 & a_1 & c_1 \\ a_2 & a_2 & c_2 \\ a_3 & a_3 & c_3 \end{vmatrix}$$

$$= a_1(a_2c_3 - c_2a_3) - a_1(a_2c_3 - c_2a_3) + c_1(a_2a_3 - a_2a_3)$$
$$= a_1a_2c_3 - a_1c_2a_3 - a_1a_2c_3 + a_1a_3c_2$$
$$= 0$$

NOTES

Theorem 4. *If every element in a row or column of a determinant is multiplied by a factor k, $k \neq 0$, then the value of the determinant is multiplied by the factor k.*

Proof. Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Let Δ' be the determinant, where k is multiplied with every element of the first column.

$$\Rightarrow \quad \Delta' = \begin{vmatrix} ka_1 & b_1 & c_1 \\ ka_2 & b_2 & c_2 \\ ka_3 & b_3 & c_3 \end{vmatrix}$$
$$= ka_1(b_2c_3 - c_2b_3) - b_1(ka_2c_3 - ka_3c_2) + c_1(ka_2b_3 - ka_3b_2)$$
$$= ka_1b_2c_3 - ka_1c_2b_3 - b_1ka_2c_3 + b_1ka_3c_2 + c_1ka_2b_3 - kc_1a_3b_2 \quad ...(i)$$

But, $\Delta = a_1b_2c_3 - a_1c_2b_3 - b_1a_2c_3 + b_1c_2a_3 + c_1a_2b_3 - c_1b_2a_3$

$$\Rightarrow \quad k\Delta = ka_1b_2c_3 - ka_1c_2b_3 - kb_1a_2c_3 + kb_1c_2a_3 + kc_1a_2b_3 - kc_1b_2a_3 \quad ...(ii)$$

This shows $(i) = (ii)$

Theorem 5. *If each constituent in any row or column consists of two or more terms, then the determinant can be expressed as the sum of two or more than two determinants.*

Proof. Let $\Delta = \begin{vmatrix} a_1 + l & b_1 & c_1 \\ a_2 + k & b_2 & c_2 \\ a_3 + n & b_3 & c_3 \end{vmatrix}$

$$= (a_1 + l)[b_2c_3 - b_3c_2] - b_1[c_3a_2 + c_3k - c_2a_3 - c_2n] + c_1[a_2b_3 + kb_3 - b_2a_3 - b_2n]$$
$$= a_1b_2c_3 - a_1b_3c_2 + lb_2c_3 - lb_3c_2 - b_1c_3a_2 - b_1c_3k + b_1c_2a_3 + b_1c_2n + c_1a_2b_3 + c_1kb_3 - c_1b_2a_3 - c_1b_2n \quad ...(i)$$

Now $\Delta = \begin{vmatrix} a_1 + l & b_1 & c_1 \\ a_2 + k & b_2 & c_2 \\ a_3 + n & b_3 & c_3 \end{vmatrix}$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} l & b_1 & c_1 \\ k & b_2 & c_2 \\ n & b_3 & c_3 \end{vmatrix} = \Delta_1 + \Delta_2 \text{ (let)}$$
$$= a_1(b_2c_3 - c_2b_3) - b_1(a_2c_3 - c_2a_3) + c_1(a_2b_3 - b_2a_3) + l(b_2c_3 - c_2b_3) - b_1(kc_3 - c_2n) + c_1(kb_3 - b_2n)$$
$$= a_1b_2c_3 - a_1c_2b_3 - b_1a_2c_3 + b_1c_2a_3 + c_1a_2b_3 - c_1b_2a_3 + lb_2c_3 - lc_2b_3 - b_1kc_3 + b_1c_2n + c_1kb_3 - c_1b_2n \quad ...(ii)$$

Hence, $(i) = (ii)$

Theorem 6. *If the elements of any row (or column) be increased or decreased by any equimultiples of the corresponding constituents of one or more of the other rows (or columns), the value of the determinant remains unaltered.*

Proof. Let $$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

NOTES

The determinant obtained, when the constituents of first columns are increased by k times, then,

$$\begin{vmatrix} a_1 + kb_1 & b_1 & c_1 \\ a_2 + kb_2 & b_2 & c_2 \\ a_3 + kb_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} kb_1 & b_1 & c_1 \\ kb_2 & b_2 & c_2 \\ kb_3 & b_3 & c_3 \end{vmatrix} \quad \text{(by Theorem 5)}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + k\begin{vmatrix} b_1 & b_1 & c_1 \\ b_2 & b_2 & c_2 \\ b_3 & b_3 & c_3 \end{vmatrix} = \Delta$$

ILLUSTRATIVE EXAMPLES

Example 1. *Evaluate* $$\begin{vmatrix} 2 & 8 & 4 \\ -5 & 6 & -10 \\ 1 & 7 & 2 \end{vmatrix}$$

Solution. Let $$\Delta = \begin{vmatrix} 2 & 8 & 4 \\ -5 & 6 & -10 \\ 1 & 7 & 2 \end{vmatrix}$$

Replacing $c_3 \to c_3 - 2c_1$, we get

$$\Delta = \begin{vmatrix} 2 & 8 & 4-4 \\ -5 & 6 & -10+10 \\ 1 & 7 & 2-2 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 8 & 0 \\ -5 & 6 & 0 \\ 1 & 7 & 0 \end{vmatrix}$$

$= 0$ (expanding by means of 3rd column)

Example 2. *Evaluate* $$\begin{vmatrix} 265 & 240 & 219 \\ 240 & 225 & 198 \\ 219 & 198 & 181 \end{vmatrix}$$

Solution. Let $$\Delta = \begin{vmatrix} 265 & 240 & 219 \\ 240 & 225 & 198 \\ 219 & 198 & 181 \end{vmatrix} \quad c_1 \to c_1 - c_3 \text{ and } c_2 \to c_2 - c_3$$

$$\Delta = \begin{vmatrix} 46 & 21 & 219 \\ 42 & 27 & 198 \\ 38 & 17 & 181 \end{vmatrix} \quad c_1 \to c_1 - 2c_2 \text{ and } c_3 \to c_3 - 10c_2$$

NOTES

$$\Delta = \begin{vmatrix} 4 & 21 & 9 \\ -12 & 27 & -72 \\ 4 & 17 & 11 \end{vmatrix} \quad r_2 \to r_2 + 3r_1 \text{ and } r_3 \to r_3 - r_1$$

$$\Delta = \begin{vmatrix} 4 & 21 & 9 \\ 0 & 90 & -45 \\ 4 & -4 & -2 \end{vmatrix}$$

$$= 4\begin{vmatrix} 90 & -45 \\ -4 & 2 \end{vmatrix}$$

$$= 4[180 - (-4)(-45)]$$

$$= 0.$$

Example 3. *Solve* $\begin{vmatrix} 1 & \omega^3 & \omega^2 \\ \omega^3 & 1 & \omega \\ \omega^2 & \omega & 1 \end{vmatrix}$.

Solution. Let $\Delta = \begin{vmatrix} 1 & \omega^3 & \omega^2 \\ \omega^3 & 1 & \omega \\ \omega^2 & \omega & 1 \end{vmatrix}$ Replacing $c_1 \to c_1 - c_2$

$$= \begin{vmatrix} 0 & 1 & \omega^2 \\ 0 & 1 & \omega \\ \omega^2 - \omega & \omega & 1 \end{vmatrix}$$

$$= \omega^2 - \omega \begin{vmatrix} 1 & \omega^2 \\ 1 & \omega \end{vmatrix}$$

$$= \omega^2 - \omega[\omega - \omega^2]$$

$$= \omega^3 - \omega^4 - \omega^2 + \omega^3$$

$$= 2 - \omega^3 . \omega - \omega^2$$

$$= 2 - \omega - \omega^2$$

$$= 3 - (1 + \omega + \omega^2) = 3 - 0 = 3.$$

Example 4. *Factorise* $\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ yz & zx & xy \end{vmatrix}$.

Solution. The given determinant is

$$= \frac{1}{xyz}\begin{vmatrix} x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \\ xyz & xyz & xyz \end{vmatrix} = \begin{vmatrix} x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \\ 1 & 1 & 1 \end{vmatrix}$$

Replacing $c_1 \to c_1 - c_2$ and $c_2 \to c_2 - c_3$

$$= \begin{vmatrix} x^2 - y^2 & y^2 - z^2 & z^2 \\ x^3 - y^3 & y^3 - z^3 & z^3 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} (x+y)(x-y) & (y+z)(y-z) \\ (x-y)(x^2+y^2+xy) & (y-z)(y^2+z^2+yz) \end{vmatrix}$$

NOTES

Taking $(x-y)$ and $(y-z)$ common, from c_1 and c_2

$$= (x-y)(y-z) \begin{vmatrix} (x+y) & (y+z) \\ (x^2+y^2+xy) & (y^2+z^2+yz) \end{vmatrix}$$

$$= (x-y)(y-z)[(x+y)(y^2+z^2+yz) - (y+z)(x^2+y^2+xy)]$$

$$= (x-y)(y-z)[xy^2 + xz^2 + xyz + y^3 + yz^2 + y^2z - yx^2 - y^3 - xy^2 - zx^2 - zy^2 - zxy]$$

$$= (x-y)(y-z)(xz^2 + yz^2 - x^2y - x^2z)$$

$$= (x-y)(y-z)[xz(z-x) + y(z^2-x^2)]$$

$$= (x-y)(y-z)(z-x)(xz+yz+xy).$$

Example 5. *Solve* $\begin{vmatrix} x+a & b & c \\ a & x+b & c \\ a & b & x+c \end{vmatrix} = 0.$

Solution. Replacing $c_1 \to c_1 + c_2 + c_3$

$$\begin{vmatrix} x+a+b+c & b & c \\ a+x+b+c & x+b & c \\ a+b+x+c & b & x+c \end{vmatrix} = 0$$

or, $$(x+a+b+c)\begin{vmatrix} 1 & b & c \\ 1 & x+b & c \\ 1 & b & x+c \end{vmatrix} = 0$$

Replacing $R_2 \to R_2 - R_3$ and $R_3 \to R_3 - R_1$

or, $$(x+a+b+c)\begin{vmatrix} 1 & b & c \\ 0 & x & -x \\ 0 & 0 & x \end{vmatrix} = 0$$

or, $$x^2(x+a+b+c) = 0$$

$$x = 0 \quad \text{or} \quad x = -(a+b+c).$$

Example 6. *Evaluate the determinant :* $\begin{vmatrix} 13 & 16 & 19 \\ 14 & 17 & 20 \\ 15 & 18 & 21 \end{vmatrix}$.

Solution. Given the determinant : $\begin{vmatrix} 13 & 16 & 19 \\ 14 & 17 & 20 \\ 15 & 18 & 21 \end{vmatrix}$

Replacing $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_2$

$$= \begin{vmatrix} 13 & 16-13 & 19 \\ 14 & 17-14 & 20 \\ 15 & 18-15 & 21 \end{vmatrix}$$

$$= \begin{vmatrix} 13 & 3 & 19 \\ 14 & 3 & 20 \\ 15 & 3 & 21 \end{vmatrix}$$

= 0 (since the two columns are identical).

NOTES

Example 7. *Show that* $\begin{vmatrix} p-q-r & 2p & 2p \\ 2q & q-r-p & 2q \\ 2r & 2r & r-p-q \end{vmatrix} = (p+q+r)^3.$

Solution. Let $|A| = \begin{vmatrix} p-q-r & 2p & 2p \\ 2q & q-r-p & 2q \\ 2r & 2r & r-p-q \end{vmatrix}$

Replacing $R_1 \to R_1 + R_2 + R_3$

$$= \begin{vmatrix} p+q+r & p+q+r & p+q+r \\ 2q & q-r-p & 2q \\ 2r & 2r & r-p-q \end{vmatrix}$$

$$= (p+q+r)\begin{vmatrix} 1 & 1 & 1 \\ 2q & q-r-p & 2q \\ 2r & 2r & r-p-q \end{vmatrix}$$

Replacing $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$

$$= (p+q+r)\begin{vmatrix} 1 & 0 & 0 \\ 2q & -q-r-p & 0 \\ 2r & 0 & -r-p-q \end{vmatrix}$$

$$\Rightarrow \qquad |A| = (p+q+r)\left[r\begin{vmatrix} -q-r-p & 0 \\ 0 & -r-p-q \end{vmatrix}\right]$$

$$= (p+q+r)\,[(-q-r-p)(-r-p-q)]$$

$$= (p+q+r)(-p-q-r)^2$$

$$= (p+q+r)(p+q+r)^2$$

$$= (p+q+r)^3$$

Hence, $\begin{vmatrix} p-q-r & 2p & 2p \\ 2q & q-r-p & 2q \\ 2r & 2r & r-p-q \end{vmatrix} = (p+q+r)^3.$

Example 8. *Prove that* $\begin{vmatrix} 1+a^2-b^2 & 2b & -2b \\ 2ab & 1-a^2-b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix} = (1+a^2+b^2)^3$

Solution. Let the given determinant be denoted by A.

Then replacing $C_1 \to C_1 - bC_2$ and $C_2 \to C_2 - aC_3$, we have

$$A = \begin{vmatrix} 1+a^2-b^2 & 0 & -2b \\ 0 & 1+a^2+b^2 & 2a \\ b(1+a^2+b^2) & -a(1+a^2+b^2) & 1-a^2-b^2 \end{vmatrix}$$

Taking common $(1 + a^2 + b^2)$ from 1st and 2nd column,

$$= (1+a^2-b^2)^2 \begin{vmatrix} 1 & 0 & -2b \\ 0 & 1 & 2a \\ b & -a & 1-a^2-b^2 \end{vmatrix}$$

NOTES

Replacing $R_3 \to R_3 - bR_1$

$$= (1+a^2+b^2)^2 \begin{vmatrix} 1 & 2a \\ -a & 1-a^2+b^2 \end{vmatrix}$$

Expanding the determinant along the first column :

$$= (1+a^2+b^2)^2 \begin{vmatrix} 1 & 2a \\ -a & 1-a^2+b^2 \end{vmatrix}$$

$$= (1 + a^2 + b^2)^2 \, [1 - a^2 + b^2 + 2a^2]$$

$$= (1 + a^2 + b^2)^2 (1 + a^2 + b^2)$$

$$= (1 + a^2 + b^2)^3$$

$$A = (1 + a^2 + b^2)^3$$

Hence, $\begin{vmatrix} 1+a^2-b^2 & 2b & -2b \\ 2ab & 1-a^2-b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix} = (1 + a^2 + b^2)^3.$

6. SOLUTIONS OF SIMULTANEOUS LINEAR EQUATIONS

A technique is given below for solving three simultaneous linear equations in three unknowns. This technique may also be applied to solve n-equations in n-unknowns.

Let us consider the system of equations whose the co-efficients are read as :

$$\left.\begin{aligned} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{aligned}\right\} \quad ...(1)$$

The co-efficients of x, y, z in the above equations may be written in the form of determinant :

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Known as co-efficient det.

If $\Delta \neq 0$, then solution of (1) is written as

$$x = \frac{\Delta_1}{\Delta}, \quad y = \frac{\Delta_2}{\Delta}, \quad z = \frac{\Delta_3}{\Delta}$$

where $\Delta_i : i = 1, 2, 3, 4$ ... is the determinant obtained from Δ by replacing the i^{th} column in Δ by d_1, d_2, d_3. Let us prove that this is infact true.

NOTES

Proof.
$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$\Rightarrow \qquad x.\Delta = \begin{vmatrix} a_1x & b_1 & c_1 \\ a_2x & b_2 & c_2 \\ a_3x & b_3 & c_3 \end{vmatrix}$$

Replacing $c_1c_1 + yc_2 + zc_3$

$$\Rightarrow \qquad x.\Delta = \begin{vmatrix} a_1x + b_1y + c_1z & b_1 & c_1 \\ a_2x + b_2y + c_2z & b_2 & c_2 \\ a_3x + b_3y + c_3z & b_3 & c_3 \end{vmatrix}$$

$$\Rightarrow \qquad x.\Delta = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$$

$$\Rightarrow \qquad x.\Delta = \Delta_1$$

where a, $$\Delta_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$$

$$\Rightarrow \qquad x = \frac{\Delta_1}{\Delta}.$$

Similarly, it can be written as

where, $$\Delta_2 = \begin{vmatrix} a_1 & d_1 & b_1 \\ a_2 & d_2 & b_2 \\ a_3 & d_3 & b_3 \end{vmatrix} \quad \text{and} \quad \Delta_3 = \begin{vmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \end{vmatrix}$$

Hence, we get

This method is known as **Cramer's Rule.**

NOTE This Cramer's rule is not applicable for the following points :

1. where $\Delta = 0$
2. if $\Delta \neq 0$, $d_1 = d_2 = d_3 = 0$, then the only solution of the system of equation will be $x = y = z = 0$
3. if $\Delta = 0$, but at least one of Δ_1, Δ_2, Δ_3 is not zero, then the system has no solution.
4. if $\Delta = \Delta_1 = \Delta_2 = \Delta_3 = 0$, then the system has infinite number of solution.

ILLUSTRATIVE EXAMPLES

Example 1. *Solve by Camer's Rule :*

$$5x - 3y = 1$$
$$3x + 2y = 12$$

Solution. The determinant of the system of equation is

$$\Delta = \begin{vmatrix} 5 & -3 \\ 3 & 2 \end{vmatrix} = 10 + 9 = 19 \neq 0$$

Now, $$\Delta_1 = \begin{vmatrix} 1 & -3 \\ 12 & 2 \end{vmatrix} = 2 + 36 = 38$$

and $$\Delta_2 = \begin{vmatrix} 5 & 1 \\ 3 & 12 \end{vmatrix} = 60 - 3 = 57$$

Hence, $$x = \frac{\Delta_1}{\Delta} = \frac{38}{19} = 2 \text{ and } y = \frac{\Delta_2}{\Delta} = \frac{57}{19} = 3.$$

Example 2. *Solve by Camer's Rule :*

$$2x - 3y = 7$$
$$3x - 2y = 3$$

Solution. The determinant of the system of equation is

$$\Delta = \begin{vmatrix} 2 & -3 \\ 3 & -2 \end{vmatrix} = -4 + 9 = 5$$

Now, $$\Delta_1 = \begin{vmatrix} 7 & -3 \\ 3 & -2 \end{vmatrix} = -14 + 9 = -5$$

and $$\Delta_2 = \begin{vmatrix} 2 & 7 \\ 3 & 3 \end{vmatrix} = 6 - 21 = -15$$

Hence, $$x = \frac{\Delta_1}{\Delta} = \frac{-5}{5} = -1 \text{ and } y = \frac{\Delta_2}{\Delta} = \frac{-15}{5} = -3$$

So, $x = -1$ and $y = -3$.

Example 3. *Solve by Camer's Rule :*

$$x - y + z = 0$$
$$2x + 2y - z = 6$$
$$x - 2y + 3z = 1$$

Solution. The determinant of the system of equation is :

$$\Delta = \begin{vmatrix} 1 & -1 & 1 \\ 2 & 2 & -1 \\ 1 & -2 & 3 \end{vmatrix}$$

$$= 1\begin{vmatrix} 2 & -1 \\ -2 & 3 \end{vmatrix} - (-1)\begin{vmatrix} 2 & -1 \\ 1 & 3 \end{vmatrix} + 1\begin{vmatrix} 2 & 2 \\ 1 & -2 \end{vmatrix}$$

$$= 1(6 - 2) + 1(6 + 1) + 1(-4 - 2)$$

$$= 4 + 7 - 6 = 5 \neq 0.$$

Hence, Camer's rule is applicable.

Now, $$\Delta_1 = \begin{vmatrix} 0 & -1 & 1 \\ 6 & 2 & -1 \\ 1 & -2 & 3 \end{vmatrix} = 0\begin{vmatrix} 2 & -1 \\ -2 & 3 \end{vmatrix} - (-1)\begin{vmatrix} 6 & -1 \\ 1 & 3 \end{vmatrix} + 1\begin{vmatrix} 6 & 2 \\ 1 & -2 \end{vmatrix}$$

$$= (18 + 1) + 1(-12 - 2)$$

$$= 19 - 14 = 5$$

Similarly, $$\Delta_2 = \begin{vmatrix} -1 & 0 & 1 \\ 2 & 6 & -1 \\ -2 & 1 & 3 \end{vmatrix} = 1\begin{vmatrix} 6 & -1 \\ 1 & 3 \end{vmatrix} - 0\begin{vmatrix} 2 & -1 \\ 1 & 3 \end{vmatrix} + 1\begin{vmatrix} 2 & 6 \\ 1 & 1 \end{vmatrix}$$

$$= (18 + 1) + 1(2 - 6)$$

$$= 19 - 4 = 15$$

and $$\Delta_3 = \begin{vmatrix} 1 & -1 & 0 \\ 2 & 2 & 6 \\ 1 & -2 & 1 \end{vmatrix} = 1\begin{vmatrix} 2 & 6 \\ -2 & 1 \end{vmatrix} - (-1)\begin{vmatrix} 2 & 6 \\ 1 & 1 \end{vmatrix} + 0\begin{vmatrix} 2 & 2 \\ 1 & -2 \end{vmatrix}$$

$$= (2 + 12) + 1(2 - 6)$$

$$= 14 - 4 = 10$$

Hence, $$x = \frac{\Delta_1}{\Delta} = \frac{5}{5} = 1, y = \frac{\Delta_2}{\Delta} = \frac{15}{5} = 3 \text{ and } z = \frac{\Delta_3}{\Delta} = \frac{10}{5} = 2.$$

Example 4. *Solve by Camer's Rule :*

$$x + y - z = -2$$
$$3x + 2y + 3z = 13$$
$$2x + 7y + 4z = 31$$

Solution. The determinant of the system of equation is :

$$\Delta = \begin{vmatrix} 1 & 1 & -1 \\ 3 & 2 & 3 \\ 2 & 7 & 4 \end{vmatrix} = 1\begin{vmatrix} 2 & 3 \\ 7 & 4 \end{vmatrix} - 1\begin{vmatrix} 3 & 3 \\ 2 & 4 \end{vmatrix} + (-1)\begin{vmatrix} 3 & 2 \\ 2 & 7 \end{vmatrix}$$

$$= 1(8 - 21) - 1(12 - 6) - 1(21 - 4)$$

$$= -13 - 6 - 17 = -36 \neq 0$$

Hence, Camer's Rule is applicable.

Now, $$\Delta_1 = \begin{vmatrix} -2 & 1 & -1 \\ 13 & 2 & 3 \\ 31 & 7 & 4 \end{vmatrix} = -2\begin{vmatrix} 2 & 3 \\ 7 & 4 \end{vmatrix} + 1\begin{vmatrix} 13 & 3 \\ 31 & 4 \end{vmatrix} + (-1)\begin{vmatrix} 13 & 2 \\ 31 & 7 \end{vmatrix}$$

$$= 38$$

$$\Delta_2 = \begin{vmatrix} 1 & -2 & -1 \\ 3 & 13 & 3 \\ 2 & 31 & 4 \end{vmatrix} = 1\begin{vmatrix} 13 & 3 \\ 31 & 4 \end{vmatrix} - 1\begin{vmatrix} 3 & 3 \\ 2 & 4 \end{vmatrix} - 2\begin{vmatrix} 3 & 13 \\ 2 & 31 \end{vmatrix}$$

$$= -96$$

$$\Delta_3 = \begin{vmatrix} 1 & 1 & -2 \\ 3 & 2 & 13 \\ 2 & 7 & 31 \end{vmatrix} = 1\begin{vmatrix} 2 & 13 \\ 7 & 31 \end{vmatrix} - 1\begin{vmatrix} 3 & 13 \\ 2 & 31 \end{vmatrix} - 2\begin{vmatrix} 3 & 2 \\ 2 & 7 \end{vmatrix}$$

$$= -130$$

Hence, $x = \frac{\Delta_1}{\Delta} = \frac{38}{-36} = \frac{-19}{18}$, $y = \frac{\Delta_2}{\Delta} = \frac{-96}{-36} = \frac{8}{3}$ and $z = \frac{\Delta_3}{\Delta} = \frac{-130}{-36} = \frac{65}{18}$.

NOTES

EXERCISE 6

1. Evaluate the followings :

(i) $\begin{vmatrix} 2 & 4 & 5 \\ 6 & 7 & 10 \\ 3 & 2 & 1 \end{vmatrix}$ (ii) $\begin{vmatrix} -1 & 2 & -3 \\ 4 & 2 & 6 \\ 3 & 2 & -3 \end{vmatrix}$

(iii) $\begin{vmatrix} a & b & c \\ d & b & l \\ k & o & p \end{vmatrix}$ (iv) $\begin{vmatrix} -1 & -2 & -4 \\ 6 & -7 & -8 \\ 9 & 10 & -2 \end{vmatrix}$

2. Prove that

(i) $\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix} = 0$ (ii) $\begin{vmatrix} 0 & a^2 & b \\ b^2 & 0 & a^2 \\ a & b^2 & 0 \end{vmatrix} = a^5 + b^5$

(iii) $\begin{vmatrix} 1 & w & w^2 \\ w & w^2 & 1 \\ w^2 & 1 & w \end{vmatrix} = 0$ (iv) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$

(v) $\begin{vmatrix} a^2+1 & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3$

3. Solve : $\begin{vmatrix} 4x & 6x+2 & 8x+1 \\ 6x+2 & 9x+3 & 12x \\ 8x+1 & 12x & 16x+2 \end{vmatrix} = 0$

4. Prove that $x = 1$ is a root of the equation :

$$\begin{vmatrix} x+1 & 3 & 5 \\ 2 & x+2 & 5 \\ 2 & 3 & x+1 \end{vmatrix} = 0$$

5. If x, y, z are different and

$$\begin{vmatrix} x & x^2 & 1+x^3 \\ y & y^2 & 1+y^3 \\ z & z^2 & 1+z^3 \end{vmatrix} = 0$$

then show that $1 + xyz = 0$

6. Solve the following simultaneous equations, using Camer's Rule :

(a) $2x - 3y = 8$
$3x + y = 1$

(b) $6x + 3y = 3$
$x + y = 8$

(c) $3x + 5y - 7z = 13$
$4x + y - 12z = 6$
$2x + 9y - 3z = 20$

(d) $x - y + 2z = 4$
$3x + y + 4z = 6$
$x + y + z = 1$

(e) $3x + 2y + 3z = 2$
$5x + 7y + 5z = 3$
$4x + 5y + 4z = 4$

Answers

3. $x = -11/97$
6. (a) $x = 1, y = -2,$ (b) $x = -7, y = 15,$
   (c) $x = 1, y = 2, z = 0,$ (d) $\Delta = 0,$
   (e) $\Delta = 0.$

SUMMARY

1. The value of the determinant is unchanged by changing its rows into columns and columns into rows.
2. If two adjacent rows or columns of a determinant are interchanged, then the numerical value of the determinant is unchanged, but its sign is reversed.
3. If two rows or columns of a determinant are identical, then the determinant vanishes.
4. If every element in a row or column of a determinant is multiplied by a factor k, $k \neq 0$, then the value of the determinant is multiplied by the factor k.
5. If each constituent in any row or column consists of two or more terms, then the determinant can be expressed as the sum of two or more than two determinants.
6. If the elements of any row (or column) be increased or decreased by any equimultiples of the corresponding constituents of one or more of the other rows (or columns), the value of the determinant remains unaltered.
7. Cramer's rule is not applicable for the following points :
   (i) Where $\Delta = 0$
   (ii) If $\Delta \neq 0$, $d_1 = d_2 = d_3 = 0$, then the only solution of the system of equation will be $x = y = z = 0$
   (iii) If $\Delta = 0$, but at least one of $\Delta_1, \Delta_2, \Delta_3$ is not zero, then the system has no solution.
   (iv) If $\Delta = \Delta_1 = \Delta_2 = \Delta_3 = 0$, then the system has infinite number of solution.

TEST YOURSELF

1. Prove that

(i) $$\begin{vmatrix} 1+a_1 & 1 & 1 \\ 1 & 1+a_2 & 1 \\ 1 & 1 & 1+a_3 \end{vmatrix} = a_1a_2a_3\left(1+\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}\right)$$

(ii) $$\begin{vmatrix} 1 & a^2+bc & a^3 \\ 1 & b^2+ca & b^3 \\ 1 & c^2+ab & c^3 \end{vmatrix} = (c-b)(c-a)(a-b)(a^2+b^2+c^2)$$

(iii) $$\begin{vmatrix} a & b & ax+by \\ b & c & bx+cy \\ ax+by & bx+cy & 0 \end{vmatrix} = (b^2-ac)(ax^2+2bxy+cy^2)$$

(iv) $$\begin{vmatrix} (x-2)^2 & (x-1)^2 & x^2 \\ (x-1)^2 & x^2 & (x+1)^2 \\ x^2 & (x+1)^2 & (x+2)^2 \end{vmatrix} = -8$$

(v) $\begin{vmatrix} 1+a & 1 & 1 & 1 \\ 1 & 1+b & 1 & 1 \\ 1 & 1 & 1+c & 1+c \\ 1 & 1 & 1 & 1 \end{vmatrix} = a_1a_2a_3\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)$

(vi) $\begin{vmatrix} x+a & b & c & d \\ a & x+b & c & d \\ a & b & x+c & d \\ a & b & c & x+d \end{vmatrix} = x^2(x+a+b+c)$

2. **Solve the following simultaneous equations, using Camer's Rule :**

(a) $2x - y + z = 0$
$x + 2y - 3z = 0$
$3x + y + 2z = 0$

(b) $2x + 3y + z = 3$
$x + 4y + z = 1$
$3x + 7y + 2z = 3$

(c) $x - 2y + 3z = 6$
$3x + y - 4z = -7$
$5x - 3y + 2z = 5$

(d) $x - 3y + z = -1$
$2x + y - 4z = -1$
$6x - 7y + 8z = 7.$

Answer

2. (a) $x = 0, y = 0, z = 0,$ (b) D = 0,
(c) D = 0, (d) $x = 1, y = 1, z = 1.$

NOTES

CHAPTER

7

PERMUTATIONS AND COMBINATIONS

LEARNING OBJECTIVES

- Definition
- To Find the Number of Permutations of n Dissimilar Things Taken r at a Time
- To Find the Total Number of Permutations of n Dissimilar Things Taken r at a Time, In which A Particular Thing Always Occurs
- Alternative Proof for ${}^nP_r$
- To Find the Number of Combinations of n Dissimilar Things, Taken r at a Time
- Complementary Combinations
- To Prove that ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$
- Alternative Proof of ${}^nC_r$
- To Find the Number of Combinations of n Things, Taken R at a Time in which (*i*) p Particular Things will Always Occur, (*ii*) p Particular Things will Never Occur
- Greatest Value of ${}^nC_r$
- To Find the Number of Permutations of n Different Things Taken r at a Time When Each Thing may be Repeated any Number of Times in any Number of Times in any Arrangement
- To Find the Number of Permutations of n Things Taken all Together when the Things are not all Different
- Circular Permutations
- Division into Groups
- To Prove that the Total Number of Ways in which a Selection can be Made of $(p + q + r)$ Things of which p are Alike, q Alike and r Alike are $(p + 1)(q + 1)(r + 1) - 1$
- Basics of Counting
- Product Rule
- Indirect Counting
- One-to-One Correspondence
- Applications to Computer Science
- The Principle of Inclusion-exclusion
- The Pigeonhole Principle
- The Extended Pigeonhole Principle
- Big O Notation

1. DEFINITION

Each of the different arrangements that can be made of a given set of things, taking some or all of them at a time, is called a *permutation*.

Each of the different groups or collections that can be made of a given set of things, taking some or all of them at a time, is called a *combination*. Thus in combinations the order of things is immaterial. We require only the number of groups, irrespective of the order in which the things appear in the group.

Thus if we have three things a, b, c and we want to take two at a time then the different arrangements possible are :

$$ab, ba$$
$$ac, ca$$
$$bc, cb$$

Thus six arrangements, of the three quantities taking two at a time are each called a permutation. Hence the number of permutations of three quantities taken two at a time is 6. But ab, ba form only one group, hence the number of groups of three quantities taken two at a time is only 3. Hence the number of combinations of three quantities taken two at a time is 3.

Let us take four quantities a, b, c, d and form arrangements taking three at a time. The different arrangements possible are :

$$abc, acb, bac, bca, cab, cba \quad ...(1)$$
$$abd, adb, bad, bda, dab, dba \quad ...(2)$$
$$acd, adc, cad, cda, dac, dca \quad ...(3)$$
$$bcd, bdc, cbd, cdb, dbc, dcb \quad ...(4)$$

Only these 24 arrangements are possible, hence the number of permutations of four things taken three at a time is 24. But each of (1), (2), (3), (4) forms one group, hence the number of combinations of four things taken three at a time is 4.

The Fundamental Principle of Counting. Some authors call this principle as the multiplications principle.

Lemma. If there are m ways of doing a thing, and when it is done in any of the m ways, if there are n ways of doing a second thing, and when the first two have been done in any of the different ways, if there are p ways of doing a third and so on, then the total number of ways in which all of them may be done is $m \times n \times p \times ...$

We illustrate the above principle by means of some examples.

ILLUSTRATIVE EXAMPLES

Example 1. *There are five trains running between Kolkata and Delhi ; in how many ways can a man go from Kolkata to Delhi and return by a different train.*

Solution. There are five trains and, therefore, the first journey from Kolkata to Delhi can be performed in 5 ways. When any one train has been used for the first journey, and one of the remaining 4 trains can be used for the return journey. Thus corresponding to each way of performing the first journey, there are 4 ways of performing the return journey.

NOTES

Since there are 5 ways of making the first journey, there are in all 5 × 4 ways of making the two journeys.

Example 2. *In how many ways can we select a boy and a girl out of party of 20 boys and 8 girls.*

Solution. A boy can be selected in 20 ways and a girl in 8 ways ; therefore, a boy and a girl can be selected in 20 × 8 ways.

Example 3. *In how many ways can 3 things be given to 4 boys, there being no restriction as to the number of things each may receive ?*

Solution. The first thing can be given to 4 boys in 4 ways. The second thing can also be given to the 4 boys in 4 ways. Thus corresponding to each of the first 4 ways, there are 4 ways of giving the second thing hence there are 4 × 4 ways of giving the first and second things.

The third thing can also be given to the 4 boys in 4 ways hence there are in all 4 × 4 × 4 ways of giving the three things.

2. TO FIND THE NUMBER OF PERMUTATIONS OF *n* DISSIMILAR THINGS TAKEN *r* AT A TIME

The number of permutations of n dissimilar things, taken r at a time, is the same as *the number of different ways of filling up r places out of n things.*

The first place may be filled up by any of the n things *i.e.*, there are n ways of filling it. When it has been filled up by any one thing, there are only $(n-1)$ things left, hence the second place can be filled up in $(n-1)$ ways, *i.e.*, there are $(n-1)$ ways of filling up the second place corresponding to everyone way of filling up the first place hence the first two places can be filled up in $n(n-1)$ ways. Similarly when the first two places have been filled up, there are only $(n-2)$ things left, hence the third place can filled up in $(n-2)$ ways corresponding to everyone way of filling up the first two places, but there are $n(n-1)$ ways of filling up the first two places, hence the first three places can be filled up in $n(n-1)(n-2)$ ways and so on.

Thus the total number of ways of filling up any number of places is equal to the *product of same number of factors which are all in* A.P. *having n as first term and* -1 as common difference.

Hence the total number of ways of filling up r places

$$= n(n-1)(n-2) \ldots \text{to } r \text{ factors}$$
$$= n(n-1)(n-2) \ldots (n-(r-1))$$
$$= n(n-1)(n-2) \ldots (n-r+1).$$

Thus the number of permutations of n dissimilar things, taking r at a time

$$= n(n-1)(n-2) \ldots (n-r+1)$$

This is generally denoted by the symbol ${}^nP_r$.

Thus $${}^nP_r = n(n-1)(n-2) \ldots (n-r+1).$$

Note. It is clear from the above that n is a positive integer and also r is a positive integer less than or at most equal to n.

Cor. 1. The number of permutations of n dissimilar things taken all at a time is

$$^nP_n = n(n-1)(n-2) \ldots \text{ to } n \text{ factors}$$
$$= n(n-1)(n-2) \ldots 3.2.1$$

This product is denoted by the symbol $\underline{|\ n}$ or $n\,!$, which is read as factorial n.

Thus $\quad n\,! = n(n-1)(n-2) \ldots 3.2.1$
$$= n(n-1)\,! = n(n-1)(n-2)\,! \text{ and so on.}$$

Again $\quad ^nP_r = n(n-1)(n-2) \ldots (n-r+1)$

$$= \frac{n(n-1)(n-2)\ldots(n-r+1)(n-r)(n-r-1)\ldots 3.2.1}{(n-r)(n-r-1)\ldots 3.2.1}$$

$$= \frac{n\,!}{(n-r)\,!}.$$

Cor. 2. Meaning of the symbol 0 !

From the above result, $^nP_r = \dfrac{n\,!}{(n-r)\,!}$

It is clear that $\quad ^nP_n = \dfrac{n\,!}{(n-n)\,!} = \dfrac{n\,!}{0\,!}$

But $\quad ^nP_n = n\,!$ hence $0\,! = 1$.

3. TO FIND THE TOTAL NUMBER OF PERMUTATIONS OF *n* DISSIMILAR THINGS TAKEN *r* AT A TIME, IN WHICH A PARTICULAR THING ALWAYS OCCURS

Let $a_1, a_2, a_3 \ldots a_n$ be n dissimilar things and let us find the number of ways in which a_1 will always occur.

Suppose a_1 occupies the first place, then there are $(n-1)$ things left out of which $(r-1)$ things are to be taken, therefore, the total number of ways in which a_1 occupies the first place is $^{n-1}P_{r-1}$. Similarly the number of ways in which a_1 occupies the second place is also $^{n-1}P_{r-1}$, and so on. Since there are r places which a_1 can occupy, the total number of permutations in which a_1 will always occur is $r \times {}^{n-1}P_{r-1}$.

Cor. The number of permutations in which a particular thing will never occur is evidently $^{n-1}P_r$.

Hence it is clear that $\quad ^nP_r = {}^{n-1}P_r + {}^{n-1}P_{r-1}$.

4. ALTERNATIVE PROOF FOR ^nP_r

^nP_r is the number of permutations of n dissimilar things taken r at a time. Suppose anyone of these things occupies the first place, then there will be $^{n-1}P_{r-1}$ permutations. Since there are in all n things, each one of which can occupy the first place, the total number of permutations is

$$n \times {}^{n-1}P_{r-1}$$

Hence $\quad {}^{n}P_r = n \times {}^{n-1}P_{r-1}$

Similarly $\quad {}^{n-1}P_{r-1} = (n-1) \times {}^{n-2}P_{r-2}$

$${}^{n-2}P_{r-2} = (n-2) \times {}^{n-3}P_{r-3}$$

... ...

... ...

and $\quad {}^{n-r+2}P_{r-2} = (n-r+2) \cdot {}^{n-r+1}P_1.$

$${}^{n-r+1}P_1 = (n-r+1).$$

Equating the products of both sides, we get

$${}^{n}P_r \times {}^{n-1}P_{r-1} \times {}^{n-2}P_{r-2} \times \ldots \times {}^{n-r+1}P_1$$

$$= n(n-1)(n-2)\ldots(n-r+1) \times {}^{n-1}P_{r-1} \times {}^{n-2}P_{r-2} \times \ldots \times {}^{n-r+1}P_1$$

$\therefore \quad {}^{n}P_r = n(n-1)(n-2)\ldots(n-r+1).$

ILLUSTRATIVE EXAMPLES

Example 1. *If* ${}^{n}P_5 = 20 \times {}^{n}P_3$, *find n.*

Solution. Here $n(n-1)(n-2)(n-3)(n-4) = 20 \times n(n-1)(n-2)$

$\therefore \quad (n-3)(n-4) = 20 \quad$ or $\quad n^2 - 7n - 8 = 0$

$\therefore \quad n = -1 \quad$ or $\quad 8$

But $\quad n = -1$ is inadmissible

$\therefore \quad n = 8.$

Example 2. *In how many ways can the letters of DELHI be arranged. How many of these will begin with D and how many will not. How many will begin with D and end with I.*

Solution. The total number of letters is 5 and all are dissimilar hence the number of ways of arranging = 5 ! = 120.

When D is always in the first place, there are only four letters left, therefore, total number of such arrangements

= 4 ! = 24.

Therefore, the number of ways in which D will not be in the first place

= 120 – 24 = 96.

If D and I occupy the first and last places, there are only 3 letters left, therefore, number of such arrangements.

= 3 ! = 6.

Example 3. *Find the number of words that can be formed with LH always together.*

Solution. Since LH are to come together they may be counted as one letter so there are only 4 letters which can be arranged in 4 ! ways. But LH can be arranged between themselves in 2 ! ways as LH and HL. Hence the total number of words

4 ! × 2 ! = 48.

Example 4. *How many arrangements can be formed from the letters of* ***section,*** *so that*

(1) *t always occupies the middle place ;*

NOTES

(2) *t is as above, and sec and ion come always before and after t respectively ;*

(3) *c t i always occupy the three middle places :*

Solution. (1) Since t is fixed up, there are only 6 letters left which can be arranged in 6 ! ways ;

(2) Letters of section can be arranged in 3 ! ways and also letters of ion in 3 ! ways, therefore, the total number of ways = 3 ! × 3 !.

(3) Taking $c\ t\ i$ as one letter, there are only 4 letters which can be arranged in 4 ! ways but $c\ t\ i$ can be arranged among themselves in 3 ! ways, therefore, the total number of arrangements = 4 ! × 3 !.

Example 5. *How many numbers of six digits can be formed from the digits 4, 5, 6, 7, 8, 9, no digits being repeated ? How many of them are not divisible by 5 ?*

Solution. The number of numbers with 6 digits = 6 !.

Out of these 6 ! numbers, those which have 5 in the unit's place are divisible by 5 and their number = 5 !.

The number of numbers not divisible by 5 = 6 ! – 5 ! = 600.

Example 6. *How many numbers less then 10,000 and divisible by 5 can be formed with the 10 digits 1, 2, 3, 4, 5, 6, 7, 8, 9, 0 each digit not occurring more than once in each number.*

Solution. The required numbers may be of one digit, two digits, three digits or four digits and each of them must end in 5 or 0, except the number of one digit which must end in 5.

The number of numbers of one digit ending in 5 is 1.

The number of numbers of two digits ending in 5 = $^9P_1 - 1$,

[∵ the number having 0 as the first figure is to be rejected]

The number of numbers of two digits ending in 0 = $^9P_1$,

The number of numbers of three digits ending in 5

$= {^9P_2} - {^8P_1}$,

[∵ the number having 0 as the first figure are $^9P_1$ in number]

The number of numbers of three digits ending in 0 = $^9P_2$

The number of numbers of four digits ending in 5

$= {^9P_3} - {^8P_2}$

[∵ the numbers having 0 as the first figure are $^8P_2$ in number]

The number of numbers of four digits ending 0 = $^9P_3$

Hence the total number

$= 1 + ({^9P_1} - 1) + ({^9P_1}) + ({^9P_2} - {^8P_1}) + ({^9P_2}) + ({^9P_3} - {^8P_2}) + ({^9P_3})$

$= 1090.$

EXERCISE 7(A)

1. Show that 2.4.6.8 ... to n factors = $2^n \times n\,!$.
2. Show that $\dfrac{(2n)!}{n!} = 1.3.5 \ldots (2n-1) \,.\, 2^n$.
3. In how many ways can the letters of LAHORE be arranged ? How many of these arrangements will begin with L ? How many will not begin with L ? How many begin with L and end with E ?

4. Find the nunber of arrangements that can be made from the letters of DELHI, (*i*) taking all together, (*ii*) taking three at a time, (*iii*) beginning with D and ending in I, (*iv*) the letter L always occupying the middle position.
5. How many words can be formed out of the letters of scholar, and how many of them will (*i*) begin with *s* and end with *r*, (*ii*) have *o* in the middle, (*iii*) have *oa* always together, (*iv*) have *oa* never together, (*v*) have sch always together ?
6. In how many ways can the letters of the word **article** be arranged so that the vowels may occupy only odd positions ?
7. Find the number of arrangements that can be made of the letters of the word youngster so that the vowels may not all be in consecutive positions in any of them.
8. How many numbers each lying between 100 and 1000 can be formed with the digits 1, 2, 3, 4, 5, 6, 0 ?
9. How many numbers each lying between 10 and 1000 can be formed with the digits 2, 3, 4, 0, 8, 9 ?
10. How many numbers each greater than 1000 can be formed with the digits 5, 6, 7, 8, 9 ?
11. How many signals can be given form four flags when :
    (*i*) all flags are used ;
    (*ii*) all the flags may or may not be used ?
12. How many changes may be rung with 4 bells out of 7 ? And how many with the whole peal ?

Answers

3. 720, 120, 600, 24 **4.** 120, 60, 6, 24 **5.** 5040, 120, 720, 1440, 3600, 720 **6.** 576
7. 332640 **8.** 180 **9.** 125 **10.** 240
11. 24, 64 **12.** 840, 5040

5. TO FIND THE NUMBER OF COMBINATIONS OF *n* DISSIMILAR THINGS, TAKEN *r* AT *A* TIME

Let x be the required number of combinations or groups each group containing r things.

The things in anyone of these x groups may be arranged amongst themselves in $r\,!$ ways. Thus each of the x groups gives $r\,!$ permutations.

Therefore, the total number of permutations we obtain from all the groups is $x \times r\,!$.

But if all the things in each group are permuted, we have all the possible permutations of n things taken r at a time.

$$\therefore \qquad x \times r\,! = {}^nP_r = n(n-1)(n-2)\ldots(n-r+1)$$

$$\therefore \qquad x = \frac{n(n-1)(n-2)\ldots(n-r+1)}{r\,!}$$

The number of combinations of n things taken r at time is usually denoted by the symbol ${}^nC_r$. Thus we have

$${}^nC_r = \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}$$

Multiplying both numerator and denominator of the right hand expression by $(n-r)!$, we get

$$^nC_r = \frac{n(n-1)(n-2)\ldots(n-r+1)(n-r)!}{r!(n-r)!} = \frac{n!}{r!(n-r)!}$$

Cor. Putting $r = n$, we get

$$^nC_n = \frac{n!}{n!.0!} = 1 \quad \because \quad 0! = 1.$$

6. COMPLEMENTARY COMBINATIONS

To prove that the number of combinations of n things taken r at a time is equal to the number of combinations of n things taken (n – r) at a time.

Evidently for every group of r things that can be taken out of n things, a group of $(n-r)$ things is left out ; thus corresponding to each group of r things there is a group of the remaining $(n-r)$ things. Hence the number of different groups of r things,

i.e., $\quad ^nC_{n-r} = {}^nC_r.$

Otherwise. Since $\quad ^nC_r = \dfrac{n!}{r!(n-r)!}$

We have by putting $(n-r)$ for r,

$$^nC_{n-r} = \frac{n!}{(n-r)!(n-(n-r))!} = \frac{n!}{r!(n-r)!}$$

i.e., $\quad ^nC_{n-r} = {}^nC_r.$

Cor. If $^nC_x = {}^nC_y$ then either $x = y$ or $x = n - y$ *i.e.*, $\quad x + y = n$

$$^nC_0 = {}^nC_{n-0} = {}^nC_n = 1.$$

7. TO PROVE THAT $^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

Suppose we have $(n+1)$ things and we take r at a time. Then the number of combinations when one particular thing is always excluded in ^nC_r and the number of combinations when that particular thing is always included is $^nC_{r-1}$. The sum of these obviously is equal to the number of all the combinations of $(n+1)$ things taken r at a time *i.e.*, $^{n+1}C_r$.

Hence $\quad ^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

Otherwise. $^nC_r + {}^nC_{r-1} = \dfrac{n!}{r!(n-1)!} + \dfrac{n!}{(r-1)!(n-r+1)!}$

$$= \frac{n!}{r(r-1)!(n-r)!} + \frac{n!}{(r-1)!(n-r+1)(n-r)!}$$

$$= \frac{n!}{(r-1)!(n-r)!}\left\{\frac{1}{r} + \frac{1}{n-r+1}\right\}$$

NOTES

$$= \frac{n!}{(r-1)!\,(n-r)!} \times \frac{n-r+1+r}{r(n-r+1)}$$

$$= \frac{n!(n+1)}{r(r-1)!\,r(n-r+1)\,(n-r)!} = \frac{(n+1)!}{r!\,(n-r+1)!} = {}^{n+1}C_r.$$

8. ALTERNATIVE PROOF OF ${}^nC_r$

Let the n things be denoted be the letters a, b, c ... etc. There are ${}^nC_r$ combinations of n things taken r at a time and each combination contains r letters, so that if all the combinations be written down, the total number of letters will be $r \times {}^nC_r$. Now all those combinations which contain 'a' also contain $(r-1)$ letters out of the remaining $(n-1)$ letters. Hence there are as many combinations containing 'a' as there are combinations of $(n-1)$ letters taken $(r-1)$ at a time *i.e.*, the number of combinations containing 'a' is ${}^{n-1}C_{r-1}$.

Similarly the number of combinations containing 'b' is also ${}^{n-1}C_{r-1}$ and so on.

Hence the total number of letter in all these combinations is $n \times {}^{n-1}C_{r-1}$.

Therefore $\quad r \times {}^nC_r = n \times {}^{n-1}C_{r-1}$.

or $$ {}^nC_r = \frac{n}{r} \times {}^{n-1}C_{r-1}$$

Changing n and r first into $(n-1)$ and $(r-1)$ respectively, into $(n-2)$ and $(r-2)$ respectively, and so on we have

$${}^{n-1}C_{r-1} = \frac{n-1}{r-1} \times {}^{n-2}C_{r-2}$$

$${}^{n-2}C_{r-2} = \frac{n-2}{r-2} \times {}^{n-3}C_{r-3}$$

....

....

$${}^{n-r+2}C_2 = \frac{n-r+2}{2} \times {}^{n-r+1}C_1$$

$${}^{n-r+1}C_1 = \frac{n-r+1}{1}$$

Multiplying the left and right sides together and cancelling like factors from each side, we get

$${}^nC_r = \frac{n(n-1)(n-2)\ldots(n-r+1)}{r(r-1)(r-2)\ldots 2.1} = \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}.$$

9. TO FIND THE NUMBER OF COMBINATIONS OF n THINGS, TAKEN r AT A TIME IN WHICH (*i*) p PARTICULAR THINGS WILL ALWAYS OCCUR, (*ii*) p PARTICULAR THINGS WILL NEVER OCCUR

(*i*) Let us set aside the p particular things and form combinations of $(r-p)$ things out of the remaining $(n-p)$ things. The number of such combinations is ${}^{n-p}C_{r-p}$. With each of these combinations we combine the p particular things, so that we get

NOTES

combination of r in each of which the p particular things will occur. Hence the required number of combinations

$$= {}^{n-p}C_{r-p}.$$

(*ii*) When we do not take the p particular things at all, we will have to form combinations of r things taken out of the remaining $(n - p)$ things, number of such combinations is evidently ${}^{n-p}C_r$.

ILLUSTRATIVE EXAMPLES

Example 1. *In how many ways can a football eleven be chosen from 14 men ? How often would a particular player be (i) always included and (ii) always excluded ?*

Solution. Here we have to form groups of 11 out of 14 and this can be done in ${}^{14}C_{11}$ ways, *i.e.*,

$${}^{14}C_3 = \frac{14\,.\,13\,.\,12}{1\,.\,2\,.\,3} = 364 \text{ ways.}$$

(*i*) When one particular player is always included, we will have to form groups of 10 out of remaining 13 and this can be done in ${}^{13}C_{10} = {}^{13}C_3 = \dfrac{13\,.\,12\,.\,11}{1\,.\,2\,.\,3} = 286$ ways.

(*ii*) When one particular player is always excluded, we will have to form groups of 11 out of the remaining 13 and this can be done in ${}^{13}C_{11} = {}^{13}C_2 = \dfrac{13\,.\,12}{1\,.\,2} = 78$ ways.

Example 2. *Out of 10 consonants and 4 vowels, how many words can be formed each containing 3 consonants and 2 vowels ?*

Solution. 3 consonants are to be chosen out of 10 and the number of such groups $= {}^{10}C_3$.

Similarly the number of groups of 2 vowels out of 4 is ${}^4C_2$.

Now combining each of first set of groups with each of the second set of groups, we get together ${}^{10}C_3 \times {}^4C_2$ groups each consisting of 3 consonants and 2 vowels.

Each of these new groups contains 5 different letters which can be arranged amongst themselves in 5! ways.

Hence the required number of words $= {}^{10}C_3 \times {}^4C_2 \times 5\,! = 86400$.

Example 3. *Find the number of ways in which 7 boys and 5 girls may be placed in a row so that no two girls may be together.*

Solution. Place the 7 boys as follows :

$$\times\, B \times B \times B \times B \times B \times B \times B \,\times$$

In order that no two girls may be placed together, they can only be placed in ×. There are 8 such positions and thus we have to choose any 5 out of these 8 positions.

Hence the required number of ways

$$= {}^8C_5 = {}^8C_3 = \frac{8\,.\,7\,.\,6}{1\,.\,2\,.\,3} = 56.$$

Example 4. *From 7 boys and 4 girls a committee of 6 is to be formed ; in how many ways can it be done when the committee contains (i) exactly 2 girls and (ii) at least 2 girls ?*

NOTES

Solution. (*i*) 2 girls can be chosen out of 4 in ${}^4C_2$ ways. Here 4 boys will be required and 4 boys can be chosen out of 7 in ${}^7C_4$ ways, hence the total number of ways $= {}^4C_2 \times {}^7C_4 = 210$.

(*ii*) The committee may be format in the following ways ;

(*a*) 2 girls + 4 boys which can be done in ${}^4C_2 \times {}^7C_4$ ways ;

(*b*) 3 girls + 3 boys which can be done in ${}^4C_3 \times {}^7C_3$ ways ;

(*c*) 4 girls + 2 boys which can be done in ${}^4C_4 \times {}^7C_2$ ways ;

Hence the total number of ways $= {}^4C_2 \times {}^7C_4 + {}^4C_3 \times {}^7C_3 + {}^4C_4 \times {}^7C_2 = 371$.

Example 5. *A boat's crew consists of 8 men, 3 of whom can only row on the stroke side and 2 only in the bow side. In how many ways can the crew be arranged ?*

Solution. Since on each side there must be four men, we require one for the stroke side and 2 for the bow side, from the remaining 3 men. This can be done in either ${}^3C_1$ or ${}^3C_2$ ways.

Now, each side consists of 4 men and they can be arranged each in 4! ways hence the total number of arrangements $= {}^3C_1 \times 4\,! = 1728$.

Example 6. *There are n points in a plane, no 3 of which are in the same straight line except p which are in a straight line. Find the number of triangles and straight lines which can be formed by joining these points.*

Solution. Three points are required to form a triangle, so from n points we should get ${}^nC_3$ triangles but p of them are in a straight line, so be lose ${}^pC_3$ triangles hence,

$$\text{number of triangles} = {}^nC_3 - {}^pC_3.$$

By joining two points we get a straight line so from n points we should get ${}^nC_2$ straight lines but p points give only one straight and not ${}^pC_2$ straight lines. Hence the number of straight lines $= {}^nC_2 - {}^pC_2 + 1$.

Example 7. *Find the number of diagonals which can be drawn in a plane figure of 16 sides.*

Solution. There are 16 angular points hence the total number of straight lines which can be drawn is ${}^{16}C_2$ but this includes the 16 sides hence the number of diagonals

$$= {}^{16}C_2 - 16 = 104.$$

Example 8. *In how many of the permutations of n things taken r at a time will p given things occur ?*

Solution. Putting aside the p things, we form groups of $(r - p)$ things, out of remaining $(n - p)$ things, the number of such groups is ${}^{n-p}C_{r-p}$. Then we include these p things in each group.

Each group now contains r things and can, therefore, be arranged in r ways.

Hence the total number of permutations

$$= {}^{n-p}C_{r-p} \times r.$$

EXERCISE 7(B)

1. Show that :

   (*i*) ${}^nC_r = \dfrac{n-r+1}{r} \times {}^nC_{r-1}$ (*ii*) ${}^nC_r = \dfrac{n}{r} \times {}^{n-1}C_{r-1}$

   (*iii*) ${}^{4n}C_{2n} : {}^{2n}C_n = \dfrac{1.3.5\ldots(4n-1)}{\{1.3.5\ldots(2-1)\}^2}$.

2. A basket contains 10 mangoes. Find how many different selections you can make of 3 mangoes so as always to include a particular mango.

3. A father with 8 children takes 3 at a time to the garden, as often as he can, without taking the same three together more than once. How often will he go and how often will each child go ?
4. Out of 16 consonants and 6 vowels how many words can be formed, each containing 2 consonants and 1 vowel ?
5. From ten books, in how many ways can selection of five be made when two specified books are always (*i*) included and (*ii*) excluded ?
6. Find the number of ways in which 8 black and 6 white balls may be placed in a row so that no two white balls may be together.
7. At an election, where every voter may vote for any number of candidates not greater than the number to be elected, there are 6 candidates and 3 members to be chosen ; in how many ways may a man vote ?
8. In a municipal corporation, there are 20 councillors and 8 directors. How many committee can be formed consisting of 5 councillors and 3 directors ?
9. From six gentlemen and four ladies a committee of five is to be formed. In how many ways can this be done so as to include at least one lady ?
10. Find the number of words, which can be formed with two different consonants and one vowel, out of 7 different consonants and 3 different vowels, the vowel to lie between two consonants.
11. In how many ways can 37 English and 35 Mathematics books be arranged in a column one above the other so that no two Mathematics books may be together ?
12. A boat's crew consists of 10 men, 3 of whom can only row on one side and two only on the other side. Find the number of ways in which the crew can be arranged.

Answers

2. 36	**3.** 56, 21	**4.** 4320	**5.** 56, 56
6. 210	**7.** 41	**8.** 868224	**9.** 246
10. 126	**11.** 8436	**12.** 144000	

10. GREATEST VALUE OF ^nC_r

$$^nC_r = \frac{n(n-1)(n-2)\dots(n-r+1)}{1.2.3\dots r}$$

$$^nC_{r-1} = \frac{n(n-1)(n-2)\dots(n-r+2)}{1.2.3\dots(r-1)}$$

$$\therefore \qquad \frac{^nC_r}{^nC_{r-1}} = \frac{n-r+1}{r}.$$

Hence $^nC_r >=< {}^nC_{r-1}$ according as $n - r + 1 >=< r$

i.e., $$^nC_r >=< {}^nC_{r-1} \text{ as } n + 1 >=< 2r$$

i.e., $$^nC_r >=< {}^nC_{r-1} \text{ as } r <=> \frac{n+1}{2}$$

(*i*) n = odd number = $2m + 1$, say

then $^nC_r >=< {}^nC_{r-1}$ according as $r <=> \dfrac{2m+2}{2} = (m+1)$.

Hence for values of r from 1 to m, each of $^nC_1 \dots < {}^nC_2 \dots {}^nC_m$ is greater than the preceding one. But when $r = m + 1$, $^nC_r = {}^nC_{r-1}$ *i.e.*, $^nC_{m+1} = {}^nC_m$ for values of r from $(m + 2)$ to n, each of $^nC_{m+2}, {}^nC_{m+3}, \dots {}^nC_n$ is less than the preceding one. Thus ^nC_m and $^nC_{m+1}$ are the greatest, *i.e.*, ^nC_r is greatest when $r = m$ or $m + 1$.

NOTES

i.e.,
$$r = \frac{n-1}{2} \quad \text{or} \quad \frac{n+1}{2}$$

(*ii*) n = even number = $2m$, say

$${}^nC_r >=< {}^nC_{r-1} \text{ according as } r <=> \frac{2m+1}{2} = \left(m + \frac{1}{2}\right).$$

Hence for values of r from 1 to m, each of ${}^nC_1, {}^nC_2 \ldots {}^nC_m$ is greater than the preceding and also for values of r from $(m + 1)$ to n, each of ${}^nC_{m+1}, {}^nC_{m+2} \ldots {}^nC_n$ is less than the preceding. Hence ${}^nC_m$ *is the greatest, i.e.,* ${}^nC_r$ *is greatest when*

$$r = m = \frac{n}{2}.$$

Example. *A person wishes to make up as many different parties as he can out of 10 friends, each party consisting of the same number. How many should he invite at time and in how many of these would the same man be found ?*

Solution. Here we have to find the greatest value of ${}^{10}C_r$. Evidently then $r = 5$. Hence the person should invite 5 friends at a time and the greatest number of parties = ${}^{10}C_5 = 252$.

The number of parties in which the same man will be found

$$= {}^9C_4 = 126.$$

11. TO FIND THE NUMBER OF PERMUTATIONS OF *n* DIFFERENT THINGS TAKEN r AT A TIME WHEN EACH THING MAY BE REPEATED ANY NUMBER OF TIMES IN ANY NUMBER OF TIMES IN ANY ARRANGEMENT

Here we have to fill up r places out of n different things when each thing can be used as often as we place. The first place may be filled up in n ways since anyone of the n things can be placed there ; when this has been done, the second place may also be filled up in n ways since anyone of the given things may be repeated. Hence the first two places can be filled up in $n \times n$ or n^2 ways. When the first two places have been filled up in anyone way, the third place may also be filled up in n ways. Therefore, the first three places can be filled up in $n^2 \times n$ or n^3 ways ; and so on. Hence the r places can be filled up in n^r ways.

ILLUSTRATIVE EXAMPLES

Example 1. *In how many ways can 3 prizes be given away to 7 boys when each boy is eligible for any of the prize ?*

Solution. The first prize can be given away in 7 ways, similarly the second also in 7 ways and the third also in 7 ways. Hence the total number of ways = $7^3 = 343$.

Example 2. *How many different numbers, each of three digits can be formed with the digits 1, 2, 3, 4, 5 each digit occurring once, twice or thrice in each number ?*

Solution. This is the same as the number of permutations of 5 things, taken 3 at a time, each thing occurring once, twice and thrice.

Hence the number = 5^3.

12. TO FIND THE NUMBER OF PERMUTATIONS OF n THINGS TAKEN ALL TOGETHER WHEN THE THINGS ARE NOT ALL DIFFERENT

Let the n things be denoted by n letters, p of them being a, q of them being b and the rest being all unlike say, c, d, e etc. Let x be the required number of permutations. In any of these x permutations, let us replace the p like letters **a** by unlike letters a_1, a_2, ... etc., different from any of the rest and let us arrange these p new letters among themselves without altering the position of any of the remaining letters. We thus have $p\,!$ new permutations and if these were made in each of the x permutations, the total number of permutations formed will be $x \times p\,!$.

In like manner, if the q letters **b** were also replaced by q unlike letters, the total number of permutations would be

$$x \times p\,! \times q\,!$$

But the letters are now all different and n in number and the total permutations $= n\,!$. Hence $x \times p\,! \times q\,! = n\,!$

$$\therefore \qquad x = \frac{n\,!}{p\,!q\,!}.$$

Any case in which things are not different, may be treated similarly.

ILLUSTRATIVE EXAMPLES

Example 1. *How many different words can be formed out of the letters of the word ALLAHABAD ? In how many of these will the vowels occupy the even places ?*

Solution. There are in all 9 letters of which 4 are As, 2 Ls and the rest all are different. Hence the number of words

$$= \frac{9\,!}{4\,!2\,!}.$$

There are four even places and 5 odd places. The 4 even places can be filled up by the 4. As in one way and the 5 odd places by consonants L, L, H, B, D in $\frac{5\,!}{2\,!}$ ways.

Hence the number of words $= 1 \times \frac{5\,!}{2\,!} = 60.$

Example 2. *Find the number of permutations of the letters of the word SERIES. How many of these begin and end with S and in how many are the vowels and the consonants placed alternately ?*

Solution. (*i*) There are 6 letters of which 2 are Ss, 2 Es and the rest are different

: number of permutations $= \frac{6\,!}{2\,!2\,!} = 180.$

(*ii*) Here we have to arrange the remaining 4 letters of which 2 are Es. Here the number $= \frac{4\,!}{2\,!} = 12.$

(*iii*) If the consonants occupy the first, third and fifth places and the vowels second, fourth and sixth places, then the number $= \frac{3!}{2!} \times \frac{3!}{2!} = 9.$

Similarly when the vowels occupy the odd places and the consonants the even places, the number = 9.

Example 3. *How many numbers greater than a million can be formed with the digits 2, 3, 0, 3, 4, 2, 3 ?*

Solution. Since each number is to be of not less than 7 digits, we will have to use all the 7 digits. Among the 7 digits, 2 are 2, and 3 are 3, hence the total number of arrangements

$$= \frac{7!}{2!3!} = 420.$$

But out of these arrangements we have to reject those that begin with 0. To get the number of such arrangements, we have to leave out zero. We then have 6 digits of which 2 are 2, 3 are 3, hence number of such arrangements

$$\frac{6!}{2!3!} = 60.$$

Hence the required number = 420 – 60 = 360.

Example 4. *In how many ways can we arrange the letters in $x^2y^4z^3$ so that all the z's should not be together.*

Solution. There are 9 letters of which 2 are x's, 4 are y's and 3 are z's.

Hence the total number of arrangements $= \frac{9!}{2!3!4!}$.

Out of these we have to reject the arrangement when z's are together.

Regarding z's as one letter, the number of such arrangements $= \frac{7!}{2!4!}$

Hence the required number $= \frac{9!}{2!4!3!} - \frac{7!}{2!4!} = 1155.$

13. CIRCULAR PERMUTATIONS

In placing a number of things round a circle or any closed curve, we regard two arrangements as different only if they are different with regard to the relative positions of the things. Here we have to consider the position of one thing relatively to others, hence we can get the desired result by fixing some particular thing in one position and then by arranging the remaining things. If there be n things, the number of arrangements is $(n - 1)!$.

To find the number of circular permutations of n things, taken r at a time.

The total number of groups is ${}^nC_r$. Out of these there are r things in any group, one thing is to be placed in a fixed position and the remaining $(r - 1)$ things can be arranged in $(r - 1)!$ ways.

Hence the total number of arrangements

$$= {}^nC_r \times (r-1)! = \frac{{}^nP_r}{r!} \times r - 1 = \frac{{}^nP_r}{r!}$$

NOTES

If we consider the clockwise and counter-clockwise arrangements as the same, the total number of arrangements

$$= \frac{1}{2} \cdot \frac{{}^nP_r}{r!}.$$

ILLUSTRATIVE EXAMPLES

Example 1. *In how many ways can n persons seat themselves at a round table ?*

Solution. Keeping one person fixed, we have to arrange the remaining $(n-1)$ persons which can be done in $(n-1)!$ ways.

If the clockwise and counter-clockwise arrangements are the same, the number of possible arrangements $= \frac{1}{2}(n-1)!$.

Example 2. *In how many ways can 6 persons be seated at a round table so that all shall not have the same neighbours in any two arrangements ?*

Solution. The number of ways of seating 6 persons = 5 !. But in clockwise and counter-clockwise arrangements same persons are neighbours and, therefore, these two arrangements are the same. Hence the required number $= \frac{1}{2} \cdot 5\,! = 60$.

Example 3. *Find the number of ways in which 3 men and 3 women can be seated at a round table so that no two women be together.*

Solution. Three men can be arranged in 2 ! ways and when they are seated three distinct places are for the 3 women. The position of the men makes it impossible for two women to sit together. Thus the number of ways of arranging the 3 women = 3 !. So that the total number of arrangements $= 2! \times 3! = 12$.

EXERCISE 7 (C)

1. How many numbers of four digits can be formed using all the digits 3, 4, 5, 6, (*i*) when no digits is to be repeated and (*ii*) when digits may be repeated.
2. From six flags of different colours how many signals can be given by hoisting them one above the other when (*i*) no flag is to used more than once in one signal and (*ii*) there is no restriction to the number of times a flag is used.
3. In how many ways can 5 prizes be given away to 4 boys when (*i*) each boy is eligible for all the prizes, (*ii*) a boy is not eligible for all prizes ?
4. In how many ways may *n* prizes be given to *n* boys,
   (*i*) so that each boy may have a prize ;
   (*ii*) when there is no restriction to the number of prizes each boy receives ;
   (*iii*) when no body should receive all the prizes ?
5. I have six friends to invite ; in how many ways can I send invitation cards to each of them if I have 3 servants to carry the cards ?

6. A servant has to post 5 letters and there are 4 letter boxes available ; in how many ways can we post the letters ?
7. How many different numbers each less then 10,000 can be formed with the digits 2, 3, 4, 5, 0 each digit occurring once, twice, etc. upto 4 times ?
8. How many different words can be formed out of the letters of the word *Constantinople* ? In how many of these will the three *ns* be consecutive ?
9. Find how many words could be made from letters of the word Orion, supposing that (*i*) the two consonants may stand in any order and (*ii*) the two consonants may not stand together ?
10. In how many ways can the letters of the word *arrange be* arranged ? How many words can be made if the two *r*'s are not allowed to come together ?
11. In how many ways can the letters of the word *plantain* be arranged so that the two a's do not come together ?
12. How many numbers greater than a million be formed with the digits 1, 2, 0, 2, 3, 1, 2 ?

Answers

1. 24, 256	**2.** 720, 6^6	**3.** 1024, 1020	**4.** $n\,!, n^n, n^n - n$
5. 3^6	**6.** 4^5	**7.** $5^4 - 1$	**8.** $\frac{14!}{2!3!2!}, \frac{12!}{2!2!}$
9. 60,360	**10.** 1260, 900	**11.** 7560	**12.** $\frac{7!}{3!2!} - \frac{6!}{3!2!}$

14. DIVISION INTO GROUPS

(*i*) To prove that number of ways in which $(p + q)$ things can be divided into two groups containing p and q things respectively is $\frac{(p+q)!}{p!q!}$. For every group of p things, selected out of $(p + q)$ things a group of q things is left. Hence the number of ways is the same as the number of groups of p things formed out of $(p + q)$ things, *i.e.*, ${}^{p+q}C_p$ $= \frac{(p+q)!}{p!q!}$.

Equal groups. If $p = q$, every two groups repeat themselves in reverse order and hence the number of different ways of division $= \frac{(2p)!}{2!(p)!^2}$.

But if $2p$ things are to be distributed equally between two persons, the number of ways

$$= \frac{(2p)!}{(p)!^2}.$$

Similarly the number of ways of dividing $(p + q + r)$ things into groups of p, q, r things respectively is $\frac{(p+q+r)!}{p!q!r!}$.

If $p = q = r$, then number of divisions $= \frac{(3p)!}{3!(p)!^3}$.

For equal distribution, the number $= \frac{(3p)!}{(p)!^3}$.

ILLUSTRATIVE EXAMPLES

NOTES

Example 1. *Find the number of ways of dealing out equally a pack of 51 cards to 3 players.*

Solution. For equal distribution, number of ways $= \frac{51!}{(17)!^3}$.

But when no regard is paid to the order or arrangements of the groups and the cards are placed in 3 heaps of 17 each, then the number of ways $= \frac{51!}{3!(17)!^3}$.

Example 2. *In how many ways can 12 different things be divided into 3 parcels of each ?*

Solution. Here we require the number of different ways of division hence the required number $= \frac{12!}{3!(4)!^3}$.

15. TO PROVE THAT THE TOTAL NUMBER OF WAYS IN WHICH A SELECTION CAN BE MADE OF $(p + q + r)$ THINGS OF WHICH p ARE ALIKE, q ALIKE AND r ALIKE ARE $(p + 1)(q + 1)(r + 1) - 1$

The p things can be disposed of in $(p + 1)$ ways, for we may take 0, 1, 2, ... p of them. Similarly q things can be disposed of in $(q + 1)$ ways, r things in $(r + 1)$ ways. Therefore, the total number of ways $= (p + 1)(q + 1)(r + 1)$. But this includes the case in which none of the things is taken, Hence rejecting this case the required number of ways

$$= (p + 1)(q + 1)(r + 1) - 1.$$

In general if the number of things be $(p + q + r + ...)$ of which p are alike, q alike, r alike and so on, the total number of ways

$$= (p + 1)(q + 1)(r + 1) ... - 1.$$

If $p = q = r ... = 1$, the things are all different, then the number of ways $= 2^n - 1$.

This result can also be proved separately as follows :

To prove that the total number of combinations of n dissimilar things by taking some or all at a time is $2^n - 1$.

Each thing may be disposed of in two ways ; we may either select it or leave it out.

Associating either of these 2 ways of disposing of any one thing with either of the 2 ways of disposing of each one of the other $(n - 1)$ things, the total number of ways $= 2 \times 2 \times ...$ to n factors $= 2^n$.

But this includes the case when everyone is simultaneously left out. Hence excluding this case, the total number $2^n - 1$

Thus ${}^nC_1 + {}^nC_2 + {}^nC_3 + ... + {}^nC_n = 2^n - 1.$

NOTES

ILLUSTRATIVE EXAMPLES

Example 1. *How many different groups of coins may be formed with a four shilling piece, a florin, a shilling and a six pence ?*

Solution. There are altogether 4 coins. Therefore, the number of groups

$$= 2^4 - 1 = 15.$$

Example 2. *How many different sums may be formed with 3 rupees. 4 half-rupees and 5 twenty five paisa pieces ?*

Solution. The required number = 4 . 5 . 6 – 1 = 119.

Example 3. *Find the number of different factors of 4200.*

Solution. $4200 = 2^3 \times 5^2 \times 3 \times 7.$

Hence number of factors = (3 + 1) (2 + 1) (1 + 1) (1 + 1) – 1 = 47, excluding 1 but including 4200 itself.

Example 4. *I have 4 mangoes and 5 oranges. In how many ways can I make a selection so to take at least one of each kind ?*

Solution. The number of ways of selecting at least one mango

$$= 2^4 - 1$$

The number of ways selecting at least one orange

$$= 2^5 - 1$$

Hence the total number required $= (2^4 - 1)(2^5 - 1) = 465.$

EXERCISE 7 (D)

1. In how many ways can 15 things be divided into groups of 9 and 6 ?
2. In how many ways can two sides of 6 players each be chosen from 12 men ?
3. In how many ways can 9 balls be put into 2 bags, neither of which can hold more than 6 ?
4. Four men play a game in which 13 cards are dealt to each. How many different distributions are possible ?
5. In how many ways can 18 different things be given away to 3 persons so that each may get 6 things ?
6. I have 8 friends, in how many ways can I invite one or more to a dinner ?
7. How many products can be formed of the factors 2, 3, 5, 7, 11 ?
8. How many different factors can 2310 have ?
9. How many products can be formed of the factors 2, 2, 3, 3, 3, 4, 5 ?
10. Find the number of factors of 22680.
11. Show that the total number of combinations of $2n$ things n of which are alike, taken n at a time is 2^n.
12. Show that the total number of ways in which a selection can be made from $3n$ things which consist of 3 groups, each containing n like things is $(n + 1)^3 - 1$.

Answers

1. $\dfrac{15!}{9!\,6!}$	**2.** $\dfrac{12!}{2!\,(6!)^2}$	**3.** $\left(\dfrac{19!}{6!\,3!}\right) + \dfrac{9!}{4!\,5!} \times 2$
4. $\dfrac{52!}{(13!)^4}$	**5.** $\dfrac{18!}{(16!)^3}$	**6.** 255
7. 26	**8.** 15	**9.** 40
10. 76.		

16. BASICS OF COUNTING

NOTES

If X is a set, let us use | X | to denote the number of elements in X.

Two Basic Counting Principles. Two elementary principles act as "building blocks" for all counting problems. The first principle essentially says that the whole is the sum of its parts ; it is at once immediate and elementary, we need only be clear on the details.

Sum Rule. The principle of disjunctive counting : If a set X is the union of disjoint nonempty subsets $S_1, S_2, S_3, ..., S_n$, then $| X | = | S_1 | + | S_2 | + | S_3 | + ... + | S_n |$, where the subsets $S_1, S_2, S_3, ..., S_n$, must have no elements in common. Again, since $X = S_1 \cup S_2 \cup S_3 \cup ... \cup S_n$, each element of X is in exactly one of the subsets of S_1. In other words, $S_1, S_2, S_3, ..., S_n$, is a partition of X. If the subsets $S_1, S_2, S_3, ..., S_n$, were allowed to overlap, than a more profound principle will be needed the *principle of inclusion and exclusion*. We will discuss this principle later in section 14.21.

Frequently, instead of asking for the number of elements in a set, some problems need that in how many ways a certain event can happen.

The difference is largely in semantics, for if A is an event, we can let X be a set of ways that A can happen and count the number of elements in set X.

If $E_1, E_2, ..., E_n$ be mutually exclusive events and E_1 can happen in e_1 ways, E_2 can happen e_2 ways, ..., E_n can happen in e_n ways, then E_1 or E_2 or ... or E_n can happen in $e_1 + e_2 + e_3 + ... + e_n$ ways. Here mutually exclusive events E_1 and E_2 mean that E_1 or E_2 can happen but both cannot happen simultaneously.

The **sum rule** can also be formulated in terms of **choices** : If an object can be selected from a reservoir in e_1 ways and an object can be selected from a separate reservoir in e_2 ways, then the selection of one object from either one reservoir or the other can be made in $e_1 + e_2$ ways.

ILLUSTRATIVE EXAMPLES

Example 1. *How many ways can we get a sum of 4 or of 8 when two distinguishable dice (say one die is red and the other is white) are rolled ? In how many ways can we get an even sum ?*

Solution. Let the out come of a 1 on the red die and a 3 on the white dic as the ordered pair (1, 3). Then the outcomes (1, 3), (2, 2), (3, 1) are the only pairs whose some is 4. Hence there are three ways to obtain the sum 4. Likewise, we obtain the sum 8 from the outcomes (2, 6), (3, 5), (4, 4), (5, 3) and (6, 2). Thus, there are 3 + 5 = 8 outcomes whose sum is 4 or 8. The number of ways to obtain the even sum is the same as the number of ways to obtain either the sum 2, 4, 6, 8, 10, or 12. There is one way to obtain the sum 2, three ways to obtain the sum 4, five ways to obtain the sum 6, five ways to obtain the sum 8, three ways to obtain a sum 10, and one way to obtain a sum 12. Hence there are 1 + 3 + 5 + 5 + 3 + 1 = 18 ways to obtain an even sum.

Example 2. *In how many ways can we get a sum of 8 when two indistinguishable dice are rolled ? In how many ways can we get an even sum ?*

Solution. We obtain the sum of 8 by the outcomes (2, 6), (3, 5), (4, 4), (5, 3) and (6, 2), but since the dice are similar, the outcomes (3, 5) and (5, 3) and as well as (2, 6) and (6, 2) are not different and thus we obtain the sum of 8 with the throw of two similar dice in only 3 ways. We can get an even sum in 1 + 2 + 3 + 3 + 2 + 1 = 12 ways.

NOTES

Example 3. *In how many ways can we draw a heart or a spade from an ordinary pack of playing cards ? An ace or a king ? A heart or an ace ? A card numbered 2 to 10 ? A numbered card or a king ?*

Solution. Since there are 13 hearts and 13 spades we may draw a heart or a spade in 13 + 13 = 26 ways. We may draw an ace or a king in 4 + 4 = 8 ways. Again we may draw a heart or an ace in 13 + 3 = 16 ways since there are only three aces that are not hearts. Now there are 9 cards numbered 2 to 10 in each of 4 suits : clubs, diamonds, hearts, spades, so we may choose a numbered card in 36 ways. Thus we may choose a numbered card or a king in 36 + 4 = 40 ways.

Note. We are counting the aces as distinct from numbered cards.

17. PRODUCT RULE

The principle of sequential counting :

If $S_1, S_2, S_3, ..., S_n$ be nonempty sets, then the number of elements in the cartesian product

$$S_1 \times S_2 \times S_3 \times ... \times S_n \text{ is the product } \prod_{i=1}^{n} |S_i|,$$

i.e.,
$$|S_1 \times S_2 \times ... \times S_n| = \prod_{i=1}^{n} |S_i|$$

Let us consider $S_1 \times S_2$ by a tree diagram, where

$$S_1 = \{a_1, a_2, a_3, a_4\} \quad \text{and} \quad S_2 = \{b_1, b_2, b_3\}.$$

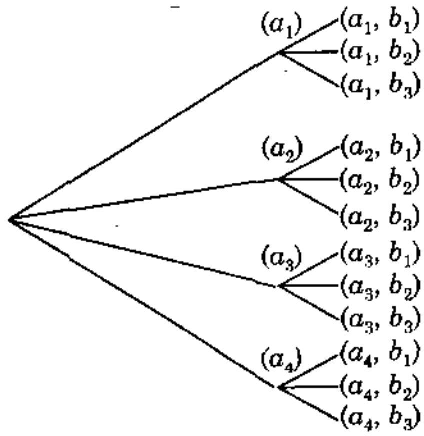
Observe that there are 4 branches in the first stage corresponding to the 4 elements of S_1 and to each of these branches there are three branches in second stage corresponding to the 3 elements of S_2, then the total number of branches is 4 × 3 = 12. The cartesian product $S_1 \times S_2$ can be written as $(a_1 \times S_2) \cup (a_2 \times S_2) \cup (a_3 \times S_2) \cup (a_4 \times S_2)$ where $(a_i \times S_2) = \{(a_i, b_1), (a_i, b_2), (a_i, b_3)\}$, $1 \le i \le 4$. Therefore, for example let $(a_4 \times S_2)$ be the fourth branch in first stage followed by each of 3 branches in the second stage.

Then
$$(a_4 \times S_2) = \{(a_4, b_1), (a_4, b_2), (a_4, b_3)\}.$$

More general example. If $a_1, a_2, ..., a_n$ are the n distinct elements of S_1 and $b_1, b_2, ..., b_m$ are the m distinct elements of S_2, then $S_1 \times S_2 = \bigcup_{i=1}^{n} (a_i \times S_2)$. Let x be an arbitrary element of $S_1 \times S_2$, then $x = (a, b)$ where $a \in S_1$ and $b \in S_2$. Thus $a = a_i$ for some i and $b = b_j$ for some j. Then $x = (a_i, b_j) \in (a_i \times S_2)$ and therefore $x \in \bigcup_{i=1}^{n} (a_i \times S_2)$.

We observe that $(a_i \times S_2)$ and $(a_j \times S_2)$ are adjoint if $i \ne j$ since if $x \in (a_i \times S_2) \cap (a_j \times S_2)$ then $x = (a_i, b_k)$ for some k and $x = (a_j, b_l)$ for some l and $(a_i, b_k) = (a_j, b_l)$ gives $a_i = a_j$ and $b_k = b_l$. But since $i \ne j$, $a_i \ne a_j$. Hence contradiction.

NOTES



Thus we conclude that $S_1 \times S_2$ is the disjoint union of the sets $(a_i \times S_2)$. Furthermore $|a_i \times S_2| = |S_2|$ since there is a one-to-one correspondence between the sets $a_i \times S_2$ and S_2, namely; $(a_i, b_j) \to b_j$. Therefore by the sum rule $|S_1 \times S_2| = \sum_{i=1}^{n} |a_i \times S_2|$

$$= |S_2| + |S_2| + |S_2| + ... + |S_2| = n|S_2| = nm.$$

Now we can determine the product rule in terms of events. If events $E_1, E_2, ..., E_n$ **can happen in** $e_1, e_2, ..., e_n$ *ways respectively, then the sequence of events* E_1 *first,* followed by E_2, ..., followed by E_n can happen in $e_1 e_2 ... e_n = \prod_{i=1}^{n} e_i$ ways.

The product rule in terms of choices. If a first object can be chosen in e_1 ways, a second in e_2 ways, ... and an nth chosen in e_n ways, then a choice of a first, second, ..., and nth object can be made in $e_1 e_2 ... e_n$ ways or $\prod_{i=1}^{n} e_i$ ways.

ILLUSTRATIVE EXAMPLES

Example 1. *Suppose that the number plates of vehicles in a certain state require 3 English letters followed by 4 digits*

(i) How many different plates can be manufactured if repetition of letters and digits are allowed ?

(ii) How many plates are possible if only the letters can be repeated ?

(iii) How many plates are possible if only the digits can be repeated ?

(iv) How many plates are possible if on repetitions are allowed as all ?

Solution. *(i)* $26^3 . 10^4$ since there are 26 possibilities for each of the 3 letters and 10 possibilities for each of 4 digits.

(ii) $26^3 . 10 . 9 . 8 . 7$ *(iii)* $26 . 25 . 24 . 10^4$ *(iv)* $26 . 25 . 24 . 10 . 9 . 8 . 7.$

Example 2. *If 2 distinguishable dice are thrown, in how many ways can they fall ? If 5 distinguishable dice are thrown, how many possible outcomes are there ? How many if 100 distinguishable dice are tossed ?*

Solution. The first die can fall in 6 ways and the second die can fall in 6 ways. Hence there are 6 . 6 = 36 outcomes when 2 dice are rolled. Also each of third, fourth and fifth die have 6 outcomes so that there are $6 . 6 . 6 . 6 . 6 = 6^5$ possible outcomes when all 5 dice are tossed. Similarly there are 6^{100} possible outcomes when 100 dice are tossed.

Example 3. *How many different number plates are there that involve 1, 2 or 3 letters followed by 4 digits ?*

Solution. The number plates with 1 letter followed by 4 digits in $26 . 10^4$ ways, with 2 letters followed by 4 digits in $26^2 . 10^4$ ways, similarly with 3 letters followed by 4 digits in $26^3 . 10^4$ ways. These separate events are mutually exclusive therefore we can use the sum rule to determine that there are $26 . 10^4 + 26^2 . 10^4 + 26^3 . 10^4$ plates with 1, 2 or 3 letters followed by 4 digits.

18. INDIRECT COUNTING

This principle is important to solve some combinational problems by counting indirectly *i.e.*, by counting the complement of a set. We give some examples of it here.

ILLUSTRATIVE EXAMPLES

Example 1. *If we draw a card from a pack of 52 cards and replace it before the next draw. In how many ways can 10 cards be drawn so that the tenth card is a repetition of a previous draw ?*

Solution. We find the ways by indirect counting. First we count the number of ways we can draw 10 cards when tenth card is not a repetition. First, choose what the 10th card will be. This can be done in 52 ways. If the first 9 cards are different from this card, then each of the 9 cards can be chosen from 51 cards. Then there are 51^9 ways to draw the first 9 cards which are different from 10th cards. Thus there, are $51^9 . 52$ ways to choose 10 cards when the first 9 cards are different from 10th cards. Hence there are $52^{10} - 51^9 . 52$ ways to draw 10 cards where the 10th is a repetition since there are 52^{10} ways to draw 10 cards with replacement.

Example 2. *In how many ways can 10 people be seated in a row so that a certain pair of them are not next to each other ?*

Solution. Since the number of people is 10, therefore there are 10! ways of seating all 10 people. Thus by indirect counting, we only count the number of ways of seating all 10 people where the certain pair of people (say A and B) are seated next to each other. If we treat the pair AB as one entity, then there are 9 entries to be arranged in 9! ways. But A and B can be seated next to each other in two different orders, as AB and BA. Therefore, there are 2. 9! ways of seating all 10 people where A and B are next to each other. Then the answer to our problem is $10! - 2 . 9!$.

19. ONE-TO-ONE CORRESPONDENCE

A one-to-one correspondence is set between the objects of the type first with those of the second type. (A one-to-one correspondence between two sets A and B is just a one to one function from A onto B). We give some illustrations.

ILLUSTRATIVE EXAMPLES

Example 1. *Let there be 101 players entered in a single elimination tennis tournament. In such a tournament, any player who loses a match must drop out, and every match ends in a victory for some player-there are no ties. In each round of the tournament, the players remaining are matched into as many pairs as possible, but if there is an odd number of players left some one receives a bye. Enough rounds are played until a single player remains who wins the tournament. How many matches must be played in total ?*

Solution. For this problem, there are two approaches, First, the straight forward approach is to analyse each round of the tournament as follows : The 50 winners and bye will go into the second round and pair into 25 matches and one bye. After this round 25 winners and the bye will go into the third round where will be exactly 13 matches. The fourth round will have six matches and a bye ; the fifth round will have 3 matches and a bye, the sixth will have 2 matches, the seventh will have 1 match and the winner of this round wins the entire tournament. Thus the total number of matches will be $50 + 25 + 13 + 6 + 2 + 1 = 100$.

There is a most useful another method to solve this problem. There is one-to-one correspondence between the number of matches and the number of losers. Each match has one and only one loser and loser is eliminated in one and only one match. Hence, the total number of matches is equal to the total number of losers. At the start there are 101 players and at the end there is one undefeated player. Therefore, there are 100 losers and hence 100 matches are required to find a winner.

Example 2. *Determine the number of subsets of a set with n elements.*

Solution. Let S_n be the number of subsets of a set with n elements. To determine the value of S_n for some values of n we see that

n	0	1	2	3	4	5
S_n	1	2	4	8	16	32

For example, if $n = 4$ then the number of elements in a set $\{a, b, c, d\}$ is 4, then there are total 16 subsets of this set when including the empty set, the four subsets with single element as $\{a\}$, $\{b\}$, $\{c\}$, $\{d\}$ and six subsets with 2 elements as $\{a, b\}$, $\{b, c\}$, $\{c, d\}$, $\{d, a\}$, $\{a, c\}$, $\{b, d\}$ and 4 subsets with three elements as $\{a, b, c\}$, $\{a, b, d\}$, $\{a, c, d\}$, $\{b, c, d\}$ and the entire set $\{a, b, c, d\}$.

In general we can say that, if n is the number of elements in a set then $S_n = 2^n$ is the number of subsets.

General Proof. Let $V = \{x_1, x_2, \ldots, x_n\}$ be the entire set. Then if T is the subset of V, assign the n-digit binary sequence $(y_1, y_2, \ldots, y_n)$ where $y_i = 1$ if $x_i \in T$ and $y_i = 0$ if $x_i \notin T$. In this way we associate a unique n-digit binary sequence to each subset of V. For example, if $T = \{x_1, x_3, x_5\}$, then the associated n-digit binary sequence is $(1, 0, 1, 0, 1, 0, 0, \ldots, 0)$ indicating $x_1 \in T$, $x_3 \in T$, $x_5 \in T$ but the other $n - 3$ elements are not in T.

Moreover, to each n-digit binary sequence there is a unique subset of V. For example, the binary sequence $(1, 0, 1, 0, 1, 0, 0, \ldots, 0)$ corresponds to the subset $\{x_2, x_4, x_6\}$. We have established a one-to-one correspondence between the collection of all n-digit binary sequence and the collection of subsets of V. Hence there are 2^n n-digit binary sequences so that there are 2^n subsets of V.

20. APPLICATIONS TO COMPUTER SCIENCE

A 2-valued Boolean function of n-variables is defined by the assignment of a value of either 0 or 1 to each of the 2^n, n-digit binary numbers.

How many Boolean functions of n variables are there ?

As there are 2 ways to assign a value to each of 2^n binary n-tuples, the method of product

there are $$\underbrace{2.2\ldots\ldots2}_{2^n \text{ factors}} = 2^{2^n}$$

ways to assign all the values, and therefore 2^{2^n} different Boolean functions of n variables.

A 2-valued Boolean function can be represented in tabular form where the n-digit binary numbers and their values are given in the table below. Such a tabular form is called the truth table of a 2-valued Boolean function. The following table is the truth table of a 2-valued Boolean function of four variables.

Four-Digit Binary Number	*Value*
0000	0
0001	1
0010	1
0011	0
0100	1
0101	0
0110	0
0111	0
1000	1
1001	0
1010	0
1011	0
1100	0
1101	0
1110	0
1111	0

A self-dual 2-valued Boolean function is one which will remain unchanged after all the 0's and 1's in the truth table are interchanged.

Now there is a question that how many self-dual 2-valued Boolean functions of n variables are there ?

The set of 2^n binary n-tuples into 2^{n-1} blocks, each block containing an n-tuple and its1's complement. In constructing a self dual function, assigning a value to either member of a block fixed the value assignments may be made for only 2^{n-1} of the 2^n n-tuples. Hence, there are $2^{2^{n-1}}$ different self-dual Boolean functions of n variables.

The applications of the 2 valued Boolean functions and self-dual Boolean functions is quite important to computer scientists who study the nature and applications of switching functions and logic design. It is important to understand their properties as well as enumerate them.

21. THE PRINCIPLE OF INCLUSION-EXCLUSION

NOTES

We already know the rule for counting the number of elements in the union of disjoint sets. However, if the sets are not disjoint the statement of sum rule is to be modified accordingly. The modified rule is generally called the *principle of inclusion-exclusion* or sometimes *sieve method*.

First Statement. If A and B are subsets of some universal set U, then

$$|A \cup B| = |A| + |B| - |A \cap B|$$

where notation "| |" stands for number of elements in the set.

Proof. We know that $A \cup B$ is the union of the three disjoint sets namely $A \cap \overline{B}$, $A \cap B$ and $\overline{A} \cap B$, so that by the sum rule,

$$|A \cup B| = |A \cap \overline{B}| + |A \cap B| + |\overline{A} \cap B| \quad \text{...(1)}$$

Now since $(A \cap \overline{B}) \cup (A \cap B) = A$ and $B = (\overline{A} \cap B) \cup (A \cap B)$

so that $|A| = |A \cap \overline{B}| + |A \cap B|$ and $|B| = |\overline{A} \cap B| + |A \cap B|$

The sum of these two relation gives

$$|A| + |B| = |A \cap \overline{B}| + |A \cap B| + |\overline{A} \cap B| + |A \cap B| \quad \text{...(2)}$$

from (1) and (2), we have

$$= |A \cup B| = |A| + |B| - |A \cap B|.$$

Remark. If $A \cap B = \phi$, then this is just the sum rule.

ILLUSTRATIVE EXAMPLES

Example 1. *In a university 200 faculty members can speak French and 50 can speak German, while only 20 can speak both French and German. How many faculty members can speak either French or German ?*

Solution. Let F be the set of faculty members who speak French and G be the set of faculty members who speak German. Then $|F| = 200$, $|G| = 50$ and $|F \cap G| = 20$

$\therefore \quad |F \cap G| = |F| + |G| - |F \cap G| = 200 + 50 - 20 = 230.$

Example 2. *From a group of 10 Professors in how many ways can a committee of 5 members be formed so that at least one of Professor X and Professor Y are included ?*

Solution. Sum rule method : The number of committees including both Professers X and Y is ${}^8C_3 = 56$. The number of committees including either Professor X or Professor Y (but not both) is ${}^8C_4 = 70$. Therefore by sum rule the total number of ways of selecting a committee of 5 including Professor X or Professor Y = 56 + 70 + 70 = 196.

(70 is taken two times once for inclusion of Professor X and second time for inclusion of Professor Y).

Aliter

By Counting Method. By counting indirectly we obtain the solution as follows. The total number of committees excluding both Professor X and Professor Y = ${}^8C_5 = 56$ and the total number of committees = ${}^{10}C_5 = 252$. Hence the number of committees including at least one of the Professor X or Professor Y = 252 – 56 = 196.

NOTES

Aliter

Principle of inclusion-exclusion Method. Total number of committees of 5 members is 252. Let A_1 and A_2 be the set of committees that include Professor X and Professor Y respectively. Then

$$|A_1| = {}^9C_4 = 126 = |A_2| \text{ and } |A_1 \cap A_2| = {}^8C_3 = 56.$$

$$\therefore \quad |A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$$
$$= 126 + 126 - 56 = 196.$$

Statement of the Principle of Inclusion-exclusion for three sets :

If A, B and C are finite sets, then

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|.$$

Theorem. If A_i are the finite subsets of universal set U, then

$$|A_1 \cup A_2 \cup \dots A_n| = \sum_{i=1}^{n} |A_i| + \sum_{i,j} |A_i \cap A_j| + \sum_{i,j} |A_i \cap A_j \cap A_k| + \dots + (-1)^{n-1} |A_1 \cap A_2 \dots \cap A_n|$$

where the second summation is taken over all 2-combinations $\{i, j\}$ of $\{1, 2, ..., n\}$, the third summation is taken over all 3-combinations $\{i, j, k\}$ of the integers $\{1, 2, ..., n\}$, also so on.

ILLUSTRATIVE EXAMPLE

Example. *In how many ways can the letters $\{5 . x, 4 . y, 3 . z\}$ be arranged so that all the letters of the same kind are not in a single block ?*

Solution. Let U be the set of $12!/[5!\,4!\,3!]$ permutations of these letters. Let A_1 be the arrangements of the letters where the 5 x's are in one block, A_2 the arrangement where the 4 y's are in single block, A_3 the arrangement where the 3 z's are in single block, Then

$$|A_1| = \frac{8!}{4!3!},\ |A_2| = \frac{9!}{5!3!},\ |A_3| = \frac{10!}{5!4!},$$

$$|A_1 \cap A_2| = \frac{5!}{3!},\ |A_1 \cap A_3| = \frac{6!}{4!},\ |A_2 \cap A_3| = \frac{7!}{5!}$$

$$|A_1 \cap A_2 \cap A_3| = 3!$$

Hence, $|\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3| = \dfrac{12!}{5!4!3!} - \left(\dfrac{8!}{4!3!} + \dfrac{9!}{5!3!} + \dfrac{10!}{5!4!}\right) + \dfrac{5!}{3!} + \dfrac{6!}{4!} + \dfrac{7!}{5!} - 3!.$

22. THE PIGEONHOLE PRINCIPLE

Theorem 1. ***(The Pigeonhole Principle).*** *If n pigeons are assigned to m pigeonholes, and $m < n$, then at least one pigeonhole contains two or more pigeons.*

Proof. Let us label the m pigeonholes with number 1 to m and the n pigeons with the numbers 1 to n. Now starting with pigeon 1, assign each pigeon in order to the pigeonhole with the same number. Having the assignment in this way to all pigeons $n - m$ pigeons that have not yet been assigned to a pigeonhole as $m < n$. Hence at least one pigeonhole will be assigned a second pigeon or more. Hence the theorem.

ILLUSTRATIVE EXAMPLES

NOTES

Example 1. *If eight people are chosen in any way from a group prove that at least two of them would have been born on the same day of the week.*

Solution. Here each person is to be assigned to the day of the week on which he or she was born. Since there are eight people and there are only seven days in a week the pigeonhole principle suggests that at least two persons must be assigned to the same day of the week.

Example 2. *Show that if any five numbers from 1 to 8 are chosen, then two of them will add upto 9.*

Solution. Let us construct four different sets, each containing two numbers that add up to 9. Let these sets be $A_1 = \{1, 8\}$, $A_2 = \{2, 7\}$, $A_3 = \{3, 6\}$, $A_4 = \{4, 5\}$. Each of the five numbers chosen belongs to one of the sets. Since there are only four sets, the pigeonhole principle tells us that two of the chosen numbers belong to the same set which add upto 9.

Example 3. *Show that if any 11 numbers are chosen from the set $\{1, 2, ..., 20\}$, then one of them will be a multiple of another.*

Solution. Any positive integer n can be written as $n = 2^r m$, where m is odd and $r \geq 0$. Let us designate m the odd part of n. If 11 numbers are chosen from the set $\{1, 2, ..., 20\}$, then two of them must have the same odd part. This follows from the pigeonhole principle because there are 11 numbers, but only 10 odd numbers between 1 and 20 that can be odd parts of these numbers.

Let n_1 and n_2 be two chosen numbers with the same odd part. Then $n_1 = 2^{r_1} m$ and $n_2 = 2^{r_2} m$, for some r_1 and r_2. If $r_1 \geq r_2$ then n_1 is a multiple of n_2 ; otherwise n_2 is a multiple of n_1.

Example 4. *Shirts numbered consecutively from 1 to 20 are worn by the 20 members of a sports team. When any 3 of these members are chosen to be a team, the sum of their shirt numbers is used as a code number for the team. Show that if any 8 of the 20 are selected, then from these 8 we may form at least two different teams having the same code number.*

Solution. Out of the 8 selected bowlers, the total number of different teams of 3 are ${}^8C_3 = 56$. The largest possible team code number is 18 + 19 + 20 = 57, and smallest possible is 1 + 2 + 3 = 6. Thus only the 52 code numbers between 6 and 57 inclusive available for the 56 possible teams. By the pigeonhole principle, at least two teams will have the same code number.

23. THE EXTENDED PIGEONHOLE PRINCIPLE

Theorem 2. *If n pigeon are assigned to m pigeonholes, then one of the pigeonholes must contain at least $[n - 1)/m] + 1$ pigeons.*

Proof. We will provide the proof by contradiction. Suppose each pigeonhole contains not more than $[(n - 1)/m]$ pigeons, then there are at most $m.\ [(n - 1)/m] \leq m\ .\ (n - 1)/m = n - 1$ pigeons in all. This is contradictory to our assumption. Hence one of the pigeonholes must contain at least $[(n - 1)/m] + 1$ pigeons.

Note. $[n/m]$ stands for the largest integer less than or equal to the rational number n/m. For example $[7/2] = 3$ etc.

ILLUSTRATIVE EXAMPLES

Example 1. *Show that if any 30 persons are selected, then a subset of 5 may be chosen such that all five were born on the same day of the week.*

Solution. There 30 persons and 7 days of a week. Let $n = 30$ and $m = 7$. Then by the extended pigeonhole principle, at least $[(30 - 1)/7] + 1$ or 5 of the people must have born on the same day of the week.

Example 2. *Show that if 30 books of Computer Science contain a total of 61327 pages, then one of them must have at least 2045 pages.*

Solution. The pages may be taken as pigeons and books as the pigeonholes. Let us assign each page to the book in which it appear. Then by the extended pigeonhole principle, one book must contain at least $[(61327 - 1)/30] + 1$ or 2045 pages.

EXERCISE 7 (E)

1. Show that if seven colours are used to paint 50 bicycles, at least 8 bicycles will be the same colour.
2. How many friends must you have to guarantee that at least five of them will have birthdays in same month ?
3. If 13 peoples are assembled in a room, show that at least 2 of them must have their birthday in the same month ?
4. Show that there must be at least 90 ways to choose six members from 1 to 15 so that all the choices have the same sum.
5. Prove that if any 14 numbers from 1 to 25 are chosen then one of them is multiple of another.
6. Show that five points are selected in a square whose sides have length 1 m, at least two of the points must be no more than 1 m apart.
7. Prove that if any 7 numbers from 1 to 12 are chosen then two of them will add upto 13.
8. Prove that if any 8 positive integers are chosen, two of them will have the same remainder when divided by 7.
9. If T is an equilateral triangle whose sides are of length 1 unit. Prove that if any 5 points are chosen laying on or inside the triangle, then two of them must be no more than $\frac{1}{2}$ unit apart.
10. Twenty cards numbered 1 through 20 are placed face down on a table. Cards are selected one at a time and turned over until 10 cards have been chosen. If two of them add upto 21, the player loses. Is it possible to win this game ?

Answer

2. 49.

24. BIG O NOTATION

Sometimes we need comparing the "sizes" of functions. Commonly called "Big O Notation" is used for this purpose particularly in analysis of the running times of algorithms. This notation is particularly interesting because it expresses a relation between functions that is neither a partial ordering nor an equivalence relation, although it is sometimes applied by mistake as if it were one or the other. Below we give the definition.

Definition. Let $f : N \to R$ be a function from the set of non-negative integers into the real numbers. $O(f)$ denotes the collection of all functions $g : N \to R$ for which there exist constants c and k (possibly different for each g) such that $\forall\ n \geq k,\ |\ g(n)\ | \leq c,\ |\ f(n)\ |$. If g is in $O(f)$ we say that g is of order f. In case of the well behaved functions the definition of Big O can be simplified as given below :

Simplified Definition. If there exists a constant λ_1 such that for every $n \geq \lambda_1$, $g(n) \geq 0$, then g is in $O(g)$ if there exist constants c and λ_2 such that for every $n \geq \lambda_2$, $g(n) \leq c\, f(n)$.

Remark. The above definition follows from the facts that $g(n)$ and $f(n)$ are both non-negative for $n \geq \lambda_1$ and therefore $|\ g(n)\ | = g(n)$ and $|\ f(n)\ | = f(n)$ for $n \geq \max(\lambda_1, \lambda_2)$.

Example. *Show that $5n^3 - 6n^2 + 4n - 2$ is in $O(n^3)$.*

Solution. We know that $|\ 5n^3 - 6n^2 + 4n - 2\ | \leq 5n^3 + 6n^3 + 4n^3 + 2n^3 \leq 17n^3$ for $n \geq 1$.

If we choose $c = 17$, $\lambda = 1$ then as per definition $5n^3 - 6n^2 + 4n - 2 \in O(n^3)$.

EXERCISE 7(F)

1. Prove that
   (*a*) $n\,! \in O(n^n)$
   (*b*) $\log_2(n\,!) \in O(n \log_2 n)$
   (*c*) $n \log_2 n \in O(\log_2 n\,!)$
   (*d*) $2^n \in O(n\,!)$
   (*e*) $n\,!$ is not in $O(2^n)$
   (*f*) $(1 + 2 + 3 + ... + n) \in O(n^2)$
   (*g*) $(1^2 + 2^2 + 3^2 + ... + n^2) \in O(n^3)$.
2. Show that the following for functions f, g and h from N into the real numbers.
   (*a*) $f \in O(g)$ if and only if $O(f) \subseteq O(g)$.
   (*b*) $f \in O(g)$ and $g \in O(f)$ if and only if the sets $O(f)$ and $O(g)$ are equal.
   (*c*) If $f \in O(g)$ and $g \in O(h)$, $f\,o(g) \in O(h)$.
   (*d*) $O(f) = O(af)$ if a is a nonzero constant.
3. Prove that the following :
   (*a*) $n^k \in O(n^{k+1})$, where k is a positive integer.
   (*b*) $2n + 3\ O(n)$
   (*c*) $9n^2 - 4n + 12 \in O(n^2)$
   (*d*) $10n^2 - 5n + 6 \in O(n^3)$
   (*e*) $3n + 2 \notin O(1)$
   (*f*) $5(2)^n + n^2 \in O(2^n)$
   (*g*) $10n^2 + 5n - 6 \notin O(n)$
   (*h*) $O(\log_a n) = O(\log_b n)$ where a and b are integers greater than 1.
4. Prove or disprove each of the following :
   (*a*) If f is in $O(g)$ and c is a positive constant, then $c.\,f$ is in $O(g)$.
   (*b*) If f_1 and f_2 are in $O(g)$ then $f_1 + f_2$ is in $O(g)$.
   (*c*) If f_1 is in $O(g_1)$ and f_2 is in $O(g_2)$ then $f_1 \cdot f_2$ is in $O(g_1 \cdot g_2)$.
   (*d*) If f_1 is in $O(g_1)$ and f_2 is in $O(g_2)$ then $f_1 + f_2$ is in $O(g_1 + g_2)$.
5. The simpler sorting algorithms are characterized by the fact that they require $f(n) \in O(n^2)$ comparisons to sort n items. On the other hand, most of the advanced sorting algorithms require $g(n) \in (n \log_2 n)$ comparisons. It is instructive to compare n^2 and $n \log_2 n$. Construct a table that compares the values of n^2 and $n \log_2 n$ for the values $n = 10, 100, 1000$ and 10000. Include the ratio $n^2/n \log_2 n$ in your table.
6. How many integral solutions are there of $x_1 + x_2 + x_3 + x_4 = 20$ if $1 \leq x_1 \leq 6$, $3 \leq x_2 \leq 7$, $5 \leq x_3 \leq 8$ and $1 \leq x_4 \leq 9$?
7. How many integral solutions are there of $x_1 + x_2 + x_3 + x_4 = 20$ if $2 \leq x_1 \leq 6$, $3 \leq x_2 \leq 7$, $5 \leq x_3 \leq 8$ and $2 \leq x_4 \leq 9$?
8. How many integers from 1 to 10^6 inclusive are neither perfect squares, perfect cubes, nor perfect fourth powers ?

SUMMARY

1. Each of the different arrangements that can be made of a given set of things, taking some or all of them at a time, is called a *permutation*.
2. If there are m ways of doing a thing, and when it is done in any of the m ways, if there are n ways of doing a second thing, and when the first two have been done in any of the different ways, if there are p ways of doing a third and so on, then the total number of ways in which all of them may be done is $m \times n \times p \times ...$
3. In placing a number of things round a circle or any closed curve, we regard two arrangements as different only if they are different with regard to the relative positions of the things. Here we have to consider the position of one thing relatively to others, hence we can get the desired result by fixing some particular thing in one position and then by arranging the remaining things. If there be n things, the number of arrangements is $(n-1)!$.

TEST YOURSELF

1. Two persons go into a railway carriage where there are six vacant seats. In how many ways can they seat themselves ?
2. You are given six books on six different subjects ; English ; Mathematics ; Sanskrit ; History ; Geography and Hindi ; in how many ways can you arrange them on a self so that the English and Sanskrit books may never come together ?
3. In how many ways can 8 men be arranged in a row of 8 chairs, if two particular men occupy the end seats.
4. Of the numbers formed by using all the figures 1, 2, 3, 4, 5, 6, 7 only once, how many are even ?
5. How many odd numbers of 5 digits can be formed with the digits 3, 2, 7, 4, 0 ?
6. Show that the number of ways in which n books can be arranged on a self so that two particular books are not together is $(n-2)(n-1)!$.
7. A self contains 15 books of which 5 are single volumes and the others from sets of 5, 3 and 2 respectively ; find in how many ways the book be arranged on the self, the volume of each set being in their due order.
8. Eight gentleman and five ladies apply for 5 different situations, 3 of which must be filled by men and 2 by women ; in how many ways can the situations be filled ?
9. In how many ways may a football eleven be arranged if 3 of them can play forward only, the remaining can play as backs or forwards, except 2, who can also play at the goal ?
10. A person has 20 friends, 12 of whom are men and the remaining being ladies. In how many ways may he invite 15 guests from among them so that 8 out of them are men ?
11. In how many ways can the letters of the word candidate be arranged without changing the order of the vowels ?
12. In how many ways can 7 Englishmen and 7 Americans sit down at a round-table no two Americans being neighbours ?
13. In how many ways can 8 persons be seated at a round table so that all shall not have the same neighbours in any two arrangements ?

Answers

1. 30	**2.** 480	**3.** 1440	**4.** 2160
5. 36	**7.** 322560	**8.** 6720	**9.** ${}^2C_1 \times {}^7C_1 \times 5! \times 5!$
10. 3960	**11.** – 1	**12.** $16! \times 7!$	**13.** $\frac{1}{2} \cdot 7!$.

CHAPTER

8

PROBABILITY

NOTES

LEARNING OBJECTIVES

- Introduction
- Probability Theory
- Mutually Exclusive Events
- Additive Law of Probability or Theorem of Total Probability
- Statistical or Empirical Definition
- Compound Events
- Independent Event
- Theorem of Compound probability
- The Probability of Impossible Event is Zero
- Generalized Theorem of Total Probability
- Theorem of Compound Probability (Another Approach)
- Independence of Two Events
- Bayes, Theorem
- Statement of Bayes, Theorem

1. INTRODUCTION

The words 'Probability' and 'Chance' are quite familiar to everyone. Many a times, we come across statements like "Probably it may rain today", Chances of his visit to the university are very few", "It is possible that he may pass the examination with good marks". In the above statements, the words *probably, chance, possible* etc., convey the sense of uncertainty about the occurrence of some event. Ordinarily, it may appear that there cannot be any exact measurement for these uncertainties, but in Mathematics, we do have methods for calculating the degree of certainty of events in numerical values, provided certain conditions are satisfied.

2. PROBABILITY THEORY

1. Random Experiment. An experiment is called a random experiment if it is such that all the possible results (outcomes) are known but it is not possible to predict the outcome of any individual experimentation in advance.

Thus

(1) Tossing a coin

(2) Drawing a card from a pack of cards

(3) Shooting a target

(4) Birth of a child

are the examples of random experiments.

2. Non-random Experiments. An experiment which is not random is known as non-random experiment thus

(1) Throwing a piece of iron in air

(2) Throwing a piece of wood in water

are the examples of non-random experiments.

3. Sample Space. Collection of all possible out-comes of a random experiment is known as the sample space and is denoted by Ω.

4. Point. Individual outcomes of a sample space are known as points or elements and are denoted by ω.

5. Event. Any desired property is known as an event and denoted by capital letters A, B, C, ... For example if experiment is throwing a dice then,

$$\Omega = \{1, 2, 3, 4, 5, 6\}.$$

Event I. Getting even numbers, so this event will be denoted by set $A = \{2, 4, 6\}$.

Event II. Getting a number which is perfect square, will be denoted by a set $B = \{1, 4\}$.

6. Conclusion. Given any random experiment we can find Ω and given any event we can find set corresponding to that event.

7. Definition. A set containing no outcome of the sample space is known as a null set and is denoted by ϕ.

Example

(1) Random experiment : A card is drawn from a pack of cards.

(2) Sample space $\Omega = \{x : x \text{ is s card}\}$.

(3) Event : The card is an ace.

This event will be denoted by a set

$$A = \{\omega : \omega \text{ is an ace}\}.$$

8. Definition. If there are infinitely many points in a sample space then it is known as a continuous sample space but if these are finite or enumerable number of points then it is known as a discrete sample space.

9. Definition. If a random experiment can give rise to n mutually exclusive and equally likely results out of which m are favourable to certain event E, then the probability of that event is given by

$$P(E) = m/n.$$

10. Theorem

(i) $P(A) \geq 0$ *(ii)* $P(\Omega) = 1$

(iii) If $A_1, A_2, A_3, \ldots A_n$ *are n disjoint sets then*

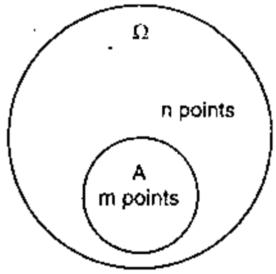
$$P(A_1 \cup A_2 \cup A_3 \cup \ldots A_n) = \sum_{i=1}^{n} P(A_i)$$

NOTES

Proof. (*i*) and (*ii*). Let the sample space Ω contain n points and let any set A contains m points.

$$P(A) = m/n > 0$$

$$P(\Omega)\ n/n = 1$$



(*iii*) Let A_1 contains m_1 points

A_2 contains m_2 points

A_n contains m_n points

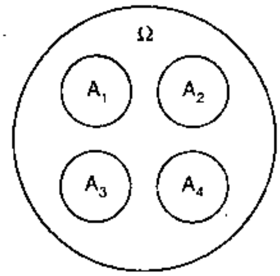
Then set $(A_1 \cup A_2 \cup A_3 \cup ...A_n)$ contains $(m_1 + m_2 + ... + m_n)$ points.

$$\therefore\ (A_1 \cup A_2 \cup A_3 \cup ...A_n) = \frac{m_1 + m_2 + m_3 + ... + m_n}{n}$$

$$= \frac{m_1}{n} + \frac{m_2}{n} + \frac{m_3}{n} ... \frac{m_n}{n}$$

$$= P(A_1) + P(A_2) + P(A_3) + ... + P(A_n)$$

$$= \sum_{i=1}^{n} P(A_i) \qquad ...(1)$$



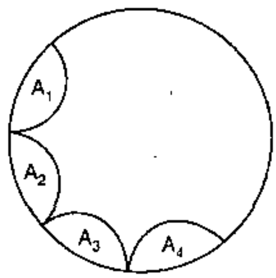
This is known as '*The Theorem of the Total probability*'.

(*i*) In particular, if $A_1, A_2, ... A_n$ are exhaustive also, then

$$(A_1 \cup A_2 \cup A_3 \cup ...A_n) = \Omega$$

or $$P(A_1 \cup A_2 \cup A_3 \cup ...A_n) = P(\Omega) = 1$$

$$P(A_1) + P(A_2) + P(A_3) + ... + P(A_n) = 1 \qquad ...(2)$$



(*ii*) We know that

$$A_1 \cap \overline{A} = \phi$$

where $\overline{A}$ is complementary set of A

and $$A \cup \overline{A} = \Omega$$

$$\therefore \qquad P(A \cup \overline{A}) = P(\Omega) = 1 \quad \text{or} \quad P(A) + P(\overline{A}) = 1$$

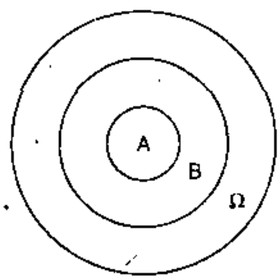
$$P(A) = 1 - P(\overline{A}).$$

11. Theorem. *If $A \subset B$, then $P(A) \le P(B)$*

Proof. $$B = A \cup (B - A)$$

$$\therefore \qquad P(B) = P(A) + P(B - A)$$

Hence $P(A) \le P(B)$ as $P(B - A)$ is non-negative.



12. Mathematical Definition. In an event A can happen in a ways and fails in b ways, all of these ways being equally likely to occur, then the probability of the happening of A is defined as $\frac{a}{a+b}$ and that of its failing is defined as $\frac{b}{a+b}$.

For example, if in a lottery there are 13 prizes and 32 blanks, then the probability that a person holding one ticket will win a prize is 13/45 and the probability of not winning is 32/45.

Explanation. When we say that a certain event happens in a ways and fails in b ways, meaning is that the chance of its happening to the chance of its failing are $a : b$. Thus if the chance of happening is represented by ka where k is an undetermined constant, the chance of its failing will be represented by kb.

Chance of happening + chance of failing = $k(a + b)$.

Now it is sure that event will happen or fail. Hence the sum of the chances of happening and failing must represent certainty. This certainty is represented by 1 and its range is from 0 to 1. So $k(a + b) = 1$ or $k = 1/(a + b)$.

Thus it is clear that probability lies between 0 and 1.

Note 1. If P is the probability of happening of certain event and q is that of failing, then

$$p = a/(a + b) \quad \text{and} \quad q = b/(a + b)$$
$$p + q = a/(a + b) + b/(a + b) = 1$$
$$q = 1 - p$$

Thus if p, the probability that the event will happen is known then that of failing can be obtained by subtracting p from 1.

Note 2. Instead of saying that the chance of the happening of an event is $\frac{a}{(a+b)}$, it is sometimes said that the odds are a to b in favour of the event or b to a against the event.

ILLUSTRATIVE EXAMPLES

Example 1. *A bag contains 7 white and 9 black balls. Find the probability of drawing a white ball.*

Solution. The total number of balls in the bag is 16.

Therefore a ball can be drawn in 16 ways.

As there are 7 white balls, hence favourable ways are 7.

Hence probability of drawing a white ball = 7/(7 + 9) = 7/16.

Example 2. *Find the chance that if a card is drawn at random from an ordinary pack, it is one of the court cards.*

Solution. From a pack of 52 cards one card can be drawn in 52 ways.

The number of favourable ways = 12 (Since king, queen and knave of each colour is to be included in court cards).

$\therefore$ Chance = 12/52 = 3/13.

Example 3. *What is the chance that a leap year, selected at random, will contain 53 Sundays ?*

Solution. A leap year consists of 366 days and hence contains 52 weeks and 2 days more. These two days may make the following 7 combinations :

(1) Monday and Tuesday
(2) Tuesday and Wednesday
(3) Wednesday and Thursday
(4) Thursday and Friday
(5) Friday and Saturday
(6) Saturday and Sunday
(7) Sunday and Monday.

Out of these seven likely cases only last two are favourable. Hence the required chance = 2/7.

Example 4. *What is the probability of throwing a 5 with an ordinary die whose faces are numbered from 1 to 6.*

NOTES

Solution. There are 6 possible ways in which the die can fall, and of these one is favourable to the event required.

∴ The probability = 1/6.

Example 5. *From a set of 19 cards, numbered 1, 2, 3, 4 ... 18, 19, one is drawn at random. Show that the chance that its number is divisible by 3 or 7 is 8/19.*

Solution. One card can be drawn in $^{19}C_1 = 19$ ways. Upto 19, six number (3, 6, 9, 12, 15, 18) are divisible by 3 two numbers (7, 14) are divisible by 7.

Therefore these are eight cards, the number of which are divisible by 3 or 7.

Hence the favourable ways $^{8}C_1 = 8$

∴ Required probability = 8/19.

EXERCISE 8(A)

1. What is the probability of throwing a number greater than 3 with an ordinary die ?
2. In a class of 15 students 6 are boys and rest are girls. Find the probability that a student selected will be a girl.
3. If three cards are drawn from a pack of 52 cards, what is the probability that all the three will be kings ?
4. What is the chance that a non-leap year should have 53 Mondays.

Answers

1. $\frac{1}{2}$
2. $\frac{3}{5}$
3. $\frac{^{4}C_1}{^{52}C_3}$
4. $\frac{1}{7}$

3. MUTUALLY EXCLUSIVE EVENTS

Two cases are said to be mutually exclusive when the happening of one of them excludes the happening of the other.

Example. Suppose we through an ordinary cubical die. It is clear that any of the six faces may be upper most when the die comes to rest. Total number of possible ways is six for a single throw. The different cases are mutually exclusive, since no two faces can be upper most at the same time.

4. ADDITIVE LAW OF PROBABILITY OR THEOREM OF TOTAL PROBABILITY

If $p_1, p_2, p_3, ..., p_n$ be separate probabilities of n mutually exclusive events then the probability p, that any of these events will happen is given by

$$p = p_1 + p_2 + ... + p_n.$$

Let $E_1, E_2, E_3, \ldots, E_n$ be the events whose probabilities are respectively $p_1, p_2, \ldots, p_n$. Let N be the total numbers of trials.

The events E_1 will happen on p_1 N occasions, the event E_2 on p_2 N occasions ..., the event E_n on P_n N occasions.

The events are mutually exclusive, hence any one of them can happen on $p_1 N + p_2 N + p_3 N + p_4 N \ldots + p_n N$ occasions.

Hence probability p is given by

$$p = (p_2 N + p_3 N + p_4 N + \ldots + p_n N)/N = p_1 + p_2 + \ldots + p_n.$$

ILLUSTRATIVE EXAMPLES

Example 1. *In a race of two persons, the probability of one is 1/5 and that of the other is 1/3. Find the probability of winning of anyone of them.*

Solution. Winning of one and the winning of the other are mutually exclusive events because both of them cannot win at the same time.

Therefore the required probability = 1/5 + 1/3 = 8/15.

Example 2. *A, B, C, D four candidates took part in a quiz contest. The probabilities of their coming first are 1/3, 1/7, 1/8 and 1/15 respectively. Find the probability of winning of anyone of them.*

Solution. Only one will come first at time. Hence the events of their coming first are mutually exclusive.

Hence required probability = 1/3 + 1/7 + 1/8 + 1/15 = = 516/840.

EXERCISE 8(B)

1. If the probability of a horse A winning a race is 1/7 and the probability of a horse B winning the same race is 1/4, what is the probability that one of the horses will win.
2. If the probabilities of four teams to win a match are 1/8, 1/17, 1/9, 1/10 separately, what is the probability that none of them wins.

Answers

1. $\frac{11}{28}$ 2. $\frac{1207}{2520}, \frac{1313}{2550}$.

5. STATISTICAL OR EMPIRICAL DEFINITION

If trials be repeated for a large number of times, say N, under the same conditions, and a certain event E happen on pN occasions, then the probability of happening the event E is defined as

$$\lim_{N \to \infty} pN/N = p.$$

6. COMPOUND EVENTS

When two or more simple events occur in connection with each other, the joint occurrence is called a compound event.

Example. Suppose we have a bag containing 4 white and 3 black balls. If we are required to find the chance in which 3 balls can be drawn are all white, it is a simple event. However if we are required to find out the chance of drawing 3 white balls and then 2 black balls, we are dealing with a compound event because it is made up of two events.

NOTES

7. INDEPENDENT EVENT

If there are two or more events such that the happening of anyone of them in no way affects the happening of any other of them, they are said to be independent.

For example, if one card is card is drawn from a pack of 52 cards and it is not replaced back and then second card is drawn, the drawing of the second card is dependent on that of the first. However, if the card drawn is replaced after the first draw, the second drawing will be independent.

8. THEOREM OF COMPOUND PROBABILITY

Let the two events E_1, E_2 have probabilities p_1 and p_2 respectively. Out of a large number N in which the event E_1 happens of p_1N occasions. The event E_1 is now to be followed by event E_2. Since the second event has the probability p_2, this would happen (along with the first) on $p_2(p_1N)$ *i.e.,* p_1p_2N occasions. Hence out of a total number of trials N. p_1p_2N are such that both events occur.

Hence the probability that both events occur

$$= \frac{p_1p_2N}{N} = p_1p_2.$$

Hence, generalizing

$$P = p_1p_2p_3 \cdots p_n.$$

Remark 1. If p is the chance that an event will happen in one trial, then the chance that it will happen in a succession of r trials is $p \cdot p \ldots r$ times $= p^r$.

Remark 2. If $p_1, p_2 \ldots p_n$ are the probabilities that certain event happen then the probability of all of these failing is $(1-p_1)(1-p_2)(1-p_3)\ldots(1-p_n)$.

Hence $1-(1-p_1)(1-p_2)(1-p_3)\ldots(1-p_n)$ is the probability in which at least one of these events must happen.

ILLUSTRATIVE EXAMPLES

Example 1. *Four cards are drawn without replacement. What is the probability that they are all aces.*

Solution. There are 4 aces in a pack of 52 cards.

The probability of drawing an ace $= \frac{4}{52}$.

Since the cards are not replaced, there remain 51 cards and 3 aces it one ace has already been drawn.

NOTES

The probability of drawing one ace second time $= \frac{3}{51}$.

Similarly the probabilities of drawing third and fourth aces are $\frac{2}{50}$ and $\frac{1}{49}$ respectively.

The events are dependent, hence the required probability

$$= \frac{4}{52} \cdot \frac{3}{51} \cdot \frac{2}{50} \cdot \frac{1}{49} = \frac{1}{270725}.$$

Example 2. *Two cards are drawn from a pack of 52 cards, what is the probability that one of them is king and the other is queen ?*

Solution. Out of 52 cards two can be drawn in $^{52}C_2$ ways.

Out of 4 kings one can be drawn in $^4C_1$ ways.

Out of 4 queens one can be drawn in $^4C_1$ ways.

Hence the probability that one is king and the other is queen

$$= \frac{^4C_1 \times {^4C_1}}{^{52}C_1} = \frac{4 \times 4}{^{52}C_1} = \frac{8}{663}.$$

Example 3. *If four whole numbers taken at random are multiplied together, show that the chance that the last digit in product is 1, 3, 7 or 9 is* $\frac{16}{625}$.

Solution. The last digits of the four whole numbers can be 0, 1, 2, 3, 4, 5, 6, 7, 8, 9.

The chance that any of the four numbers is divisible by 2 or 5 is $\frac{6}{10} = \frac{3}{5}$.

Hence the chance that it is not divisible by 2 or 5 is

$$1 - \frac{3}{5} = \frac{2}{5}.$$

The chance that all of the four numbers are not divisible by 2 or 5 is

$$= \left(\frac{2}{5}\right)\left(\frac{2}{5}\right)\left(\frac{2}{5}\right)\left(\frac{2}{5}\right) = \frac{16}{625}.$$

This is the chance that the last digit in the product will not be 0, 5, 2, 4, 6, 8 and consequently the chance that the last digit in the product is 1, 3, 7 or 9.

9. THE PROBABILITY OF IMPOSSIBLE EVENT IS ZERO

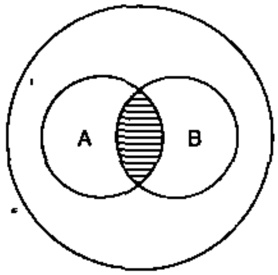
Let ϕ be impossible event.

Now,
$$\Omega \cup \phi = \Omega$$
$$P(\Omega \cup \phi) = P(\Omega) = 1$$
$$P(\Omega) + P(\phi) = 1$$
$$1 + P(\phi) = 1$$
$$P(\phi) = 0.$$

10. GENERALIZED THEOREM OF TOTAL PROBABILITY

NOTES

For any two events A and B, the probability that either A or B or both occur is given by



$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$A = (A \cap B) \cup (A - B)$$
$$B = (A \cap B) \cup (B - A)$$
$$P(A) = P(A \cap B) + P(A - B)$$
$$P(B) = P(A \cap B) + P(B - A)$$
$$P(A) + P(B) = P(A \cap B) + P(A - B) + P(A \cap B) + P(B - A)$$
$$A \cup B = (A - B) \cup (A \cap B) \cup (B - A)$$
$$P(A \cup B) = P(A - B) + P(A \cap B) + P(B - A)$$
$$P(A) + P(B) = P(A \cap B) + P(A \cup B)$$

or $$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

If we consider $A_1, A_2, \ldots, A_n$, events then by mathematical induction it can also be proved that,

$$P(A_1 \cup A_2 \cup \ldots \cup A_n) = \sum_{i=1}^{n} P(A_i) - \sum_{i \neq j=1}^{n} P(A_i \cap A_j) + \sum_{i \neq j \neq k=1}^{n} P(A_i \cap A_j \cap A_k) + \ldots\ldots + (-1)^{n-1} P(A_1 \cap A_2 \cap \ldots \cap A_n).$$

Corollary. $P(A \cup B) \leq 1$

$$\therefore \quad = P(A) + P(B) - P(A \cap B) \leq 1$$

or $$P(A \cap B) \geq P(A) + P(B) - 1.$$

If we consider A, B, C events then

$$P(A \cap B \cap C) = P(A \cap D) \text{ where } D = B \cap C$$
$$\geq P(A) + P(D) - 1$$
$$\geq P(A) + P(B) + P(C) - 2$$

likewise, $P(A_1 \cap A_2 \cap A_3 \cap \ldots \cap A_n) \geq P(A_1) + P(A_2) + P(A_3) + \ldots + P(A_n) - (n-1).$

Example. *In the game of bridge find the probability that at least one player gets the complete suit.*

Solution. Let A_1 denote the event that player No. (1) gets the complete suit.

Let A_2 denote the event that player No. (2) gets the complete suit.

Let A_3 denote the event that player No. (3) gets the complete suit.

And Let A_4 denote the event that player No. (4) gets the complete suit.

$$A = A_1 \cup A_2 \cup A_3 \cup A_4$$
$$P(A) = P(A_1 \cup A_2 \cup A_3 \cup A_4)$$

$$= \sum_{i=1}^{4} P(A_i) - \sum_{i \neq j=1}^{4} P(A_i \cap A_j) + \sum_{i \neq j \neq k=1}^{4} P(A_i \cap A_j \cap A_k) - P(A_1 \cap A_2 \cap A_3 \cap A_4)$$

$$P(A_i) = \frac{13!\,39!}{52!} \cdot 4$$

$$P(A_i \cap A_j) = \frac{13!\,13!\,26!}{52!} \cdot 4.3$$

NOTES

$$P(A_i \cap A_j \cap A_k) = \frac{13!\,13!\,13!\,13!}{52!} \,.\, 4.3.2$$

$$P(A_1 \cap A_2 \cap A_3 \cap A_4) = \frac{13!\,13!\,13!\,13!}{52!} \,.\, 4.3.2.1$$

$$\because \quad P(A) = \frac{4.4.13!\,39!}{52!} - 6\,.\,\frac{13!.13!.26!}{52!}\,.\,12 + 4\,.\,\frac{(13!)^2\,.\,24}{52!} - \frac{(13!)^2\,.\,24}{52!}$$

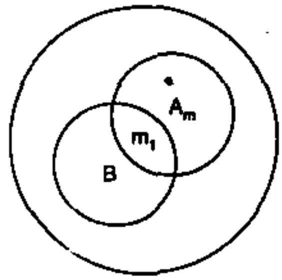
$$= \frac{13!}{52!}\,[39!.16 - 72.13!\,26! + 72\,(13!)^2].$$

11. THEOREM OF COMPOUND PROBABILITY (Another Approach)

Statement. The probability of simultaneous occurrence of two events A and B is given by the product of unconditional probability of B, assuming that A had already been occurred.

i.e., $P(A \cap B) = P(A)\,P(B/A)$ where

P(B/A) is conditional probability of B, given A has already occurred.



Proof. Let there be n points in Ω, m point in A and m_1 points belonging to A and B both.

$$P(A \cap B) = \frac{m_1}{n} = \frac{m}{n}\frac{m_1}{m} \quad \text{where} \quad m > 0$$

$$= P(A)\,P(B/A).$$

Example. *A die is thrown, given that the even number turns up. What is the probability that it is '2' ?*

Solution. Let A be the event in which even numbers turn up.

B is the event that '2' truns up.

$$P(A) = \frac{3}{6} = \frac{1}{2}$$

$$(A \cap B) = B$$

$$P(A \cap B) = P(B) = \frac{1}{6}$$

$$P(B/A) = \frac{P(A \cap B)}{P(A)} \frac{\frac{1}{6}}{\frac{1}{2}} = \frac{1}{3}.$$

12. INDEPENDENCE OF TWO EVENTS

Two events a and b are said to be independeent if

$$P(A \cap B) = P(A)\,P(B) \quad \text{or} \quad P(B)\,.\,P(B/A).$$

Example. A coin is tossed twice. Let A be the event that head occurs in the first toss and B is the event that head occurs in second toss.

NOTES

$$P(A) = \frac{1}{2}$$

$$P(B) = \frac{1}{2}$$

$$P(A \cap B) = P(A)\,P(B) = \frac{1}{2}.\frac{1}{2} = \frac{1}{4}.$$

13. BAYES' THEOREM

Suppose we are given that the ball drawn is white and we are interested in finding the probability of the event that the ball was drawn from the 1st urn or 2nd urn etc., then the addition theorem and multiplication theorem fail to find the probability as the probability of urn drawn will depend upon the colour of the ball drawn.

14. STATEMENT OF BAYES' THEOREM

Let $E_1, E_2, ..., E_n$ be a mutually exclusive and exhaustive event, with non-zero probabilities, of a random experiment. If A be any arbitrary event of the sample space of the above experiment with P(A) > 0, then

$$P(E_i/A) = \frac{P(E_i)\,P(A/E_i)}{\sum_{j=1}^{n} P(E_j)\,P(A/E_j)}.$$

Proof. Let S be the sample space of the random experiment.

$$S = E_1 \cup E_2 \cup ... \cup E_n$$

Now
$$A = S \cap A = (E_1 \cup E_2 \cup ... \cup E_n) \cap A$$
$$= (E_1 \cap A) \cup (E_2 \cap A) \cup ... \cup (E_n \cap A))$$
$$P(A) = (E_1 \cap A) + P(E_2 \cap A) + ... + P(E_n \cap A))$$
$$= P(E_1)\,P(A/E_1) + P(E_2)\,P(A/E_2) + ... + P(E_n)\,P(A/E_n)$$
$$= \sum_{j=1}^{n} P(E_j)\,P(A/E_j)$$

Now
$$P(E_i/A) = \frac{P(A \cap E_i)}{P(A)} = \frac{P(E_i \cap A)}{P(A)}$$

$$P(E_i/A) = \frac{P(E_i)\,P(A/E_i)}{\sum_{j=1}^{n} P(E_j)\,P(A/E_j)}.$$

NOTES

Remark. If $n = 2$, then

$$P(E_1/A) = \frac{P(E_1)\,P(A/E_1)}{P(E_1)\,P(A/E_1) + P(E_2)\,P(A/E_2)}$$

$$P(E_2/A) = \frac{P(E_2)\,P(A/E_2)}{P(E_1)\,P(A/E_1) + P(E_2)\,P(A/E_2)}.$$

ILLUSTRATIVE EXAMPLES

Example 1. *In a bolt factory, machines A, B and C manufacture 60%, 25% and 15% respectively. Of the total of their output 1%, 2% and 1% are defective bolts. A bolt is drawn at random from the total production and found to be defective. From which machine, the defective bolt is expected to have been manufactured.*

Solution. Let E_1, E_2, E_3 be the events of drawing a bolt produced by machine A, B and C respectively. Let D be the event of drawing a defective bolt.

We have $P(E_1) = \frac{60}{100}$, $P(E_2) = \frac{25}{100}$, $P(E_3) = \frac{15}{100}$

$$P(D/E_1) = \frac{1}{100},\ P(D/E_2) = \frac{2}{100},\ P(D/E_3) = \frac{1}{100}.$$

The events E_1, E_2 and E_3 are mutually exclusive and exhaustive.

By Bayes theorem,

P (defective bolt is produced by machine A)

$$= P(E_1/D) = \frac{P(E_1)\,P(D/E_1)}{P(E_1)\,P(D/E_1) + P(E_2)\,P(D/E_2) + P(E_3)\,P(D/E_3)}$$

$$= \frac{\frac{60}{100} \times \frac{1}{100}}{\left(\frac{60}{100} \times \frac{1}{100}\right) + \left(\frac{25}{100} \times \frac{2}{100}\right) + \left(\frac{15}{100} \times \frac{1}{100}\right)}$$

$$= \frac{60}{60 + 50 + 15} = \frac{60}{125} = \frac{12}{25}.$$

P (defective bolt is produced by machine B)

$$= P(E_2/D) = \frac{P(E_2)\,P(D/E_2)}{P(E_1)\,P(D/E_1) + P(E_2)\,P(D/E_2) + P(E_3)\,P(D/E_3)}$$

$$= \frac{\frac{25}{100} \times \frac{2}{100}}{\left(\frac{60}{100} \times \frac{1}{100}\right) + \left(\frac{25}{100} \times \frac{2}{100}\right) + \left(\frac{15}{100} \times \frac{1}{100}\right)}$$

$$= \frac{50}{60 + 50 + 15} = \frac{50}{125} = \frac{10}{25}.$$

P (defective bolt is produced by machine C)

$$= P(E_3/D) = \frac{P(E_3)\,P(D/E_3)}{P(E_1)\,P(D/E_1) + P(E_2)\,P(D/E_2) + P(E_3)\,P(D/E_3)}$$

NOTES

$$= \frac{\frac{15}{100} \times \frac{1}{100}}{\left(\frac{60}{100} \times \frac{1}{100}\right) + \left(\frac{25}{100} \times \frac{2}{100}\right) + \left(\frac{15}{100} \times \frac{1}{100}\right)}$$

$$= \frac{15}{60 + 50 + 15} = \frac{15}{125} = \frac{3}{25}.$$

Since the probability in case of machine A is the largest, the defective bolt has been drawn from the output of the machine A.

EXERCISE 8 (C)

1. A coin is tossed three times. Find the chance of getting head and tail alternately.
2. If four cards are drawn from a pack, what is the probability that there will be one of each suit ?
3. A problem is given to three students whose chance of solving are 1/2, 1/3, 1/4 respectively. What is the probability that problem will be solved ?

Answers

1. $\frac{1}{4}$ 2. $\frac{2197}{20825}$ 3. $\frac{3}{4}$

SUMMARY

1. **Random Experiment.** An experiment is called a random experiment if it is such that all the possible results (outcomes) are known but it is not possible to predict the outcome of any individual experimentation in advance.
2. **Non-random Experiments.** An experiment which is not random is known as non-random experiment thus

   (1) Throwing a piece of iron in air

   (2) Throwing a piece of wood in water

   are the examples of non-random experiments.
3. **Sample Space.** Collection of all possible out-comes of a random experiment is known as the sample space and is denoted by Ω.
4. **Point.** Individual outcomes of a sample space are known as points or elements and are denoted by ω.
5. **Event.** Any desired property is known as an event and denoted by capital letters A, B, C, ... For example if experiment is throwing a dice then,

   $$\Omega = \{1, 2, 3, 4, 5, 6\}.$$
6. ***Conclusion.*** Given any random experiment we can find Ω and given any event we can find set corresponding to that event.
7. **Definition.** A set containing no outcome of the sample space is known as a null set and is denoted by ϕ.
8. If there are infinitely many points in a sample space then it is known as a continuous sample space but if these are finite or enumerable number of points then it is known as a discrete sample space.

NOTES

9. If a random experiment can give rise to n mutually exclusive and equally likely results out of which m are favourable to certain event E, then the probability of that event is given by

$$P(E) = m/n.$$

TEST YOURSELF

1. A card is drawn from an ordinary pack and a player bets that it is a spade or an ace. What is the probability of his winning ?
2. The chance of one event happening is the square of the chance of happening a second event, but the odds against the first are the cube of the odds against the second. Find the chance of each.
3. If three squares are chosen at random on a chess-board, find the chance that they should be in diagonal line.
4. Two dice are thrown. Find the probability that both of them show the same face.
5. Bag A contains 2 white and 3 red balls and bag B contains 4 white and 5 red balls. One ball is drawn at random from one of the bags and found to be red. Find the probability it was drawn from beg B.

Answers

1. $\frac{4}{13}$
2. $\frac{1}{9}, \frac{1}{3}$.
3. $\frac{7}{744}$
4. $\frac{1}{6}$
5. $\frac{25}{52}$

SECTION D

9. Graph Theory
10. Digraphs : Applications

CHAPTER

9

GRAPH THEORY

NOTES

LEARNING OBJECTIVES

- Introduction
- Simple Graph
- Incidence and Degree
- Regular Graph
- Isolated Vertex, Pendent Vertex and Null Graph
- Some More Initial Concepts
- Digraph
- Isomorphism
- Some Structures Based on Connectivity
- Complementarity
- Trees
- Distances of Trees
- Tree Enumeration
- Spanning Trees
- Fundamental Cycles
- Concept of Traversability
- Eulerian Graphs

1. INTRODUCTION

The last two decades have witnessed a practical utility of Graph Theory, particularly among computer scientists, applied mathematicians and engineers. We intend to present selected topics from the theory of graphs, the choice being guided by what is said in above sentence. It goes without saying that while discussing the subject likely to be new to most of the intended readers, basic concepts of the subject and notations along with terminology which describes them is needed to be made clear.

Definition. A graph G consists of a set of objects $V = \{v_1, v_2, ...\}$ called vertices and another set $E = \{e_1, e_2, ...\}$, whose elements are said to be edges such that each edge e_k is identified with an unordered pair of vertices (v_i, v_j). We write G = (V, E) to express the two parts of G. The vertices v_i and v_j are called the end vertices of the edge e_k.

Remark. The edge having the same vertex as both its end vertices is called a self-loop or simply a loop. Whenever more than one edge is associated with a given pair of vertices, then such edges are said to be the parallel edges. Following is a graph having self loop as well as parallel edges.

NOTES

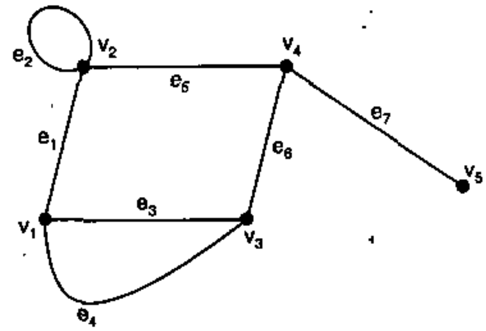


Fig. 1. Graph with five vertices and seven edges.

2. SIMPLE GRAPH

A graph which has neither self loops nor parallel edges is said to be a simple graph.

Note. 1. In drawing a graph, it does not matter that an edge is a straight line segment or curved line, long or short. Only the incidence between the edges and vertices is important.

2. A vertex is also referred to as a node, a function, a point, 0-cell or an 0-simplex.

3. An edge is also referred as a branch, a line, an element, a *b*-cell, an arc and a 1-simplex.

4. A graph with a finite number of vertices as well as a finite number of edges is called a finite graph ; otherwise, it is an infinite graph.

3. INCIDENCE AND DEGREE

Let v_i be an end vertex of some edge e_j then v_i and e_j are said to be incident with (on or to) each other. The number of edges incident with a vertex v, in a graph is called the degree or valence of v and is denoted by writing $d(v)$.

Theorem 1. *The number of vertices of odd degree in a graph is always even.*

Proof. Let the n vertices be denoted by v_i $(i = 1, 2, ..., n)$ in a graph G. Let v_j be the vertices having even degrees and v_k the vertices having odd degrees.

We know that in a graph the sum of the degree of all vertices in the graph is twice the number of edges in G. Therefore

$$\sum_{i=1}^{n} d(v_i) = 2e \qquad ...(1)$$

Again, $$\sum_{i=1}^{n} d(v_i) = \sum_{\text{even}} d(v_j) + \sum_{\text{odd}} d(v_k) \qquad ...(2)$$

Since left hand side of this equation is even in view of (1) and first term on the right hand side is also even, the second term on the right hand side of equation (2) must be even *i.e.*,

$$\sum_{\text{odd}} d(v_k) = \text{an even number} \qquad ...(3)$$

In equation (3) each $d(v_k)$ is odd, but the total sum is even. Hence the total number of terms in the sum must be even to make the sum an even number. This proves the theorem.

NOTES

4. REGULAR GRAPH

A graph in which all vertices are of equal degree is said to be a regular graph.

5. ISOLATED VERTEX, PENDENT VERTEX AND NULL GRAPH

A vertex of degree 0 is called isolated. A vertex of degree 1 is said to be pendant. An edge incident with a pendent vertex is itself called pendant.

A graph (V, E), in which E = ϕ, *i.e.*, a graph without any edges is termed a null graph. It is to be carefully noted that V cannot be empty in any graph. A vertex of degree greater than 1 is said to be interval. Two adjacent edges are said to be in series if their common vertex is of degree two.

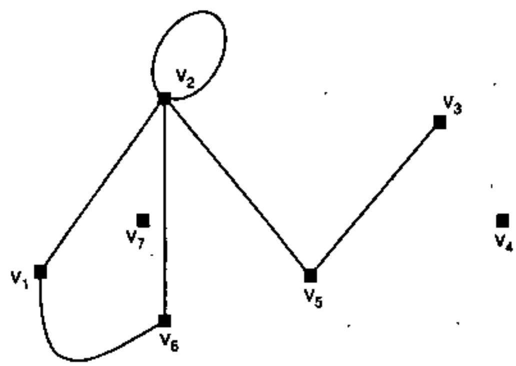
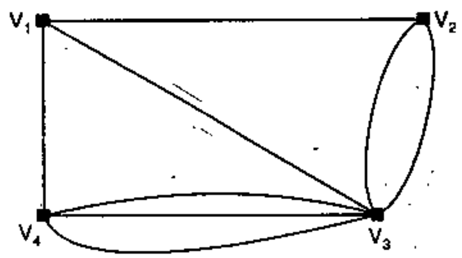


Fig. 2. Graph containing isolated vertices, a pendent vertex and series edges.

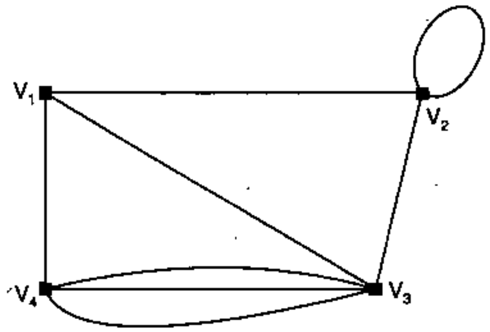
6. SOME MORE INITIAL CONCEPTS

Multi-graph. The multi-graph is an ordered pair of sets (V, E), where V is finite and non-empty, and E is a class of unordered pairs of distinct elements of V whose repetitions in the class are allowed. Following is the example of a multi-graph



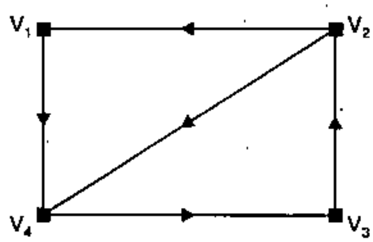
Each edge in a graph and in a multi-graph joins two distinct vertices. Whenever this law is not observed the multi-graphs are said to be pseudographs.

Pseudographs. A pseudograph is an ordered pair of sets (V, E), whose V is finite and non-empty and E is a class of unordered pairs of elements of E whose repetitions in the class are allowed. Following is an example of pseudograph.



7. DIGRAPH

We know that a graph (V, E), is defined by an unordered pair of vertices of V. In some cases it is desired to give each edge an orientation or direction. When this is done the graph is called a directed graph or **digraph** in short. Thus *digraph is defined to be an ordered pair of sets (V, A), where V is a finite and non-empty set and A is a set of ordered pairs of distinct elements of V.* We call the elements of A as arcs, otherwise the terminology is same as for graphs. Adjoining is figure of a digraph.



Note. If $(u. v)$ denoted by uv in short is an arc of a digraph, u is said to be a predecessor of v and v is called a successor of u.

A graph G = (V, E) or a digraph D = (V, A) is called weighted if there exists a function w : $E \to R$ or $w : A \to R$, R being the set of real numbers, which assigns a real number, called a weight to each edge of E, or arc of A. We write this fact by the symbol w_{uv} where $u, v \in E$.

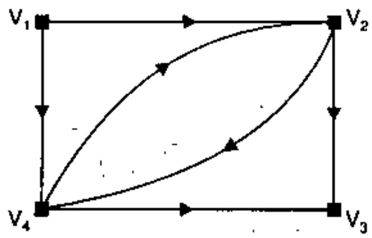
Remark. There is a special class of digraphs termed as networks.

Note. The number of arcs directed away from a vertex v, in a digraph is called the out degree of v and is denoted by writing $od(v)$. The number of arcs directed towards a vertex v, in a digraph is termed the in degree of u and is denoted by $id\ (u)$. Thus for any vertex in a digraph

$$d(u) = od\ (u) + id\ (u)$$

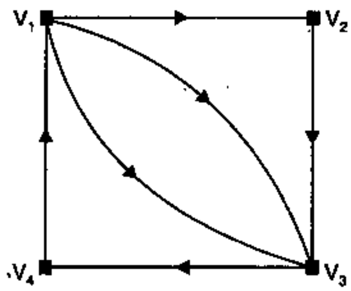
Following are some of the concepts used in the definition.

Let $a = (u_i, v_j)$ be an arc in a digraph (V, A) then a is said to be directed away from u_i and directed towards v_j. Any vertex which has no arcs directed towards it is said to be a *source*. A vertex which has no arcs directed away from it is called a *sink*. There may be digraphs without sink or source. A **network** is a digraph which possesses exactly one source and exactly one sink. The vertices of a network are said to be nodes. In most of the applications of networks, there is at least one item flowing from the source to the sink in the system represented by the network concerned. Adjoining figure represents a network.



NOTES

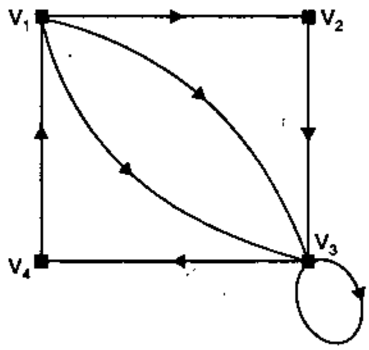
Like graphs, it is possible to relax the constraints in the definition of a digraph that no more than one arc can join any pair of vertices. The structures thus obtained are called multi digraphs. Thus a multi digraph is an ordered pair of sets (V, A), where V is finite and nonempty and A is a class of ordered pairs of distinct elements of V whose repetition in the class are allowed. Adjoining is the figure of a multi digraph.



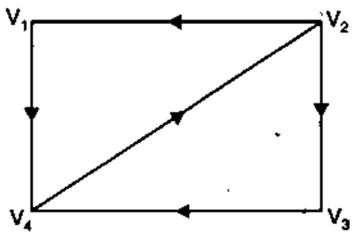
As in the case of graphs, here too we need each arc, in digraph and in a multi digraph, to join two distinct vertices.

If this constraint is relaxed then the structures (with reference to discussions on graphs) are said to be pseudo digraphs.

Definition. A pseudo digraph is an ordered pair of sets (V, A), where V is finite and nonempty and A is a class of unordered pairs of elements of A where repetitions in the class are permitted. The adjoining graph depicts the pseudo digraph where an arc of the form (u, u), $u \in V$ is said to be a loop.



Definition. *Oriented* graph is a digraph (V, A), possessing at most one of the pair of arcs $u_i v_j$ and $u_j v_i$, for every pair of vertices $u_i, v_j \in V$. Following is the figure for *oriented* graph.



Definition. A graph (V, E) is said to have been *labeled* when its vertices in V are distinguished one from other by naming them separately.

8. ISOMORPHISM

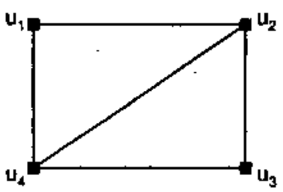
This concept has been explained with reference to graph theory. The concept is same as the reader is aware with it.

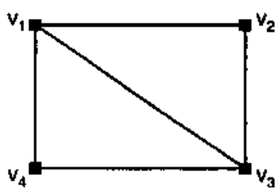
An **automorphism** of a graph is an isomorphism of G with itself.

Isomorphic Graphs. Two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are said to be isomorphic if there exists a one-one, onto mapping $f: V_1 \to V_2$, such that $uv \in E_1 \Leftrightarrow f(u) f(v) \in E_2$ and is denoted by writing $G_1 \cong G_2$, Thus

$G_1 \cong G_2 \Leftrightarrow$ [Two vertices are adjacent in G_1 if and only if their images under f are adjacent in G_2] *i.e.*, f preserves adjacency.

Following graphs are isomorphic to each other as the mapping $f(u_i) = v_{i+1}$ (mod u), $i = 1, 2, 3, 4$ exists for them which preserves adjacency.





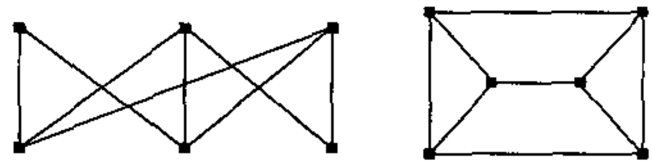
Isomorphic graphs possess the following characteristics :

(1) the number of vertices is same

(2) the number of edges is same and

(3) there is an equal number of vertices of any degree.

The above properties are necessary for isomorphism but are not sufficient. This can be realized from the following figure where these properties are satisfied but graphs are not isomorphic.



Non-isomorphic graphs

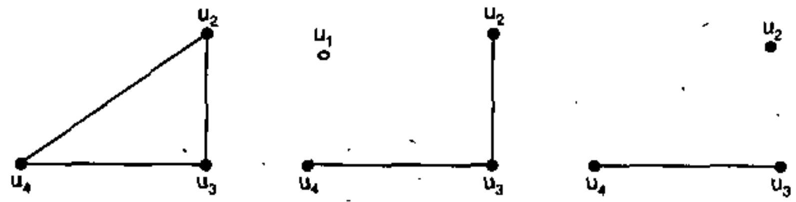
Subgraph :

Definition 1. Let G = (V, E) be a graph. Let U be a nonempty subset of V. The graph whose vertex set is U and whose edge set is E′ *i.e.*, the edge set comprising of exactly the same edges of E which join vertices in U is called an **induced subgraph** of G.

Definition 2. Let G = (V, E) be a graph. Let F be a proper subset of E. The graph (V, F) is said to be a proper spanning subgraph of G.

Definition 3. Let G = (V, E) be a graph. A structure made up of the union of a induced subgraph and a proper spanning subgraph is said to be subgraph of G *i.e.*, let U be *a* nonempty subset of V and F be a nonempty subset of E then the graph (U, F) is termed as subgraph of G.

Following figures exhibit the above subgraphs of figure given on left of the upper most figures.



An induced graph A spanning graph A subgraph

Remark 1. When a vertex v (or the vertices of a subgraph G_1) and all the edges incident with it are removed from a graph G, the resulting graph is denoted by $G - v$ (or $G - G_1$). If an edge e, is removed from a graph, the resulting graph is denoted by $G - e$. If an edge (or set of edges E′), not already present in a graph G, is added to G, then the resulting graph is denoted by $G + e$ ($G + E'$).

Connectivity. Connectivity is an important concept of graph theory. We introduce here some concepts related to this aspect of graph structure.

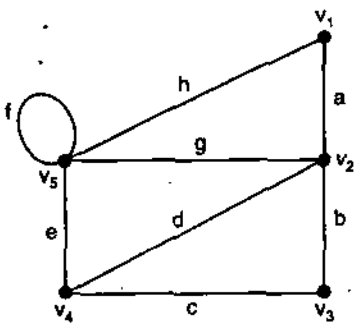
Walk. A walk in a graph G is a finite alternating sequence of vertices and edges of G beginning and ending with vertices, such that each edge is incident with the vertices preceding and following it. The form of the sequence can be expressed as

$$\{v_1, \{v_1, v_2\}, v_2, \{v_2, v_3\}, v_3, \ldots\ldots, v_{n-1}, \{v_{n-1}, v_n\}, v_n\}$$

Here the walk joins v_1 and v_n. If there is no ambiguity, the walk is expressed by :

$$\{v_1, v_2, v_3 \ldots\ldots, v_n\}.$$

Remark 1. A vertex may appear more than once in a walk but no edge appears more than once. A walk is said to be **closed** if $v_1 = v_n$ and **open** otherwise. A walk is termed a trail if all of its edges are distinct. An open walk in which no vertex appears more than once is called a **path**. The number of edges in a path is called the **length** of a path.



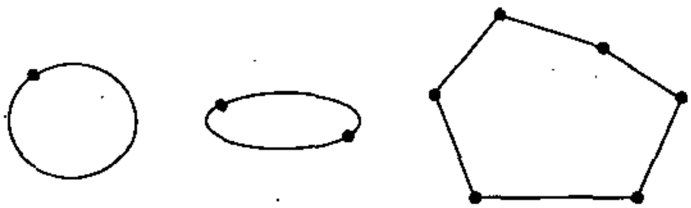
It is evident from the definition that a path does not intersect itself. Also an edge which is not a self loop is a path of length one. Further a self-loop can be included in a walk but not in a path. The terminal vertices of a path are of degree one and rest of the vertices (intermediate vertices) are of degree two.

Assignment. Specify on open walk in the adjoining figure.

Remark 2. *The degree is counted only with respect to the edges included in the path* and not the entire graph containing the path.

Circuit. A closed walk in which no vertex except the initial and final, appears more than once is said to be a circuit. In other words, a circuit is a closed, non-intersecting walk.

Following are the graphs denoting a circuit.

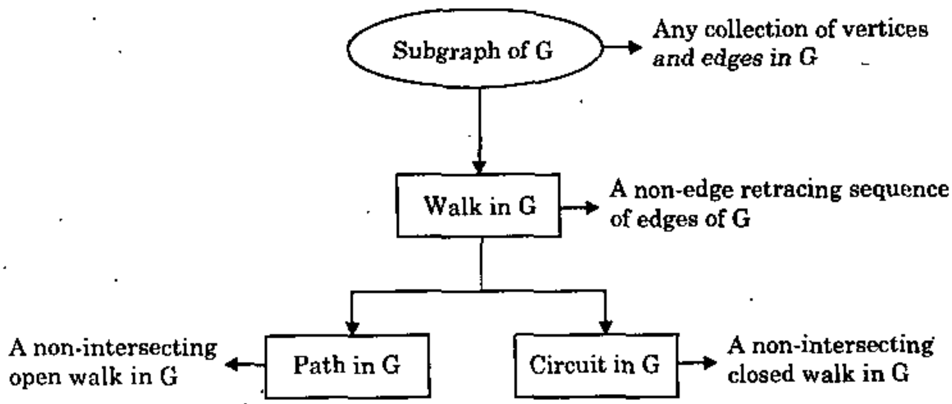


Note 1. Every vertex in a circuit has degree 2.

2. If the circuit is a subgraph of any graph, then we have to count degrees contributed by the edges in the circuit only.

Remark. A *circuit* is also termed as a **cycle, elementary cycle, circular path** and **polygon.**

In electrical Engineering circuit is sometimes termed as a loop, of course when it is not to confuse with *self-loop. Every self-loop is a circuit* but the converse is not true.



The concept of connectivity is obvious to every one by nature. Reachability to every vertex from any vertex by traversing the edges is connectedness of a graph. Formal definition is as follows :

NOTES

Definition. A graph G is said to be *Connected* if there is at least one path between every pair of vertices in G. Otherwise it is disconnected. The graph on page 266/7 (remark. 1) is a demonstration of a connected graph while the following is disconnected graph.

As is clear from the definition a disconnected graph has two or more connected graphs. Each of these connected subgraphs is said to be a *component*. The concept of the component may be looked in another way as follows :

Let v_i be a vertex in the disconnected graph G. In fact all vertices of G are not joined to v_i by paths to v_i. Vertex v_i and all the vertices of G having paths to v_i, along with all the edges incident on them form a component. Clearly a component of a graph itself is a graph.

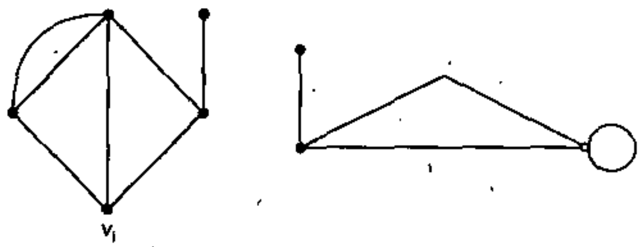


Fig. 10.3 Example of disconnected graph having two components.

Remark. A graph in which every pair of its n vertices is directly connected by an edge is said to be complete and is denoted by K_n.

Theorem 2. *A graph G is disconnected iff its vertex set V can be partitioned into nonempty disjoint subsets V_1 and V_2 such that there exists no edge in G whose one end vertex is in subset V_1 and other end in subset V_2.*

Proof. Let us suppose that such a partition exists. Let these be two arbitrary vertices v_1 and v_2 such that v_1 belongs to V_1 and v_2 belongs to V_2. There is no path between vertices v_1 and v_2 otherwise there will exist at least one edge having its one end vertex in V_1 and other in V_2. Hence G is not connected.

Conversely let G be a disconnected graph. Let there be a vertex v_1 in G. Consider V_1 as the set of all vertices which are joined by paths to v_1. Since G is disconnected, the set V_1 does not contain all the vertices of G. Therefore the remaining vertices of G form a nonempty set V_2 such that no vertex in V_1 is joined to any other vertex in V_2 by an edge. Hence the partition exists.

Theorem 3. *A simple graph with n vertices and k components can have atmost $(n-k)(n-k+1)/2$ edges.*

Proof. Let the number of vertices in each of the k-components of a graph G be $n_1, n_2, \ldots, n_k$. Thus we have $n_1 + n_2 + \ldots + n_k = n$, $n_i \geq 1$

We know that $\sum_{i=1}^{k} (n_i - 1) = n - k$.

Now squaring on both sides,

$$\left(\sum_{i=1}^{k} (n_i - 1)\right)^2 = n^2 + k^2 - 2nk ;$$

or $$(n_1-1)^2+(n_2-1)^2+\ldots+(n_k-1)^2+\sum_{i=1}^{k}\sum_{\substack{j=1\\ i\neq j}}^{k}(n_i-1)(n_j-1)=n^2+k^2-2nk$$

or $$\sum_{i=1}^{k}(n_i^2-2n_i)+k+\textit{non-negative terms}=n^2+k^2-2nk \text{ (since } (n_i-1)\geq 0,\ \forall\, i)$$

Therefore $$\sum_{i=1}^{k} n_i^2 \leq n^2+k^2-2nk-k+2n$$
$$= n^2-(k-1)(2n-k) \qquad \ldots(1)$$

Again the maximum number of edges in the i^{th} component of G which is a simple connected graph is $\frac{1}{2}n_i(n_i-1)$. Hence, the maximum number of edges in G is

$$\frac{1}{2}\sum_{i=1}^{k}(n_i-1)n_i=\frac{1}{2}\left(\sum_{i=1}^{k}n_i^2\right)-\frac{n}{2}$$
$$\leq \frac{1}{2}(n^2-(k-1)(2n-k))-\frac{n}{2}$$
$$=\frac{1}{2}(n-k)(n-k+1).$$

Euler Graphs. Graph theory originated in 1736 with Euler's famous paper in which he solved the Königsberg bridge problem. In fact he posed a more general problem and solved the same. This was as given below. In what way a graph G is possible such that there exists a closed walk running through every edge of G exactly once. This type of *walk* is called an *Euler line* and a graph that contains an Euler line is said to be an *Euler graph.* This can be stated formally as :

If some closed walk in a graph contains all the edges of the graph, the walk is said to be an Euler line and the graph an Euler graph.

The walk is always connected. Since the Euler line, which is of course a walk, contains all the edges of the graph, an Euler graph is always connected except for the case of existence of any isolated vertices. In fact isolated vertices do not affect the properties of an Euler graph and therefore, it is assumed that Euler graphs do not have any isolated vertices and hence are connected.

Now we provide an important theorem which will enable us to know whether or not a given graph is an Euler graph.

Theorem 4. *A given connected graph G is an Euler graph if and only if all vertices of G are of even degree.*

Theorem 5. *If a graph has exactly two vertices of odd degree then they must be connected by a path.*

Proof. Let G be a graph with all its vertices of even degree, except for two vertices namely v_1 and v_2 which are of odd degree. Consider the component C of graph G to which v_1 belongs . We know that C must have an even number of vertices of odd degree. Thus v_2 must belong to C which is the only other vertex of odd degree. Therefore v_1 and v_2 are in the same component. Since a component is a connected graph there must be a path joining v_1 and v_2.

9. SOME STRUCTURES BASED ON CONNECTIVITY

Definition. The complement $\overline{G}$, of a graph G = (V, E) is the graph with the vertex set V such that two vertices are adjacent in $\overline{G}$ if and only if these vertices are not adjacent in G.

Note. A graph G is called self complementary if $G \approx \overline{G}$.

Definition. A graph G = (V, E) is n-partite, $n > 1$, if it is possible to partition V into n subsets : $V_1, V_2, \ldots, V_n$ (called partite sets) such that every edge of E joins a vertex of V_i to a vertex of V_j, $i \neq j$.

When $n = 2$, then we call these graphs as **bipartite** graphs.

Theorem 6. *A graph is bipartite if and only if it contains no odd cycles.*

Proof. Necessity of the condition. Let G = (V, E) be a connected bipartite graph with bipartite sets V_1 and V_2. Let there exist a cycle $(v_1, v_2, \ldots\ldots, v_k), v_1$ in G.

Let $v_1 \in V_1$ then $v_i \in V_1$ if i is odd, and $v_i \in V_2$, if i is even. Therefore k is even. Hence if a graph is bipartite then there do not exist cycles.

Sufficiency of the condition. Suppose a vertex $v_i \in V$. Let V_1 be the subset of V such that it contains v_1 and all vertices of G whose shortest paths from v_1 have an even number of edges. Let V_2 contain the remaining vertices of G. Since every cycle in G is even, every edge of G joins a vertex in V_1 to a vertex in V_2. This is true because if there exist two adjacent vertices (say v_2 and v_3), in V_1, then the cycle A (say) so made up by adjoining the shortest path from v_1 to v_2, the edge v_2v_3, and the shortest path from v_3 to v_1, is odd which is a contradiction. Hence the result.

10. COMPLEMENTARITY

Theorem 7. *Any graph and its complement cannot both be disconnected.*

Proof. Let $u, v \in \overline{G}$, the complement of a disconnected graph G. If u, v belong to different components of G, then u and v are adjacent in $\overline{G}$. If u and v belong to the same component say G_i, of G, then let w be a vertex of some other component, say G_j, of G. Then both u and w, and v and w are adjacent in $\overline{G}$. In either case, u is connected to v by a path in G. Thus $\overline{G}$ is connected.

EXERCISE 9 (A)

1. Show that the maximum number of edges in a simple graph with n vertices is $n(n-1)/2$.
2. Draw all simple graphs of one, two, three and four vertices.
3. Use a graph theoretic approach to show that, in any set of n people ($n > 1$), there are at least two people with exactly the same number of friends within the set.
4. Show that the maximum degree of any vertex in a simple graph with n vertices is $n - 1$.
5. Name 10 situations (games, real-life problem, activities etc.) that can be represented by means of graphs. Explain what the vertices and the edges denote.
6. Prove that the simple graph with n vertices must be connected if it has more than $[(n-1)(n-2)/2]$ edges.

11. TREES

The trees are very important graphs because of their applications to Computer Science, Operations Research, Molecular Biology, Industrial Engineering and Social Sciences etc. This is mainly because of the fact that most of the graph theory applications involve trees directly or indirectly.

Definition. A tree is a connected acyclic graph. In other words a connected graph without circuits is a tree. We call its edges as branches. Following are the graphs of a tree with eight vertices.

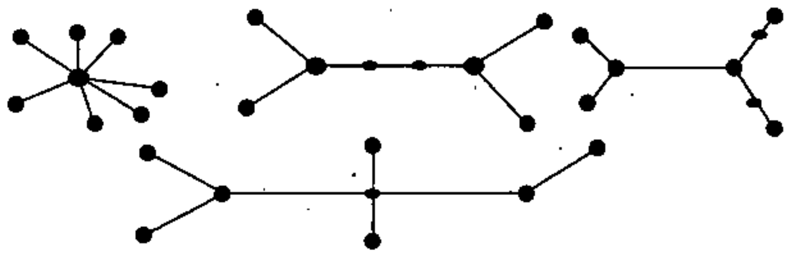


Fig. 4. Trees with 8 vertices.

We can use the trees to model blood relation in Social Sciences, sequential decision or sorting in computer science, minimally connected communication networks and the anatomy of structures having various levels and numerous other phenomena in a wide range of cases.

We give below some characteristics of the trees based on elementary observations :

1. The removal of any single edge from a tree makes it a disconnected graph.

2. In a connected graph which is not a tree, it is always possible to find at least one edge after whose removal the graph will remain connected.

Remark. Based on the above two observations, a tree can be characterized as a minimally connected graph in the sense that it does not possess a proper spanning subgraph which is connected.

Theorem 8. *There is exactly one path between every pair of vertices in a graph G if and only if G is a tree.*

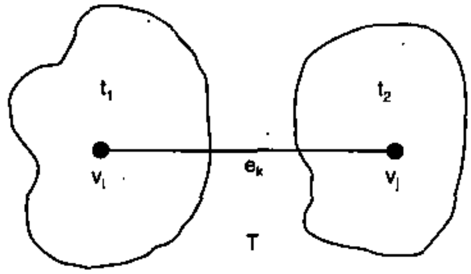
Proof. The necessity of the condition. The existence of a path between every pair of vertices implies that G is connected. If G is connected, now suppose G is cyclic then there would exist at least one pair of vertices in G having the property that there are at least two distinct paths between them. This is contrary to the hypothesis that there is exactly one path between every pair of vertices of G. Hence G is acyclic. Therefore, G is a tree.

The sufficiency of the condition. If G is a tree, it is connected graph and as such there is at least one path between every pair of its vertices. Suppose there exists a pair of vertices in G such that there is at least two distinct paths between them. Then the union of these two paths will provide a cycle. This is not possible as tree is acyclic graph. Therefore, there is exactly one path between every pair of vertices in G.

Theorem 9. *A tree with n vertices has (n – 1) edges.*

Proof. We prove this theorem by induction on the number of vertices. We easily see that the theorem is true for $n = 1$ as there will be no edge in this case. The theorem is true for $n = 2$ and 3 also.

Now let us consider a tree T with n vertices. In T let e_k be an edge joining the vertices v_i and v_j.

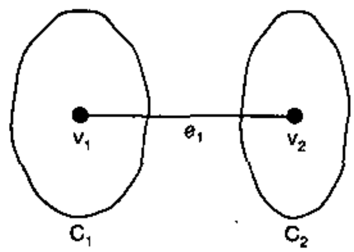


We know that there is only one path between v_i and v_j *i.e.*, e_k only. Therefore removing of e_k will make the graph disconnected as can be seen in adjoining figure. Also $T - e_k$ consists of exactly two components and as there are no circuits in T, each of the components t_1 and t_2 is a tree. Each of t_1 and t_2 contains vertices less than n and therefore by induction each contains one edge less than the number of vertices in it. Thus $T - e_k$ consists of $n - 2$ edges and n vertices implying that T has exactly $n - 1$ edges.

Theorem 10. *An acyclic graph which has one less edge than its vertices must be connected.*

Proof. Suppose there exists a circuitless disconnected graph G with n vertices and $(n - 1)$ edges: G consists of two or more acyclic components.

Suppose it contains two components C_1 and C_2 (without loss of generality). Let us add an edge e_1 between vertices v_1 in C_1 and v_2 in C_2. Since there was no path between v_1 and v_2 in G adding e_1 does not add any circuit to G. Thus $G \cup e_1$ is a circuitless connected graph *i.e.*, a tree with n vertices and n edges, which is not possible because only $n - 1$ edges are possible. Hence the theorem.



Corollary. Connected graph G is a tree if and only if G has one less edge than it has vertices.

Based on the above observations alternative definitions of a tree are as follows :

Definition 1. A tree is a connected graph with n vertices and $(n - 1)$ edges.

Definition 2. A tree is an acyclic graph with n vertices and $(n - 1)$ edges.

Definition 3. A tree is a graph in which there is exactly one path between every pair of its vertices.

Definition 4. A tree is an acyclic graph having the property that if any two of its vertices which are not adjacent are joined directly by an edge then the resulting graph possesses exactly one cycle.

More theorems on trees :

Theorem 11. *Any tree with at least two vertices has at least two pendant vertices.*

Proof. Let the number of vertices in a given tree be n (>1). We know that the tree has $(n - 1)$ edges. Thus the degree sum of the tree is $2(n - 1)$. This degree sum is to be divided among the n vertices. Because a tree is connected it cannot possess a vertex of 0 degree. Therefore each vertex contributes at least 1 to the above sum. Therefore there must be at least two vertices of degree exactly 1.

Definition. *An acyclic graph is termed a forest.*

The components of a forest are trees. Removal of a single edge from a tree creates a forest of exactly two trees. Removal of a further edge will create a forest of three trees. Continuing in this way the removal of any $(k-1)$ edges from a tree will create a forest of k trees. The ultimate result of this removal process is a forest of n trees, each being an isolated vertex. Thus we have the following theorem :

Theorem 12. *A forest of k trees which has a total of n vertices has (n – k) edges.*

Proof. The proof follows from the preceding remarks.

12. DISTANCES OF TREES

Definition. The *distance* d_{uv}, between two vertices u and v of a graph G, is equal to the number of edges in the path with the shortest number of edges connecting u and v.

For example, the distance between a pair of distinct vertices selected at random from the graph of figure of 10.11 is 1, or 2, or 3. We know that there is exactly one path between any two vertices in a tree. This makes these distances relatively easy to calculate for trees. Since the distance function we have defined, $d : V \to R \cup \{0\}$, is non-negative, symmetric and obeys the triangle inequality, it is a metric in the normal topological sense.

We are now in the position to develop rigorously the concept of centrality.

In the figure of 10.11, it can be observed that the two vertices with degree greater than 1 are somewhat more central, in a certain sense, than the pendant vertices.

Definitions. The *eccentricity* $e(v)$, of a vertex v of a connected graph $G = (V, E)$, is defined as $\max_{u \in V} d_{uv}$.

The *radius* $r(G)$, of a connected graph G is equal to the eccentricity of the vertex in G which is of minimum eccentricity.

The *diameter* $d(G)$, of a connected graph G is equal to the eccentricity of the vertex in G which is of maximum eccentricity.

A vertex v, in a connected graph G is termed *central* if $e(v) = r(G)$.

The *centre* of a connected graph G is the set of all central vertices in G.

Each vertex is labeled with its eccentricity in the top tree in figure given below. It is evident from the figure, this graph has a centre comprising of two vertices, each labelled C. Clearly a graph can have a centre with cardinality greater than 1. In fact a graph can have an arbitrary number of central points. The fact that the tree in the following figure has two central points raises the question as to how many central points are possible in any tree. The following theorem provides the answer to this question.

Theorem 13. *Every tree has a centre comprising of either one vertex or two adjacent vertices.*

Proof. The result is clear for the first two trees in figure of 11. We will show that any tree T has the same centre as a tree T′, obtained from it by removing from T all of its pendant vertices. Also the eccentricity of any vertex v, in T is equal to the distance from v to some pendant vertex in T. Therefore eccentricity of each vertex in T′ is one less than the eccentricity of that vertex in T. Therefore the vertices of minimum eccentricity in T will have minimum eccentricity in T′. Therefore T and T′ have the same centre. Let us successively remove the pendant vertices all at once from T. If the process is repeated successively, we have a sequence of trees having the same centre, infact the same centre as of T. This process leads to one of two results : either the first or the second of the two trees in figure of 11 finally remain. In either case the vertices of this final tree comprise of the centre of T. Hence the result.

This process is illustrated in the lower diagrams of figure given below. By definitions, the radius of the original tree in this figure is 4. It is easily seen that the diameter of the tree is 7. Thus we see that the diameter of a tree is not necessarily twice its radius.

Binary Trees. We turn now to a brief discussion of concepts which are essential to the application of trees in various references.

Definitions. A tree in which exactly one vertex is distinguished is called a *rooted tree.*

The distinguished vertex in a rooted tree is called its root.

All the rooted trees with five vertices are shown in figure 6, with the root distinguished by a square rather than a disk. Sometimes trees are turned into diagraphs by assigning an orientation to each edge away from the root, to make it an arc. Ordinary trees are often called *free* to emphasize the fact that they do not possess a root. One of the most common classes of rooted trees is *binary* as defined below.

Definition. A *binary tree* is a rooted tree with at least three vertices in which the root is of degree 2 and all other internal vertices are of degree 3.

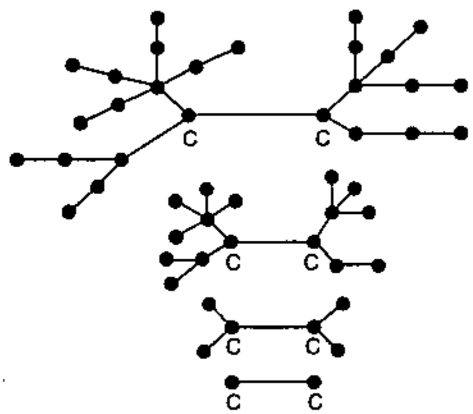


Fig. 5. Trees with two central points.

We now state some theorems about binary trees.

Theorem 14. *Every binary tree has an odd number of vertices.*

Proof. Apart from the root, every vertex in a binary tree is of odd degree. We know that in any graph, there must be an even number of such odd vertices. Therefore when the root (which is of even degree) is added to this number the total number of vertices must be odd.

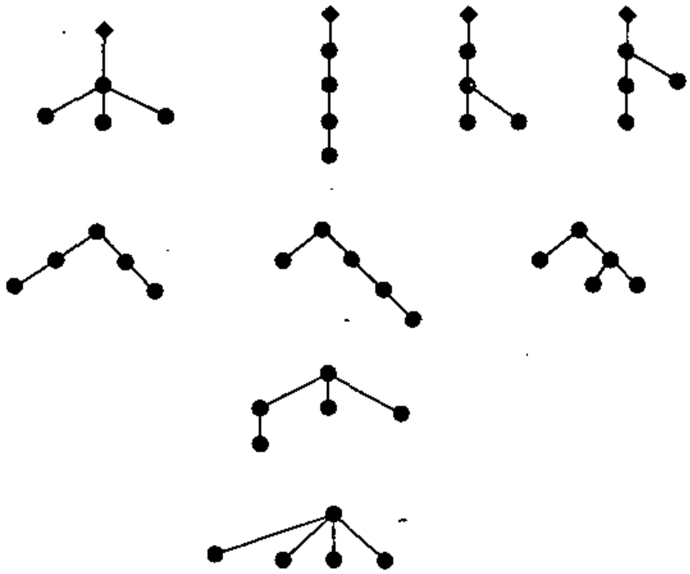


Fig .6. The rooted trees with 5 vertices.

Theorem 15. *There are* $\frac{1}{2}$ $(n + 1)$ *pendant vertices in any binary tree with n vertices.*

Proof. Suppose that T is a binary tree with n vertices. Let m be the number of pendant vertices in T. There are then $(n - m)$ internal vertices in T and $(n - m - 1)$ vertices of degree 3. Thus there are $\frac{1}{2}$ $(3(n - m - 1) + 2 + m)$ edges in T. However since T has n vertices, it has $(n - 1)$ edges. In equating these two expressions and solving for m, we have $m = \frac{1}{2}(n + 1)$

13. TREE ENUMERATION

We know that Cayley introduced the concept of a tree and enumeration of all rooted trees. Since then, enumerative methods for counting various classes of graphs, including trees, have been developed, but are still far from completely scientific. The father of this field was Redfield (1927) whose contributions foreshadowed much of the later work by Polya (1937, 1940). Here without going to all such details we state a few results and the power of current enumeration methods by referring to the counting of one special class of trees being used as the application of graph theory in molecular evolution. This in turn is used in computer science in various algorithms. A further illustration of graph enumeration, occur in the counting of maximal planar graphs.

Theorem 16. *There are* $2^{^nC_2}$ *labelled graphs with n vertices.*

Proof. Consider a graph with its vertices labelled 1, 2, ..., n. In any such graph, each of the ${}^nC_2$ possible edges is either present or absent. Hence there are $2^{{}^nC_2}$ labelled graphs with n vertices.

NOTES

Cayley (1889) stated the corresponding result for the special class comprising labelled trees :

Theorem 17. There are n^{n-2} labelled trees with n vertices.

The proof is beyond the scope of the text and can be seen in any reference text of graph theory.

Enumerating Phylogenetic Trees. Biologists often represent postulated evolutionary relationship between existing biological species by means of a tree. Such a diagram, linking related species to a common ancestor, is called a *phylogenetic tree or phylogeny.*

Definition. A phylogeny is a tree T = (V, E), together with a set {1, 2, ..., n} of labels and a function f, mapping the labels into V, where every vertex of degree less than three must be in the image of f.

It is clear from this definition that some vertices may possess multiple labels and some vertices possess no labels at all.

Definitions. The magnitude of a phylogeny is defined to be n, the number of its labels.

The order of a phylogeny is defined to be its number of vertices.

A planted phylogeny is a rooted tree which is a phylogeny having a pendant vertex which is distinguished and is called its root.

This root represents the common ancestor of all the species in the labeling set and thus is not given one of the labels of the set and is not counted in calculating the order of the phylogeny. In the search for phylogenies which are optimal, in the sense of satisfying criteria of scientific models of evolution, it is of interest to know how many possible feasible phylogenies exist for a given set of n labelled species. However there are various classes of phylogenies. They are of the following type :

(*i*) can be binary,

(*ii*) can have all vertices of degree three or more unlabelled,

(*iii*) can have only singleton labels,

(*iv*) can have no vertices of degree two.

Combinations of the requirement or absence of each of the above for conditions, ostensibly creates 16 classes of phylogenies. However four classes are empty as there are no vertices of degree two (other than the root) in a binary phylogeny. Hence there are 12 nonempty classes, as defined in Table given below.

The number of phylogenies for all 12 classes can be calculated exactly for given n, along with the mean and variance of the number of vertices in the phylogenies and the asymptotic behaviour of the exact numbers, by using the existing methods.

Class Degree	Must be Binary	Only Vertices of Degree 1 or 2 Labelled	All Labels Singleton	Vertices of 2 Allowed
1	NO	NO	NO	YES
2	NO	NO	YES	YES
3	YES	YES	YES	NO
4	YES	YES	NO	NO
5	YES	NO	YES	NO
6	YES	NO	NO	NO
7	NO	YES	YES	NO
8	NO	YES	NO	NO
9	NO	NO	YES	NO
10	NO	YES	YES	YES
11	NO	NO	NO	NO
12	NO	YES	NO	YES

Table : The 12 classes of phylogenies.

NOTES

14. SPANNING TREES

Let us now consider trees as subgraphs of larger graphs. Consider a connected graph G = (V, E), which contains a subgraph which is the tree T = (V′, E′). The edges E′, of T are *called branches* and the degree of G which are not in T are called chords (both relative to T).

Definition. If V′ = V then T is said to be a *spanning tree* of the graph G.

In other words, a spanning tree contains all the vertices of the graph of which it is a subgraph. It is also clear that only connected graphs have spanning trees as subgraphs. In fact, if a connected graph G, has a unique spanning tree then G is itself a tree and that spanning tree is G itself. Also, each component of an arbitrary graph G contains at least one spanning tree. A collection of such spanning trees, one for each component of G, is called a *spanning forest*. This is shown in figure 7. Each spanning tree in the spanning forest is exhibited by thick lines. The following theorem provides the number of chords in any graph with a spanning tree.

Theorem 18. *A connected graph containing n vertices and edges will have, relative to any of its spanning trees, (e – n + 1) edges that are not members of that spanning tree.*

Proof. The proof follows immediately from the definition of a spanning tree and from the fact that a tree with n vertices has $(n - 1)$ edges.

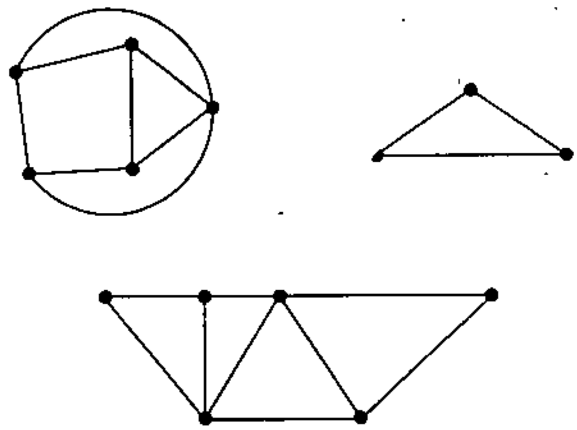


Fig .7. A spanning forest.

15. FUNDAMENTAL CYCLES

We have introduced a spanning tree in a connected graph by the successive removal of edges from its cycles. If we begin with a connected acyclic graph, namely a tree, we can add edge to join two of its vertices that are not directly connected, to create a cycle. This fact coincides with definition of a tree. Suppose that the tree T, is a spanning subgraph of a connected graph G, with n vertices and e edges. One can identify the chords of G with regard to T. The addition of a chord to T will create exactly one cycle with regard to T, called a *fundamental cycle.*

Therefore, for any of its spanning trees, the graph G has as many fundamental cycles as it has chords, namely $(e - n + 1)$. Each spanning tree of G will be associated with it a set of $(e - n + 1)$ fundamental cycles. We now illustrate these ideas through the figure on next page. At the top of the figure is a connected graph and directly below it are two of its spanning trees with edge set : $\{d, a, b\}$ and $\{d, c, b\}$ respectively. Adding the chord e, to the first spanning tree creates the cycle (a, b, e).

Adding the other chord c, to that spanning tree creates the fundamental cycle $< a, b, c, d >$. Repeating an analogous process for the second spanning tree, we create a different set of fundamental cycles, namely the cycles : $< a, b, c, d >$ and $< e, c, d >$. These sets of fundamental cycles have exactly one cycle in common namely $< a, b, c, d >$. Thus we see that the set of fundamental cycles is very much dependent on the spanning tree selected. As an exercise, one should identify the other six spanning trees for the original graph and the set of fundamental cycles that each one creates. Because the original graph had two chords, there are always two fundamental cycles in the set of fundamental cycles created by any spanning tree. The number of fundamental cycles is constant for any connected graph.

In many applications we may like to analyse the cycles of a connected graph. The simple way to do that is to create a spanning tree and to create a set of fundamental cycles from it. It means that all the cycles of a graph can be created by performing a special type of simple matrix arithmetic on the elements in the set of fundamental cycles. This fact was discovered by Kirchhoff, in his analysis of electrical networks. Following figures illustrate the Fundamental cycles in a connected graph.

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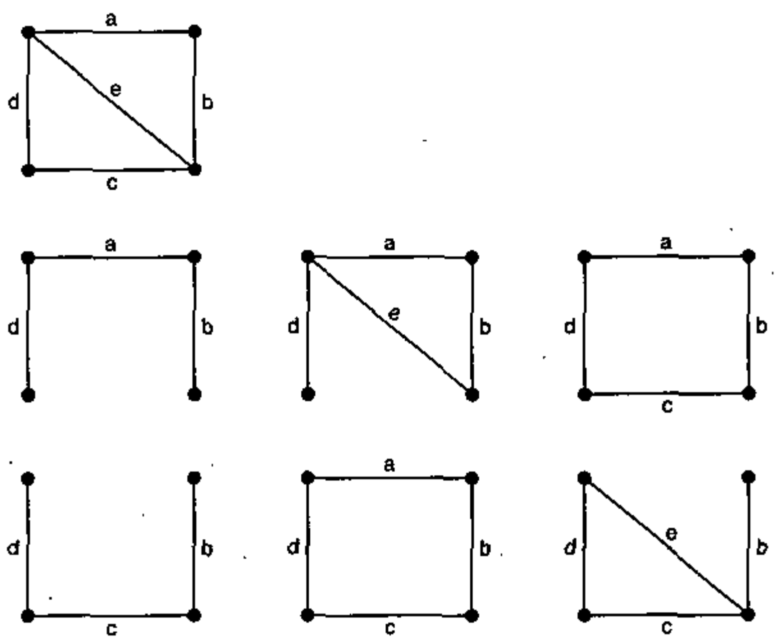

Fig. 8. Fundamental cycles in a connected graph.

Concludingly the class of graphs called trees, are of great importance for their applications. From a mathematical point of view often a tree represents a simple type of graph which is worth analysing first in order to judge whether or not a particular proposition is likely to be true. In order to provide the tools for some of these applications, some elementary theorems concerning trees, tree distances, binary trees, tree enumeration, spanning trees and fundamental cycles have already been introduced.

EXERCISE 8 (B)

1. Draw the 20 roots trees with 6 vertices.
2. Draw the 23 unlabelled trees with 8 vertices.
3. *Draw a tree which has radius 5 and diameter 10.*
4. Prove that every connected graph processes a spanning tree as a subgraph.
5. Prove that any subgraph G′, of a connected graph G, is contained in a spanning tree of G if and only if G′ is acyclic.
6. A star is a tree which has at most 1 non-pendant vertex. Show that a tree with more than two vertices is a star if and only if it has diameter two.
7. Prove that each edge of connected graph G, belongs to at least one spanning tree of G.
8. Prove that each spanning tree of a connected graph G, contains all the pendant edges of G.

16. CONCEPT OF TRAVERSABILITY

We know two puzzles that can be represented by graphs. The problems are : Königsberg bridge problem and Hamilton's game. There may be other problems also such as "puzzle with multi coloured cubes etc., which may be analysed by using graph theoretic concepts. This can be done by connecting the puzzle into corresponding either an Eulerian trend or Hamiltonian cycle within an appropriate graph. Now we introduce the concept of Eulerian graphs.

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17. EULERIAN GRAPHS

The origin of Eulerian graphs is due to Königsberg bridge problem. Leonard Euler is said to be the father of graph theory because the subject originated with his famous paper on Könibergs problem. Given a connected graph G, then a closed trail containing every edge of G exactly once is said to be an Euler trail. An open trail containing every edge of G exactly once is said to be an open Euler trail. Below we provide some definitions because many applications involve finding Eulerian trails in multi graphs rather than in graphs.

Definition 1. A multi graph containing an Euler trail is termed as Eulerian.

2. A multi graph containing an open Euler trail is said to be unicursal.

Theorem 19. *The following statements are equivalent for any connected multi graph G.*

(*i*) *G is Eulerian*

(*ii*) *Every vertex of G is of even degree*

(*iii*) *The set of edges of G can be partitioned in cycles.*

Proof. If a connected multigraph G is Eulerian then every vertex of G has even degree. Let G be an Eulerian multigraph. Let T be an Euler trail in G. In tracing this trail, each time a vertex is met, it contributes 2 to the degree of that vertex. Because each edge of G occurs exactly once in T, every vertex must have even degree. Thus secondly if every vertex of a connected multi graph G has even degree then the set of edges of G can be partitioned into cycles. Let G be a connected multigraph in which every vertex has even degree. Since G is connected and contains at least one vertex, every vertex has degree at least 2. Therefore G contains a cycle. The removal of the edges of this cycle results in a spanning subgraph G (say) in which every vertex has even degree. If G has no edges then the result follows, if G still has edges, the above process is repeated on G. This results in a graph in which all of the vertices are still even. This process is repeated until a multigraph with no edges is obtained. Each step of the above process creates a cycle and the totality of these cycles represents a partition of the edges of G into disjoint cycles.

Finally if the set of edges in a connected multi graph G can be partitioned into cycles,.then G is Eulerian. Suppose that G is a connected multi graph such that its edges can be partitioned into cycles. Let C be one of the cycles in this partitioned. If G consists solely of C, then G is Eulerian. If not, there exists another cycle C′, (say) with a vertex v, which is common in C too. At least one such cycle C′ must exist because G is connected. The walk which begins at v, and comprises of the cycles C and C′ in succession is a closed trail consisting of the edges of these two cycles. By continuing this logic we construct a closed trail containing all the edges of G. Hence the theorem.

Hamiltonian Graphs. We know that in Hamilton's game there is a cycle which contains each vertex of a given graph. To find such cycles in any graph we need some foundation. To meet this end we begin with some definitions.

Definition 1. A graph with a spanning cycle is said to be a Hamiltonian graph.

2. A spanning cycle in a graph is called a Hamiltonian cycle.

Hamiltonian posed a problem as to how it can be ensured that a given graph is Hamiltonian or not. Many mathematicians have been trying to find an elegant characterization to this effect but there is no simple characterization of Hamiltonian graphs equivalent to theorem 19 for Euler multi graphs. Below we give a theorem on Hamiltonian graphs.

Theorem 20. Let *G = (V, E) be a connected graph with n vertices, where n > 2. Let u and v be a pair of distinct non-adjacent vertices of G such that*

$$d(u) + d(v) \geq n.$$

Then G + uv is Hamiltonian if and only if G is Hamiltonian.

Proof. The *necessity* of the theorem is evident from the definition of Hamiltanian graphs. We prove now the *sufficiency*. Suppose that G is a graph with n vertices containing non-adjacent vertices u and v, such that G + uv is Hamiltonian. Suppose that in G :

$$d\,(u) + d\,(v) \geq n$$

Suppose G is not Hamiltonian. As per our assumption, there exists a Hamiltonian cycle of (G + uv) containing the edge uv. Thus there is a path P, in G from : u to v : $(u = u_1, u_2, ..., u_n = v)$ in G, containing every vertex of G.

If $u_1 u_i \in$ E, $2 \leq i \leq n$, then $u_{i-1} u_n \notin$ E, because otherwise

$$< u_1, u_i, u_{i+1}, ..., u_n, u_{i-1}, u_{i-2}, ... u_n >$$

is a Hamiltonian cycle of G. Therefore for each vertex of $\{u_2, u_3, ... , u_n\}$ adjacent to u_1 there is a vertex of $\{u_1, u_2, ..., u_{n-1}\}$ not adjacent to u_n.

Hence $\quad d(u_n) \leq (n-1) - d(u_1) \quad$ or $\quad d(u) + d(v) \leq n - 1.$

This is a contradiction to our assumption. Hence G is Hamiltonian.

The concept of adding an edge discussed in the theorem 21 given below leads us to a useful definition concerning Hamiltonicity.

Definition. The closure of a graph G containing n vertices, denoted by C(G), is the graph obtained from G by successively joining pairs of non-adjacent vertices whose sum of degrees is at least n, (in the graph obtained at each step of joining) until it is not possible to join any further pairs. This closure operation, which can be shown to be well defined, is illustrated in figure given on the next page. Now a theorem is provided for a sufficiency condition for Hamiltonicity.

Theorem 21. *A graph is Hamiltonian if and only if its closure is Hamiltonian.*

Proof. By the above theorem and the definition of closure the proof is obvious. We now give a theorem for a sufficient condition for Hamiltonian graphs.

Theorem 22. *Let G be a graph with at least 3 vertices. If C(G) is complete, then G is Hamiltonian.*

Proof. The proof is immediate by above theorem and the fact that each complete graph with at least 3 vertices is Hamiltonian.

There are a number of corollaries of theorem 22 which also provide sufficient conditions for Hamiltonicity, although they are weaker than theorem 22 itself.

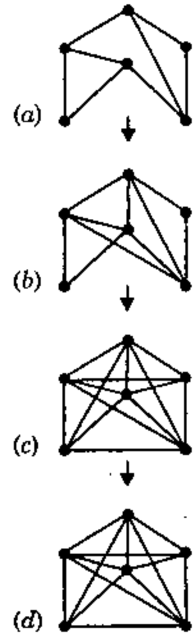

Fig. 9. The closure operation.

Corollary 1. If G is a graph with at least p vertices, where $p \geq 3$, such that for every integer i, with $1 \leq i \leq p/2$, the number of vertices of degree not exceeding i is less than i, then G is Hamiltonian.

Corollary 2. If G is a graph with at least p vertices, where $p \geq 3$, such that for all distinct non-adjacent vertices u and v,

$$d(u) + d(v) \geq p,$$

Then G is Hamiltonian.

Corollary 3. If G is a graph with at least p vertices where $p \geq 3$, such that $d(v) \geq p/2$ for every vertex v of G, then G is Hamiltonian.

So far we discussed first the need for a closed trail in a graph G, containing all of the edges of G exactly once. We then turned to finding an open trail containing all of the edges of G. Similarly, we now focus our attention for finding a path which covers all the vertices of a graph.

Definition. A spanning path in a graph is called a *Hamiltonian path.*

If a graph G, is Hamiltonian then it must contain Hamiltonian paths. These paths can be obtained by deleting exactly one edge from any Hamiltonian cycle. Conditions for a graph to contain a Hamiltonian path can be derived from the corollaries to Theorem 22.

There is a relationship between Eulerian and Hamiltonian graphs. An illustration of this is given in Figure 10, where (a) contains a graph which is both Eulerian and Hamiltonian, (b) contains a graph which is Eulerian but not Hamiltonian, (c) contains a graph which is Hamiltonian but not Eulerian, and finally (d) contains a graph which is neither Eulerian nor Hamiltonian.

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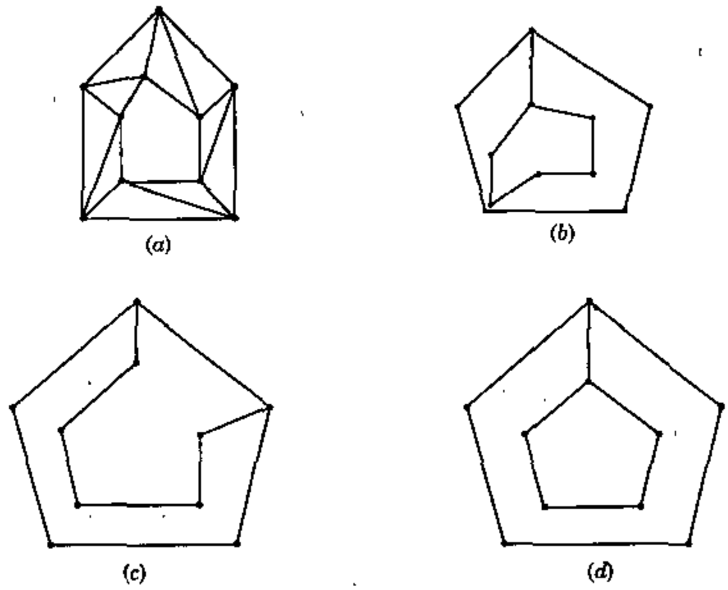


Fig .10. Little relationship between Eulerian and Hamiltonian graphs.

EXERCISE 9 (C)

1. Prove that a connected graph is unicursal if and only if it has exactly two vertices of odd degree.
2. Characterize graphs which possess Hamiltonian paths but not Hamiltonian cycles.
3. Is it possible to move a knight on a chessboard such that it touches each square exactly once and returns back to the original square from where it started ?
4. Characterize graphs which are both Eulerian and Hamiltonian.
5. Characterize graphs which are unicursal but not Eulerian.

SUMMARY

1. **A graph** G consists of a set of objects $V = \{v_1, v_2, ...\}$ called vertices and another set $E = \{e_1, e_2, ...\}$, whose elements are said to be edges such that each edge e_k is identified with an unordered pair of vertices (v_i, v_j). We write G = (V, E) to express the two parts of G. The vertices v_i and v_j are called the end vertices of the edge e_k.
2. A graph which has neither self loops nor parallel edges is said to be a simple graph.
3. **Multi-graph.** The multi-graph is an ordered pair of sets (V, E), where V is finite and non-empty, and E is a class of unordered pairs of distinct elements of V whose repetitions in the class are allowed. Following is the example of a multi-graph
4. **Pseudographs.** A pseudograph is an ordered pair of sets (V, E), whose V is finite and non-empty and E is a class of unordered pairs of elements of E whose repetitions in the class are allowed. Following is an example of pseudograph.
5. **Walk.** A walk in a graph G is a finite alternating sequence of vertices and edges of G beginning and ending with vertices, such that each edge is incident with the vertices preceding and following it. The form of the sequence can be expressed as

   $\{v_1, \{v_1, v_2\}, v_2, \{v_2, v_3\}, v_3, ..., v_{n-1}, \{v_{n-1}, v_n\}, v_n\}$.
6. **Circuit.** A closed walk in which no vertex except the initial and final, appears more than once is said to be a circuit. In other words, a circuit is a closed, non-intersecting walk.

NOTES

7. **Euler Graphs.** Graph theory originated in 1736 with Euler's famous paper in which he solved the Königsberg bridge problem. In fact he posed a more general problem and solved the same. This was as given below. In what way a graph G is possible such that there exists a closed walk running through every edge of G exactly once. This type of *walk* is called an *Euler line* and a graph that contains an Euler line is said to be an *Euler graph.*

TEST YOURSELF

1. Prove that any two simple connected graphs with n vertices, all of degree two, are isomorphic.
2. Let a, b and c be three distinct vertices in a graph. There is a path between a and b and also there is a path between b and c. Prove that there is a path between a and c.
3. Prove that a connected graph G remains connected after removing an edge e_i from G, if and only if e_i is in some circuit in G.
4. Draw a connected graph that becomes disconnected when any edge is removed from it.
5. Under what conditions is the diameter of a tree equal to twice its radius ?
6. Prove that the diameter of any tree is equal to the number of edges in its path with the maximum number of edges.
7. Prove that a subgraph of a connected graph G, is a maximal acyclic subgraph of G if and only if it is a spanning tree of G.
8. A connected graph $G = (V, E)$, is said to be minimally connected if, for each edge $e \in E$, $G - e$ is disconnected. Prove that a connected graph is minimally connected if and only if it is a tree.

C H A P T E R

10

DIGRAPHS : APPLICATIONS

NOTES

LEARNING OBJECTIVES

- Concept of Connectivity
- Concept of Traversability
- Directed Trees
- Some Useful Digraph Matrices
- The Principle of Directional Duality
- Tournaments
- Strong Tournaments
- Acyclic Tournaments
- Tournament Winning Analysis
- Coverings and Colourings
- Covering, Independence and Domination
- Colouring
- Matching

1. CONCEPT OF CONNECTIVITY

Now we intend to discuss applications of digraphs.

Let that D = (V, E), be a digraph. Then a walk in D is an alternating sequence of vertices and arcs of D : $\langle v_0, a_1, v_1, ..., a_n, v_n \rangle$, in which each arc a_i is $v_{i-1}\, v_i$. The walk is called *closed* when $v_0 = v_n$, and *spanning* if $\{v_0, v_1, ..., v_n\}$ = V. A walk is said to be a *trial* if all of its arcs are distinct, a *path* if all of its vertices are distinct and a *cycle* if it contains at least three vertices and all its vertices are distinct except for the fact that $v_0 = v_n$. A digraph is said to be cyclic if it contains a cycle and acyclic otherwise. If there exists a path in D from one of its vertices u, to another v, then v is said to be reachable from u.

Definitions. If arc uv is a member of a digraph, its *converse* is defined to be arc vu. The converse digraph D^c, of digraph D, has the same vertex set as D, and an arc is in D^c if and only if its converse is in D.

A *semi-walk* is an alternating sequence : $\langle v_0, a_1, v_1, \dots, a_n, v_n \rangle$ of vertices and arcs where each arc a_i is either: $v_{i-1}\, v_i$ or its converse. A semi walk is called a semi-trail if all of its arcs are distinct, a semi-path if all of its vertices are distinct, and a *semi-cycle* if it contains at least three vertices and all of its vertices are distinct except for the fact that $v_0 = v_n$.

Various Types of Connectivity in a Digraph.

Definitions. A digraph is said to be strongly connected or strong, if every two of its distinct vertices, u and v say, are such that u is reachable from v and v is reachable from u. D is unilaterally connected or unilateral, if either u is reachable from v or v is reachable from u, and is weakly connected, or weak, if u and v are joined by a semi-path. Figure 1 shows : (*a*) a strong digraph, (*b*) a unilateral digraph, and (*c*) a weak digraph.

Every strongly connected digraph is unilateral and every unilateral digraph is weak, but there are weak digraphs which are not unilateral, and unilateral digraphs which are not strong.

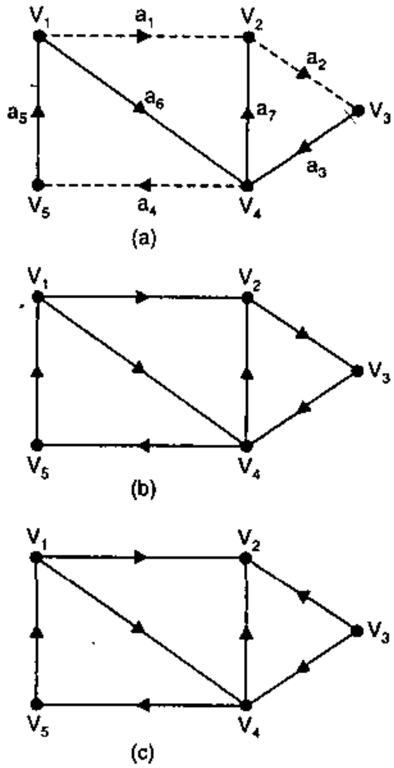


Fig. 1. Strong, unilateral and weak digraphs.

A digraph is called *disconnected* if it is not weak.

The theorems for analysing the connectivity of digraphs.

Theorem 1. *A digraph is strong if and only if it has spanning closed walk.*

Proof. Necessity of the condition. Let D = (V, E) be a strong digraph with V = $\{v_1, v_2, \dots, v_n\}$, then there is a walk from each vertex in V to each other vertex V. Thus there exist in D, walks: $W_1, W_2, \dots, W_{n-1}$ such that the first vertex of W_i is v_i and the last vertex of W_i is v_{i+1}, for $i = 1, 2, \dots, n-1$. There also exists a walk, say W_n, with first vertex v_n and last vertex v_1. Then the walk obtained by traversing the walks : W_1, $W_2, \dots, W_n$ in succession, is a spanning closed walk of D.

Sufficiency of the condition. If u and v be two distinct vertices of V and if v follows u in any spanning closed walk, say W of D then there exists a sequence of the arcs of W constituting a walk from u to v. If u follows v in W then there is a walk from u to the least vertex of W and a walk from that vertex to v. A walk from u to v can be constructed by traversing these two walks in succession.

Theorem 2. *A digraph D is unilateral if and only if it has a spanning walk.*

Proof. Necessity of the condition. Let D be a unilateral digraph. Suppose W is a walk in D containing the maximum number of vertices. Let W begin at vertex v_1 of D and ends at vertex v_2 of D. If W is a spanning walk the result is proved. Assume that W is not a spanning walk. Then there exists a vertex say (u), of D that is not in W. Also there cannot exist in D a walk from u to v_1 or a walk from v_2 to u. Since D is unilateral and does not possess a walk from u to v_1, D must possess a walk from v_1 to u.

Suppose l is the least vertex of W from which a walk from l to u exists in D. Let U be a walk from l to u in D. Let m be the vertex in D which is the immediate successor of the last appearance of l in W. D does not possess a walk from s to u, however because D is unilateral, there is a walk say, Y, from u to m in D. Let us traverse W from v_1 to the last appearance of l, then traverse u, then traverse Y to vertex m and then traverse W to v_2. This represents a walk from v_1 to v_2 which has more distinct vertices than W. This is a contradiction. Hence W is a spanning walk of D.

Sufficiency of the condition. Sufficiency of the condition is obvious.

If D = (V, E) is a unilateral digraph then $od(u) = 0$ for at most one $u \in V$ and $id(v) = 0$ for at most one $v \in V$, $u \neq v$.

Theorem 3. *A digraph is weak if and only if it has a spanning semi-walk.*

Proof. Necessity of the condition. Let D = (V, E) be a weak digraph with V = $\{v_1, v_2, ..., v_n\}$. Since D is weak there is a semi-walk W_i say from v_i to v_{i+1} in D for $i = 1, 2, ..., n - 1$. The semi-walk obtained by traversing the semi-walks : $W_1, W_2, ..., W_n$, in succession is a spanning semi-walk of D.

Sufficiency of the condition. Let D be a digraph containing a spanning semi-walk W say.

Let v_1 and v_2 be any two distinct vertices of D. Since W is spanning, v_1 and v_2 belong to W. The part of W which begins at any appearance of $v_1(v_2)$ and ends at any appearance of $v_2(v_1)$ represents a semi-walk from v_1 to v_2 or from v_2 to v_1 in D. Hence D is weak.

Definition. A digraph D′ = (V′, A′), is termed a subdigraph of a digraph D = (V, A), if $V' \subseteq V$ and $A' \subseteq A$ and D′ is a digraph.

Just as there are three concepts of connectively in the theory of digraphs there are also three kinds of components.

Definitions. A strong component *in a digraph D, is a maximal strong subdigraph* of D. A unilateral component in a digraph D, is a maximal unilateral subdigraph of D.

A weak component in a digraph D, is a maximal weak subdigraph of D.

The digraph in figure 1(*b*) has a strong component induced by the vertex set $\{v_2, v_3, v_4\}$. The digraph in figure 1(*c*) has a unilateral component induced by the vertex set $\{v_1, v_4, v_2\}$ and a weak component which is the digraph itself.

Every vertex and every arc of a digraph D, belongs to exactly one weak component of D and to at least one unilateral component of D. Also, every vertex belongs to exactly one *strong component of D. Every arc a, belongs to exactly component* of D is a does not belong to a cycle of D.

Definition. Let D be a digraph with strong components : $S_1, S_2, ..., S_p$. The condensation $D^* = (V^*, A^*)$, of D is a digraph with $V^* = \{S_1, S_2, ..., S_p\}$ and where $S_i S_j$ is an arc of A^* if and only if there exists an arc uv in D for $u \in S_i$ and $v \in S_j$.

The condensation of the digraph in Figure 1 (*a*) is just a single vertex. The condensation of the digraph in Figure 1(*b*) comprises of $V^* = \{S_1, S_2, S_3\}$, and $A^* = \{(S_1, S_2), (S_1, S_3), (S_2, S_3)\}$, where $S_1 = \{v_1\}$, $S_2 = \{v_2, v_3, v_4\}$ and $S_3 = \{v_5\}$. The condensation of the digraph in Figure 1(*c*) is a digraph itself.

These are the examples of some general observations. Let D^* be the condensation of the digraph D.

(*i*) D^* is acyclic

NOTES

(*ii*) If D* is strong the D* comprises a single vertex (and no arcs).

(*iii*) D* comprises a unique spanning path and is unilateral if and only if D is unilateral.

Definition. A cut-set in a digraph D = (V, A), is a set of arcs of A, which constitute a cut-set in the multigraph G = (V, E), obtained from D by removing the orientation from each arc of A.

Let us now discuss the concept of digraph traversability.

2. CONCEPT OF TRAVERSABILITY

Definition. A digraph D is known as Eulerian if it contains a closed trail which traverses every arc of D exactly once. Such a trail is termed an Euler trail. D is called unicursal if it contains an open Eular trail. Following distinguished digraphs as Eulerian and unicursal.

Theorem 4. *A digraph D = (V, A), is Eulerian if and only if D is connected and for each of its vertices v, id(v) = od(v).*

Proof. See any reference text.

Corollary. Let D = (V, A) be a weak digraph. D is unicursal if and only if D contains vertices u and v such that : $od(u) = id(u) + 1$, $id(v) = od(v) + 1$, and $od(w) = id(w)$, for all $w \in$ V where $w \neq u, v$. Here D has an open Euler trail which begins at u and ends at v.

Definition. A digraph is called Hamiltonian if it has a cycle containing all of the vertices of D.

There are a number of results that are analogous to those proved for Hamiltonian graphs in earlier. For example, let D be a strong digraph with n vertices ($n \geq 3$) such that for every pair u and v, of distinct non-adjacent vertices of D, $d(u) + d(v) \geq 2n - 1$. D is Hamiltonian.

3. DIRECTED TREES

We know that a tree is a connected acyclic graph. A directed tree can be defined analogously.

Definition. A directed tree is a weak digraph that does not contain a semi-cycle.

There is one particular type of directed tree that is of importance in network analysis, computer science, enumeration and other fields of applied graph theory. It is said to be an *arborescence.*

Definition. A directed tree is called an *arborescence* if it contains exactly one vertex, called the root, with no arcs directed towards it, and if all the arcs on any semi-path are directed away from the root.

One can make a number of observations about any arborescence D as detailed below :

(1) Every vertex in D, other than the root, has exactly one arc directed towards it.

(2) There is a path from the root of D to every other vertex in D.

(3) The root r, of d has the property that every other vertex in D is accessible from r, and r is not accessible from any other vertex of D.

4. SOME USEFUL DIGRAPH MATRICES

NOTES

Let D = (V, A) be a digraph whose arcs have been labelled and whose cycles have been both labelled and given an arbitrary orientation.

Definition. The semi-cycle matrix $C = (c_{ij})$ of a digraph D is a matrix in which $c_{ij} = 1$ if cycle Z_i of D is such that arc a_j is directed in the same way as the orientation of Z_i, $c_{ij} = -1$ if Z_i contains arc a_j directed in the opposite way to the orientation of Z_i otherwise $c_{ij} = 0$.

As in the case of graphs the arcs of a connected digraph D, not part of a given spanning directed tree T, of D are called chords. When a chord is added to T it creates a fundamental semi-cycle (which may be a cycle).

We now illustrate these concepts with digraph in figure 1(a). Here T is specified by the arcs in thick lines, with branches: a_3, a_5, a_6 and a_7. The chords with respect to T are shown as the dotted lines: a_1, a_2 and a_4. The fundamental semi-cycles with respect to T are :

Chord	Semi-cycle	
a_1	$\langle a_1, a_7, a_6\rangle$,	
a_2	$\langle a_2, a_3, a_7\rangle$,	(a cycle),
a_4	$\langle a_4, a_5, a_6\rangle$,	(a cycle).

The rows of the cycle matrix C of a digraph D, containing a specified spanning tree T, which correspond to its fundamental semi-cycles with respect to T, constitute a submatrix of C, called the fundamental semi-cycle matrix C_f with respect to T.

If D has n vertices and e arcs then there are $m = e - n + 1$ fundamental semi-cycles with respect to any specified spanning tree. In the same way as with graphs, it is possible to generate all the semi-cycles of D from linear combinations of the fundamental semi-cycles. This is usually carried out by row manipulation of C_f, where we use ordinary (rather than modulo 2) arithmetic.

A natural orientation for the fundamental semi-cycles is provided by the direction of the chords which create them, with respect to a given spanning tree. We now illustrate this for the graph in Figure 1(a), with T defined as before. Then

$$C_f = \begin{matrix} Z_1 \\ Z_2 \\ Z_3 \end{matrix}\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The cycle matrix C, for this digraph can be generated by taking linear combinations of C_f. Although it is termed the cycle matrix, some of its rows represent semi-cycles which are not cycles, as indicated by (– 1) entries.

$$C = \begin{matrix} Z_1 \\ Z_2 \\ Z_3 \\ Z_4 \\ Z_5 \\ Z_6 \end{matrix}\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & -1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 \end{bmatrix} \begin{matrix} \\ \\ (= Z_1 + Z_3) \\ (= Z_1 + Z_2) \\ (= Z_1 + Z_2 + Z_3). \\ \\ \end{matrix}$$

Remarks.

(1) An arc in D that does not belong to any semi-cycle is represented by a column of zeros.

(2) The number of nonzero entries in a row r, equals the number of arcs in the semi-cycle that r represents.

It is also possible to prove a result which is analogous to Theorem 15 of last chapter.

Theorem 5. *Let d be a digraph with incidence matrix B, and semi-cycle matrix C, in which the columns are arranged according to the same order of arcs. Then*

$$BC^T = 0, \quad and \quad CB^T = 0,$$

where the matrix arithmetic is carried out in the field of real numbers.

Proof. Let us consider the pth row of B and the qth row of C. The q-th semi-cycle, say c_q, either : (a) does not , or (b) does possess an arc incident with vertex, say v_p, represented by the pth row of B. If (a), the product of the two rows is zero. If (b), there are exactly two arcs say a_i and a_j of the qth semi-cycle incident with v_p. There are following four possibilities:

(1) a_i and a_j are both incident towards v_p,

(2) a_i and a_j are both incident away from v_p,

(3) the directions of both a_i and a_j are compatible with the orientation of c_q and

(4) the directions of both a_i and a_j are incompatible with the orientation of c_q.

It can be easily checked that, in all four cases, the product of the pth row of B and qth row of C is zero.

Rank. The rank of a digraph is defined to be the rank of its incidence matrix.

Following are the further remarks about the relationship between B and C for a digraph with n vertices and e arcs, where the rank of a matrix M, is denoted by r(M) :

(1) $r(B) + r(C) = e$,

(2) $r(B) = n - 1$ for any weak digraph, and

(3) $r(C) = e - n + 1$. (From (1) and (2) on subtraction),

(4) Let us now return to C_f for the digraph in Figure 1(a) and permute its columns so as to create a matrix of the form : $[I_\mu : C_t]$.

$$C_f = \begin{array}{c} \begin{matrix} a_1 & a_4 & a_2 & & a_5 & a_6 & a_7 & a_3 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & : & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & : & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & : & 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

As in the case of the cycle matrix, we can define the cut-set matrix $K = (k_{ij})$, fort any weak digraph D = (V, A), in which the rows correspond to the cut-sets of D and the columns to the arcs of D. Each cut-set must be given an arbitrary orientation. Let k_i be the ith cut-set of D. Suppose that k_i partitions V into nonempty vertex sets V_i' and V_i''. The orientation can be defined to be either : from V_i' to V_i'' or from V_i'' to V_i'. Suppose that the orientation is chosen to be from V_i' to V_i''. Then the orientation of an arc a_j of cut-set is the same as that of k_i if a_j is of the form $v_a v_b$ where $v_a \in V_i'$ and $v_b \in V_i''$ and the opposite, otherwise. Then $k_{ij} = 1$ if arc a_j of cut-set k_i has the same orientation as k_i, $k_{ij} = -1$, if arc a_j of cut-set k_i has the opposite orientation to k_i, and $k_{ij} = 0$ otherwise.

Following are some remarks about K, the cut-set matrix of a weak digraph D, with n vertices and e arcs :

(1) A permutation of the rows or columns corresponds to a relabelling of the cut-sets and arcs of D respectively,

(2) $r(K) \geq r(B)$,

(3) $r(K) \geq n - 1$, By the remark (2).

(4) If the arcs of D are arranged in the same column order in C and K, then

$$CK^T = 0 \quad \text{and} \quad KC^T = 0,$$

where the matrix arithmetic is carried out on the real numbers.

(5) $r(C) + r(K) \leq e$,

(6) Because D is a weak digraph, $r(C) = e - n + 1$ and $r(K) \leq n - 1$.

(7) $r(K) = n - 1$ (By the remarks (3) and (4)).

Similar to the cycle matrix, we find *it convenient to define a fundamental cut-set* matrix (which has linearly independent rows) of a digraph D.

Evidently, the removal of an arc, say $a = v_p v_q$, (also called a branch) of a spanning directed tree of *D, partitions the vertices of a digraph D into two disjoint sets,* say V_1 and V_2.

The cut-set created by the removal of a is called either :

(1) directed away from V_1 and towards V_2 if $v_p \in V_1$ and $v_q \in V_2$ or

(2) directed away from V_1 and towards V_2 if $v_p \in V_2$ and $v_q \in V_1$

This type of a cut-set is termed a fundamental cut-set. In fact, not all the chords in k_i necessarily have *the same orientation as* $v_q v_p$. *If* $v_q v_p$, *is directed away from a* vertex in V_1, there may exist a chord in k_i which is directed towards a vertex in V_1. The orientation of a cut-set on the basis of the direction of the branch giving rise to it constitutes a natural way of orienting cut-sets. If all the chords of k_i are oriented as $v_q v_p$ is, then k_i is said to be directed.

Referring to the digraph in figure 1(a), with T defined earlier, the fundamental cut-sets with respect to T are :

Branch	cut-set
a_3	$\{a_3, a_2\}$
a_5	$\{a_5, a_4\}$
a_6	$\{a_6, a_1, a_4\}$
a_7	$\{a_7, a_1, a_2\}$.

A fundamental cut-set is created from K, the cut-set matrix of a weak digraph with given directed spanning tree T, by deleting from K, all rows which do not correspond to fundamental cut-sets with respect to T. Therefore K_f is an $(n - 1) \times e$ submatrix of K such that *each row represents a unique fundamental cut-set with respect* to T.

As in the case of the fundamental semi-cycle matrix, the rows of any fundamental *cut-set matrix* K_f, *can be permuted to create a matrix of the form:* $K_f = [K_c : I_{n-1}]$. Here K_c is an $(n - 1) \times (e - n + 1)$ matrix whose columns correspond to the chords of T and I_{n-1} is the identity matrix of order $n - 1$ whose columns correspond to the branches of T. Let D be a weak digraph with given directed spanning tree T. We now deduce relationships among C_f, K_f and B_r. (Here B_r is the reduced incidence matrix in which an arbitrary row has been removed in order to make its rows linearly independent). We know that

$$C_f = [I_m : C_t], \qquad \text{...(1)}$$

and $$K_f = [K_c : I_{n-1}], \qquad \text{...(2)}$$

where t corresponds to the branches of T and c to the chords of T. We assume that the arcs are assembled in the same order in (1), (2) and in B_r. We begin by partitioning B_r into :

$$B_r = [B_c : B_t],$$

where B_c is an $(n-1) \times (e-n+1)$ submatrix whose columns correspond to the chords of T and B_t is an $(n-1) \times (n-1)$ submatrix whose columns correspond to the branches of T. Because

$$BC^T = 0,$$

We can deduce (by analogous argument) that

$$K_c + C_t^T = 0.$$

Hence $\quad K_c = -C_t^T \quad$ ...(3)

and therefore $\quad K_c = B_t^{-1}, B_c. \quad$ ...(4)

To illustrate these relationship we consider the digraph in Figure 1 (a), with the spanning tree T,

$$B_r = [B_c : B_t] = \begin{array}{c} \begin{array}{ccccccc} a_1 & a_4 & a_2 & & a_5 & a_6 & a_7 & a_3 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & : & -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & : & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & : & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & : & 0 & -1 & 1 & -1 \end{bmatrix} \end{array}$$

$$C_f = [I_3 : C_t] = \begin{array}{c} \begin{array}{ccccccc} a_1 & a_4 & a_2 & & a_5 & a_6 & a_7 & a_3 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & : & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & : & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & : & 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

$$K_f = [K_c : I_4] = \begin{array}{c} \begin{array}{ccccccc} a_1 & a_4 & a_2 & & a_5 & a_6 & a_7 & a_3 \end{array} \\ \begin{bmatrix} 0 & -1 & 0 & : & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & : & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & : & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & : & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Here the last row of B, corresponding to vertex v_5, has been removed to create B_f We now form linear combinations of the rows of K_f to create all 10 rows of K, representing all of the cut-sets of the digraph in figure 1(a).

$$K = \begin{array}{c} \begin{array}{ccccccc} a_1 & a_4 & a_2 & a_5 & a_6 & a_7 & a_3 \end{array} \\ \begin{bmatrix} 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & -1 \\ 0 & -1 & 1 & 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \end{array} \begin{array}{c} \\ k_1 \\ k_2 \\ k_3 \\ k_4 \\ k_1 - k_2 \\ k_2 - k_3 + k_4 \\ k_2 - k_1 - k_3 \\ k_1 - k_2 + k_3 - k_4 \\ k_2 - k_3 \\ k_3 - k_4 \end{array}$$

Thus we have similar observations :

(1) Given B_r, we can construct C_f.

(2) Given B_r, we can construct K_f.

(3) Given C_r, we can construct K_f.

(4) Given K_f, we can construct C_f.

NOTES

We define a semi-path matrix for a digraph similarly as that for the path matrix of a graph.

Semi-Path Matrix. The semi-path matrix $P(u, v) = (p_{ij})$, of a digraph $D = (V, A)$, where $u, v \in V$, is the matrix with each row representing a distinct semi-path from u to v and the columns representing the arcs of D, in which $p_{ij} = 1$ if ith semi-path contains the jth arc, $p_{ij} = -1$ if ith semi-path contains the converse of the jth arc, otherwise $p_{ij} = 0$.

The matrix $P(v_3, v_5)$, for the digraph of figure 1(*a*) is

$$P(v_3, v_5) = \begin{array}{c} \begin{matrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 \end{matrix} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & -1 & 0 & -1 & 0 \\ 1 & 1 & 0 & -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Remarks.

(1) If $P(u, v)$ contains a column of all zeros then the vertex that it represents does not belong to any of the semi-paths between u and v.

(2) If $P(u, v)$ contains a column of all unit entries then the vertex that it represents belongs to every semi-path between u and v.

(3) The number of non-zero entries in any row of $P(u, v)$ is equal to the number of arcs in the semi-path represented by the row.

5. THE PRINCIPLE OF DIRECTIONAL DUALITY

We now state a specialization of the principle of directional duality for digraphs:

The principle of directional duality for digraphs.

In order to illustrate this principle we first give the following theorem.

Theorem 6. *An acyclic digraph has at least one vertex with no arcs incident towards it.*

Proof. Consider the first vertex of any maximal path in the digraph. This vertex cannot have any arcs adjacent it, otherwise the path would not be maximal to the digraphs would contain a cycle. Hence the theorem is proved.

Theorem 7. *An acyclic graph has at least one vertex with no arcs incident away from it.*

Proof. The proof follows directly by applying the principle of directional duality to the proof of **Theorem 6.**

6. TOURNAMENTS

We know that a tournament usually means a kind of competition in sports. We use the word in a sense which corresponds to what are generally called round-robin tournaments, where each pair of contestants (individual or team) play with each other exactly once. We assume that exactly one of the pair wins. Such sporting tournaments (and the ranking of objects in the social sciences by comparing them two at a time) can be modelled by the theory of digraphs. *Each contestant is represented by a unique* vertex of a digraph D. If the competitor represented by vertex v_i, beats the competitor represented by vertex v_j, then D will possess arc $v_i v_j$. This provides the following definition.

Definition. A tournament is an oriented complete graph.

Figure 2 shows all the tournaments on 1, 1, 3, or 4 vertices.

Theorem 8. *For any vertex v, in a tournament with n vertices,*

$$id(v) + od(v) = n - 1.$$

Proof. There is exactly one arc between u and each of the $n - 1$ other vertices.

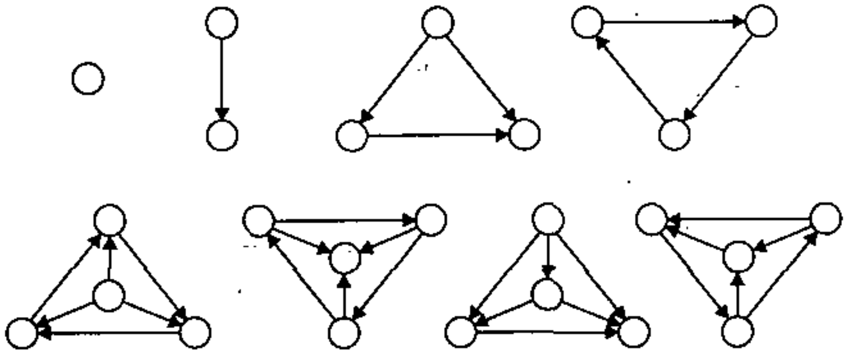


Fig. 2. All tournaments with at most four vertices.

Corollary 2. Any tournament with n vertices has $\frac{1}{2}\, n(n-1)$ arcs.

If u and v are vertices in any tournament T then either (u, v) or (v, u) is an arc of T. Hence either $< u, v >$ or $< v, u >$ is a walk in T and thus either v is reachable from u or u is reachable from v. Therefore T is unilateral.

Since every tournament is unilateral, the results of 1 may be applied to tournaments. Also the extra properties of tournaments enable us to strengthen these results considerably. The results of 1 show that the strong components of a tournament D = (V, A), may be put in order : $C_1, C_2, \ldots, C_k$ in such a way that if $u \in C_i$ and $v \in C_j$ and if $i > j$, then $(u, v) \in$ A. Hence if we know the components and their order, we know the results of each comparison between objects in different components. The special nature of tournaments leads to a theorem on the possible size of a component.

Theorem 9. *No component in a tournament consists of exactly two vertices.*

Proof. In any digraph D = (V, A), with $u, v \in$ V, $\{u, v\}$ is a component if and only if there is a walk from u to v and a walk from v to u, none of them passing through any other vertex. This both (u, v) and (v, u) are arcs. But these two arcs cannot both appear in a tournament.

Thus the component structure of a tournament, can be described by the cardinalities of its components and the sequence in which they occur. Each tournament has a component size sequence, consisting of the cardinalities of the components in the order in which they occur in the spanning path of the condensation of the tournament. The sum of these numbers is equal to the number of vertices in the tournament and the number 2 cannot occur. These are the only restrictions. The tournaments in Figure 2 have component cardinality sequences as given below :

1 ; 1, 1 ; 1, 1, 1 ; 3 ; 1, 3 ; 3, 1 ; 1, 1, 1, 1 and 4, respectively.

The nature of tournaments also implies that subdigraphs of the tournament share the property, as stated in the following theorem.

Theorem 10. *Consider a subdigraph S of any tournament and the underlying graph G, (say) obtained from S by removing the orientations of the arcs of S. If G is a complete then S is a tournament.*

Proof. From the definition of a tournament the result is obvious.

Definition. A subdigraph of a tournament is said to be a *subtournament.*

It is immediate consequence of theorem 10 that every component of a tournament is a subtournament. It is also strong. We now discuss strong tournaments.

7. STRONG TOURNAMENTS

The first, fourth and last tournaments in figure 2 are strong. Such tournaments exist for any number of vertices except two.

Theorem 11. *If T is a strong tournament with n vertices, and $3 \le k \le n$, then there is a cycle in T which passes through exactly k vertices.*

Proof. See any reference text.

To demonstrate this theorem we consider the strong tournament shown in figure 3. Theorem says that we can choose any vertex of the tournament and there will be a cycle of length k, for each k, $3 \le k \le n$, through that vertex. In Figure 3 we choose vertex 1 and find cycles of lengths : 3, 4, 5, 6 and 7 through vertex 1.

To find the initial cycle of length 3 through vertex 1, let

$$W = \{6, 7\} \quad \text{and} \quad X = \{2, 3, 4, 5\}.$$

We note that (2, 7) is an arc beginning in X and ending in W, so a cycle of length 3 is $\langle 1, 2, 7, 1\rangle$, with $k = 3$, $u_1 = 1$, $u_2 = 2$ and $u_3 = 7$. We observe that (1, 3) and (3, 2) are both arcs, giving the cycle $\langle 1, 3, 2, 7, 1\rangle$.

At the next stage there is no $v \in \{4, 5, 6\}$ such that $(1, v)$ and $(v, 2)$ are both arcs, but (3, 6), (6, 2) are arcs, so $\langle 1, 3, 6, 2, 7, 1\rangle$ is the next cycle. Then (3, 4) and (4, 6) are arcs and the next cycle is $\langle 1, 3, 4, 6, 2, 7, 1\rangle$.

Finally (3, 5) and (5, 4) are arcs so that $\langle 1, 3, 5, 4, 6, 2, 7, 1\rangle$ is the spanning cycle.

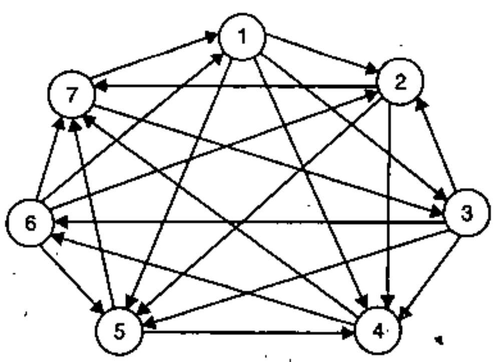


Fig. 3. A strong tournament.

Theorem 11 allows us to strengthen Theorem 2 that a unilateral digraph has a spanning walk, for the special case of tournaments.

Theorem 12. *Every tournament has a spanning path.*

Proof. Each component of a tournament is a strong tournament and has a spanning cycle. If component C_i, has a single vertex, this cycle is $\langle x \rangle$, where $\{x\} = C_i$, otherwise it has the form $\langle x_i,..., y_i, x_i \rangle$. Further, (y_i, x_{i+1}) is an arc for each pair of consecutive components C_i, C_{i+1}. If we use y_i as an alternative name for x_i in components with only one vertex,

$$\langle x_1,..., y_1, x_2,..., y_2,..., x_k,..., y_k \rangle$$

is a spanning path of the tournament. Hence the result.

Tournaments can be applied in the scheduling of round-robin sporting competitions in those sports in which each competing team has its own home ground. For justice we may expect each team to play half its matches at home and half away. We now discuss how to attempt to arrange this. Since the number of matches each competitor plays is one less than the number, say n, of competitors, this will only be possible if the number of competitors is odd. One rule is to assign each competitor a unique integer between 1 and n. Then if i and j are two competitors, their match is played on i's home ground if the difference $| i - j |$ is odd and at j's home ground if $| i - j |$ is even. It apparent that this gives each team half its matches at home and half away. When $p = 7$, the matches are (home competitor, away competitor) : (1, 2), (3, 1), (1, 4), (5, 1), (1, 6), (7, 1), (2, 3), (4, 2), (2, 5), (6, 2), (2, 7), (2, 4), (5, 3), (3, 6), (7, 3), (4, 5), (6, 4), (4, 7), (5, 6), (7, 5), (6, 7).

8. ACYCLIC TOURNAMENTS

On the extreme end of strong tournaments, a tournament may be acyclic, as are the first, second, third and fifth tournaments of figure 2. Acyclic tournaments are important, since they correspond to consistent rankings and are easily detected.

Definition. A logical numbering of a digraph $D = (V, A)$, with n vertices is a one-to-one correspondence $f : V \rightarrow \{1, 2,..., n\}$, such that if $uv \in A$ then $f(u) < f(v)$.

Theorem 13. *An acyclic tournament $T = (V, A)$, with n vertices has only one logical numbering and one spanning path. If f is the logical numbering of T and if, $f(v) = i$ for some $v \in V$. Then $od(v) = n - i$ and $id(v) = i - 1$.*

Proof. Every tournament has a spanning path by Theorem 12. Let $\langle v_1, v_2,..., v_p \rangle$ be a spanning path of an acyclic tournament. Then if v_i have number i, the numbering is consistent with the arcs of the spanning path, and since the digraph is acyclic, the arc between v_i and v_j must be form the one with the smaller number to the one with the larger. This fixes the indegree and outdegree also at the required values. But the indegree and outdegree are not dependent on the spanning path or the logical numbering. Hence both spanning path and logical numbering are unique.

This relationship between the outdegrees and between indegrees (either directly implies the other) can be used to prove a tournament to be acyclic.

Theorem 14. *Let T be a tournament with p vertices, and suppose that the outdegrees of the vertices are $p - 1, p - 2,..., 2, 1, 0$, then T is acyclic.*

Proof. Let the vertices be numbered in such a way that v_i has outdegree $p - i$ and therefore, indegree $i - 1$. Suppose v_1 has indegree 0, hence far all k, the arcs incident with v_1 are $v_1 v_k$. In particular, v_1v_2 is an arc. Since v_2 has indegree 1, this is the only arc into v_2. Hence the arc between v_2 and v_3 is v_2v_3. The two arcs into v_3 are v_1v_3 and v_2v_3 so the arc between v_3 and v_4 is v_3v_4 and so on. This may be converted into a formal inductive proof using the subtournament defined by $V \setminus \{v_1, v_2,..., v_m\}$, $m = 1, 2,..., p$.

Note. Outdegree Analysis. We have seen that an acyclic tournament can be recognized from the list of outdegree (or equivalently from the indegrees). Now we will show that the whole component structure of a general tournament can be obtained from the list of outdegrees. We shall reduce the process to a mechanical calculation. Suppose that the components are C_1, C_2,..., C_k in non-increasing order of cardinality and that the cardinalities of these components are $c_1, c_2,..., c_k$ respectively.

Theorem 15. *If T is a tournament and u is a vertex in component C_i of T, then*

$$c_{i+1} + c_{i+2} + ... + c_k \le od(u) \le c_{i+1} + c_{i+2} + ... + c_{k-1}$$

and
$$c_1 + c_2 + ... + c_{i-1} \le id(u) \le c_1 + c_2 + ... + c_{i-1}.$$

Proof. Every member of $C_i \cup C_{i+1} \cup ... \cup C_k$ is a successor of u. On the other hand, at least one member of C_i, namely u itself, is not a successor of u and if $C_i \neq \{u\}$ there will be another also. The proof for indegree is similar.

Theorem 16. *If u and v are vertices in a tournament and $u \in C_i$ but $v \in C_j$ with $i < j$ then*

$$od(u) > od(v) \text{ and } id(u) < id(v).$$

Proof. Left to the reader.

Theorem 17. *If u and v are vertices in a tournament and $od(u) = od(v)$, then u and v belong to the same component.*

Proof. Left to the reader.

Suppose we partition the vertices of a tournament into two nonempty disjoint sets M and N, *i.e.*, $M \cup N = V$ and $M \cap N = \phi$.

Let M and N have m and n vertices respectively. Consider the sum of the outdegrees of the vertices in M. This is the number of arcs which begin and end in M. There are $\frac{1}{2} m (m - 1)$ such arcs. Then there are the arcs which begin in M and end in N. There may no such arcs or as many as mn, the latter occurring only if every member of M is a precursor of every member of N. Hence we have $\frac{1}{2} m (m - 1) \le \sum_{u \in M} od(u) \le \frac{1}{2} m (m + 2n - 1)$.

Theorem 18. *Let T be a tournament on p vertices and let v be its vertex set. If $V = M \cup N$, where $M \cap N = \phi$, then $M = C_1 \cup C_2 \cup ... \cup C_i$ and $N = C_{i+1} \cup C_{i+2} \cup ... \cup C_k$, for some i, if and only if the sum of the outdegrees of vertices in M is $\frac{1}{2} m (2p - m - 1)$, where $C_1, C_2,..., C_k$ are the components in order.*

Proof. See any reference text.

NOTES

9. TOURNAMENT WINNING ANALYSIS

1. Construct a table with five columns, headed : vertex, *od* , $p - m$, def and cum.
2. In column vertex list the vertices in non-increasing order of outdegree.
3. In column *od* list the corresponding outdegrees.
4. Let p be the number of vertices in the tournament. In column $p - m$ write $p - 1$ in the first row, decreasing by 1 at each row to 0 in the bottom row.
5. In each row subtract the entry under *od* from the entry under $p - m$ and enter the result in the column def.
6. Copy the first entry in column def into column cum. To obtain the remaining entries in the cum column add the entry to the left to the entry above.
7. Rule across the table under every 0 in the cum column. As a check, the final entry must be 0 and none can be negative.
8. The lines partition the vertices into the components.

For an illustration, we analyse the tournament of figure 4 in Table 1 given below the figure.

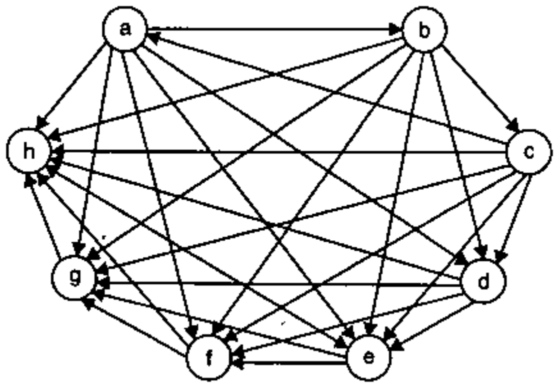


Fig. 4. Tournament wins analysis.

Vertex	*od*	$p - m$	*def*	*cum*
a	6	7	1	1
b	6	6	0	1
c	6	5	– 1	0
d	4	4	0	0
e	2	3	1	1
f	2	2	0	1
g	1	1	0	1
h	1	0	– 1	0

Table 1. A tournament winning analysis

The components are:

$\{a, b, c\}, \{d\}, \{e, f, g, h\}.$

NOTES

From the list of components we can reconstruct most of the individual arcs. To be precise, we know the direction of all arcs whose beginning and end are in different components. If we are provided with a little more information we can reconstruct the remaining arcs by similar chain of reasoning.

Let us illustrate for this case as follows. Suppose that we are given the components, outdegrees and the facts that ab and he are arcs. Within the component $\{a, b, c\}$, each vertex is the beginning of one arc and the end of one arc. If ab is an arc, therefore, so are bc and ca. Within $\{e, f, g, h\}$ the argument is a little more complex. As he is an arc and h is of outdegree 1, the other arcs at h are fh and gh. Since e is the end of one arc within the subtournament defined by $\{e, f, g, h\}$, ef and eg are arcs. This leaves the arc between f and g, which must be fg to balance the outdegrees.

This type of logic can be used in general but each component must be treated separately and in general, the number of arcs needed before a complete solution can be obtained, increases with the number of vertices in the component.

EXERCISE 10 (A)

1. Prove that every edge in a digraph belongs either to a directed circuit or a directed cut-set.
2. Define and analyse the directed Hamiltonian circuit and semi-Hamiltonian circuit in a digraph.
3. Prove that an n-vertex digraph is strongly connected if and only if the matrix M, defined by

   $M = X + X^2 + X^3 + ... + X^n$ has no zero entry, X is the adjacency matrix.

10. COVERINGS AND COLOURINGS

The covering is a concept of interest, both from the theoretical and practical points of view. A vertex (edge) of a graph is said to cover the edges (vertices) with which it is incident.

Now we discuss is that for a given graph G, what is the minimum number of edges (vertices) needed to cover all the vertices (edges) of G ? Conversely for a given graph G, what is the maximum number of edges (vertices) which are mutually non adjacent ? Such sets of edges (vertices) are called independent.

Definition. A set of edges (vertices) in a graph G, is said to be dominant if every edge (vertex) of G either belongs to the set or is adjacent to a member of it.

We know the four colour Theorem, which says that it is possible to colour the regions of any map drawn in a plane so that every pair of adjacent regions have different colours. One can view the colouring of such maps in graph theoretic terms by creating a planar graph for any planar map. This graph will have a vertex for every region of the map and the pair of its vertices are directly joined by an edge whenever the regions they represent in the map have a common boundary. The colouring of a region of the map is equivalent to associating a colour with the corresponding vertex of the graph. A graph constructed in this way will have the same colouring structure as the map it represents.

A concept closely related to that of edge independence, is *matching*. Matching is of great importance with graph theoretic applications in view. This is because in many

NOTES

practical problems, one is often endeavouring to assign one set of objects to another set, in a one-to-one fashion. Such examples are of workers to machines or children to schools. Such an assignment is often called a matching and can be modelled in graph theoretic terms.

The topics covering, colouring and matching are of practical importance in Operations Research and computer shink specially.

11. COVERING, INDEPENDENCE AND DOMINATION

Definitions. An edge $\{u, v\}$ in a graph is termed to cover its incident vertices u and v.

A vertex in a graph is said to cover the edges with which it is incident.

If $G = (V, E)$ is a graph and $E' \subseteq E$, then E' is said to be an edge cover of G and to cover G if for each vertex $v \in V$, there in the sense that it covers all the edges of the graph but no proper subset of it does so. However U does not correspond to the vertex covering number because $U_1 = \{v_5\}$ is the unique minimum vertex cover and thus $\alpha_0 = I_1$. Also $\alpha_1 = 4$ because the set of edges $\{e_1, e_2, e_3, e_4\}$ is the minimum edge cover.

In general, the edges (vertices) of any spanning tree, Hamiltonian path or unicursal path of any connected graph G, constitute an edge (vertex) cover of G.

Let G be a graph with vertex set V. We can make some remarks about edge coverings in G.

C(1) An edge covering of G can always be found so long as G does not contain an isolated vertex.

C(2) If $| V | = n$, where $(n > 1)$, then any edge covering of G will contain at least $n/2$ edges. If $G = K_n$, then $\alpha_1 = [(n + 1)/2]$.

C(3) Every edge covering includes every pendant edge.

C(4) It is possible to remove a subset of edges (possibly empty) from any edge covering of G in order to create a minimal (but not necessarily minimum) edge covering of G.

C(5) Minimal edge coverings are acyclic.

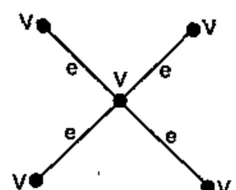
The similar remarks are about vertex coverings.

C(6) A vertex covering exists for any graph G.

C(7) If $G = K_n$, then $\alpha_0 = n - 1$.

A vertex covering may have only a single element. This is true for any star. A graph which possesses a unique vertex called the centre with which every edge is incident. Figure 5(a) depicts a star with centre v_5.

C(8) It is possible to remove a subset of vertices (possibly empty) from any vertex covering in order to create a minimal (but not necessarily minimum) vertex covering.



(a)

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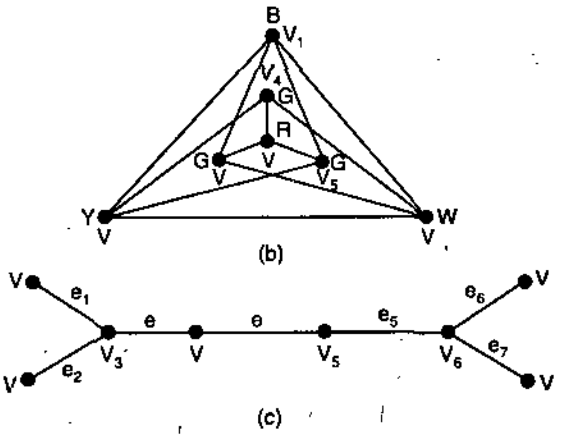

Fig. 5. (*a*) Cover, Independence and Dominance (*b*) Colouring (*c*) Matching.

Theorem 19. *An edge covering in a graph does not contain a path of at least three edges if and only if it is minimal.*

Proof. Let us consider an edge covering containing a path of at least three edges. The second edge of the path can be removed, leaving an edge covering. Hence the original covering is not minimal.

Suppose now that there exists an edge covering which does not contain a path of at least three edges. In this case, each component of the graph is a star. Hence it is impossible to remove an edge from an edge covering of a star, the edge covering is minimal.

We now consider the concept of independence.

Definitions. If G = (V, E) is a graph and E′ ⊆ E, then E′ is said to be an edge independent if no two edges of E′ are adjacent.

If G = (V, E) is a graph and U ⊆ V, then U is said to be an vertex independent if no two edges of U are adjacent.

For a given graph G, the cardinality of the set of edges of G which is the largest vertex-independent set of G is said to be the edge dependence number of G and is denoted by $\beta_1(G)$ or β_1.

For a given graph G, the cardinality of the set of edges of G which is the largest vertex-independent set of G is called the vertex dependence number of G and is denoted by $\beta_0(G)$ or β_0.

An independent set is termed maximal if none of its proper supersets is independent. An independent set in a graph G is called maximum if there is no independent set in G with a greater number of elements.

We demonstrate these ideas with the graph in figure 12.5(*a*). Any one of the edges of the graph constitutes an edge-independent set. Such a singleton set is at once maximal and maximum.

The set $\{v_5\}$ is a maximal vertex-independent set. However it is not maximum because of the existence of the $\{v_1, v_2, v_3, v_4\}$. Thus in this graph $\beta_0 = 4$ and $\beta_1 = 1$.

We make some remarks about edge-independent sets in any graph G.

I(1) An edge-independent set can always be found in G contains at least one edge. Any single edge of G constitutes such a set.

I(2) If G = K_n, the complete graph on n vertices, then $\beta_1 = [n/2]$, the integer part of $n/2$.

I(3) Every pendant edge in G belongs to at least one maximal edge-independent set.

I(4) It is possible to add a subset of edges (possibly empty) to any edge-independent set in G in order to create a maximal (but not necessarily maximum) edge-independent set in G.

We can make similar observations about vertex-independent sets in G.

I(5) A vertex-independent set exists for G. (Any single vertex of G constitutes such a set).

I(6) If $G = K_n$ then $\beta_0 = 1$.

I(7) Every pendant vertex in G belongs to at least one maximal vertex-independent set.

I(8) It is possible to add a subset of vertices (possibly empty) to any vertex-independent set in G in order to create a maximal (but not necessarily maximum) vertex-independent set in G.

We now give theorems linking the concepts of covering and independence.

Theorem 20. *For any connected graph G, with n vertices and e edges in which $n + e > 1$,*

$$\alpha_0(G) + \beta_0(G) = \alpha_1(G) + \beta_1(G) = n.$$

Proof. Let G be a connected graph with at least one edge.

(1) We first show that $\alpha_1(G) + \beta_1(G) = n$.

Let E_1 be an edge-independent set of β_1 edges in G. An edge cover of G, E′ say, can be constructed as E_1 together with an additional edge for each vertex of G not covered by any edge in E_1. Since

$$|E_1| + |E'| \leq n \quad \text{and} \quad |E'| \geq \alpha_1,$$

we obtain $\alpha_1 + \beta_1 \leq n$.

To establish the reverse inequality, let $\overline{E}$ be a minimum edge cover of G. By definitions, $\overline{E}$ does not possess an edge whose two incident vertices are incident with edges in $\overline{E}$. Thus $\overline{E}$ comprises a set of subsets of edges of G where each subset constitutes a star. A set of edges, say $\overline{E}$ comprising exactly one edge from each star, is edge-independent.

But $|E'| + |\overline{E}| = n$, and $|\overline{E}| \leq \beta_1$

Hence $\alpha_1 + \beta_1 \geq n$.

(2) Now we show that $\alpha_0 + \beta_0 = n$.

Let V_1 be any maximum independent set of β_0 vertices of G. Because no edge in G joins two vertices of V_1, the $n - \beta_0$ remaining vertices of G constitute a vertex cover of G.

Therefore, $\alpha_0 \leq n - \beta_0$

However if V′ is a minimum vertex cover of G, then no edge can join any two of the $n - \alpha_0$ vertices of G. Therefore V/V′ is independent. Thus

$$\beta_0 \geq n - \alpha_1.$$

Theorem 21. *If G is a bipartite graph then $a_0(G)$ (G) $= b_1(G)$.*

Proof. The proof is omitted.

We now take up the notion of dominance.

Definition. An edge (vertex) in a graph G is called to dominate those other edges (vertices) in G with which it is adjacent.

If G = (V, E) is a graph and $E_1 \subseteq E$, (U < V) the E_1(U) is said to be an edge (vertex) dominating set for G if every edge of E (vertex of V) either belongs to E_1(U) or is dominated by an edge of E_1 (vertex U).

For a given graph G = (V, E), the cardinality of the edge (vertex)-dominating set with the least number of elements is called the edge (vertex) dominating number of G and is denoted by σ_1(G) or σ_1 (by σ_0(G) or σ_0).

A dominating set is termed minimal if none of its proper subsets are dominating.

A dominating set of graph G is called minimum if there is no dominating set of G with a smaller number of elements.

Let us look into the graph in figure 5(*a*) in order to illustrate the notion of dominance. Any one of the edges of the graph constitutes an edge-dominating set, which is at once minimal and minimum. The set $\{v_1, v_2, v_3, v_4\}$ is a maximal vertex-dominating set. However it is not minimum as $\{v_5\}$ is dominating. Thus $\sigma_1 = \sigma_0 = 1$.

In general, the edges (vertices) of any spanning tree, Hamiltonian path, or unicursal path of any connected graph G, constitute an edge (vertex) – dominating set of G.

Let G = (V, E) be a graph. We now some observations about edge-dominating sets for G.

D(1) An edge-dominating set can always be found in G contains at least one edge.

D(2) If G = K_n, then $\sigma_1 = [n/2]$.

D(3) If G is connected with at least one internal edge, then every minimum edge- dominating set for G does not contain any pendant edges.

D(4) It is possible to remove a subset of edges (possibly empty) from any edge-dominating set for G in order to create a minimal (but not necessarily minimum) edge-dominating set of G.

D(5) A minimal edge-dominating set for G is not necessarily an edge-independent set of G.

D(6) Every maximal edge-independent set of G is an edge-dominating set for G. (See Theorem 22).

D(7) An edge-independent set of G is an edge-dominant set for G only if it is maximal.

Similar observations about vertex-dominating sets for G can be realized.

D(8) A vertex-dominating set for G exists.

D(9) If G is complete, then $\sigma_0 = 1$.

D(10) If G is connected with at least one internal vertex, then every minimum vertex-dominating set for G in order to create a minimal (but not necessarily minimum) vertex-dominating set for G.

D(11) It is possible to remove a subset of vertices (possibly empty) from any vertex-dominating set for G in order to create a minimal (but not necessarily minimum) vertex-dominating set for G.

D(12) A minimal vertex-dominating set for G is not necessarily a vertex-independent set of G.

D(13) Every maximal vertex-independent set of G is a vertex-dominating set for G. (This is established in most proofs of Theorem 23).

D(14) A vertex-independent set of G is an vertex-dominant set for G only if it is maximal.

Two theorems linking the notions of independence and dominance.

Theorem 22. *For every graph G = (V, E) with at least one edge,*

$$\sigma_1(G) \leq \beta_1(G).$$

Proof. Let E_1 be a maximum edge-independent set of G. By assumption and observation I(1), E_1 can always be found. By definition, $|E_1| = \beta_1(G)$.

Every edge $e \in E/E_1$, is adjacent to some edge of E_1 otherwise $E_1 \cup \{e\}$ is an independent set of edges with cardinality $\beta_1 + 1$. This is impossible as E_1 is assumes maximum. Hence E_1 is an edge-dominating set for G. Hence the result.

Theorem 23. *For every graph G = (V, E),*

$$\sigma_0(G) \leq \beta_0(G).$$

Proof. Similar to that of Theorem 22.

We now come to graph vertex colouring.

12. COLOURING

Definition. A colouring of a graph G, is an assignment of colours to its vertices so that no two adjacent vertices in G have the same colour.

We notice from this definition that a (vertex) colouring of a graph is very much akin to concepts introduced in the previous section. Since a colouring requires a partitioning of the set of vertices of a graph into non-adjacent subsets, it has similarities with the identification of a vertex-independent set. Figure 5(*b*) shows a colouring of a graph. Analysis of this colouring raises the obvious question. Can the vertices be re-assigned colours so that less than five colours can be used in colouring of this graph? Since the graph is planar, by the Four Colour Theorem, we know that it is possible to find a colouring with four colours. Now question arises, whether four is the minimum number of colours that can be used ? In fact what is the minimum number of colours that must be used in any colouring of this graph ? It turns out that four is indeed the minimum number of colours required, and is termed the chromatic number of the graph. This prompt us to give some more definitions.

Definitions. The set of vertices assigned any one colour in a colouring of a graph is called a *colour class.*

By definition, a colour class in a vertex-coloured graph is a vertex-independent set.

A colouring of a graph which assigns *c* colours to the vertices of a graph is termed a *c-colouring*.

Note. A c-colouring partitions the vertices of a graph into *c* colour classes.

The *chromatic number* of a graph G is the minimum number χ, for which G has a path χ-colouring and is denoted by χ (G) or χ.

A graph G is *c-colourable* if $\chi(G) \leq c$ and *c*-chromatic if $\chi(G) = c$.

Thus the colouring of the graph in figure 5(*b*) shows that it is 5-colourable. In fact it is 4-chromatic.

Due to practical importance we consider only connected graphs in studying colouring. Let G = (V, E) be a connected graph. We have some observations about any colouring of G as follows:

VC(1) *G is 1-chromatic if and only if it has no edges.*

VC(2) G is c-chromatic, where $c > 1$, if and only if G has at least one edge.

VC(3) G = K_n, then G is an n-chromatic.

VC(4) If G contains K_n as a subgraph then G is c-chromatic, where $c \geq n$.

VC(5) If G is an n-cycle then G is 3-chromatic if n is odd and 2-chromatic if n is even.

VC(6) If G is bipartite, G is 2–chromatic.

VC(7) If G is a tree with at least one edge then G is 2-chromatic.

VC(8) G is 2-colourable if and only if G has no cycles with an odd number of edges.

VC(9) The chromatic number of G is at most one greater than the degree, say d_{max}, of its vertex with minimum degree.

VC(10) If G does not have a complete subgraph with (d_{max} + 1) vertices then the chromatic number of G is at most d_{max}.

VC(11) A colouring of G partitions V into vertex-independent sets.

From the last observation we have :

Theorem 24. *If G = (V, E) is a graph with n vertices and chromatic number χ, and has been assigned a colouring with χ colours, then $\chi\, \beta_0\, (G) \geq n$.*

Proof. This follows easily from the fact that the largest number of vertices in G with the same colour cannot exceed the vertex-independence number β_0 (G) of G.

We give below the more practical method of actually finding the chromatic number of a given graph G = (V, E). We could do this by finding the minimum number of maximal vertex-independent sets which together from a partition of V. For the graph in figure 5(a) this partition is : $\{v_1, v_2, v_3, v_4\}$ and $\{v_5\}$. The number of sets equals the chromatic number of the graph and thus χ = 2. From the graph in figure 5(b) such a partition is $\{v_1, v_4\}$, $\{v_2, v_5\}$, $\{v_3, v_6\}$ and $\{v_7\}$. Thus in this case, χ = 4. This analysis leads us to what is known as the chromatic partitioning problem.

Given a connected graph G = (V, E), partitioning of V into the minimum possible number of colour classes.

We know that each colour class corresponds to a maximal vertex-independent set. Thus the vertex sets we have just talked for the graphs in figure 5(a) and (b) constitute chromatic partitioning for those graphs. The chromatic partition given for the graph in figure 5(a) is unique. The complete list of chromatic partitionings for the graph in figure 5(b) is :

$\{v_1, v_4\}$	$\{v_2, v_5\}$	$\{v_3, v_6\}$	$\{v_7\}$
$\{v_1, v_7\}$	$\{v_2, v_5\}$	$\{v_3, v_6\}$	$\{v_4\}$
$\{v_1, v_4\}$	$\{v_2, v_7\}$	$\{v_3, v_6\}$	$\{v_5\}$
$\{v_1, v_4\}$	$\{v_2, v_5\}$	$\{v_3, v_7\}$	$\{v_6\}$
$\{v_4, v_5\}$	$\{v_2, v_7\}$	$\{v_3, v_6\}$	$\{v_1\}$
$\{v_4, v_6\}$	$\{v_2, v_5\}$	$\{v_3, v_7\}$	$\{v_1\}$
$\{v_1, v_7\}$	$\{v_4, v_5\}$	$\{v_3, v_6\}$	$\{v_2\}$
$\{v_1, v_4\}$	$\{v_5, v_6\}$	$\{v_3, v_7\}$	$\{v_2\}$
$\{v_1, v_7\}$	$\{v_2, v_5\}$	$\{v_4, v_6\}$	$\{v_3\}$
$\{v_1, v_4\}$	$\{v_2, v_7\}$	$\{v_5, v_6\}$	$\{v_3\}$

Hence there are many 4-colourings of this graph. A graph for which there is only a single χ-colouring is termed uniquely colourable. An example of a uniquely colourable graph is given in figure 5(*a*). The graph colouring from an algorithmic point of view will be discussed in chapter 13. We need defining chromatic polynomial $P_n(\lambda)$ of a graph with n vertices which on using λ, or fewer colours gives the number of ways of properly colouring the graph. If, c_i denotes these ways using exactly i colours, then

$$P_n(\lambda) = \sum_{i=1}^{n} c_i \, {}^{\lambda}C_i .$$

13. MATCHING

Definition. If G = (V, E) is a graph and $E_1 \subseteq E$, then E_1 is said to be a matching if E_1 is edge-independent.

The term matching in the above definition is used because the two vertices incident with an edge e, in E_1 are associated, paired or matched by e.

Definitions. A matching in a graph is called maximal if it is maximal edge-independent. A matching in a graph is said to be maximum if it is maximum edge-independent.

As an example, consider the graph in figure 5(*c*). M = $\{e_3, e_5\}$ is edge-independent. It matches the vertices : v_3 and v_4, and also v_5 and v_6. However M is also maximal, as no further edge can be added to it without eliminating its edge-independence. Thus M is not maximum, a property possessed by the set $\{e_1, e_4, e_6\}$.

The reader may have noticed a number of facts about the graph in Figure 5(*c*). Firstly it is bipartite, with vertex partition : $\{e_1, e_2, e_4, e_6\}$ and $\{e_3, e_5, e_7, e_8\}$. Secondly, the number of edges in any maximum matching is 3, which is equal to the vertex covering number, *i.e.*, $\alpha_0 = \beta_1$. This is true in general for any bipartite graph, as stated in the following theorem.

Theorem 25. *If G is a bipartite graph, then number of edges in any of its maximum matchings β_1 (G) is equal to its vertex covering number α_0 (G).*

Proof. The proof is left to the reader.

In order to characterize maximum matchings, we give following definitions :

Definitions. Let G be a graph with matching ME and *a* path P. Then P is called alternating (with respect to M) if its edges are alternately in M and not in M.

An alternating path P, with respect to a matching M is said to be augmenting if its end vertices are not incident with any edge of M. Such end vertices are termed weak.

A matching M, in a graph G is said to be unaugmentable if G does not have an augmenting path with respect to M.

We now have the following theorem :

Theorem 26. *A matching in a graph G is unaugmentable if and only if it is maximum.*

Proof. See any reference text.

Definition. A matching M, in a graph G is said to be perfect, or a 1-factor or a dimmer covering, if some edge of M is incident with every vertex of G.

NOTES

At this stage we introduce some more notations.

A subgraph of a graph G = (V, E) obtained from G by deleting the vertices in U together with their incident edges, is represented by G – U.

Also the set of components of a graph G, having an odd number of vertices is denoted by $o(G)$.

Theorem 27. *A graph G = (V, E) has a perfect matching if and only if*

$$o(G - U) \leq |U|, \forall U \subseteq V.$$

Proof. The proof is left to the reader.

Despite the fact that it has an even number of vertices (an obvious necessary condition for the existence of a perfect matching) the graph in figure 6(*a*) does not have a perfect matching. However the graph in figure 6(*b*) does have a perfect matching as indicated by the thick lines.

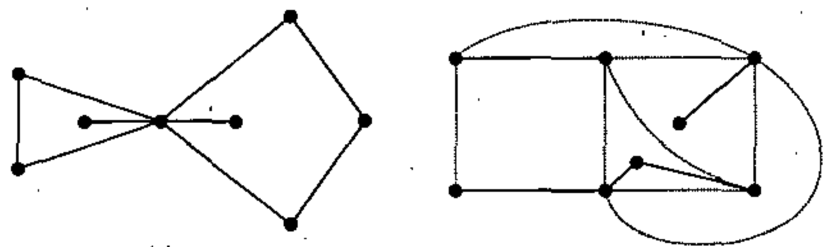

Fig. 6. (*a*) A graph which does not contain a perfect matching
(*b*) A graph containing a perfect matching.

Many of the matching problems in operations research can be modelled by bipartite graphs. We now give a definition as given below.

Definition. A matching M, in a bipartite graph G = (V, E), with bipartite vertex partition V_1 and V_2 is said to be complete in V_1 if there is one edge of M incident with every vertex of V_2.

This means that a matching M, in a bipartite graph G = (V, E), with bipartite vertex partition V_1 and V_2, is complete in $V_1 \subset V$ if every vertex in V_1 is matched by M (to a vertex in V_2).

A complete matching in $V_1 = \{v_1, v_2, v_3\}$ is shown as the solid lines in figure 7(*a*). In fact this matching is a maximum matching, like complete matchings. However there are maximal matchings which are not complete. One such case is shown in the solid lines in figure 7(*b*), where $V_1 = \{v_1, v_2, v_3\}$.

Now there is a question. When does a bipartite graph possess a complete matching in V_1? Evidently a necessary but not sufficient condition is that $|V_1| \leq |V_2|$.

Following theorem gives a necessary and sufficient condition for complete matching to be satisfied.

Theorem 28. *If G is a bipartite graph with bipartite vertex partition : V_1 and V_2, G has a matching that is complete in V_1 if and only if every subset of p vertices of V_1 is collectively adjacent to at least p vertices in V_2 for all $p = 1, 2, \ldots, |V_1|$.*

Proof. Sufficiency is obvious.

Necessity of the condition can be proved by using induction for *p*.

Let $V_1 = \{v_1, v_2, v_3\}$. It is instructive to apply Theorem 28 to establish that the graph in figure 7(*b*) has a matching that is complete in V_1, but that the graph in figure 7(*c*) does not have. The former fact is established because each vertex of V_1 in figure 7(*b*) is adjacent to at least two vertices and all three vertices in V_1 are collectively adjacent to four vertices in V_2. The latter fact is established because, in figure 7(*c*), vertices v_1 and v_3 are collectively adjacent to only a single vertex of V_2.

NOTES

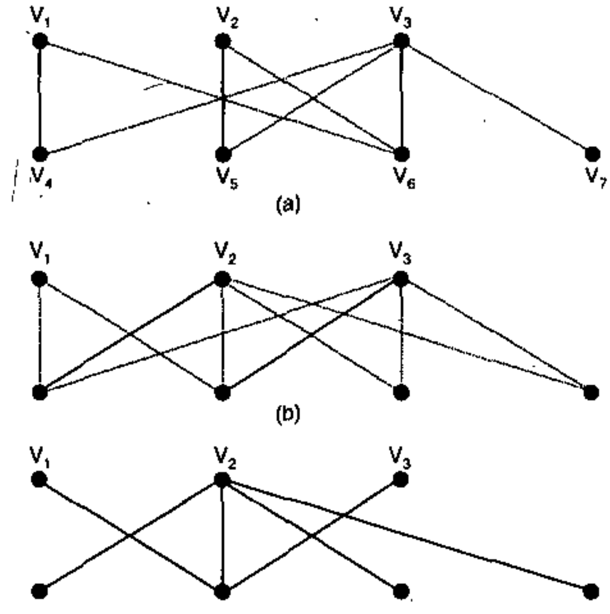

Fig. 7. (*a*) A complete matching (*b*) A maximal but not complete matching (*c*) A graph without a complete matching.

We can use such checking of collective adjacency to establish whether a given bipartite graph has a complete matching or not. But, such checking is very laborious for large, general bipartite graphs.

We now need a check whether a complete matching exists.

Corollary 3. *If G is a bipartite graph with bipartite vertex partition V_1 and V_2 and there exists an integer p such that*

$$\min_{v \in V_1} \{d(v)\} \geq p \geq \min_{v \in V_2} \{d(v)\}$$

then G has a complete matching in V_1.

Proof. Let G be a bipartite vertex partition say V_1 and V_2. Let $U \subseteq V_1$ where $|U| = q$. By assumption these q vertices are incident with at least pq edges. Since G is bipartite, each of the pq edges is incident with a vertex in V_2. However by assumption, that the degree of each vertex in V_2 is at most p. Thus these pq edge are incident with at least q vertices in V_2. Thus the q vertices in U are collectively adjacent to at least q vertices in V_2. Now the result follows from Theorem 28.

Note. Corollary provides a sufficient condition for a bipartite graph to have a complete matching, but it is not necessary. The graph in figure 8(*a*) does not obey the condition for the above corollary because $d(v_1) < d(v_6)$, where $V_1 = \{v_1, v_2, v_3\}$. Even then it has a complete matching in V_1, shown by the solid lines.

If a bipartite graph does not have a complete matching in certain applications (such as the assignment of jobs to workers) it is sometimes necessary to find a maximal matching. Let introduce a new definition at this junction.

Definition. If G is a bipartite graph with bipartite vertex partition : V_1 and V_2, then the deficiency of G, written $\delta(G) = \max \{p - q\}$, where the maximum is taken over all subsets of p vertices of V_1 which are collectively incident on a subset of q vertices of V_2.

Now theorem 28 can be restated as.

Theorem 29. *If G is a bipartite graph with bipartite vertex partition : V_1 and V_2, G has a matching that is complete in V_1 if and only if $\delta(G) \le 0$.*

Proof. As for Theorem 28.

This theorem can be used to check whether or not a bipartite graph has a complete matching. It is to be noted that the graphs in figure 7(*a*) and (*b*) have $\delta = 1$ and thus do not have complete matching. The reason for choosing the term deficiency for δ is apparent from the following theorem.

Theorem 30. *If G is a bipartite graph with bipartite vertex partition : V_1 and V_2, such that $\delta(G) \ge 0$, then the maximum number of vertices in V_1 that can be matched by any matching of G is $|V_1| - \delta(G)$.*

Proof. The proof is left to the reader.

Application of Theorem 30 to the graph in Figure 7(*c*) implies that any maximum matching contains exactly two edges. We now state the relationship between

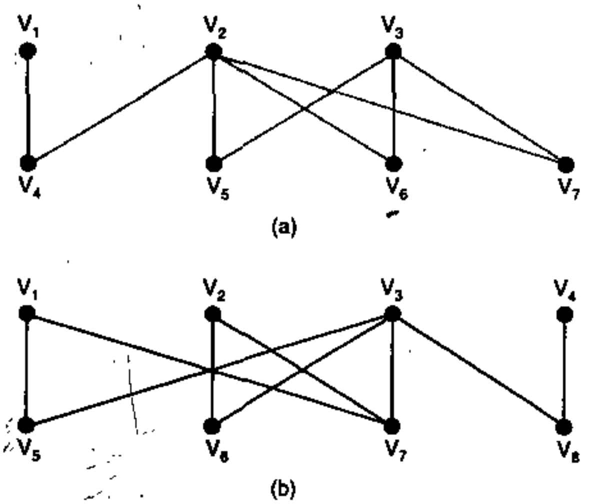

Figure 8. (*a*) A counter example to the necessity condition in Theorem 28 (*b*) A graph used to illustrate the relationship between a matching and the adjacency matrix of the graph.

a matching and the adjacency matrix of a bipartite graph. Let G be bipartite graph with bipartite vertex partition: V_1 and V_2. Suppose that $|V_1| = n_1$ and $|V_2| = n_2$. Then G is completely specified by the $n_1 \times n_2$ submatrix A_1, of the adjacency matrix of G, comprising the rows of A corresponding to the vertices of V_1 and the columns of A corresponding to the vertices of V_2.

We now have some observations about A_1 with regard to any matching M of G.

- M(1) There is a one-to-one correspondence between the edges of M and the unit entries of A_1.
- M(2) No row or column of A_1 has more than one unit entry corresponding to an edge of M.
- M(3) M is perfect if the unit entries of A_1 comprise an identity matrix within a permutation of rows.
- M(4) M is complete in V_1 if there is exactly one unit entry in each row of A_1 corresponding to the edge of M.

M(5) M is maximal if no further unit entries of A_1 can be added to those unit entries of A_1 associated with the edges of M without creating a row or column of A_1 with at least two such entries.

We now *illustrate these relationships* via the graph in figure 8(*b*).

Let $V_1 = \{v_1, v_2, v_3\}$ and $V_2 = \{v_4, v_5, v_6, v_7\}$, then

$$A_1 = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_5 & v_6 & v_7 & v_8 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

If $M = \{v_1 v_5, v_2 v_6, v_3 v_7, v_4 v_8\}$, then M is complete, as M corresponds to the leading diagonal of A_1, thus there is exactly one corresponding entry in each row of A_1. Since there is also exactly one corresponding entry in each column of A_1, M is perfect.

If $M = \{v_1 v_7, v_3 v_8, v_2 v_6\}$, then M is maximal as no further edge can be added to M without one of the first three rows of A_1 or one of the last three columns of A_1 possessing two unit entries corresponding to the new M. We will detail matching from an algorithmic point of view in chapter 13.

EXERCISE 10 (B)

1. Show that if a bipartite graph has any circuits, they must all be of even lengths.
2. Prove that the chromatic number of a graph will not exceed by more than one the maximum degree of the vertices in a graph.
3. Show that a simple graph with n vertices and more than $[n^2/4]$ edges can not be bipartite graph.

SUMMARY

1. If arc uv is a member of a digraph, its *converse* is defined to be arc vu. The converse digraph D^c, of digraph D, has the same vertex set as D, and an arc is in D^c if and only if its converse is in D.
2. A digraph $D' = (V', A')$, is termed a subdigraph of a digraph $D = (V, A)$, if $V' \subseteq V$ and $A' \subseteq A$ and D' is a digraph.
3. A strong component in a digraph D, is a maximal strong subdigraph of D. A unilateral component in a digraph D, is a maximal unilateral subdigraph of D.
4. Let D be a digraph *with strong components* : $S_1, S_2, \ldots, S_p$. The *condensation* $D^* = (V^*, A^*)$, of D is a digraph with $V^* = \{S_1, S_2, \ldots, S_p\}$ and where $S_i S_j$ is an arc of A^* if and only if there exists an arc uv in D for $u \in S_i$ and $v \in S_j$.
5. A cut-set in a digraph $D = (V, A)$, is a set of arcs of A, which constitute a cut-set in the multigraph $G = (V, E)$, obtained from D by *removing the orientation from each arc of* A.
6. A digraph D is known as Eulerian if it contains a closed trail which traverses every arc of D exactly once. Such a trail is termed an Euler trail. D is called unicursal if it contains an open Eular trail. Following distinguished digraphs as Eulerian and unicursal.
7. A directed tree is a weak digraph that does not contain a semi-cycle.

   There is one particular type of directed tree that is of importance in network analysis, computer science, enumeration and other fields of applied graph theory. It is said to be an *arborescence*.

8. A directed tree is called an *arborescence* if it contains exactly one vertex, called the root, with no arcs directed towards it, and if all the arcs on any semi-path are directed away from the root.
9. A set of edges (vertices) in a graph G, is said to be dominant if every edge (vertex) of G either belongs to the set or is adjacent to a member of it.

NOTES

TEST YOURSELF

1. Is it possible for two nonisomorphic digraphs to have the same reachability matrix R ? Explain.
2. Prove that every Euler digraph (without isolated vertices) is strongly connected. Also show, by constructing a counter example that the converse is not true.
3. Show that the chromatic number of a graph G cannot exceed by the diameter (*i.e.*, the length of the longest path) of G by more than one.
4. Show that the absolute value of the second coefficient of λ^{n-1} in the chromatic polynomial $p_n(\lambda)$ of a graph is equal to the number of edges in the graph.