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Syllabus

SPECIAL FUNCTION AND MECHANICS
SC-115

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UNIT 1

POWER SERIES SOLUTIONS OF D.E.

STRUCTURE

- Power Series Method
- Power Series Solution
  - Summary
  - Student Activity
  - Test Yourself

LEARNING OBJECTIVES

After going through this unit you will learn :

- What is a power series ?
- How to find the power series solution of a differential equation.

• 1.1. POWER SERIES METHOD

This method is very effective for the to linear homogeneous differential equation with variable coefficients. This method gives the solution of the differential equations in the form of a power series. Therefore, an infinite series of the form

$$\sum_{m=0}^{\infty} a_m x^m = a_0 + a_1 x + a_2 x^2 + \dots + a_m x^m + \dots$$

is called a **power series**. This power series is said to be **convergent** at a point x if

$$\lim_{n \to \infty} \sum_{m=0}^{n} a_m x^m$$

exists. It is clear that the above series is always convergent at $x = 0$. To explain this method clear, let us consider a general homogeneous differential equation of second order

$$y'' + P(x)\, y' + Q(x)\, y = 0.$$

The solution y of this given differential equation is assumed in the form of a power series as above with undetermined coefficient and these coefficients are determined by putting that series and the series for the derivatives of y into the given differential equation.

Ordinary and Singular Points :

Let us consider a general homogeneous linear differential equation of order two :

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)\, y = 0$$

or
$$y'' + P(x)\, y' + Q(x)\, y = 0. \qquad \dots(1)$$

The main concept about the solution of (1) is that the behaviour of the solutions near a point $x = x_0$ depends on the behaviour of $P(x)$ and $Q(x)$ near this point x_0. If $P(x)$ and $Q(x)$ are analytic at this point x_0, then power series method is applicable in some neighbourhood of x_0. Then this point x_0 is called an *ordinary point* of the differential equation (1). Thus we can say that every solution of (1) is analytic at x_0. If x_0 is not an ordinary point, then this point x_0 is called a *singular point.*

Regular Singular Points :

In the above section, we have seen that if one of the coefficient functions $P(x)$ and $Q(x)$ is not differentiable at x_0 then this point is called a *singular point.* Thus a point x_0 of the differential equation (1) is called regular if the functions $(x - x_0)\, P(x)$ and $(x - x_0)^2\, Q(x)$ are analytic at $x = x_0$.

If a singular point x_0 is located at the origin, then the general form of an analytic function at $x = x_0 = 0$ is $\sum_{m=0}^{\infty} a_m x^m$.

This implies that the origin will definitely be a singular point of (1) of $P(x)$ and $Q(x)$ have at least one of the coefficients with negative subscripts non-zero. In this case we assume the solution of the differential equation (1) of the form

$$y = x^n \sum_{m=0}^{\infty} a_m x^m = \sum_{m=0}^{\infty} a_m x^{m+n}$$

where n may be a negative integer or may be a fraction or even an irrational number.

• 1.2. POWER SERIES SOLUTION

(1) Solution near an ordinary point :

Consider the differential equation

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)\,y = 0 \quad \text{...(1)}$$

Let us take a trial solution of the form

$$y = \sum_{n=0}^{\infty} C_n x^n \quad \text{...(2)}$$

$$\Rightarrow \quad \left.\begin{aligned} \frac{dy}{dx} &= \Sigma n\, C_n\, x^{n-1} \\ \text{and} \quad \frac{d^2y}{dx^2} &= \Sigma n\,(n-1)\, C_n\, x^{n-2} \end{aligned}\right] \quad \text{...(3)}$$

Also, by letting $P(x)$ and $Q(x)$ are not polynomial in x, we can expand them as

$$P(x) = \sum_{n=1}^{\infty} p_n x^n \text{ and } Q(x) = \sum_{n=0}^{\infty} q_n \cdot x^n. \quad \text{...(4)}$$

Now putting all these values in equation (1), we get the required solution.

(2) Solution near a regular singular point :

Here, we assume a trial series solution of the type

$$y = x^m (C_0 + C_1 x + C_2 x^2 + \ldots) \quad \text{...(1)}$$

$$= x^m \cdot \sum_{n=0}^{\infty} C_n x^n, \text{ where all } C_i\text{'s constant with } C_i \neq 0.$$

To find the values of m and C's, we proceed as follows :

(i) Put the value of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in the given differential equation

(ii) By equating to zero the coefficients of the lowest power of x, get a quadratic equation in m, which is called indicial equation.

(iii) To find the values of the equations $C_1, C_2, \ldots,$ etc. in terms of C_0, equating to zero the coefficients of other powers of x.

(iv) The nature of the root can be determine as follows :

(A) If roots of the indicial equation are equal :

Let $m = n$, be two equal roots. Then putting $m = m_1$, in y and in $\frac{\partial y}{\partial m}$ we may get the two independent solutions.

(B) If roots of the indicial equation unequal and not differing by an integer :

If the indicial equation has two unequal roots $m = m_1$ and m_2 which do not differ by an integer, then by putting $m = m_1$ and m_2 in the series we get two independent solutions.

(C) If the roots of the indicial equation differing by an integer an making the coefficients of some powers of x in the series for y infinity :

Let $m = m_1$ and m_2 be two roots of the indicial equation which differ by an integer and some of the coefficents of powers of k in the series for y infinity for $m = m_2$.

Here put $C(m - m_2)$ for C_0, then we get two independent solutions for $m = m_2$. Then proceed as in case I.

(D) If the roots of the indicial equation differing by an integer and making a coefficient of the series for y indeterminate :

If $m = m_1$ and m_2 $(m_1 > m_2)$ are two roots of the indicial equation which differ by an integer. If one of the coefficients of the series for y becomes indeterminate when $m = m_2$, the complete solution is given by putting $m = m_2$ in y, which have two arbitrary constants.

SOLVED EXAMPLES

Example 1. *Solve* $x\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + xy = 0.$

Solution. Here, the given equation is

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} + xy = 0. \quad \text{...(1)}$$

Putting $y = x^m$ in the LHS of (1), we get

$$xm(m-1)x^{m-2} + mx^{m-1} + x \,.\, x^m = x^{m+1} + m^2x^{m-1}.$$

Clearly, the common difference of the powers is $(m+1) - (m-1)$ *i.e.*, 2.

Let $$y = \sum_{r=0}^{\infty} C_r x^{m+2r} = C_0 x^m + C_1 x^{m+2} + C_2 x^{m+4} + \ldots \quad \text{...(2)}$$

is the solution of (1).

Then, we have

$$\frac{dy}{dx} = \sum_{r=0}^{\infty} C_r (m+2r) x^{m+2r-1}$$

$$\frac{d^2y}{dx^2} = \sum_{r=0}^{\infty} C_r (m+2r)(m+2r-1)^{m+2r-2}.$$

Put all these values in (1), we get

$$\sum_{r=0}^{\infty} C_r [(m+2r)(m+2r-1)x^{m+2r-1} + (m+2r)x^{m+2r-1} + x^{m+2r+1}] = 0$$

$$\Rightarrow \quad \sum_{r=0}^{\infty} C_r [x^{m+2r-1} + (m+2r)^2 x^{m+2r-1}] = 0.$$

Equating to zero, the coefficient of the lowest power of x *i.e.*, of x^{m-1}, we have

$$C_0 m^2 = 0$$

which is the required indicial equation.

Since $C_0 \neq 0$, therefore $m = 0, 0$ are two equal roots.

Now equating to zero the coefficient of the general term *i.e.*, of x^{m+2p+1}, we get

$$C_p + (m + 2p + 2)^2 C_{p+1} = 0$$

$$\Rightarrow \quad C_{p+1} = -\frac{1}{(m+2p+2)^2} C_p. \quad \text{...(3)}$$

Putting $p = 0, 1, 2, \ldots$, in (3), we get

$$C_1 = -\frac{1}{(m+2)^2} C_0, \quad C_2 = -\frac{1}{(m+4)^2} C_1 = (-1)^2 \frac{1}{(m+2)^2 (m+4)^2} C_0$$

$$C_3 = -\frac{1}{(m+6)^2} C_2 = (-1)^3 \frac{1}{(m+2)^2 (m+4)^2 (m+6)^2} C_0 \ldots \text{ and so on.}$$

Put all these values in (2), we get

$$y = C_0 x^m \left[1 - \frac{x^2}{(m+2)^2} + \frac{x^4}{(m+2)^2 (m+4)^2} - \ldots\right]. \quad \text{...(4)}$$

Putting $m = 0$, we get

$$y = C_0\left[1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 . 4^2} - \frac{x^6}{2^2 . 4^2 . 6^2}\right] + \ldots \qquad \ldots(5)$$

$= C_0 . u$ (say), which is the first solution of the given equation (1)

when $\quad u = 1 - \dfrac{x^2}{2^2} + \dfrac{x^4}{2^2 . 4^2} - \dfrac{x^6}{2^2 . 4^2 . 6^2} + \ldots .$

Since, there are two equal values of m, therefore, second solution can not be obtained from (4).

Now, from (4)

$$\frac{dy}{dx} = C_0\left[mx^{m-1} - \frac{(m+2)\,x^{m+1}}{(m+2)^2} + \frac{(m+4)\,x^{m+3}}{(m+2)^2\,(m+4)^2} - \ldots\right]$$

$$\Rightarrow \frac{d^2y}{dx^2} = C_0\left[m\,(m-1)\,x^{m-2} - \frac{(m+2)\,(m+1)}{(m+2)^2}\,x^m + \frac{(m+4)\,(m+3)\,x^{m+2}}{(m+2)^2\,(m+4)^2} - \ldots\right].$$

Put above two values in (1), we get

$$\text{LHS} = xC_0\left[m\,(m-1)\,x^{m-2} - \frac{(m+2)\,(m+1)}{(m+2)^2}\,x^m + \frac{(m+4)\,(m+3)\,x^{m+2}}{(m+2)^2\,(m+4)^2} - \ldots\right]$$

$$+ C_0\left[mx^{m-1} - \frac{(m+2)\,x^{m+1}}{(m+2)^2} + \frac{(m+4)\,x^{m+3}}{(m+2)^2\,(m+4)^2} - \ldots\right]$$

$$+ xC_0\left[x^m - \frac{x^{m+2}}{(m+2)^2} + \frac{x^{m+4}}{(m+2)^2\,(m+4)^2} - \ldots\right]$$

$$= C_0\,m^2\,x^{m-1}.$$

$$\therefore \left[x\frac{d^2}{dx^2} + \frac{d}{dx} + x\right]y = C_0\,m^2\,x^{m-1}.$$

Differentiating both sides, partially, w.r.t. m, we get

$$\frac{\partial}{\partial m}\left[x\frac{d^2}{dx^2} + \frac{d}{dx} + x\right]y = \frac{\partial}{\partial m}\left(C_0\,m^2\,x^{m-1}\right)$$

$$\Rightarrow \qquad \left[x\frac{d^2}{dx^2} + \frac{d}{dx} + x\right]\left(\frac{\partial y}{\partial m}\right) = C_0 . 2mx^{m-1} + C_0\,m^2\,x^{m-1}\log x.$$

Putting $m = 0$, we get

$$\left[x\frac{d^2}{dx^2} + \frac{d}{dx} + x\right]\left[\frac{\partial y}{\partial m}\right] = 0$$

$\Rightarrow \quad \left(\dfrac{\partial y}{\partial m}\right)_{m=0}$ satisfy the equation (1), therefore it is also a solution of (1).

Differentiating (4), partially, w.r.t., m we get

$$\frac{\partial y}{\partial m} = C_0\,x^m\log x\left[1 - \frac{x^2}{(m+2)^2} + \frac{x^4}{(m+2)^2\,(m+4)^2} - \ldots\right]$$

$$+ C_0\,x^m\left[\frac{2x^2}{(m+2)^3} + \left\{\frac{-2}{(m+2)^3\,(m+4)^2} + \frac{-2}{(m+2)^2\,(m+4)^3}\right\}x^4 + \ldots\right].$$

Putting $m = 0$, we get

$$\left(\frac{\partial y}{\partial m}\right)_{m=0} = C_0\log x\left[1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 . 4^2} - \ldots\right] + C_0\left[\frac{x^2}{2^2} + \left\{\frac{-2}{2^3 . 4^2} + \frac{-2}{2^2 . 4^3}\right\}x^4 \ldots\right]$$

$$= bu \log x + b\left[\frac{x^2}{2^2} - \frac{3}{2^3 . 4^2}x^4 + \dots\right]$$

$$= bv \text{ (say)}$$

where $v = u \log x + \left[\frac{x^2}{2^2} - \frac{3}{2^3 . 4^2}x^4 + \dots\right]$ and b is any arbitrary.

Constant which replaces C_0.

Hence, the required general solution of (1) is given by

$$y = au + bv$$

where a and b are arbitrary constants.

Example 2. *Solve the following Legendre's equation*

$$(1 - x^2)\, y'' - 2xy' + p\,(p + 1)\, y = 0$$

in descending powers of x.

Solution. Here, the given equation can be written as

$$(1 - x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + p\,(p + 1)\, y = 0. \qquad \dots(1)$$

Putting $y = x^m$ in the LHS of (1), we get

$$(1 - x^2)\, m\,(m - 1)\, x^{m-2} - 2x \,.\, mx^{m-1} + p\,(p + 1)\, x^m$$

or $$(-m^2 - m + p^2 + p)\, x^m + m\,(m - 1)\, x^{m-2}.$$

Clearly, the common difference of the powers is $m - (m - 2)$ *i.e.*, 2.

Let the solution of (1) in descending powers of x be

$$y = C_0 x^m + C_1 x^{m-2} + C_2 x^{m-4} + \dots = \sum_{r=0}^{\infty} C_r x^{m-2r} \qquad \dots(2)$$

$$\Rightarrow \quad \frac{dy}{dx} = \sum_{r=0}^{\infty} C_r\,(m - 2r)\, x^{m-2r-1}$$

and $$\frac{d^2y}{dx^2} = \sum_{r=0}^{\infty} C_r\,(m - 2r)\,(m - 2r - 1)\, x^{m-2r-2}.$$

Put all these values in (1), we get

$$\sum_{r=0}^{\infty} C_r\,[(1 - x^2)\,(m - 2r)\,(m - 2r - 1)\, x^{m-2r-2} - 2x\,(m - 2r)\, x^{m-2r-1} + p\,(p + 1)\, x^{m-2r} = 0$$

$$\Rightarrow \quad \sum_{r=0}^{\infty} C_r\,[\{-(m - 2r)\,(m - 2r - 1) - 2\,(m - 2r) + p\,(p + 1)\}\, x^{m-2r} + (m - 2r)\,(m - 2r - 1)\, x^{m-2r-2}] = 0$$

$$\Rightarrow \quad \sum_{r=0}^{\infty} C_r\,[\{p^2 - (m - 2r)^2 + (p - m + 2r)\}\, x^{m-2r} + (m - 2r)\,(m - 2r - 1)\, x^{m-2r-2}] = 0$$

$$\Rightarrow \quad \sum_{r=0}^{\infty} C_r\,[(p - m + 2r)\,(p + m - 2r + 1)\, x^{m-2r} + (m - 2r)\,(m - 2r - 1)\, x^{m-2r-2}] = 0.$$

Equating to zero, the coefficients of the highest power of x *i.e.*, x^m, we get the initial equation as

$$C_0\,(p - m)\,(p + m + 1) = 0.$$

Since $C_0 \neq 0$, therefore, we get

$$m = p, -(p + 1).$$

Now, equating to zero the coefficients of x^{m-2r}, we get

$$C_r\,(p - m + 2r)\,(p + m - 2r + 1) + (m - 2r + 2)\,(m - 2r + 1)\, C_{r-1} = 0$$

$$\Rightarrow \quad C_r = \frac{(m - 2r + 2)\,(m - 2r + 1)}{(p - m + 2r)\,(p + m - 2r + 1)}\, C_{r-1}.$$

Putting $r = 1, 2, \ldots$, we get

$$C_1 = -\frac{m(m-1)}{(p-m+2)(p-m-1)} C_0,$$

$$C_2 = -\frac{(m-2)(m-3)}{(p-m+4)(p+m-3)} C_1$$

$$= (-1)^2 \frac{m(m-1)(m-2)(m-3)}{(p-m+2)(p-m+4)(p+m-1)(p+m-3)} C_0$$

... ... and so on.

Put all these values in (2), we get

$$y = C_0 \left[x^m - \frac{m(m-1)}{(p-m+2)(p+m-1)} x^{m-2} + \frac{m(m-1)(m-2)(m-3)}{(p-m+2)(p-m+4)(p+m-1)(p+m-3)} x^{m-4} - \ldots \right].$$

Now, putting $m = p, -(p+1)$ successively, we get

$$y = C_0 \left[x^p - \frac{p(p-1)}{2(2p-1)} x^{p-2} + \frac{p(p-1)(p-2)(p-3)}{2 \cdot 4 \cdot (2p-1)(2p-3)} x^{p-4} - \ldots \right]$$

$$= au \text{ (say)}$$

which is one solution of the given equation.

Also,

$$y = C_0 \left[x^{-p-1} + \frac{(p+1)(p+2)}{2(2p+3)} x^{-p-3} + \frac{(p+1)(p+2)(p+3)(p+4)}{2 \cdot 4 (2p+3)(2p+5)} x^{-p-5} + \ldots \right]$$

$$= bv \text{ (say)}.$$

Here, the required solution of the given equation is $y = au + bv$, where a and b are arbitrary constants.

• SUMMARY

- Power series : $y = \sum_{m=0}^{n} a_m x^m$

• STUDENT ACTIVITY

1. Define ordinary and singular points of D.E.

$$y'' = P(x)\, y' + Q(x)\, y = 0$$

2. Solve : $x \dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + xy = 0.$

• TEST YOURSELF

1. Solve $\frac{d^2y}{dx^2} - 2x^2\frac{dy}{dx} + 4xy = x^2 + 2x + 2$ in powers of x.

2. Solve $x\frac{d^2y}{dx^2} + \frac{dy}{dx} + xy = 0.$

3. Solve $x\frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + 2y = 0.$

Objective evaluations

Fill in the blanks :

1. The series $\sum_{m=0}^{k} a_m x^m$ is a power series if $k =$ ....
2. D.E. $y'' + P(x)\, y' + Q(x)\, y = R$ is homogeneous of $R =$ ....

True or False

1. D.E. $y'' + P(x)\, y' + q(x)\, y = 0$ is homogeneous. *(T/F)*
2. Series $\sum_{m=0}^{\infty} a_m x^{m+n}$ is called quasi-power series. *(T/F)*

Multiple Choice Questions

1. Ordinary point for D.E. $y'' + y = 0$ are/is :

   (a) $[-1, 1]$ (b) set of all rals

   (c) 0 (d) 1

ANSWERS

1. $y = C_0\left(1 - \frac{2}{3}x^3 - \frac{2}{45}x^6 \ldots\right) + C_1\left(x - \frac{1}{6}x^4 - \frac{1}{63}x^7\right) + x^2 + \frac{1}{3}x^3 + \frac{1}{12}x^4 + \frac{1}{45}x^6 + \ldots$

2. $y = au + bv$, where $u = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 . 4^2} - \frac{x^6}{2^2 . 4^2 . 6^2} + \ldots$

and $v = u \log x + \left[\frac{x^2}{2^2} - \frac{3}{2^3 . 4^2}x^4 + \ldots\right]$

3. $y = au + bv$, where $u = 1 - 2x + \frac{3}{2\,!}x^2 - \frac{4}{3\,!}x^3 + \ldots$

$v = bu \log x + b\left[2\left(2 - \frac{1}{2}\right)x - \frac{3}{2\,!}\left(-\frac{1}{3} + 2 + \frac{1}{2}\right)x^2 + \ldots\right]$

Fill in the blanks

1. $K = \infty$ **2.** $R = 0$

True or False

1. T **2.** T

Multiple Choice Questions

1. (b)

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UNIT 2

LEGENDRE'S FUNCTIONS

STRUCTURE

LEARNING OBJECTIVES

After going through this unit you will learn :

- What is Legendre Differential equation ?
- The power series solution of Legendre D.E. is the Legendre polynomials.
- How to generate Legendre polynomial.
- What are their orthogonal properties and recurrence relations ?

2.1. LEGENDRE'S D.E.

Consider a homogeneous linear differential equation of order two of the form

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0 \qquad \ldots(1)$$

where n is a real number. This differential equation is known as Legendre's differential equation, and any solution of (1) *is called a Legendre function.*

Solution of Legendre Equation

Dividing (1) by $(1-x^2)$, we get

$$\frac{d^2y}{dx^2} - \frac{2x}{1-x^2}\frac{dy}{dx} + n(n+1)\cdot\frac{1}{1-x^2}y = 0$$

Now compare this equation with the standard form

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0$$

$$\therefore \qquad P(x) = -\frac{2x}{1-x^2}, \quad Q(x) = \frac{n(n+1)}{1-x^2}.$$

It is trivially obtained that $P(x)$ and $Q(x)$ are analytic at $x = 0$, so, for finding the solutions of (1) we apply the power series method. Let us assume the solution of (1)

$$y = \sum_{m=0}^{\infty} a_m x^m \qquad \ldots(2)$$

Now differentiating (2) w.r.t. x one time and then two times, we get,

$$\frac{dy}{dx} = \sum_{m=1}^{\infty} m a_m x^{m-1} \qquad \ldots(3)$$

and $$\frac{d^2y}{dx^2} = \sum_{m=2}^{\infty} m(m-1)\, a_m\, x^{m-2} \quad \ldots(4)$$

Substitute the values of y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ from (2), (3) and (4) into equation (1), we get

$$(1-x^2) \sum_{m=2}^{\infty} m(m-1)\, a_m\, x^{m-2} - 2x \sum_{m=1}^{\infty} m a_m\, x^{m-1} + n(n-1) \sum_{m=0}^{\infty} a_m\, x^m = 0$$

or $$\sum_{m=2}^{\infty} m(m-1)\, a_m\, x^{m-2} - \sum_{m=2}^{\infty} m(m-1)\, a_m\, x^m - 2\sum_{m=1}^{\infty} m a_m\, x^m + n(n+1) \sum_{m=0}^{\infty} a_m\, x^m = 0$$

or $$\{2.1\, a_2 + 3.2\, a_3 x + 4.3\, a_4 x^2 + \ldots + (r+2)(r+1)\, a_{r+2}\, x^r + \ldots\}$$
$$- \{2.1\, a_2\, x^2 + 3.2\, a_3 x^3 + \ldots + r(r-1)\, a_r\, x^r + \ldots\} - 2\{a_1 x + 2a_2 x^2 + \ldots + r a_r\, x^r + \ldots\}$$
$$+ n(n+1)\{a_0 + a_1 x + \ldots + a_r\, x^r + \ldots\} = 0. \quad \ldots(5)$$

If equation (2) is a solution of (1), then equation (5) must be an identity in x. Thus in (5) the sum of the coefficients of each power of x must be zero. We therefore obtain

$$2a_2 + n(n+1)\, a_0 = 0,$$
$$6a_3 + \{-2 + n(n+1)\}\, a_1 = 0,$$
$$12a_4 + \{-2a_2 - 4a_2 + n(n+1)\, a_2\} = 0$$

$\therefore$ $$a_{n-2} = -\frac{n(n-1)}{2(2n-1)} \cdot \frac{(2n)!}{2^n (n!)^2} \qquad \left(\because\ a_n = \frac{(2n!)}{2^n (n!)^2}\right)$$
$$= -\frac{n(n-1)\, 2n\,.\,(2n-1)\,.\,(2n-2)!}{2(2n-1)\,.\,2^n\,.\,n!\,.\,n(n-1)\,.\,(n-2)!}$$
$$= -\frac{n(n-1)\, 2n\,.\,(2n-1)\,.\,(2n-2)!}{2(2n-1)\, 2^n\, n\,.\,(n-1)!\,.\,n(n-1)\,.\,(n-2)!}$$
$$= -\frac{(2n-2)!}{2^n (n-1)!\,.\,(n-2)!}.$$

Similarly, $$a_{n-4} = -\frac{(n-2)(n-3)}{4(2n-3)}\, a_{n-2}$$
$$= -\frac{(n-2)(n-3)}{4(2n-3)} \cdot \frac{-(2n-2)!}{2^n (n-1)!\,(n-2)!}$$
$$= \frac{(n-2)(n-3)\,.\,(2n-2)(2n-3)(2n-4)!}{4(2n-3)\, 2^n (n-1)\,.\,(n-2)!\,(n-2)(n-3)\,.\,(n-4)!}$$
$$= \frac{(2n-4)!}{2^n\,.\,(2)!\,(n-2)!\,(n-4)!}$$

Continuing in this way, we get in general,

$$a_{n-2m} = \frac{(-1)^m (2n-2m)!}{2^n (m)!\,(n-m)!\,(n-2m)!}, \quad n-2m \geq 0$$

Thus we obtain the first kind of Legendre polynomial of degree n and it is denoted by $P_n(x)$ which is given as

$$P_n(x) = \sum_{m=0}^{N} a_{n-2m}\, x^{n-2m}.$$

in general, we obtain

$$(r+2)(r+1)\, a_{r+2} + \{-r(r-1) - 2r + n(n+1)\}\, a_r = 0$$

for $r = 2, 3, 4, \ldots$

or $$(r+2)(r+1)\, a_{r+2} + (n-r)(n+r+1)\, a_r = 0$$

or $$a_{r+2} = -\frac{(n-r)(n+r+1)}{(r+2)(r+1)}\, a_r, \quad r = \{0, 1, 2, \ldots\} \quad \ldots(6)$$

This equation (6) is known as recursion formula. Now finding the coefficients successively for $r = 0, 1, 2, 3 \ldots$

$$a_2 = -\frac{n(n+1)}{2.1} a_0 = -\frac{n(n+1)}{(2)!} a_0$$

$$a_3 = -\frac{(n-1)(n+2)}{3.2} a_1 = -\frac{(n-1)(n+2)}{(3)!} a_1$$

$$a_4 = -\frac{(n-2)(n+3)}{4.3} a_2$$

$$= -\frac{(n-2)(n+3)}{4.3} \cdot \frac{-n(n+1)}{2.1} a_0$$

$$= \frac{(n-2)n(n+1)(n+3)}{(4)!} a_0$$

$$a_5 = -\frac{(n-3)(n+4)}{5.4} a_3$$

$$= \frac{(n-3)(n-1)(n+2)(n+4)}{(5)!} a_1$$

$$\vdots$$

etc.

We observed from above coefficients that all the even numbered coefficients are obtained in terms of a_0 while all odd numbered coefficients are obtained in terms of a_1. Thus we obtain the solution as

$$y = a_0 y_1(x) + a_1 y_2(x)$$

where
$$y_1(x) = 1 - \frac{n(n+1)}{(2)!} x^2 + \frac{(n-2)n(n+1)(n+3)}{(4)!} x^4 - \dots$$

and
$$y_2(x) = x - \frac{(n-1)(n+2)}{(3)!} x^3 + \frac{(n-3)(n-1)n(n+2)(n+4)}{(5)!} x^5 - \dots.$$

These both series are convergent if $|x| < 1$. Sometimes, we have observed that the parameter n in the Legendre's differential equation will be nonnegative. Then, from recursion formula (6), we obtain

$$a_{r+2} = 0, \text{ when } r = n \text{ i.e. } a_{n+2} = a_{n+4} = \dots = 0$$

Hence we can say that if n is even, $y_1(x)$ becomes a polynomial of degree n whereas n is odd $y_2(x)$ becomes a polynomial of degree n. Therefore if $y_1(x)$ is multiplied by some constant, then this polynomial is called Legendre's polynomial of **first kind** and if $y_2(x)$ is multiplied by some constant, then $y_2(x)$ is called Legendre's polynomial of **second kind.** Now to obtain first kind of Legendre's polynomial we proceed as follows :

The recursion formula in (6) may be written as

$$a_r = -\frac{(r+2)(r+1)}{(n-r)(n+r+1)} a_{r+2} \quad \text{for } r \le n-2$$

Also all a's may express in terms of the coefficient a_n which is the coefficient of the highest power of x of the polynomial. This a_n is an arbitrary and choose $a_n = 1$ when $n = 0$ and $a_n = \frac{1.3.5 \dots (2n-1)}{n!} = \frac{2n!}{2^n . (n)^2 !}$ for all $n = 1, 2, 3 \dots$. This a_n is chosen in such a way that the values of all those polynomial will be 1 when $x = 1$. Now finding the coefficients as follows :

$$a_{n-2} = -\frac{n(n-1)}{2(2n-1)} a_n$$

or
$$P_n(x) = \sum_{m=0}^{N} \frac{(-1)^m (2n-2m)!}{2^n (m)!(n-m)!(n-2m)!} . x^{n-2m}$$

where
$$N = \begin{cases} n/2; & \text{if } n \text{ is even} \\ (n-1)/2, & \text{if } n \text{ is odd.} \end{cases}$$

• 2.2. GENERATING FUNCTIONS OF LEGENDRE POLYNOMIAL $P_n(x)$

The functions of the type, $\frac{1}{\sqrt{1-2xt+t^2}}$ *generates Legendre polynomial* $P_n(x)$, *is called generating function. Thus we obtained*

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)\, t^n$$

Proof. $$\text{L.H.S.} = \frac{1}{\sqrt{1-2xt+t^2}}$$

$$= \frac{1}{\sqrt{1-s}} \qquad (\because\ s = 2xt - t^2)$$

$$= (1-s)^{-1/2} \qquad \text{(Expand by binomial theorem)}$$

$$= 1 + \frac{1}{2}s + \frac{1.3}{2.4}s^2 + \frac{1.3.5}{2.4.6}s^3 + \ldots + \frac{1.3.5\ldots(2n-3)}{2.4.6\ldots(2n-2)}s^{n-1} + \frac{1.3.5\ldots(2n-1)}{2.4.6\ldots(2n)}s^n + \ldots$$

since $s = 2xt - t^2$

$$\therefore \quad s^n = (2xt-t^2)^n = t^n(2x-t)^n$$

$$= t^n\,[{}^nC_0(2x)^n - {}^nC_1(2x)^{n-1}t + \ldots].$$

Similarly,

$$s^{n-1} = t^{n-1}\left[{}^{n-1}C_0(2x)^{n-1} - {}^{n-1}C_1(2x)^{n-2}t + \ldots\right]$$

and $$s^{n-2} = t^{n-2}\left[{}^{n-2}C_0(2x)^{n-2} - {}^{n-2}C_1(2x)^{n-3}t + {}^{n-2}C_2(2x)^{n-4}t^2 \ldots\right]$$

$\vdots$

etc.

substitute these value in the above equation, we get

$$\text{L.H.S.} = 1 + \frac{1}{2}t(2x-t) + \frac{1.3}{2.4}t^2(2x-t)^2 + \frac{1.3.5}{2.4.6}t^3(2x-t)^3 + \ldots$$

$$\ldots + \frac{1.3.5\ldots(2x-5)}{2.4.6\ldots(2x-4)}t^{n-2}\left[{}^{n-2}C_0(2x)^{n-2} - {}^{n-2}C_1(2x)^{n-3}t + {}^{n-2}C_2(2x)^{n-4}t^2 + \ldots\right]$$

$$+ \frac{1.3.5\ldots(2n-3)}{2.4.6\ldots(2n-2)}t^{n-1}\left[{}^{n-1}C_0(2x)^{n-1} - {}^{n-1}C_1(2x)^{n-2}t + \ldots\right]$$

$$+ \frac{1.3.5\ldots(2n-1)}{2.4.6\ldots(2n)}t^n\left[{}^nC_0(2x)^n - {}^nC_1(2x)^{n-1}t + \ldots\right]$$

Now collecting the coefficients of t^n

$$= \frac{1.3.5\ldots(2n-1)}{2.4.6\ldots(2n)}.\,{}^nC_0(2x)^n - \frac{1.3.5\ldots(2n-3)}{2.4.6\ldots(2n-2)}\,{}^{n-1}C_1(2x)^{n-2} + \frac{1.3.5\ldots(2n-5)}{2.4.6\ldots(2n-4)}\,{}^{n-2}C_2(2x)^{n-4} - \ldots$$

$$= \frac{1.3.5\ldots(2n-1)}{2.4.6\ldots(2n)}.\,2^n x^n - \frac{1.3.5\ldots(2n-3)}{2.4.6\ldots(2n-2)}.\frac{(n-1)}{(1)!}2^{n-2}.x^{n-2} + \frac{1.3.5\ldots(2n-5)}{2.4.6\ldots(2n-4)}.\frac{(n-2)(n-3)}{(2)!}2^{n-4}.x^{n-4} - \ldots$$

$$= \frac{1.3.5\ldots(2n-1)}{2.4.6\ldots 2n}2^n\left[x^n - \frac{2n(n-1)}{(2n-1)2^2}x^{n-2} + \frac{2n(2n-2)(n-2)(n-3)}{(2n-1)(2n-3)(2)!.2^4}.x^{n-4} - \ldots\right]$$

$$= \frac{1.3.5\ldots(2n-1)}{(n)!}\left[x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2.4(2n-1)(2n-3)}.x^{n-4} - \ldots\right]$$

$$= P_n(x).$$

Hence we obtained

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)\, t^n.$$

2.3. RODRIGUE'S FORMULA

The expression for $P_n(x)$, given by

$$P_n(x) = \frac{1}{2^n\,(n)!}\frac{d^n}{dx^n}(x^2-1)^n$$

*is called **Rodrigue's Formula.***

Proof. Since $P_n(x)$ is a Legendre Polynomial whose expression is given as

$$P_n(x) = \sum_{m=0}^{[n/2]} \frac{(-1)^m\,(2n-2m)!}{2^n\,(m)!\,(n-m)!\,(n-2m)!}\,.\,x^{n-2m} \qquad \ldots(1)$$

where $[n/2]$ is an integral value of $n/2$ not exceed $n/2$, rearrange (1), we get

$$P_n(x) = \sum_{m=0}^{[n/2]} \frac{(-1)^m}{2^n\,(m)!\,(n-m)!}\,.\,\frac{(2n-2m)!}{(n-2m)!}\,.\,x^{n-2m}$$

$$= \sum_{m=0}^{[n/2]} \frac{(-1)^m}{2^n\,(m)!\,(n-m)!}\,.\,\frac{d^n}{dx^n}x^{2n-2m}$$

$$\left(\because\ \frac{d^r}{dx^r}x^{2n-2m} = \frac{(2n-2m)!}{(2n-2m-r)!}\,.\,x^{2n-2m-r}\right)$$

$$= \frac{1}{2^n\,.\,(n)!}\sum_{m=0}^{[n/2]} \frac{(n)!}{(m)!\,(n-m)!}\,.\,\frac{d^n}{dx^n}(x^2)^{n-m}\,.\,(-1)^m$$

Now extending the range of m from 0 to n. To do so no change will occur in the above expression, because nth derivatives of those terms whose degree are less than n will be zero. Thus above expression can be written as

$$= \frac{1}{2^n\,(n)!}\frac{d^n}{dx^n}\sum_{m=0}^{n} \frac{(n)!}{(m)!\,(n-m)!}(x^2)^{n-m}(-1)^m$$

$$= \frac{1}{2^n\,(n)!}\frac{d^n}{dx^n}\sum_{m=0}^{n} {}^nC_m\,(x^2)^{n-m}(-1)^m \qquad \left(\because\ {}^nC_m = \frac{(n)!}{(m)!\,(n-m)!}\right)$$

$$= \frac{1}{2^n\,(n)!}\frac{d^n}{dx^n}\left[{}^nC_0\,(x^2)^n - {}^nC_1\,(x^2)^{n-1} + {}^nC_2\,(x^2)^{n-2} + \ldots + {}^nC_n\,(-1)^n\right]$$

$$= \frac{1}{2^n\,(n)!}\frac{d^n}{dx^n}(x^2-1)^n \qquad \text{(By Binomial theorem)}$$

Hence $\quad P_n(x) = \dfrac{1}{2^n\,(n)!}\dfrac{d^n}{dx^n}(x^2-1)^n.$

2.4. LAPLACE INTEGRAL FOR $P_n(x)$

(i) Laplace's First Integral for $P_n(x)$:

$$P_n(x) = \frac{1}{\pi}\int_0^{\pi}[x \pm \sqrt{x^2-1}\cos\theta]^n\,d\theta$$

where n is any positive integer.

Proof. Since we know that

$$\int_0^{\pi}\frac{d\theta}{a \pm b\cos\theta} = \frac{\pi}{\sqrt{a^2-b^2}}\,,\ \text{where } a^2 > b^2$$

let us taking $a = 1 - tx$ and $b = t\sqrt{x^2-1}$, then

$$a^2 - b^2 = (1-tx)^2 - t^2(x^2-1)$$

$$= 1 + t^2x^2 - 2tx - t^2x^2 + t^2 = 1 - 2tx + t^2.$$

Thus (1) becomes

$$\int_0^{\pi} \frac{d\theta}{(1 - tx) \pm t\sqrt{x^2 - 1}\cos\theta} = \frac{\pi}{\sqrt{1 - 2tx + t^2}} \quad \ldots(2)$$

since generating function gives

$$\frac{1}{\sqrt{1 - 2xt + t^2}} = \sum_{n=0}^{\infty} P_n(x)\, t^n$$

$\therefore$(2) becomes

$$\pi \sum_{n=0}^{\infty} P_n(x)\, t^n = \int_0^{\pi} \frac{d\theta}{1 - tx \pm t\sqrt{x^2 - 1}\cos\theta}$$

$$= \int_0^{\pi} \frac{d\theta}{[1 - t\{x \mp \sqrt{x^2 - 1}\cos\theta\}]}$$

$$= \int_0^{\pi} [1 - t\{\mp \sqrt{x^2 - 1}\cos\theta\}]^{-1}\, d\theta$$

$$= \int_0^{\pi} (1 - ts)^{-1}\, d\theta, \qquad \text{where } s = x \mp \sqrt{x^2 - 1}\cos\theta$$

$$= \int_0^{\pi} (1 + ts + t^2s^2 + \ldots + t^ns^n + \ldots)\, d\theta$$

$$= \int_0^{\pi} \sum_{n=0}^{\infty} t^n s^n\, d\theta$$

$$= \sum_{n=0}^{\infty} \int_0^{\pi} s^n t^n\, d\theta$$

$$= \sum_{n=0}^{\infty} t^n \int_0^{\pi} [x \mp \sqrt{x^2 - 1}\cos\theta]^n\, d\theta$$

$$\therefore \quad \pi \sum_{n=0}^{\infty} P_n(x)\, t^n = \sum_{n=0}^{\infty} t^n \int_0^{\pi} [x \mp \sqrt{x^2 - 1}\cos\theta]^n\, d\theta$$

$$\therefore \quad \pi P_n(x) = \int_0^{\pi} [x \pm \sqrt{x^2 - 1}\cos\theta]^n\, d\theta$$

or

$$P_n(x) = \frac{1}{\pi}\int_0^{\pi} [x \pm \sqrt{x^2 - 1}\cos\theta]^n\, d\theta.$$

(ii)Laplace's Second integral for $P_n(x)$

$$P_n(x) = \frac{1}{\pi}\int_0^{\pi} \frac{d\theta}{\sqrt{[x \pm \sqrt{(x^2 - 1)}\cos\theta]^{n+1}}}$$

where n is any positive integer.

Proof. Since we know that

$$\int_0^{\pi} \frac{d\theta}{a \pm b\cos\theta} = \frac{\pi}{\sqrt{a^2 - b^2}}, \text{ where } a^2 > b^2 \quad \ldots(1)$$

Here taking $a = xt - 1$, and $b = t\sqrt{x^2 - 1}$, then $a^2 - b^2 = 1 - 2xt + t^2$

$\therefore$(1) becomes

$$\int_0^{\pi} \frac{d\theta}{(xt-1) \pm t\sqrt{x^2-1}\cos\theta} = \frac{\pi}{\sqrt{1-2xt+t^2}} \qquad \ldots(2)$$

since
$$\sum_{n=0}^{\infty} P_n(x)\, t^n = \frac{1}{\sqrt{1-2xt+t^2}}$$

$\therefore$ (2) becomes,

$$\pi \sum_{n=0}^{\infty} P_n(x)\, t^n = \int_0^{\pi} \frac{d\theta}{[-1+t\{x \pm \sqrt{x^2-1}\cos\theta\}]}$$

$$= \int_0^{\pi} [t\{x \pm \sqrt{x^2-1}\cos\theta\} - 1]^{-1}\, d\theta$$

$$= \int_0^{\pi} (ts-1)^{-1}\, d\theta, \quad \text{where } s = x \pm \sqrt{x^2-1}\cos\theta$$

$$= \int_0^{\pi} \frac{1}{ts}\left(1 - \frac{1}{ts}\right)^{-1} d\theta$$

$$= \int_0^{\pi} \frac{1}{ts}\left(1 + \frac{1}{ts} + \frac{1}{t^2s^2} + \ldots + \frac{1}{t^ns^n} + \ldots\right) d\theta$$

$$= \int_0^{\pi} \left(\frac{1}{ts} + \frac{1}{t^2s^2} + \ldots + \frac{1}{t^{n+1}s^{n+1}} + \ldots\right) d\theta$$

$$= \int_0^{\pi} \sum_{n}^{\infty} \frac{1}{t^{n+1}s^{n+1}}\, d\theta$$

$$\therefore \quad \pi \sum_{n=0}^{\infty} P_n(x)\, t^n = \sum_{n=0}^{\infty} \cdot \frac{1}{t^{n+1}} \int_0^{\pi} \frac{d\theta}{[x \pm \sqrt{x^2-1}\cos\theta]^{n+1}}$$

or
$$\pi \cdot \frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} \frac{1}{t^{n+1}} \int_0^{\pi} \frac{d\theta}{[x \pm \sqrt{x^2-1}\cos\theta]^{n+1}}$$

or
$$\frac{\pi}{t} \cdot \frac{1}{\sqrt{1 - 2x \cdot \frac{1}{t} + \frac{1}{t^2}}} = \sum_{n=0}^{\infty} \frac{1}{t^{n+1}} \int_0^{\pi} \frac{d\theta}{[x \pm \sqrt{x^2-1}\cos\theta]^{n+1}}$$

or
$$\frac{\pi}{t} \sum_{n=0}^{\infty} \frac{1}{t^n} P_n(x) = \sum_{n=0}^{\infty} \frac{1}{t^{n+1}} \int_0^{\pi} \frac{d\theta}{[x \pm \sqrt{x^2-1}\cos\theta]^{n+1}}$$

or
$$\pi \sum_{n=0}^{\infty} \frac{1}{t^{n+1}} P_n(x) = \sum_{n=0}^{\infty} \frac{1}{t^{n+1}} \int_0^{\pi} \frac{d\theta}{[x \pm \sqrt{x^2-1}\cos\theta]^{n+1}}$$

$$\therefore \quad \pi P_n(x) = \int_0^{\pi} \frac{d\theta}{[x \pm \sqrt{x^2-1}\cos\theta]^{n+1}}$$

Hence
$$\boldsymbol{P_n(x) = \frac{1}{\pi} \int_0^{\pi} \frac{d\theta}{[x \pm \sqrt{x^2-1}\cos\theta]^{n+1}}}.$$

• 2.5. ORTHOGONAL PROPERTIES OF LEGENDRE'S POLYNOMIALS

(i) $\displaystyle\int_{-1}^{1} P_m(x)\, P_n(x)\, dx = 0$, **when** $m \neq n$.

(ii) $\displaystyle\int_{-1}^{1} [P_n(x)]^2\, dx = \frac{2}{2n+1}$, **when** $m = n$.

Proof. (i) Legendre differential equation is

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0$$

or
$$\frac{d}{dx}\left\{(1-x^2)\frac{dy}{dx}\right\} + n(n+1)y = 0 \quad \ldots(1)$$

since $P_m(x)$ and $P_n(x)$ are the solutions of (1), so, we have

$$\frac{d}{dx}\left\{(1-x^2)\frac{dP_m(x)}{dx}\right\} + m(m+1)P_m(x) = 0 \quad \ldots(2)$$

and
$$\frac{d}{dx}\left\{(1-x^2)\frac{dP_n(x)}{dx}\right\} + n(n+1)P_n(x) = 0 \quad \ldots(3)$$

Now multiplying (2) by $P_n(x)$ and (3) by $P_m(x)$ and then subtract, we get

$$\frac{d}{dx}\left\{(1-x^2)\frac{dP_m(x)}{dx}\right\}P_n(x) - \frac{d}{dx}\left\{(1-x^2)\frac{dP_n(x)}{dx}\right\}P_m(x)$$
$$+ [m(m+1) - n(n+1)]P_m(x)P_n(x) = 0 \quad \ldots(4)$$

Integrating (4) w.r.t. x from $x = -1$ to $x = +1$, we get

$$\int_{-1}^{1}\frac{d}{dx}\left\{(1-x^2)\frac{dP_m(x)}{dx}\right\}P_n(x)\,dx - \int_{-1}^{1}\frac{d}{dx}\left\{(1-x^2)\frac{dP_n(x)}{dx}\right\}P_m(x)\,dx$$
$$+ (m-n)(m+n+1)\int_{-1}^{1}P_m(x)P_n(x)\,dx = 0 \quad \ldots(5)$$

Let
$$I_1 = \int_{-1}^{1}\frac{d}{dx}\left\{(1-x^2)\frac{dP_m(x)}{dx}\right\}P_n(x)\,dx$$

and
$$I_2 = \int_{-1}^{1}\frac{d}{dx}\left\{(1-x^2)\frac{dP_n(x)}{dx}\right\}P_m(x)\,dx.$$

$\therefore$ (5) becomes

$$I_1 - I_2 + (m-n)(m+n+1)\int_{-1}^{1}P_m(x)P_n(x)\,dx = 0. \quad \ldots(6)$$

Now solving I_1 and I_2

$$\therefore \quad I_1 = \int_{-1}^{1}\frac{d}{dx}\left\{(1-x^2)\frac{dP_m(x)}{dx}\right\}P_n(x)\,dx$$

$$= P_n(x)\left[(1-x)^2\frac{dP_m(x)}{dx}\right]_{-1}^{1} - \int_{-1}^{1}\frac{dP_n(x)}{dx}(1-x^2)\frac{dP_m(x)}{dx}\,dx$$

(By Integration by parts)

$$= 0 - \int_{-1}^{1}(1-x^2)\frac{dP_n(x)}{dx}\cdot\frac{dP_m(x)}{dx}\,dx$$

$$I_1 = -\int_{-1}^{1}(1-x^2)\frac{dP_n(x)}{dx}\cdot\frac{dP_m(x)}{dx}\,dx.$$

Taking I_2,

$$I_2 = \int_{-1}^{1}\frac{d}{dx}\left\{(1-x^2)\frac{dP_n(x)}{dx}\right\}P_m(x)\,dx$$

$$= \left[P_m(x)\cdot(1-x^2)\frac{dP_n(x)}{dx}\right]_{-1}^{1} - \int_{-1}^{1}(1-x^2)\frac{dP_m(x)}{dx}\cdot\frac{dP_n(x)}{dx}\,dx$$

$$= 0 - \int_{-1}^{1} (1-x^2)\frac{dP_m(x)}{dx}\cdot\frac{dP_n(x)}{dx}\,dx$$

$$\therefore \qquad I_2 = -\int_{-1}^{1} (1-x^2)\frac{dP_m(x)}{dx}\cdot\frac{dP_n(x)}{dx}\,dx.$$

Thus $I_1 - I_2 = 0$. Now (6) becomes

$$0 + (m-n)(m+n+1)\int_{-1}^{1} P_m(x)\,P_n(x)\,dx = 0$$

if $m \neq n$, then

$$\int_{-1}^{1} P_m(x)\,P_n(x)\,dx = 0.$$

Proof. (ii) $\int_{-1}^{1} [P_n(x)]^2\,dx = \frac{2}{2n+1}$, if $m = n$.

Since we know that

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)\,t^n$$

or

$$\frac{1}{\sqrt{1-2xt+t^2}} = P_0(x) + tP_1(x) + t^2P_2(x) + \ldots + t^nP_n(x) + \ldots$$

Squaring of both sides, we get

$$\frac{1}{1-2xt+t^2} = [P_0(x) + tP_1(x) + t^2P_2(x) + \ldots + t^nP_n(x) + \ldots]^2$$

$$= [P_0(x)]^2 + [tP_1(x)]^2 + [t^2P_2(x)]^2 + \ldots + [t^nP_n(x)]^2 + \ldots$$
$$+ 2\,[tP_0(x)\,P_1(x) + t^2\,P_0(x)\,P_2(x) + \ldots + t^n\,P_0(x)\,P_n(x) + \ldots$$
$$\ldots + t^3P_1(x)\,P_2(x) + t^4P_1(x)\,P_3(x) + \ldots + t^{n+1}P_1(x)\,P_n(x) + \ldots]$$

$$= \sum_{n=0}^{\infty} t^{2n}[P_n(x)]^2 + 2\sum_{\substack{m,n=0\\ m\neq n}}^{\infty} t^{m+n}\,P_m(x)\,P_n(x)$$

$$\therefore \qquad \frac{1}{1-2xt+t^2} = \sum_{n=0}^{\infty} t^{2n}[P_n(x)]^2 + 2\sum_{\substack{m,n=0\\ m\neq n}}^{\infty} t^{m+n}\,P_m(x)\,P_n(x)$$

Integrating both sides w.r.t. x from $x = -1$ to 1, we get

$$\int_{-1}^{1}\frac{dx}{1-2xt+t^2} = \int_{-1}^{1}\sum_{n=0}^{\infty} t^{2n}[P_n(x)]^2\,dx + 2\int_{-1}^{1}\sum_{\substack{m,n=0\\ m\neq n}}^{\infty} t^{m+n}\,P_m(x)\,P_n(x)\,dx$$

$$= \sum_{n=0}^{\infty} t^{2n}\int_{-1}^{1}[P_n(x)]^2\,dx + 2\sum_{\substack{m,n=0\\ m\neq n}}^{\infty} t^{m+n}\int_{-1}^{1}P_m(x)\,P_n(x)\,dx$$

$$= \sum_{n=0}^{\infty} t^{2n}\int_{-1}^{1}[P_n(x)]^2\,dx + 0$$

$$\left(\because \int_{-1}^{1} P_m(x)\,P_n(x)\,dx = 0\right) \text{ when } m \neq n$$

$$\therefore \sum_{n=0}^{\infty} t^{2n}\int_{-1}^{1}[P_n(x)]^2\,dx = \int_{-1}^{1}\frac{dx}{1-2xt+t^2}$$

$$= -\frac{1}{2t}\Big[\log(1-2xt+t^2\Big]_{-1}^{1}$$

$$= -\frac{1}{2t}[\log(1-2t+t^2) - \log(1+2t+t^2)]$$

$$= -\frac{1}{2t}\{\log(1-t)^2 - \log(1+t)^2]$$

$$= -\frac{1}{2t}\left[\log\left(\frac{1-t}{1+t}\right)^2\right]$$

$$= -\frac{1}{t}\left[\log\frac{1-t}{1+t}\right] = \frac{1}{t}\left[\log\frac{1+t}{1-t}\right]$$

$$= \frac{1}{t}[\log(1+t) - \log(1-t)]$$

$$= \frac{1}{t}\left[\left\{t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \dots\right\} - \left\{-t - \frac{t^2}{2} - \frac{t^3}{3} - \frac{t^4}{4} - \dots\right\}\right]$$

$$= \frac{1}{t}\left[2t + \frac{2t^3}{3} + \frac{2t^5}{5} + \dots\right]$$

$$= 2\left[1 + \frac{t^2}{3} + \frac{t^4}{5} + \dots\right] = 2\sum_{n=0}^{\infty}\frac{t^{2n}}{2n+1}$$

$$\therefore \quad \sum_{n=0}^{\infty} t^{2n}\int_{-1}^{1}[P_n(x)]^2\,dx = \sum_{n=0}^{\infty}\frac{2}{2n+1}\cdot t^{2n}$$

Hence $$\int_{-1}^{1}[P_n(x)]^2\,dx = \frac{2}{2n+1}.$$

• 2.6. RECURRENCE RELATIONS

(I) $(2n+1)\,x\,P_n = (n+1)\,P_{n+1} + n\,P_{n-1}$.

Proof. Since we know that

$$(1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} t^n P_n(x). \qquad \dots(1)$$

Differentiating (1) both sides w.r.t. 't', we get

$$-\frac{1}{2}(1-2xt+t^2)^{-3/2}\cdot(-2x+2t) = \sum_{n=1}^{\infty} nt^{n-1}P_n(x)$$

or $$\frac{(x-t)(1-2xt+t^2)^{-1/2}}{(1-2xt+t^2)} = \sum_{n=1}^{\infty} nt^{n-1}P_n(x)$$

or $$(x-t)(1-2xt+t^2)^{-1/2} = (1-2xt+t^2)\sum_{n=1}^{\infty} nt^{n-1}P_n(x)$$

or $$(x-1)\sum_{n=0}^{\infty} t^nP_n(x) = (1-2xt+t^2)\sum_{n=1}^{\infty} nt^{n-1}P_n(x) \qquad \text{[from (1)]}$$

or $$x\sum_{n=0}^{\infty} t^nP_n(x) - \sum_{n=0}^{\infty} t^{n+1}P_n(x) = \sum_{n=1}^{\infty} nt^{n-1}P_n(x) - 2x\sum_{n=1}^{\infty} nt^nP_n(x) + \sum_{n=1}^{\infty} nt^{n+1}P_n(x)$$

$$x(P_0(x) + tP_1(x) + \dots + t^nP_n(x) + \dots) - (tP_0(x) + t^2P_1(x) + \dots + t^nP_{n-1}(x) + \dots)$$
$$= (P_1(x) + 2tP_2(x) + \dots + (n+1)\,t^n P_{n+1}(x) + \dots)$$
$$- 2x\,(tP_1(x) + 2t^2P_2(x) + \dots + nt^nP_n(x) + \dots)$$
$$+ (t^2P_1(x) + 2t^3P_2(x) + \dots + (n-1)\,t^nP_{n-1}(x) + \dots)$$

Taking the coefficient of t^n both sides, we get

$$xP_n(x) - P_{n-1}(x) = (n+1)P_{n+1}(x) - 2nx\,P_n(x) + (n-1)P_{n-1}(x)$$

or $$(2n+1)\,x\,P_n(x) = (n+1)P_{n+1}(x) + n\,P_{n-1}(x)$$

or $$\mathbf{(2n+1)\,x\,P_n = (n+1)P_{n+1} + n\,P_{n-1}}.$$

(II) $\boldsymbol{nP_n = xP'_n - P'_{n-1}}$**, where** $\boldsymbol{P'_n \equiv \frac{dP_n}{dx}}$ **etc.**

Proof. Since we have

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} t^n P_n(x). \quad \ldots(1)$$

Differentiating (1) both sides w.r.t. 't' and w.r.t. x, respectively, we get

$$(x-t)(1-2xt+t^2)^{-3/2} = \sum_{n=1}^{\infty} nt^{n-1} P_n(x) \quad \ldots(2)$$

and $$t(1-2xt+t^2)^{-3/2} = \sum_{n=0}^{\infty} t^n P'_n(x)$$

From (2) and (3), we get

$$(x-t)\sum_{n=0}^{\infty} t^n P'_n(x) = t\sum_{n=1}^{\infty} nt^{n-1} P_n(x)$$

or $$x\sum_{n=0}^{\infty} t^n P'_n(x) - \sum_{n=0}^{\infty} t^{n+1} P'_n(x) = \sum_{n=1}^{\infty} nt^n P_n(x)$$

or $$x(P'_0(x) + tP'_1(x) + \ldots + t^n P'_n(x) + \ldots) - (tP'_0(x) + t^2P'_1(x) + \ldots + t^{n-1}P_{n-1}(x) + \ldots)$$
$$= tP_1(x) + 2t^2 P_2(x) + \ldots + nt^n P_n(x) + \ldots .$$

Taking the coefficients of t^n of both sides, we get

$$xP'_n(x) - P'_{n-1}(x) = n\,P_n(x)$$

or $$\boldsymbol{nP_n = xP'_n - P'_{n-1}}.$$

(III) $\boldsymbol{(2n+1)P_n = P'_{n+1} - P'_{n-1}}$.

Proof. From recurrence relation (I), we have

$$(2n+1)\,xP_n = (n+1)P_{n+1} + nP_{n-1}.$$

Differentiating this w.r.t. 'x' of both sides, we get

$$(2n+1)P_n + (2n+1)\,xP'_n = (n+1)P'_{n+1} + nP'_{n-1}.$$

From recurrence relation II, we have

$$nP_n = xP'_n - P'_{n-1}$$

or $$xP'_n = nP_n + P'_{n-1}$$

substitute this value of xP'_n into (1), we get

$$(2n+1)P_n + (2n+1)(nP_n + P'_{n-1}) = (n+1)P'_{n+1} + nP'_{n-1}$$

or $$(n+1)(2n+1)P_n = (n+1)P'_{n+1} - (2n+1)P'_{n-1} + nP'_{n-1}$$
$$= (n+1)P'_{n+1} - (n+1)P'_{n-1}$$

$\therefore$ $$(2n+1)P_n = P'_{n+1} - P'_{n-1}.$$

(IV) $\boldsymbol{(n+1)P_n = P'_{n+1} - xP'_n}$

Proof. From recurrence relations II and III, we have

$$nP_n = xP'_n - P'_{n-1}$$

and $$(2n+1)P_n = P'_{n+1} - P'_{n-1} \quad \ldots(2)$$

substract (1) from (2), we get

$$(2n+1)P_n - nP_n = P'_{n+1} - xP'_n$$

or $$(n+1)P_n = P'_{n+1} - xP'_n$$

(V) $\boldsymbol{(1-x^2)P'_n = n(P_{n-1} - xP_n)}$.

Proof. From recurrence relations (II) and (IV), we have

$$nP_n = xP'_n - P'_{n-1} \quad \ldots(1)$$

and $$(n+1)P_n = P'_{n+1} - xP'_n \quad \ldots(2)$$

Putting $(n-1)$ in place of n in (2), we get

$$nP_{n-1} = P'_n - xP'_{n-1} \quad \ldots(3)$$

Now multiplying (1) by x and subtract from (3), we get

$$nP_{n-1} - nxP_n = P'_n - x^2 P'_n$$

or $$n(P_{n-1} - xP_n) = (1-x^2) P'_n$$

or $$(1-x^2) P'_n = n(P_{n-1} - xP_n)$$

(VI) $(1-x^2) P'_n = (n+1)(xP_n - P_{n+1})$.

Proof. *From recurrence relations (I) and (V), we have*

$$(2n+1) xP_n = (n+1) P_{n+1} + nP_{n-1} \quad \ldots(1)$$

and $$(1-x^2) P'_n = n(P_{n-1} - xP_n) \quad \ldots(2)$$

substitute the value of nP_{n-1} from (1) into (2), we get

$$(1-x^2) P'_n = (2n+1) xP_n - (n+1) P_{n+1} - nxP_n$$

$$= (n+1) xP_n - (n+1) P_{n+1}.$$

$\therefore$ $$(1-x^2) P'_n = (n+1)(xP_n - P_{n+1}).$$

Beltrami's Relation :

The following relation

$$\mathbf{(2n+1)(x^2-1) P'_n = n(n+1)(P_{n+1} - P_{n-1})}$$

*is known as **Beltrami's Relation**.*

Proof. From recurrence relations (V) and (VI), we have

$$(1-x^2) P'_n = n(P_{n-1} - xP_n) \quad \ldots(1)$$

and $$(1-x^2) P'_n = (n+1)(xP_n - P_{n+1}) \quad \ldots(2)$$

Eliminating xP_n from (1) and (2), we get

$$\frac{(1-x^2) P'_n}{n} + \frac{(1-x^2) P'_n}{n+1} = P_{n-1} - P_{n+1}$$

or $$\frac{(n+1)(1-x^2) P'_n + n(1-x^2) P'_n}{n(n+1)} = P_{n-1} - P_{n+1}$$

or $$(2n+1)(1-x^2) P'_n = n(n+1)(P_{n-1} - P_{n+1})$$

or $$(2n+1)(x^2-1) P'_n = n(n+1)(P_{n+1} - P_{n-1}).$$

• 2.7. CHRISTOFFEL'S EXPANSION

The following series

$$\mathbf{P'_n = (2n-1) P_{n-1} + (2n-5) P_{n-3} + (2n-9) P_{n-5} + \ldots + l}$$

where $$l = \begin{cases} 3P_1, & \text{if } n \text{ is even} \\ P_0, & \text{if } n \text{ is odd} \end{cases}$$

is known as Christoffel's Expansion.

Proof. From recurrence relation (III), we have

$$(2n+1) P_n = P'_{n+1} - P'_{n-1}$$

$\therefore$ $$P'_{n+1} = (2n+1) P_n + P'_{n-1} \quad \ldots(1)$$

Now putting $(n-1)$ in place of n in (1), we get

$$P'_n = (2n-1) P_{n-1} + P'_{n-2} \quad \ldots(2)$$

Now putting $(n-2)$, $(n-4)$, $(n-6)$, ... in place of n in (2), we get

$$P'_{n-2} = (2n-5) P_{n-3} + P'_{n-4} \quad \ldots(3)$$

$$P'_{n-4} = (2n-9) P_{n-5} + P'_{n-6} \quad \ldots(4)$$

$$P'_{n-6} = (2n-13) P_{n-7} + P'_{n-8} \quad \ldots(5)$$

$$\vdots$$

$$P'_2 = 3P_1 + P'_0, \quad \text{if } n \text{ is even.}$$

Adding (2), (3), (4), (5), ... , we get

$$P'_n = (2n-1) P_{n-1} + (2n-5) P_{n-3} + (2n-9) P_{n-5} + (2n-13) P_{n-7} + \ldots + 3P_1 + P'_0$$

$$= (2n-1) P_{n-1} + (2n-5) P_{n-3} + (2n-9) P_{n-5} + \ldots + 3P_1$$

If n is odd, then

$$P'_n = (2n-1) P_{n-1} + (2n-5) P_{n-3} + (2n-9) P_{n-5} + \ldots + 5P_2 + P'_1$$

$$= (2n-1) P_{n-1} + (2n-5) P_{n-3} + (2n-9) P_{n-5} + \ldots + 5P_2 + P_0 \quad (\because\ P_0 = 1 = P'_1)$$

Hence, we obtained Christoffel's Expansion.

Christoffel's Summation Formula :

The following summation

$$\sum_{K=0}^{n} (2K+1) P_K(x) P_K(y) = (n+1)\left[\frac{P_{n+1}(x) P_n(y) - P_{n+1}(y) P_n(x)}{(x-y)}\right]$$

is known as Christoffel's summation.

Proof. From Recurrence relation I, we have

$$(2K+1)\, x\, P_K(x) = (K+1) P_{K+1}(x) + KP_{K-1}(x) \qquad \ldots(1)$$

and
$$(2K+1)\, y\, P_K(y) = (K+1) P_{K+1}(y) + K P_{K-1}(y) \qquad \ldots(2)$$

Multiplying (1) by $P_K(y)$ and (2) by $P_K(x)$ and then subtract, we get

$$(2K+1)(x-y) P_K(x) P_K(y) = (K+1)[P_{K+1}(x) P_K(y) - P_K(x) P_{K+1}(y)] + K\{P_{K-1}(x) P_K(y) - P_K(x) P_{K-1}(y)].$$

Taking summation from $K = 0$ to $K = n$, we get

$$(x-y) \sum_{K=0}^{n} (2K+1) P_K(x) P_K(y)$$

$$= \sum_{K=0}^{n} (K+1)[P_{K+1}(x) P_K(y) - P_K(x) P_{K+1}(y)] + \sum_{K=0}^{n} K[P_{K-1}(x) P_K(y) - P_K(x) P_{K-1}(y)]$$

$$= \{[P_1(x) P_0(y) - P_0(x) P_1(y)] + 2[P_2(x) P_1(y) - P_1(x) P_2(y)] + 3[P_3(x) P_2(y) - P_2(x) P_3(y)] + \ldots + n[P_n(x) P_{n-1}(y) - P_{n-1}(x) P_n(y)] + (n+1)[P_{n+1}(x) P_n(y) - P_n(x) P_{n+1}(y)]\}$$
$$+ \{[P_0(x) P_1(y) - P_1(x) P_0(y)] + 2[P_1(x) P_2(y) - P_2(x) P_1(y)] + 3[P_2(x) P_3(y) - P_3(x) P_2(y)] + \ldots + (n-1)[P_{n-2}(x) P_{n-1}(y) - P_{n-1}(x) P_{n-2}(y)] + n[P_{n-1}(x) P_n(y) - P_n(x) P_{n-1}(y)]\}$$

$$= (n+1)[P_{n+1}(x) P_n(y) - P_n(x) P_{n+1}(y)] \quad \text{(All the terms cencel except above)}$$

$$\therefore \quad \sum_{K=0}^{n} (2K+1) P_K(x) P_K(y) = (n+1)\left[\frac{P_{n+1} P_n(y) - P_n(x) P_{n+1}(y)}{(x-y)}\right].$$

SOLVED EXAMPLES

Example 1. *Prove that* $|P_n(x)| < 1$, *when* $-1 < x < 1$.

Solution. From Laplace first integral for $P_n(x)$, we have

$$P_n(x) = \frac{1}{\pi}\int_0^{\pi} [x \pm \sqrt{x^2-1}\cos\theta]^n\, d\theta. \qquad \ldots(1)$$

Now taking

$$|[x \pm \sqrt{x^2-1}\cos\theta]| = |x \pm i\sqrt{1-x^2}\cos\theta|$$
$$= \sqrt{x^2 + (1-x^2)\cos^2\theta} = \sqrt{1-(1-x^2)\sin^2\theta}$$

$$\therefore \quad |x \pm \sqrt{x^2-1}\cos\theta| < 1 \text{ except } \theta = 0 \text{ and } \theta = \pi.$$

From (1), we have

$$|P_n(x)| = \left| \frac{1}{\pi}\int_0^{\pi} [x \pm \sqrt{x^2-1}\cos\theta]^n \, d\theta \right|$$

$$\leq \frac{1}{\pi}\int_0^{\pi} | x \pm \sqrt{x^2-1}\cos\theta |^n \, d\theta$$

$$< \frac{1}{\pi}\int_0^{\pi} 1 \, . \, d\theta = \frac{1}{\pi} . \pi = 1$$

$\therefore \qquad |P_n(x)| < 1.$

Example 2. *Show that* $P_n(-x) = (-1)^n P_n(x)$ *and* $P'_n(-x) = (-1)^{n+1} P'_n(x)$.

Solution. (i) Since we have

$$P_n(x) = \sum_{m=0}^{[n/2]} \frac{(-1)^m (2n-2m)!}{2^n (m)! (n-m)! (n-2m)!} . x^{n-2m}$$

putting $-x$ in place of x, we get

$$P_n(-x) = \sum_{m=0}^{[n/2]} \frac{(-1)^m (2n-2m)!}{2^n (m)! (n-m)! (n-2m)!} . (-x)^{n-2m}$$

$$= \sum_{m=0}^{[n/2]} \frac{(-1)^m (2n-2m)!}{2^n (m)! (n-m)! (n-2m)!} . (-1)^{n-2m} . x^{n-2m}$$

$$= (-1)^n \sum_{m=0}^{[n/2]} \frac{(-1)^m (2n-2m)!}{2^n (m)! (n-m)! (n-2m)!} . x^{n-2m} \qquad (\because \ (-1)^{-2m} = 1)$$

$$= (-1)^n P_n(x).$$

Hence $\qquad \boldsymbol{P_n(-x) = (-1)^n P_n(x)}.$

(ii) To show $\ P'_n(-x) = (-1)^{n+1} P'_n(x)$

From above result we have

$$P_n(-x) = (-1)^n P_n(x).$$

Differentiating both sides w.r.t. 'x', we get

$$-P'_n(-x) = (-1)^n P'_n(x)$$

or
$$P'_n(-x) = (-1)^{n+1} P'_n(x).$$

Hence proved the result.

Example 3. *Show that* $P_n(1) = 1$ *and* $P_n(-1) = (-1)^n$.

Solution. Since we have

$$\sum_{n=0}^{\infty} P_n(x) t^n = (1 - 2xt + t^2)^{-1/2} \qquad \ldots(1)$$

putting $x = 1$ of both sides

$$\sum_{n=0}^{\infty} P_n(1) t^n = (1 - 2t + t^2)^{-1/2} = (1-t)^{-1}$$

or
$$[P_0(1) + tP_1(1) + \ldots + t^n P_n(1) + \ldots] = [1 + t + t^2 + \ldots + t^n + \ldots].$$

Taking the coefficient of t^n of both sides, we get

$$P_n(1) = 1$$

Hence proved.

Next putting $x = -1$ in (1), we get

$$\sum_{n=0}^{\infty} P_n(-1) t^n = (1 + 2t + t^2)^{-1/2} = (1+t)^{-1}$$

or
$$[P_0(-1) + tP_1(-1) + \ldots + t^n P_n(-1) + \ldots] = [1 - t + t^2 - \ldots + (-1)^n t^n + \ldots].$$

Comparing the coefficient of t^n of both sides, we get

$$P_n(-1) = (-1)^n.$$

Hence proved.

Example 4. *Prove that* $\int_{-1}^{1} x^2 P_{n+1} P_{n-1}\, dx = \frac{2n(n+1)}{(2n-1)(2n+1)(2n+3)}$.

Solution. From Recurrence relation I, we have

$$(2n+1)\, x P_n = (n+1) P_{n+1} + n P_{n-1}.$$

Putting $(n-1)$ and $(n+1)$ in place of n respectively, we get

$$(2n-1)\, x P_{n-1} = n P_n + (n-1) P_{n-2} \quad \text{...(2)}$$

$$(2n+3)\, x P_{n+1} = (n+2) P_{n+2} + (n+1) P_n \quad \text{...(3)}$$

Multiplying (2) and (3), we get

$$\begin{aligned}(2n-1)(2n+3)\, x^2 P_{n+1} P_{n-1} &= (nP_n + (n-1) P_{n-2})((n+2) P_{n+2} + (n+1) P_n) \\ &= n(n+2) P_n P_{n+2} + n(n+1)(P_n)^2 \\ &\quad + (n-1)(n+2) P_{n-2} P_{n+2} + (n^2-1) P_{n-2} P_n.\end{aligned}$$

Now integrating from $x = -1$ to $x = 1$ w.r.t 'x', we get

$$\begin{aligned}(2n-1)(2n+3)\int_{-1}^{1} x^2 P_{n+1} P_{n-1} &= n(n+2)\int_{-1}^{1} P_n P_{n+2}\, dx \\ &\quad + n(n+1)\int_{-1}^{1} [P_n]^2\, dx + (n-1)(n+2)\int_{-1}^{1} P_{n-2} P_{n+2}\, dx \\ &\quad + (n^2-1)\int_{-1}^{1} P_{n-2} P_n\, dx \\ &= n(n+1)\int_{-1}^{1} [P_n]^2\, dx + 0 + 0 + 0 \\ &= n(n+1)\left[\frac{2}{2n+1}\right] \quad \text{(By orthogonal properties)}\end{aligned}$$

$$\therefore \quad \int_{-1}^{1} x^2 P_{n+1} P_{n-1}\, dx = \frac{2n(n+1)}{(2n-1)(2n+1)(2n+3)}.$$

Example 5. *Prove that* $\int_{-1}^{1} (x^2-1) P_{n+1} P'_n\, dx = \frac{2n(n+1)}{(2n+1)(2n+3)}$.

Solution. Since we have

$$(2n+1)(x^2-1) P'_n = n(n+1)(P_{n+1} - P_{n-1}) \quad \text{(Beltrami's result)}$$

Now multiplying by P_{n+1} and then integrating from $x = -1$ to 1, we get

$$\begin{aligned}(2n+1)\int_{-1}^{1} (x^2-1) P_{n+1} P'_n\, dx &= n(n+1)\int_{-1}^{1} [P_{n+1}]^2\, dx - n(n+1)\int_{-1}^{1} P_{n+1} P_{n-1}\, dx \\ &= n(n+1)\left[\frac{2}{2n+3}\right] - 0 \quad \text{(By orthogonal properties)}\end{aligned}$$

$$(2n+1)\int_{-1}^{1} (x^2-1) P_{n+1} P'_n\, dx = \frac{2n(n+1)}{(2n+3)}$$

$$\therefore \quad \int_{-1}^{1} (x^2-1) P_{n+1} P'_n\, dx = \frac{2n(n+1)}{(2n+1)(2n+3)}.$$

Example 6. *Show that* $\int_{-1}^{1} x\, P_n\, P_{n-1}\, dx = \frac{2n}{4n^2 - 1}$.

Solution. From Recurrence relation I, we have

$$(2n+1)\, x\, P_n = (n+1)\, P_{n+1} + n\, P_{n-1}. \qquad \ldots(1)$$

Multiplying (1) by P_{n-1} and then integrate from $x = -1$ to 1

$$(2n+1)\int_{-1}^{1} xP_nP_{n-1}dx = (n+1)\int_{-1}^{1} P_{n+1}P_{n-1}dx + n\int_{-1}^{1} [P_{n-1}]^2\, dx$$

$$= 0 + n\left[\frac{2}{2n-1}\right] \qquad \text{(By orthogonal properties)}$$

$$= \frac{2n}{2n-1}$$

$$\therefore \quad \int_{-1}^{1} x\, P_n\, P_{n-1}\, dx = \frac{2n}{(2n+1)(2n-1)} = \frac{2n}{4n^2-1}.$$

Example 7. *Show that* $\int_{-1}^{1} \frac{P_n(x)}{\sqrt{1-2xt+t^2}}\, dx = \frac{2t^n}{2n+1}$.

Solution. Since we have

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)\, t^n$$

or

$$\frac{1}{\sqrt{1-2xt+t^2}} = P_0(x) + tP_1(x) + \ldots + t^n P_n(x) + t^{n+1} P_{n+1}(x) + \ldots$$

Now multiplying this equation by $P_n(x)$ and then integrating from $x = -1$ to 1, we get

$$\int_{-1}^{1} \frac{P_n(x)}{\sqrt{1-2xt+t^2}}\, dx = \int_{-1}^{1} P_0(x)\, P_n(x)\, dx + t\int_{-1}^{1} P_1(x)\, P_n(x)\, dx + \ldots$$

$$+ t^n \int_{-1}^{1} [P_n(x)]^2\, dx + t^{n+1}\int_{-1}^{1} P_{n+1}(x)\, P_n(x)\, dx + \ldots$$

$$= t^n \int_{-1}^{1} [P_n(x)]^2\, dx \qquad \text{(All integral except one is zero)}$$

$$= t^n\left[\frac{2}{2n+1}\right]$$

$$\therefore \int_{-1}^{1} \frac{P_n(x)}{\sqrt{1-2xt+t^2}}\, dx = \frac{2t^n}{2n+1}.$$

• SUMMARY

- Legendre's D.E. : $(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)\, y = 0$.
- Legendre's Polynomial :

$$P_n(x) = \sum_{m=0}^{N} a_{n-2m}\, x^{n-2m}$$

where $\quad a_{n-2} = -\frac{n(n-1)}{2(2n-1)}\, a_n, \quad a_n = \frac{(2n)!}{2^n (n!)^2}$

and $\quad N = \begin{cases} n/2, & \text{if } n \text{ is even} \\ (n-1)/2, & \text{if } n \text{ is odd.} \end{cases}$

- Generating function : $\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)\, t^n$.

- Rodrigue's formula : $P_n(x) = \frac{1}{2^n n!} \cdot \frac{d^n}{dx^n} [(x^2-1)^n]$
- Laplace first integral for $P_n(x)$:

$$P_n(x) = \frac{1}{\pi}\int_0^{\pi} [x \pm (\sqrt{x^2-1}) \cos\theta]^n \, d\theta$$

- Laplace second integral for $P_n(x)$:

$$P_n(x) = \frac{1}{\pi}\int_0^{\pi} \frac{d\theta}{\sqrt{[x \pm (\sqrt{x^2-1})\cos\theta]^{n+1}}}$$

- Orthogonal properties of $P_n(x)$:

$$\int_{-1}^{1} P_m(x)\, P_n(x)\, dx = \begin{cases} 0, & m \neq n \\ \frac{2}{2n+1}, & m = n \end{cases}$$

- Recurrence Relations
  (i) $(2n+1)\, x\, P_n = (n+1)\, P_{n+1} + nP_{n-1}$
  (ii) $nP_n = xP'_n - P'_{n-1}$
  (iii) $(2n+1)\, P_n = P'_{n+1} - P'_{n-1}$
  (iv) $(n+1)\, P_n = P'_{n+1} - xP'_n$
  (v) $(1-x^2)\, P'_n = n\,(P_{n-1} - xP_n)$
  (vi) $(1-x^2)\, P'_n = (n+1)\,(xP_n - P_{n+1})$

• STUDENT ACTIVITY

1. Solve that : $\int_{-1}^{1} P_m(x)\, P_n(x)\, dx = 0, \quad m \neq n$

2. Prove that : $(2n+1)\, xP_n = (n+1)\, P_{n+1} + nP_{n-1}$

• TEST YOURSELF

1. Show that $\frac{1-t^2}{(1-2xt+t^2)^{3/2}} = \sum_{n=0}^{\infty} (2n+1)\, P_n(x)\, t^n$.
2. Prove that $\int_{-1}^{1} (P'_n)^2\, dx = n\,(n+1)$.
3. Show that $2P_2(x) - 3P_1(x)\, P_1(x) + 1 = 0$.

4. Prove that $P'_{n+1} + P'_n = \sum_{r=0}^{n} (2r+1) P_r(x)$.

5. Prove that $\int_{-1}^{1} x^n P_n(x)\,dx = \frac{2^{n+1}((n)!)^2}{(2n+1)!}$.

6. Prove that

(i) $\int_{-1}^{1} P_n(x)\,dx = 0, \quad n \neq 0$ (ii) $\int_{-1}^{1} P_0(x)\,dx = 2$.

7. Find the value of the integrals

(i) $\int_{-1}^{1} x^{99} P_{100}(x)\,dx$ (ii) $\int_{-1}^{1} x^2 P_2(x)\,dx$.

8. Prove that

(i) $P'_n(1) = \frac{1}{2} n(n+1)$ (ii) $P'_n(-1) = (-1)^{n-1} \frac{1}{2} n(n+1)$.

ANSWERS

1. (i) 0 (ii) $\frac{4}{15}$.

OBJECTIVE EVALUATION

Fill in the blanks :

1. The solution of Legendre's D.E. is known as ....
2. $P_n(x)$, the Legendre's polynomial has a degree .... if n is even.
3. $|P_n(x)| <$ .... if $-1 < x < 1$.
4. $\int_{-1}^{1} P_0(x)\,dx =$ ....

True or False

1. The equation $P_n(x) = 0$ has its all roots real. (T/F)
2. $P_n(1) = 0$.
3. $\int_{-1}^{1} P_n(x)\,dx = 0.\ n \neq 0$. (T/F)

Multiple Choice Questions (MCQ's) :

1. $P_n(x)$ is an even function if n equals :

(a) -1 (b) 0 (c) 3 (d) 4

2. $P_1(x)$ equals :

(a) $\frac{x^2}{2}$ (b) x (c) 1 (d) $-x$

3. $\int_{-1}^{1} (P_n(x))^2\,dx$ equals :

(a) $\frac{1}{2n+1}$ (b) $\frac{1}{n}$ (c) $\frac{2}{n+1}$ (d) $\frac{2}{2n-1}$

ANSWERS

Fill in the Blanks :

1. Legendre's Function 2. Even 3. 1 4. 2

True or Flase :

1. T 2. F 3. T

MCQ

1. (d) 2. (b) 3. (c)

UNIT 3

BESSEL'S FUNCTIONS

STRUCTURE

- Bessel's D.E. and its solution
- General Solution
- Linear Dependence
- Definition of $J_n(x)$, when $n = 0$
- Generating function for $J_n(x)$
- Recurrence Relations
  - Summary
  - Student Activity
  - Test Yourself

LEARNING OBJECTIVES

After going through this unit you will learn :

- What is Bessel's Differential equation ?
- The power series solution of Bessel's D.E. is the Bessel's function.
- How to generate Bessel's functions ?
- What are their recurrence solutions ?

3.1. BESSEL'S FUNCTION

The homogeneous linear differential equation of the form

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)\, y = 0 \qquad ...(1)$$

is known as Bessel's differential equation, where n is a non-negative real number.

Solution of the Bessel's Functions :

Change the differential equation (1) into standard form by dividing (1) by x^2.

$$\therefore \qquad \frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(1 - \frac{n^2}{x^2}\right) y = 0 \qquad ...(2)$$

Now compare this differential equation with following equation

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)\, y = 0$$

$$\therefore \qquad P(x) = \frac{1}{x}, \quad Q(x) = \left(1 - \frac{n^2}{x^2}\right).$$

It is obvious from $P(x)$ and $Q(x)$ that $x = 0$ is a singular point which is located at the origin. Therefore we assume the solution of (1) in the form of a power series of the following type

$$y = \sum_{m=0}^{\infty} a_m x^{m+r} \quad (a_0 \neq 0) \qquad ...(3)$$

Differentiating (3) w.r.t. x, we get

$$\frac{dy}{dx} = \sum_{m=0}^{\infty} a_m (m + r)\, x^{m+r-1} \qquad ...(4)$$

Again differentiating (4) w.r.t. x, we get

$$\frac{d^2y}{dx^2} = \sum_{m=0}^{\infty} a_m (m+r)(m+r-1) x^{m+r-2}. \quad \ldots(5)$$

Now substitute the values of $y, \frac{dy}{dx}, \frac{d^2y}{dx^2}$ from (3), (4) and (5) into (1), we have

$$x^2 \sum_{m=0}^{\infty} a_m (m+r)(m+r-1) x^{m+r-2} + x \sum_{m=0}^{\infty} a_m (m+r) x^{m+r-1} + (x^2 - n^2) \sum_{m=0}^{\infty} a_m x^{m+r} = 0$$

$$\sum_{m=0}^{\infty} a_m (m+r)(m+r-1) x^{m+r} + \sum_{m=0}^{\infty} a_m (m+r) x^{m+r} - \sum_{m=0}^{\infty} a_m n^2 x^{m+r} + \sum_{m=0}^{\infty} a_m x^{m+r+2} = 0 \quad \ldots(6)$$

equation (6) will be an identity if the equation (3) is a solution of (1), then coefficient of each terms in (6) will be zero. Thus taking the coefficient of x^r, x^{r+1}

$$a_0 r(r-1) + a_0 r - n^2 a_0 = 0 \quad \ldots(7)$$

$$a_1 (r+1) r + a_1 (r+1) - n^2 a_1 = 0. \quad \ldots(8)$$

In general taking the coefficients of x^{s+r}

$$a_s (s+r)(s+r-1) + a_s (s+r) - n^2 a_s + a_{s-2} = 0 \quad \ldots(9)$$

for $\qquad s = 2, 3, 4, \ldots$

From (7), we have

$$r(r-1) + r - n^2 = 0 \qquad (\because a_0 \neq 0)$$

or $$r^2 - n^2 = 0$$

or $$r = n, -n.$$

From (8), we have

$$[(r+1) r + (r+1) - n^2] a_1 = 0$$

For any value of $r = n, -n$, we get

$$a_1 = 0$$

From (9), we have

$$a_s [(s+r)(s+r-1) + s + r - n^2] + a_{s-2} = 0$$

or $$a_s [(s+r)^2 - n^2] + a_{s-2} = 0$$

or $$a_s (s+r-n)(s+r+n) + a_{s-2} = 0 \quad \ldots(10)$$

For case if $r = n$, then (1) becomes

$$a_s (s)(s+2n) + a_{s-2} = 0$$

or $$a_s = -\frac{1}{s(s+2n)} \cdot a_{s-2}$$

Putting $s = 2, 3, 4, 5, \ldots$

$$a_2 = -\frac{1}{2(2+2n)} a_0$$

$$a_3 = -\frac{1}{3(3+2n)} a_1 = 0 \qquad (\because a_1 = 0)$$

$$a_4 = -\frac{1}{4(4+2n)} a_2 = -\frac{1}{4(4+2n)} - \frac{1}{2(2+2n)} a_0$$

$$= (-1)^2 \frac{1}{2 \cdot 4 (2+2n)(4+2n)} a_0$$

$$\vdots$$

etc.

We observed that $a_1 = a_3 = a_5 = \ldots = 0$. Since a_0 is arbitrary. Let us choose

$$a_0 = \frac{1}{2^n \Gamma(n+1)}$$

where $\Gamma(n+1)$ is a the **Gamma function**, therefore, we know that $\Gamma(n+1) = n\,\Gamma(n)$ and if n is positive integer, $\Gamma(n+1) = (n)\,!$. Thus,

$$a_2 = -\frac{1}{2(2+2n)} a_0$$

$$= -\frac{1}{2^2(1+n)} \cdot \frac{1}{2^n \Gamma(n+1)}$$

$$= -\frac{1}{2^{n+2}\Gamma(n+2)}$$

$$a_4 = (-1)^2 \frac{1}{2^4 \cdot (2)\,!\,(1+n)(2+n)} \cdot \frac{1}{2^n \Gamma(n+1)}$$

$$= (-1)^2 \frac{1}{2^{n+4} \cdot (2)\,!\,\Gamma(n+3)}$$

and so on. Now From (3), we have

$$y = \sum_{m=0}^{\infty} a_m x^{m+r}$$

$$= a_0 x^r + a_1 x^{1+r} + a_2 x^{2+r} + a_3 x^{3+r} + a_4 x^{4+r} + \ldots$$

$$= a_0 x^r + a_2 x^{2+r} + a_4 x^{4+r} + \ldots$$

$$= a_0 x^n + \left\{-\frac{1}{2^{n+2}\Gamma(n+2)} x^{n+2}\right\} + \frac{1}{2^{n+4}(2)\,!\,\Gamma(n+3)} x^{n+4} + \ldots$$

$$= \frac{1}{2^n \Gamma(n+1)} x^n - \frac{1}{2^{n+2}(1)\,!\,\Gamma(n+2)} x^{n+2} + \frac{1}{2^{n+4}(2)\,!\,\Gamma(n+3)} x^{n+4} + \ldots$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m x^{n+2m}}{2^{n+2m}(m)\,!\,\Gamma(n+m+1)}.$$

This solution is known as Bessel's function, which is denoted by $J_n(x)$. This function is also known as **Bessel's function of first kind.**

$$\therefore \qquad J_n(x) = \sum_{m=0}^{\infty} (-1)^m \frac{x^{n+2m}}{2^{n+2m}(m)\,!\,\Gamma(n+m+1)},$$

For case if $r = -n$, we have

$$J_{-n}(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{-n+2m}}{2^{-n+2m}(m)\,!\,\Gamma(-n+m+1)} \qquad \ldots(12)$$

• 3.2. GENERAL SOLUTIONS

The solution of the Bessel's differential equation of the type

$$y(x) = AJ_n(x) + BJ_{-n}(x)$$

where A and B are arbitrary constants, is called general solution.

• 3.3. Linear Dependence :

For an integer $r = n$, the Bessel's functions $J_n(x)$ and $J_{-n}(x)$ are linearly dependent, because

$$J_{-n}(x) = (-1)^n J_n(x) \qquad \textit{for } n = 1, 2, \ldots$$

Proof. Since

$$J_{-n}(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{-n+2m}}{2^{-n+2m}(m)\,!\,\Gamma(-n+m+1)}, \qquad \ldots(1)$$

if n is a positive integer, then the gamma functions in the coefficients of first n terms becomes infinite and coefficients of (1) becomes zero. Thus the summation will start at $m = n$ and in this case $\Gamma(-n+m+1) = (m-n)\,!$.

From (1), we now have,

$$J_{-n}(x) = \sum_{m=n}^{\infty} \frac{(-1)^m x^{-n+2m}}{2^{-n+2m}(m)!\,(m-n)!}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^{n+k} x^{n+2k}}{2^{n+2k}(k)!\,(n+k)!} \qquad (\because\ m = n+k)$$

$$= (-1)^n \sum_{k=0}^{\infty} \frac{(-1)^k x^{n+2k}}{2^{n+2k}(k)!\,\Gamma(n+k+1)}$$

$\therefore$ $$J_{-n}(x) = (-1)^n J_n(x).$$

• 3.4. DEFINITION OF $J_n(x)$, WHEN $n = 0$

Putting $n = 0$ in the Bessel's differential equation, we get

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} + xy = 0 \qquad \ldots(1)$$

Let us asume the solution

$$y = \sum_{m=0}^{\infty} a_m x^{m+r} \quad (a_0 \neq 0) \qquad \ldots(2)$$

$\therefore$ $$\frac{dy}{dx} = \sum_{m=0}^{\infty} a_m (m+r)\, x^{m+r-1}$$

and $$\frac{d^2y}{dx^2} = \sum_{m=0}^{\infty} a_m (m+r)(m+r-1)\, x^{m+r-2}.$$

Substitute these values in (1), we get

$$x\sum_{m=0}^{\infty} a_m (m+r)(m+r-1)\, x^{m+r-2} + \sum_{m=0}^{\infty} a_m (m+r)\, x^{m+r-1} + x\sum_{m=0}^{\infty} a_m x^{m+r} = 0$$

or $$\sum_{m=0}^{\infty} a_m (m+r)(m+r-1)\, x^{m+r-1} + \sum_{m=0}^{\infty} a_m (m+r)\, x^{m+r-1} + \sum_{m=0}^{\infty} a_m x^{m+r+1} = 0 \qquad \ldots(3)$$

If (2) is the solution of (1), then (3) will be an identity. Thus coefficients of each terms will be zero. So that taking the coefficients of x^{r-1}, we get

$$a_0 r(r-1) + a_0 r = 0$$

or $$r^2 a_0 = 0$$

or $$r = 0 \qquad (\because\ a_0 \neq 0)$$

Now taking the coefficient of x^r, we have

$$a_1(1+r)r + a_1(1+r) = 0$$

or $$a_1(1+r)^2 = 0$$

or $$a_1 = 0 \qquad (\because\ r = 0)$$

In general, taking the coefficient of x^{m+r}

$$a_{m+1}(m+r+1)(m+r) + a_{m+1}(m+r+1) + a_{m-1} = 0$$

or $$a_{m+1}(m+r+1)^2 + a_{m-1} = 0$$

or $$a_{m+1} = -\frac{a_{m-1}}{(m+r+1)^2}$$

For the case $r = 0$,

$$a_{m+1} = -\frac{a_{m-1}}{(m+1)^2}$$

Putting $m = 1, 2, 3, 4, 5, \ldots$

$$a_3 = -\frac{a_1}{9} = 0 \qquad (\because a_1 = 0)$$

$$a_2 = -\frac{a_0}{2^2}$$

$$a_4 = -\frac{a_2}{4^2} = \frac{(-1)^2 a_0}{2^2 . 4^2}$$

$$a_5 = 0 \quad \text{etc.}$$

Thus we obtained $a_1 = a_3 = a_5 = \ldots = 0$. Hence,

$$y = a_0\left(1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 . 4^2} - \frac{x^6}{2^2 . 4^2 . 6^2} + \ldots\right)$$

If $a_0 = 1$, then $y = J_0(x)$.

$$\mathbf{J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 . 4^2} - \frac{x^6}{2^2 . 4^2 . 6^2} + \ldots}$$

$J_0(x)$ is also known as **Bessel's function of order zero.**

• 3.5. GENERATING FUNCTION FOR $J_n(x)$

The function of the form

$$e^{\left[\frac{1}{2}x\left(t - \frac{1}{t}\right)\right]}$$

generates $J_n(x)$, if taking coefficient of t^n. Thus this function is known as Generating function for $J_n(x)$.

Proof. Expand $e^{\left[\frac{1}{2}x\left(t - \frac{1}{t}\right)\right]}$.

$$\therefore \qquad e^{\left[\frac{1}{2}x\left(t - \frac{1}{t}\right)\right]} = e^{\frac{xt}{2}} . e^{-\frac{x}{2t}}$$

$$= \left[1 + \frac{xt}{2} + \frac{1}{(2)!}\left(\frac{xt}{2}\right)^2 + \ldots + \frac{1}{(n)!}\left(\frac{xt}{2}\right)^n + \frac{1}{(n+1)!}\left(\frac{xt}{2}\right)^{n+1} + \frac{1}{(n+2)!}\left(\frac{xt}{2}\right)^{n+2} + \ldots\right]$$

$$. \left[1 - \frac{x}{2t} + \frac{1}{(2)!}\left(\frac{x}{2t}\right)^2 + \ldots + \frac{(-1)^n}{(n)!}\left(\frac{x}{2t}\right)^n + \frac{(-1)^{n+1}}{(n+1)!}\left(\frac{x}{2t}\right)^{n+1} + \frac{(-1)^{n+2}}{(n+2)}\left(\frac{x}{2t}\right)^{n+2} + \ldots\right]$$

Now collecting the coefficient of t^n, in above expression obtained after multiplication,

$$= \frac{1}{(n)!}\left(\frac{x}{2}\right)^n - \frac{1}{(n+1)!}\left(\frac{x}{2}\right)^{n+2} + \frac{1}{(n+2)!} \cdot \frac{1}{(2)!}\left(\frac{x}{2}\right)^{n+4} + \ldots$$

$$= \sum_{m=0}^{\infty} (-1)^m \cdot \frac{1}{(m)!\,(n+m)!} \cdot \left(\frac{x}{2}\right)^{n+2m}$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m x^{n+2m}}{2^{n+2m}(m)!\,\Gamma(m+n+1)}$$

$$= J_n(x).$$

$$\therefore \qquad e^{\left[\frac{1}{2}x\left(t - \frac{1}{t}\right)\right]} = \sum_{n=0}^{\infty} t^n J_n(x).$$

If taking the coefficient of t^{-n}, we get

$$= \frac{(-1)^n}{(n)!}\left(\frac{x}{2}\right)^n + \frac{(-1)^{n+1}}{(n+1)!}\left(\frac{x}{2}\right)^{n+2} + \frac{(-1)^{n+2}}{(n+2)!} \cdot \frac{1}{(2)!}\left(\frac{x}{2}\right)^{n+4} + \ldots$$

$$= (-1)^n \left[\frac{1}{(n)!} \left(\frac{x}{2}\right)^n - \frac{1}{(n+1)!} \left(\frac{x}{2}\right)^{n+2} + \frac{1}{(n+2)!} \cdot \frac{1}{(2)!} \left(\frac{x}{2}\right)^{n+4} - \dots \right]$$

$$= (-1)^n \sum_{m=0}^{\infty} \frac{(-1)^m x^{n+2m}}{2^{n+2m} (m)! \, \Gamma(n+m+1)}$$

$$= (-1)^n J_n(x)$$

$$= J_{-n}(x) \qquad (\because \quad J_{-n}(x) = (-1)^n J_n(x))$$

Hence we obtained

$$e^{\left[\frac{1}{2} x \left(t - \frac{1}{t}\right)\right]} = \sum_{n=-\infty}^{\infty} t^n J_n(x).$$

• 3.6. RECURRENCE RELATION FOR $J_n(x)$

(I) $$x J_n'(x) = n J_n(x) - x J_{n+1}(x)$$

where $$J_n'(x) = \frac{d J_n(x)}{dx}$$

Proof. Since we have

$$J_n(x) = \sum_{m=0}^{\infty} (-1)^m \frac{1}{(m)! \, \Gamma(n+m+1)} \cdot \left(\frac{x}{2}\right)^{n+2m} \qquad \dots(1)$$

Differentiating (1) w.r.t. x, we get

$$J'_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m (n+2m)}{(m)! \, \Gamma(n+m+1)} \cdot \frac{1}{2} \cdot \left(\frac{x}{2}\right)^{n+2m-1}$$

or $$x J'_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m (n+2m)}{(m)! \, \Gamma(n+m+1)} \cdot \left(\frac{x}{2}\right)^{n+2m}$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m (n+2m)}{(m)! \, \Gamma(n+m+1)} \cdot \left(\frac{x}{2}\right)^{n+2m}$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m n}{(m)! \, \Gamma(n+m+1)} \cdot \left(\frac{x}{2}\right)^{n+2m} + \sum_{m=0}^{\infty} \frac{(-1)^m 2m}{(m)! \, \Gamma(n+m+1)} \cdot \left(\frac{x}{2}\right)^{n+2m}$$

$$= n J_n(x) + \sum_{m=0}^{\infty} \frac{(-1)^m . 2}{(m-1)! \, \Gamma(n+m+1)} \cdot \frac{x}{2} \cdot \left(\frac{x}{2}\right)^{n-1+2m}$$

$$= n J_n(x) + x \sum_{m=0}^{\infty} \frac{(-1)^m}{(m-1)! \, \Gamma(n+m+1)} \cdot \left(\frac{x}{2}\right)^{n-1+2m}$$

$$= n J_n(x) + x \sum_{m=1}^{\infty} \frac{(-1)^m}{(m-1)! \, \Gamma(n+m+1)} \cdot \left(\frac{x}{2}\right)^{n-1+2m} \qquad \left(\because \quad \frac{1}{(-1)!} = 0\right)$$

$$= n J_n(x) + x \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{(k)! \, \Gamma(n+1+k+1)} \cdot \left(\frac{x}{2}\right)^{n+1+2k}$$

$$= n J_n(x) - x J_{n+1}(x).$$

$\therefore$ $$x J'_n(x) = n J_n(x) - x J_{n+1}(x).$$

(II) $$x J'_n(x) = -n J_n(x) + x J_{n-1}(x).$$

Proof. Since we have

$$J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(m)! \, \Gamma(n+m+1)} \cdot \left(\frac{x}{2}\right)^{n+2m} \qquad \dots(1)$$

Differentiating (1) w.r.t. x, we get

$$J'_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m (n+2m)}{(m)!\,\Gamma(n+m+1)} \cdot \frac{1}{2} \cdot \left(\frac{x}{2}\right)^{n+2m-1}$$

or
$$x\,J'_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m (n+2m)}{(m)!\,\Gamma(n+m+1)} \cdot \left(\frac{x}{2}\right)^{n+2m}$$
$$= \sum_{m=0}^{\infty} \frac{(-1)^m (2n+2m-n)}{(m)!\,\Gamma(n+m+1)} \cdot \left(\frac{x}{2}\right)^{n+2m}$$
$$= -n \sum_{m=0}^{\infty} \frac{(-1)^m}{(m)!\,\Gamma(n+m+1)} \cdot \left(\frac{x}{2}\right)^{n+2m} + \sum_{m=0}^{\infty} \frac{(-1)^m\, 2(n+m)}{(m)!\,\Gamma(n+m+1)} \cdot \left(\frac{x}{2}\right)^{n+2m}$$
$$= -n\, J_n(x) + \sum_{m=0}^{\infty} \frac{(-1)^m\, 2}{(m)!\,\Gamma(n+m)} \cdot \frac{x}{2} \cdot \left(\frac{x}{2}\right)^{n+2m-1}$$
$$= -n\, J_n(x) + x \sum_{m=0}^{\infty} \frac{(-1)^m}{(m)!\,\Gamma(n-1+m+1)} \cdot \left(\frac{x}{2}\right)^{n-1+2m}$$
$$= -n\, J_n(x) + x\, J_{n-1}(x).$$

$\therefore$ $$x\,J'_n(x) = -n\,J_n(x) + x\,J_{n-1}(x).$$

(III) $\boldsymbol{2J'_n(x) = J_{n-1}(x) - J_{n+1}(x).}$

Proof. From recurrence relations I and II, we have

$$x\,J'_n(x) = n\,J_n(x) - x\,J_{n+1}(x) \qquad \ldots(1)$$
$$x\,J'_n(x) = -n\,J_n(x) + x\,J_{n-1}(x) \qquad \ldots(2)$$

Adding (1) and (2), we get

$$2x\,J'_n(x) = x\,J_{n-1}(x) - x\,J_{n+1}(x)$$

$\therefore$ $$2J'_n(x) = J_{n-1}(x) - J_{n+1}(x).$$

(IV) $\boldsymbol{2n\,J_n(x) = x\,[J_{n-1}(x) + J_{n+1}(x)].}$

Proof. From recurrence relations I and II, we have

$$x\,J'_n(x) = n\,J_n(x) - x\,J_{n+1}(x)$$
$$x\,J'_n(x) = -n\,J_n(x) + x\,J_{n-1}(x) \qquad \ldots(2)$$

From (1) and (2), we get

$$n\,J_n(x) - x\,J_{n+1}(x) = -n\,J_n(x) + x\,J_{n-1}(x)$$

or
$$2n\,J_n(x) = x\,[J_{n-1}(x) + J_{n+1}(x)].$$

(V) $\boldsymbol{\frac{d}{dx}[x^{-n} J_n(x)] = -x^{-n} J_{n+1}(x).}$

Proof.
$$\text{L.H.S.} = \frac{d}{dx}[x^{-n} J_n(x)]$$
$$= x^{-n} J'_n(x) - n\,x^{-n-1} J_n(x)$$
$$= x^{-n-1}[x\,J'_n(x) - n\,J_n(x)]$$
$$= x^{-n-1}[-x\,J_{n+1}(x)] \qquad \text{(from recurrence relation I)}$$
$$= -x^{-n} J_{n+1}(x)$$
$$= \text{R.H.S.}$$

$\therefore$ $$\frac{d}{dx}[x^{-n} J_n(x)] = -x^{-n} J_{n+1}(x).$$

(VI) $\boldsymbol{\frac{d}{dx}[x^{n} J_n(x)] = x^{n} J_{n-1}(x).}$

Proof.
$$\text{L.H.S.} = \frac{d}{dx}[x^{n} J_n(x)]$$
$$= x^{n} J'_n(x) + n\,x^{n-1} J_n(x) = x^{n-1}[x\,J'_n(x) + n\,J_n(x)]$$

$= x^{n-1} [x J_{n-1}(x)]$

$= x^n J_{n-1}(x). =$ R.H.S.

$\therefore \quad \frac{d}{dx} [x^n J_n(x)] = x^n J_{n-1}(x).$

SOLVED EXAMPLES

Example 1. *Show that $J_n(x)$ is even and odd function for even n and for odd n respectively.*

Solution. Since we have

$$J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(m)!\,\Gamma(m+n+1)} \cdot \left(\frac{x}{2}\right)^{n+2m} \qquad \ldots(1)$$

Putting $-x$ in place of x, we get

$$J_n(-x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(m)!\,\Gamma(m+n+1)} \cdot \left(-\frac{x}{2}\right)^{n+2m}$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m}{(m)!\,\Gamma(n+m+1)} \cdot (-1)^{n+2m} \cdot \left(\frac{x}{2}\right)^{n+2m}$$

$$= (-1)^n \sum_{m=0}^{\infty} \frac{(-1)^m}{(m)!\,\Gamma(n+m+1)} \cdot \left(\frac{x}{2}\right)^{n+2m}$$

$$= (-1)^n J_n(x).$$

(i) If n is even, then $(-1)^n = 1$

$\therefore \quad J_n(-x) = J_n(x)$

$\therefore \quad J_n(x)$ is even.

(ii) If n is odd, then $(-1)^n = -1$

$\therefore \quad J_n'(-x) = -J_n(x)$

$\therefore \quad J_n(x)$ is odd.

Example 2. *Show that $J_0'(x) = -J_1(x)$.*

Solution. From recurrence relation I, we have

$$x J_n'(x) = n J_n(x) - x J_{n+1}(x)$$

Putting $n = 0$, we get

$$x J'_0(x) = -x J_1(x)$$

$$\therefore \quad J'_0(x) = -J_1(x).$$

Example 3. *Prove that* $\frac{d}{dx}[J_n^2 + J_{n+1}^2] = 2\left(\frac{n}{x} J_n^2 - \frac{n+1}{x} J_{n+1}^2\right)$.

Solution. $\quad$ L.H.S. $= \frac{d}{dx}[J_n^2 + J_{n+1}^2]$

$$= 2J_n J'_n + 2J_{n+1} J'_{n+1} \qquad \ldots(1)$$

from recurrence relation I, we have

$$x J'_n = n J_n - x J_{n+1}$$

$$\therefore \quad J'_n = \frac{n}{x} J_n - J_{n+1} \qquad \ldots(2)$$

From recurrence relation II, we have

$$x J'_n = -n J_n + x J_{n-1}$$

or $$J'_n = -\frac{n}{x} J_n + J_{n-1}.$$

Putting $(n+1)$ in place of n, we get

$$J'_{n+1} = -\frac{n+1}{x} J_{n+1} + J_n \qquad \ldots(3)$$

substitute the values of J'_n and J'_{n+1} from (2) and (3) into (1), we get

$$\text{L.H.S.} = 2J_n\left[\frac{n}{x}J_n - J_{n+1}\right] + 2\left[-\frac{n+1}{x}J_{n+1} + J_n\right]J_{n+1}$$

$$= 2\frac{n}{x}J_n^2 - 2J_nJ_{n+1} - 2\frac{(n+1)}{x}J_{n+1}^2 + 2J_nJ_{n+1}$$

$$= 2\left(\frac{n}{x}J_n^2 - \frac{n+1}{x}J_{n+1}^2\right) = \text{R.H.S.}$$

Hence $\frac{d}{dx}[J_n^2 + J_{n+1}^2] = 2\left(\frac{n}{x}J_n^2 - \frac{n+1}{x}J_{n+1}^2\right)$.

Example 4. *Prove that* $\frac{d}{dx}(x\,J_nJ_{n+1}) = x\,(J_n^2 - J_{n+1}^2)$.

Solution. $\text{L.H.S.} = \frac{d}{dx}(x\,J_n\,J_{n+1})$

$$= xJ_n\,J'_{n+1} + xJ'_n\,J_{n+1} + J_n\,J_{n+1} \qquad \ldots(1)$$

From recurrence relations I and II, we have

$$xJ'_n = nJ_n - x\,J_{n+1} \qquad \ldots(2)$$

and
$$xJ'_n = -n\,J_n + x\,J_{n-1} \qquad \ldots(3)$$

putting $(n+1)$ in place of n in (3), we get

$$x\,J'_{n+1} = -(n+1)\,J_{n+1} + x\,J_n \qquad \ldots(4)$$

Substitute the values of xJ'_n and xJ'_{n+1} from (2) and (4) into (1), we get

$$\therefore \quad \text{L.H.S.} = J_n[-(n+1)\,J_{n+1} + xJ_n] + J_{n+1}[nJ_n - xJ_{n+1}] + J_n\,J_{n+1}$$

$$= -n\,J_n\,J_{n+1} - J_n\,J_{n+1} + xJ_n^2 + nJ_n\,J_{n+1} - xJ_{n+1}^2 + J_n\,J_{n+1}.$$

$$= x\,J_n^2 - x\,J_{n+1}^2$$

$$= x\,(J_n^2 - J_{n+1}^2)$$

$$= \text{R.H.S.}$$

Hence $\frac{d}{dx}(xJ_n\,J_{n+1}) = x\,(J_n^2 - J_{n+1}^2)$.

Example 5. *Prove the followings* :

(*i*) $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\,.\,sin\,.\,x$

(*ii*) $J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\,.\,cos\,.\,x.$

Solution. (i) Since we have

$$J_n(x) = \frac{x^n}{2^n\,\Gamma(n+1)}\left[1 - \frac{x^2}{2\,.\,(2n+2)} + \frac{x^4}{2\,.\,4\,(2n+2)\,(2n+4)}\ldots\right]. \qquad \ldots(1)$$

Putting $n = 1/2$ in (1), we get

$$J_{1/2}(x) = \frac{x^{1/2}}{2^{1/2}\,\Gamma\left(1+\frac{1}{2}\right)}\left[1 - \frac{x^2}{2\,.\,3} + \frac{x^4}{2\,.\,4\,.\,3\,.\,5} - \ldots\right]$$

$$= \sqrt{\frac{x}{2}}\,.\,\frac{1}{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}\left[1 - \frac{x^2}{(3)\,!} + \frac{x^4}{(5)\,!} - \ldots\right]$$

$$= \sqrt{\frac{2}{x}}\,.\,\frac{1}{\Gamma\left(\frac{1}{2}\right)}\left[x - \frac{x^3}{(3)\,!} + \frac{x^5}{(5)\,!} - \ldots\right]$$

$$= \sqrt{\frac{2}{\pi x}}\,.\,\sin x \quad \left(\because\ \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \text{ and } \sin\theta = \theta - \frac{\theta^3}{(3)\,!} + \frac{\theta^5}{(5)\,!} - \ldots\right)$$

(ii) Putting $n = -\frac{1}{2}$ in (i), we get

$$J_{-1/2}(x) = \frac{x^{-1/2}}{2^{-1/2}\,\Gamma\left(1-\frac{1}{2}\right)}\left[1 - \frac{x^2}{1\,.\,2} + \frac{x^4}{1\,.\,2\,.\,3\,.\,4} - \ldots\right]$$

$$= \sqrt{\frac{2}{x}}\,.\,\frac{1}{\Gamma\left(\frac{1}{2}\right)}\left[1 - \frac{x^2}{(2)\,!} + \frac{x^4}{(4)\,!} - \ldots\right]$$

$$= \sqrt{\frac{2}{\pi x}}\left[1 - \frac{x^2}{(2)\,!} + \frac{x^4}{(4)\,!} - \ldots\right]$$

$$= \sqrt{\frac{2}{\pi x}}\,.\cos x \qquad \left(\because \cos\theta = 1 - \frac{\theta^2}{(2)\,!} + \frac{\theta^4}{(4)\,!} - \ldots\right)$$

$$\therefore \qquad J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\,.\cos x.$$

Example 6. *Prove that*

(*i*) $[J_{1/2}(x)]^2 + [J_{-1/2}(x)]^2 = \frac{2}{\pi x}$

(*ii*) $J_{-3/2}(x) = -\sqrt{\frac{2}{\pi x}}\left(\frac{1}{x}\cos x + \sin x\right)$.

Solution. (i) In Ex. 6, we proved that

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\,.\sin x$$

and

$$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\,.\cos x.$$

Squaring these and add, we get

$$[J_{1/2}(x)]^2 + [J_{-1/2}(x)]^2 = \frac{2}{\pi x}(\sin^2 x + \cos^2 x)$$

$$= \frac{2}{\pi x}.$$

(ii) Since we know that

$$2n\,J_n(x) = x\,[J_{n-1}(x) + J_{n+1}(x)]$$

or

$$J_{n-1}(x) = \frac{2n}{x}J_n(x) - J_{n+1}(x).$$

Now putting $n = -1/2$, we get

$$J_{-3/2}(x) = \frac{2\left(-\frac{1}{2}\right)}{x}J_{-1/2} - J_{1/2}(x)$$

$$= -\frac{1}{x}J_{-1/2}(x) - J_{1/2}(x). \qquad \ldots(1)$$

Putting the values of

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\,.\sin x$$

and

$$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\,.\cos x \text{ into (1), we get}$$

$$J_{-3/2}(x) = -\sqrt{\frac{2}{\pi x}}\left[\frac{1}{x}\cos x + \sin x\right].$$

• SUMMARY

- Bessel's D.E. :

$$x^2y'' + xy' + (x^2 - n^2)\, y = 0.$$

- Bessel's Function of first kind :

$$J_n(x) = \sum_{m=0}^{\infty} (-1)^m \frac{x^{n+2m}}{2^{n+2m}\, m!\, \Gamma(n+m+1)}$$

- $J_0(x) = 1 - \dfrac{x^2}{2^2} + \dfrac{x^4}{2^2 . 4^2} - \dfrac{x^6}{2^2 . 4^2 . 6^2} + \ldots\ldots$
- Generating function for $J_n(x)$:

$$e^{\frac{x}{2}\left(t - \frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} t^n J_n(x)$$

- Recurrence Relations

(i) $x J'_n(x) = nJ_n(x) - xJ_{n+1}(x)$

(ii) $xJ'_n(x) = -nJ_n(x) + xJ_{n-1}(x)$

(iii) $2J'_n(x) = J_{n-1}(x) - J_{n+1}(x)$

(iv) $2nJ_n(x) = x\,[J_{n-1}(x) + J_{n+1}(x)]$

(v) $\dfrac{d}{dx}[x^{-n} J_n(x)] = -x^{-n} J_{n+1}(x)$

(vi) $\dfrac{d}{dx}[x^n J_n(x)] = x^n J_{n-1}(x).$

• STUDENT ACTIVITY

1. Prove that : $x J'_n(x) = n J_n(x) - xJ_{n+1}(x)$.

2. Prove that :

$$\frac{d}{dx}(x J_n J_{n+1}) = x\,(J_n^2 - J_{n+1}^2)$$

• TEST YOURSELF

1. Prove that $4J''_n(x) = J_{n-2}(x) - 2J_n(x) + J_{n+2}(x)$.

2. Prove that $J_n J'_{-n} - J_{-n} J'_n = -\dfrac{2 \sin n\pi}{\pi x}$

hence deduce that $\dfrac{d}{dx}\left[\dfrac{J_{-n}}{J_n}\right] = -\dfrac{2 \sin n\pi}{\pi x J_n^2}$.

3. Prove that

(i) $J_2 = J''_0 - \dfrac{1}{x} J'_0$ (ii) $J_2 - J_0 = 2 J''_0$.

4. Prove that

(i) $J_{3/2}(x) = \sqrt{\dfrac{2}{\pi x}} \left[\dfrac{1}{x} \sin x - \cos x\right]$

(ii) $J_{-5/2}(x) = \sqrt{\dfrac{2}{\pi x}} \left[\left(\dfrac{3 - x^2}{x^2}\right) \cos x + \dfrac{3}{x} \sin x\right]$

(iii) $J_{5/2}(x) = \sqrt{\dfrac{2}{\pi x}} \left[\left(\dfrac{3 - x^2}{x^2}\right) \sin x - \dfrac{3}{x} \cos x\right]$.

OBJECTIVE EVALUATION

Fill in the blanks :

1. $J_{-n}(x) = (-1)^n$ ....
2. $J_0'(x) =$ ....
3. $J_n(x)$ is even function if n is ....

True or False

1. $J_{-n}(x) = (-1)^n J_{n+1}(x)$. **(T/F)**
2. $|J_0(x)| \le 1,\ n \ge 1$. **(T/F)**
3. $[J_{1/2}(x)]^2 + [J_{-1/2}(x)]^2 = \dfrac{2}{\pi x}$ **(T/F)**

Multiple Choice Questions (MCQ's)

1. $(-1)^n J_n(x)$ equals :

(a) $J_n(x)$ (b) $J_{-n}(x)$ (c) $J_{n-1}(x)$ (d) $J_{n+1}(x)$

2. $x[J_{n-1} + J_{n+1}]$ equals :

(a) $2n J_{n-1}$ (b) nJ_n (c) $2n J_n$ (d) $2n J_{n+1}$

ANSWERS

Fill in the blanks :

1. $J_n(x)$ **2.** $-J_1(x)$ **3.** even

True or False

1. F **2.** T **3.** T

MCQ

1. (b) **2.** (c)

UNIT 4

AN INTRODUCTION TO PARTIAL DIFFERENTIAL EQUATIONS

STRUCTURE

- P.D.E.
- Order and Degree
- Classification of Partial Differential equation
- Solution of P.D.E
- Linear partial differential equation of first order
- Derivation of P.D.E. by elimination of arbitary constants.
- Derivation of P.D.E. by elimination of arbitrary functions.
- Solution of standard forms
  - ❑ Summary
  - ❑ Student Activity
  - ❑ Test Yourself

LEARNING OBJECTIVES

After going through this unit you will learn :

- What is P.D.E. ?
- How to find its order and degree ?
- How to find its solution ?

• 4.1. P.D.E.

Here, we have already discussed the differential equations, with number of independent variables are two or more. In such cases, any dependent variable is likely to be a function of more than one variable, so that it possesses not ordinary derivatives with respect to a single variable but partial derviatives with respect to several variables. The partial differential equation implies necessarily the existence of more than one independent variables. We shall usually take z as dependent variable and x, y as independent variables and throughout the chapter we shall denote the partial derivatives $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, \frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial x \partial y}$ and $\frac{\partial^2 z}{\partial y^2}$ by p, q, r, s and t respectively.

Definition. *The equation of the type*

$$F\left(\frac{\partial z}{\partial x}, \ldots, \frac{\partial^2 z}{\partial x^2}, \ldots, \frac{\partial^2 z}{\partial x \partial y}, \ldots\right) = 0$$

*is called a **partial differential equation.***

• 4.2. ORDER AND DEGREE

Order. The order of the partial differential equation is the order of its highest derivative.

(i) First order PDE. A first order partial differential equation for a function $z = f(x, y)$ contains at least one of the partial derivatives $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$, but no partial derivative of order higher than one.

For example :

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0.$$

(ii) Second order PDE. A second order partial differential equation for $z = f(x, y)$ contains at least one of the partial derivatives $\frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial y^2}, \frac{\partial^2 z}{\partial x \partial y}$, but no partial derivatives of order higher than two.

For examples :

(i) $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$

(ii) $\frac{\partial z}{\partial t} - C\frac{\partial^2 z}{\partial x^2} = 0.$

REMARK

➤ The second order partial differential equation may also contain first order term like $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ etc.

Degree of PDE :

The degree of partial differential equation is the power of the highest derivative in the equation.

For Examples :

(i) $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$

(ii) $\frac{\partial z}{\partial t} - C\frac{\partial^2 z}{\partial x^2} = 0$

(iii) $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0$

(iv) $\frac{\partial^2 z}{\partial t^2} = C^2\frac{\partial^2 z}{\partial y^2}$

(v) $\left(\frac{\partial z}{\partial x}\right)^3 + \frac{\partial z}{\partial x} = 0.$

Equations (i), (ii), (iii) and (iv) are PDEs of degree one, and the equation (v) is a PDE of degree 3.

• 4.3. CLASSIFICATION OF PARTIAL DIFFERENTIAL EQUATIONS

(A) Linear and Non-linear Partial Differential Equations :

A partial differential equation is said to be **linear** if :

(i) It is of the first degree in the dependent variable and its partial derivatives.

(ii) It does not contain the product of dependent variables and either of its partial derivatives.

and **(iii)** It does not contain any transcendental function.

For examples :

(i) $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$

(ii) $\frac{\partial T}{\partial t} - K\frac{\partial^2 T}{\partial t^2} = 0$

(iii) $\frac{\partial^2 u}{\partial t^2} = C^2\frac{\partial^2 u}{\partial y^2}$

(iv) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y)$

The above all equations are linear.

Non-Linear PDE :

A partial differential equation, which is not linear is called non-linear equation.

For example :

(1) $\left(\frac{\partial f}{\partial x}\right)^3 + \frac{\partial f}{\partial t} = 0.$

Quási-Linear :

Consider a non-linear equation

$$R_1 r + S_1 s + T_1 t = V_1 \qquad \ldots(1)$$

where R_1, S_1, T_1 and V_1 are the functions of p and q as well as of x, y and z. Then, we observe that, it has a certain formal resemblence to a linear equation. Due to this resemblence with linear equation, equation (1) is said to be quasi-linear equation.

(B) Homogeneous and Non-homogeneous Equations :

A linear partial differential equation can be classified as follows :

(i) Homogeneous linear equation

(ii) Non-homogeneous linear equation

(i) Homogeneous linear equation :

If each term of a partial differential equation contains either the dependent variable (or unknown function) or one of its partial derivatives, it is said to be homogeneous.

For examples :

(i) $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$

(ii) $\frac{\partial^2 z}{\partial t^2} = c^2 \frac{\partial^2 z}{\partial y^2}.$

(ii) Non-homogeneous linear equation :

An equation, which is not homogeneous is called non-homogeneous linear equation.

For examples :

(i) $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = f(x, y)$

(ii) $\frac{\partial^3 u}{\partial x^3} + 2\frac{\partial^3 u}{\partial x\, \partial y^2} - 6\left(\frac{\partial u}{\partial y}\right)^4 = 0.$

• 4.4. SOLUTION OF PDE

A solution of PDE in some region R of the space of independent variables is a function all of whose partial derivatives appearing in the equation exist in some domain containing R and which satisfies the equation everywhere in R.

• 4.5. LINEAR PARTIAL DIFFERENTIAL EQUATION OF FIRST ORDER

A differential equation involving partial derivatives p and q only, no higher derivative is called of order 1. If the degree of p and q are unity, then it is called a linear partial differential equation of order one.

Some Basic Definitions :

(i) Complete Integral. Let us consider the partial differential equation

$$f(x, y, z, p, q) = 0$$

where x, y are independent variable, and z is dependent while $p = \frac{\partial z}{\partial x}, q = \frac{\partial z}{\partial y}$, then

A relation of type $F(x, y, z, a, b) = 0$ containing as many arbitrary constants as the number of independent variables in the above partial differential equation is called complete integral.

(ii) Particular Integral. In the complete integral $F(x, y, z, a, b) = 0$ giving the particular values to the constants a and b, we get the particular integral.

(iii) Singular Integral. The envelope of the surfaces given by the complete integral $F(x, y, z, a, b) = 0$ is called singular integral. Therefore, the singular integral is obtained by eliminating a and b from

$$F(x, y, z, a, b) = 0, \frac{\partial F}{\partial a} = 0 \text{ and } \frac{\partial F}{\partial b} = 0.$$

(iv) General Integral. Let $u = u(x, y, z)$ and $v = v(x, y, z)$ be two functions of x, y and z, then the solution of the differential equation $pP + qQ = R$ of the types $f(u, v) = 0$ is called the general integral. This also, can be taken as $u = f(v)$ or $v = f(u)$.

• 4.6. DERIVATION OF A PARTIAL DIFFERENTIAL EQUATIONS BY THE ELIMINATION OF ARBITRARY CONSTANTS

Consider the equation

$$F(x, y, z, a, b) = 0 \qquad \ldots(1)$$

where, a and b are arbitrary constant. *Differentiating (1) partially with respect to x, regarding z as a function of two independent variables x and y,* we get

$$\frac{\partial F}{\partial x} + p\frac{\partial F}{\partial z} = 0 \text{ and } \frac{\partial F}{\partial y} + q\frac{\partial F}{\partial z} = 0 \qquad \ldots(2)$$

By the elimination of a and b from (1) and (2), we shall get an equation of the type

$$F(x, y, z, p, q) = 0 \qquad \ldots(3)$$

which is the required partial differential equation of the first oder.

SOLVED EXAMPLES

Example 1. *Construct a partial differential equation, by eliminating a, b and c from*

$$z = a(x + y) + b(x - y) + abt + c.$$

Solution. Here, the given equation is

$$z = a(x + y) + b(x - y) + abt + c \qquad \ldots(1)$$

Now, differentiating (1) partially with respect to x, y and t, we get

$$\frac{\partial z}{\partial x} = a + b, \frac{\partial z}{\partial y} = a - b, \frac{\partial z}{\partial t} = ab \qquad \ldots(2)$$

Now, using

$$(a + b)^2 - (a - b)^2 = 4ab$$

$$\Rightarrow \qquad \left(\frac{\partial z}{\partial x}\right)^2 - \left(\frac{\partial z}{\partial y}\right)^2 = 4\frac{\partial z}{\partial t}$$

which is the required partial differential equation.

• 4.7. DERIVATION OF A PARTIAL DIFFERENTIAL EQUATION BY THE ELIMINATION OF AN ARBITRARY FUNCTION

Let u and v be any two functions of x, y, z connected by the relation

$$\phi(u, v) = 0 \qquad \ldots(1)$$

Now, it is to be shown that on the elimination of the arbitrary function ϕ from (1), a partial differential equation will be formed and moreover, this equation will be linear.

Differentiating (1) partially with respect to x any y, regarding z as independent variables, we have

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial x}\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial x}\right) = 0$$

$$\Rightarrow \qquad \frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}\right) = 0 \qquad \ldots(2)$$

and

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial y}\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial y}\right) = 0$$

$$\Rightarrow \qquad \frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z}\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial y} + q\frac{\partial v}{\partial z}\right) = 0 \qquad \ldots(3)$$

Now, eliminating $\frac{\partial \phi}{\partial u}, \frac{\partial \phi}{\partial v}$ between (2) and (3) by the method of determinant, we get

$$\begin{vmatrix} \left(\dfrac{\partial u}{\partial x} + p\dfrac{\partial u}{\partial z}\right) & \left(\dfrac{\partial v}{\partial x} + p\dfrac{\partial v}{\partial z}\right) \\ \left(\dfrac{\partial u}{\partial y} + q\dfrac{\partial u}{\partial z}\right) & \left(\dfrac{\partial v}{\partial y} + q\dfrac{\partial v}{\partial z}\right) \end{vmatrix} = 0$$

$$\Rightarrow \quad \left(\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial z} - \frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial y}\right)p + \left(\frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial z}\right)q = \frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial x}$$

$$\Rightarrow \quad \frac{\partial (u, v)}{\partial (y, z)}p + \frac{\partial (u, v)}{\partial (z, x)}q = \frac{\partial (u, v)}{\partial (x, y)}$$

which is the linear PDE of first order and first degree in p and q which can also be written as

$$Pp + Qq = R$$

where, $\quad P = \dfrac{\partial (u, v)}{\partial (y, z)}, \; Q = \dfrac{\partial (u, v)}{\partial (z, x)}$ and $R = \dfrac{\partial (u, v)}{\partial (x, y)}.$

SOLVED EXAMPLES

Example 1. *By means of a partial differential equation, eliminate the arbitrary function from the equation*

$$x + y + z = f(x^2 + y^2 + z^2). \qquad \ldots(1)$$

Solution. Differentiating (1) partially w.r.t. x and y, we get

$$(1 + p) = f'(x^2 + y^2 + z^2) \,.\, (2x + 2zp) \qquad \ldots(2)$$

and
$$(1 + q) = f'(x^2 + y^2 + z^2) \,.\, (2y + 2zq) \qquad \ldots(3)$$

From (2) and (3), we have

$$\frac{(1 + p)}{(2x + 2zp)} = \frac{(1 + q)}{2y + 2zq}$$

$$\Rightarrow \quad (1 + p)(y + zq) = (1 + q)(x + zp)$$

$$\Rightarrow \quad (y - z)\,p + (z - x)\,q = (x - y),$$

which is the required PDE.

Example 2. *Eliminate the arbitrary functions f and g from*

$$y = f(x - at) + g(x + at).$$

Solution. Here, the given equation is

$$y = f(x - at) + g(x + at) \qquad \ldots(1)$$

$$\Rightarrow \quad \frac{\partial y}{\partial x} = f'(x - at) + g'(x + at)$$

and
$$\frac{\partial^2 y}{\partial x^2} = f''(x - at) + g''(x + at) \qquad \ldots(2)$$

Now
$$\frac{\partial y}{\partial t} = f'(x - at)\,.\,(-a) + g'(x + at)\,(a)$$

$$\Rightarrow \quad \frac{\partial^2 y}{\partial t^2} = f''(x - at)\,(-a)^2 + g''(x + at)\,(a)^2$$

$$= a^2\,[f''(x - at) + g''(x + at)]$$

$$= a^2\,\frac{\partial^2 y}{\partial x^2} \qquad \text{[using (2)]}$$

$$\Rightarrow \quad \frac{\partial^2 y}{\partial t^2} = a^2\,\frac{\partial^2 y}{\partial x^2}$$

which is the required PDE.

• TEST YOURSELF

Form a PDE, by eliminating arbitrary constants for the following equations :

1. $z = (x + a)(y + b)$.
2. $z = ax + by + ab$.
3. $z = ax + a^2y^2 + b$.

4. $f(x+y+z, x^2+y^2-z^2)=0$.
5. $lx+my+nz=f(x^2+y^2+z^2)$.

ANSWERS

1. $z=pq$ 2. $z=px+qy+pq$
3. $q=2yp^2$ 4. $(y+z)\,p-(z+x)\,q=x-y$
5. $(l+np)\,y+z\,(lq-mp)=(m+nq)\,x$

• 4.8. SOLUTION OF STANDARD FORMS (NON-LINEAR EQUATIONS)

In this section, we shall deal with some special types of equations which can be solved easily by some special methods, other than the general method.

Standard Form (I) :

Equation involving only p and q and no x, y, z :

The complete integral of equations of the type $f(p, q)=0$ i.e., in which x, y, z do not occur, is

$$z=ax+by+c \qquad \ldots(1)$$

where a and b are connected by the relation

$$f(a, b)=0 \qquad \ldots(2)$$

Since, we have $p=\dfrac{\partial z}{\partial x}=a$ and $q=\dfrac{\partial z}{\partial y}=b$, which on substitution in (2) becomes the given equation.

Let us suppose from (2), $b=g(a)$ and replacing c by $\phi(a)$, the general solution is obtained by eliminating 'a' between the following equation

$$z=ax+g(a)\,y+\phi(a) \qquad \ldots(3)$$

Differentiating (3) with respect to a, we get

$$0=x+yg'(a)+\phi'(a) \qquad \ldots(4)$$

Now, to find the singular integral, differentiate

$$z=ax+g(a)\,y+c$$

with respect to a and c, we get

$$0=x+y\,g'(a)$$

and $$0=1$$

$0=1 \Rightarrow$ there is no singular solution.

Standard Form (II) :

Equation involving only p, q and z.

The equations which do not contain x and y i.e., which are of the form

$$f(z, p, q)=0 \qquad \ldots(1)$$

Equation (1), *can be solved in the following way* :

Write $X=x+ay$, where a is an arbitrary constant and assume z to be function of $(x+ay)$ *i.e.*, of X alone.

$\therefore$ $$z=f(X)=f(x+ay)$$

$\Rightarrow$ $$p=\frac{\partial z}{\partial x}=\frac{dz}{dX}\cdot\frac{\partial X}{\partial x}=\frac{dz}{dX}\cdot 1$$

and $$q=\frac{\partial z}{\partial y}=\frac{dz}{dX}\cdot\frac{\partial X}{\partial y}=a\cdot\frac{dz}{dX}.$$

Now, the equation (1), becomes

$$F\left(z, \frac{dz}{dX}, a\frac{dz}{dX}\right)=0$$

which is an ordinary differential equation of the first order and can be integrated. So, the complete integral will be known.

If $f=0$ is the complete integral involving two constants a and b, then replacing b by $g(a)$, the general integral is obtained by eliminating a form

$$f=0,\ \frac{df}{da}=0.$$

The **singular integral** is obtained by eliminating a and b from

$$f = 0,\ \frac{\partial f}{\partial a} = 0 \text{ and } \frac{\partial f}{\partial b} = 0.$$

SOLVED EXAMPLES

Example 1. *Solve* $p^2 + q^2 = 1$.

Solution. The given equation is of the form

$$f(p, q) = 0$$

The solution is given by $\quad z = ax + by + c$

where a, and b are related by $\quad f(a, b) = 0$

$$\Rightarrow \quad a^2 + b^2 = 1$$

$$\Rightarrow \quad b = \sqrt{(1 - a^2)}$$

Hence, the complete integral is

$$z = ax + \sqrt{(1 - a^2)}\ y + c$$

For the general integral write $c = \phi(a)$

Then it is obtained by eliminating a from

$$z = ax + \sqrt{(1 - a^2)}\ y + \phi(a)$$

and

$$0 = x + \frac{-a}{\sqrt{(1 - a^2)}} y + \phi'(a).$$

Example 2. *Solve* $x^2p^2 + y^2q^2 = z^2$.

Solution. Here, the given equation can be written as

$$\left(\frac{x}{z} \cdot \frac{\partial z}{\partial x}\right)^2 + \left(\frac{y}{z} \cdot \frac{\partial z}{\partial y}\right)^2 = 1 \quad \ldots(1)$$

Putting $\quad \frac{1}{z} dz = dZ \quad i.e., \quad z = e^Z$

$$\frac{1}{x} dx = dX \quad i.e., \quad x = e^X$$

and $\quad \frac{1}{y} dy = dY \quad i.e., \quad y = e^Y$

in (1), we get

$$\left[\frac{\partial Z}{\partial X}\right]^2 + \left[\frac{\partial Z}{\partial Y}\right]^2 = 1$$

which is of the type $f(p, q) = 0$.

Therefore, the complete integral is given by

$$Z = aX + bY + c_1$$

where a and b are related by $a^2 + b^2 = 1$

$$\Rightarrow \quad b = \sqrt{(1 - a^2)}$$

$$\Rightarrow \quad z = aX + \sqrt{(1 - a^2)}\ Y + c_1$$

$$\Rightarrow \quad \log z = a \log x + \sqrt{(1 - a^2)} \log y + c_1$$

To find the general solution put $a = \cos\theta$

$$\Rightarrow \quad \log z = \cos\theta \log x + \sin\theta \log y + \log c$$

$$\Rightarrow \quad z = cx^{\cos\theta} \cdot y^{\sin\theta}.$$

Now, we eliminate θ from

$$z = g(\theta)\, x^{\cos\theta}\, y^{\sin\theta}$$

and

$$0 = g'(\theta)\, x^{\cos\theta}\, y^{\sin\theta} + g(\theta)\, x^{\cos\theta}\, y^{\sin\theta}\, (-\sin\theta) \log_e x + g(\theta)\, x^{\cos\theta}\, y^{\sin\theta} \cos\theta \log_e y$$

which is the required general solution.

To find singular integral, we eliminate θ and c, from

$$z = cx^{\cos\theta} \cdot y^{\sin\theta}$$

$\Rightarrow$ $$\frac{\partial z}{\partial \theta} = -c \sin\theta\, x^{\cos\theta} y^{\sin\theta} \log_e x + c\cos\theta \cdot x^{\cos\theta} \cdot y^{\sin\theta} \log_e y = 0$$

and $$\frac{\partial z}{\partial c} = x^{\cos\theta} \cdot y^{\sin\theta} = 0$$

$\Rightarrow$ $z = 0$ is the singular integral of the given equation.

Example 3. *Find the complete integral of* $p^3 + q^3 = 27z$.

Solution. Here, the given equation is

$$p^3 + q^3 = 27z$$

which is in the standard form

$$f(p, q, z) = 0$$

Put $$X = x + ay$$

$\Rightarrow$ $$z = f(X) = f(x + ay)$$

$\Rightarrow$ $$p = \frac{\partial z}{\partial x} = \frac{dz}{dX}$$

and $$q = \frac{\partial z}{\partial y} = a\frac{dz}{dX}.$$

We may take $\frac{dz}{dX}$ in place of $\frac{\partial z}{\partial x}$ because z is a function of x only.

Hence, the given equation reduces to

$$(1 + a^3)\left(\frac{dz}{dX}\right)^3 = 27z$$

$\Rightarrow$ $$(1 + a^3)^{1/3} \frac{dz}{dX} = 3z^{1/3}$$

$\Rightarrow$ $$(1 + a^3)^{1/3} \cdot \frac{2}{3} z^{-1/3}\, dz = 2dX$$

On integrating, we get

$$z^{2/3} (1 + a^3)^{1/3} = 2X + c = 2(X + b)$$

$\Rightarrow$ $$(1 + a^3)\, z^2 = 8(x + ay + b)^3 \qquad \ldots(1)$$

which is the complete integral of the given equation.

To find the singular integral, differentiating (1) partially with respect to a and b, we get

$$3a^2 z^2 = 24y\,(x + ay + b)^2 \qquad \ldots(2)$$

and $$0 = 24\,(x + ay + b)^2 \qquad \ldots(3)$$

By eliminating a, b from (1); (2) and (3), we get

$$z = 0$$

which is the required singular solution.

• TEST YOURSELF

1. Solve $q = 3p^2$.
2. Solve $p^2 + q^2 = npq$.
3. Solve $\sqrt{p} + \sqrt{q} = 1$.
4. Find the complete integral of $p^2 = zq$.
5. Solve $pz = (1 + q^2)$.
6. Solve $9(p^2 z + q^2) = 4$.

ANSWERS

1. $z = ax + 3a^2 y + c$. 2. $z = \frac{ax + n \pm \sqrt{(n^2 - 4)}}{2} \cdot ay + c$

3. $z = ax + (1 - \sqrt{a})^2 y + c$ 4. $z = be^{(ax + a^2 y)}$

5. $z^2 \mp [z\sqrt{(z^2 - 4a^2)} - 4a^2 \log\{z + \sqrt{(z^2 - 4a^2)}\}] = 4x + 4ay + 2c$

6. $(z + a^2)^3 = (x + ay + b)^2$

Standard Form III :

Equation of the form $f_1(x, p) = f_2(y, q)$.

If the given equation is of the type $f_1(x, p) = f_2(y, q)$ ...(1)

then, first write $f_1(x, p) = f_2(y, q) = c_1$. ...(2)

Now, solving (2) for q and p, we get

$$p = \frac{\partial z}{\partial x} = g_1(x, c_1)$$

and
$$q = \frac{\partial z}{\partial y} = g_2(y, c_1).$$

Now
$$dz = pdx + qdy$$
$$= g_1(x, c_1)\,dx + g_2(y, c_1)\,dy$$

which gives

$$z = \int g_1(x, c_1)\,dx + g_2(y, c_1)\,dy + b$$

The general solution may be obtained from this complete integral also, there is no singular solution.

Standard From IV :

Equation of the form $z = px + qy + f(p, q)$.

The equation $z = px + qy + f(p, q)$

*which is analogous to **Clairaut's form,** has for its complete integral.*

$$z = ax + by + f(a, b) \qquad \text{...(2)}$$

For
$$\frac{\partial z}{\partial x} = p = a \text{ and } \frac{\partial z}{\partial y} = q = b$$

In order to obtain the general solution, put $b = g(a)$

Therefore, $z = ax + y\,g(a) + f\{a, g(a)\}$ ...(3)

Differentiating (3) with respect to a, we get

$$0 = x + y\,g'(a) + f'(a) \qquad \text{...(4)}$$

Now, eliminate a from (3) and (4) and get the required general solution.

To obtain the singular solution, differentiating (2) with respect to a and b, which gives

$$0 = x + \frac{\partial f}{\partial a} \qquad \text{...(5)}$$

$$0 = y + \frac{\partial f}{\partial b} \qquad \text{...(6)}$$

and eliminate a and b between the equations (2), (5) and (6).

SOLVED EXAMPLES

Example 1. *Solve* $p^2 + q^2 = x + y$.

Solution. Here, the given equation can be written as

$$p^2 - x = y - q^2.$$

Let us write

$$p^2 - x = y - q^2 = a$$

$\Rightarrow$
$$p = \sqrt{(x + a)} \text{ and } q = \sqrt{(y - a)}$$

Now, putting the values of p and q in

$$dz = pdx + qdy$$

we get
$$dz = \sqrt{(x + a)}\,dx + \sqrt{(y - a)}\,dy.$$

On integrating, we have

$$z = \frac{2}{3}(x + a)^{3/2} + \frac{2}{3}(y - a)^{3/2} + b.$$

Example 2. *Solve* $z^2(p^2+q^2) = x^2+y^2$.

Solution. Here, the given equation is

$$z^2(p^2+q^2) = x^2+y^2.$$

Replace $zdz = dZ$

$$\Rightarrow \qquad \frac{z^2}{2} = Z.$$

Therefore, the given equation becomes

$$P^2+Q^2 = x^2+y^2, \text{ where } P = \frac{dZ}{dx} \text{ and } Q = \frac{dZ}{dy}$$

$$\Rightarrow \qquad P^2 - x^2 = y^2 - Q^2.$$

Let us write $\qquad P^2 - x^2 = y^2 - Q^2 = a$

$$\Rightarrow \qquad P = \sqrt{(a+x^2)} \text{ and } Q = \sqrt{(y^2-a)}.$$

Now, putting the values of P and Q in

$$\begin{aligned} dZ &= Pdx + Qdy \\ &= \sqrt{(a+x^2)}\, dx + \sqrt{(y^2-a)}\, dy. \end{aligned}$$

On integrating, we have

$$Z = \frac{x}{2}\sqrt{(a+x^2)} + \frac{a}{2}\log\{x+\sqrt{(a+x^2)}\} + \frac{y}{2}\sqrt{(y^2-a)} - \frac{a}{2}\log\{y+\sqrt{(y^2-a)}\} + b$$

$$\Rightarrow \qquad z^2 = x\sqrt{(a+x^2)} + a\log\{x+\sqrt{(a+x^2)}\} + y\sqrt{(y^2-a)} - a\log\{y+\sqrt{(y^2-a)}\} + c.$$

Example 3. *Solve* $z = px+qy+c\sqrt{(1+p^2+q^2)}$.

Solution. Here, the given equation is of the standard form IV. Therefore, the complete solution is

$$z = ax+by+c\sqrt{(1+a^2+b^2)} \qquad \ldots(1)$$

To find the singular solution, differentiating (1) partially with respect to a and b, we have

$$0 = x + \frac{ac}{\sqrt{(1+a^2+b^2)}} \Rightarrow a = \frac{-x}{\sqrt{(c^2-x^2-y^2)}} \qquad \ldots(2)$$

and

$$0 = y + \frac{bc}{\sqrt{(1+a^2+b^2)}} \Rightarrow b = -\frac{y}{\sqrt{(c^2-x^2-y^2)}} \qquad \ldots(3)$$

which gives $\qquad x^2+y^2 = \dfrac{(a^2+b^2)c^2}{1+a^2+b^2}$

$$\Rightarrow \qquad (c^2-x^2-y^2) = \frac{c^2}{1+a^2+b^2}$$

$$\Rightarrow \qquad (1+a^2+b^2) = \frac{c^2}{(c^2-x^2-y^2)}. \qquad \ldots(4)$$

Now using (2), (3) and (4), (1) becomes

$$\begin{aligned} z &= \frac{-x^2}{\sqrt{(c^2-x^2-y^2)}} - \frac{y^2}{\sqrt{(c^2-x^2-y^2)}} + \frac{c^2}{\sqrt{(c^2-x^2-y^2)}} \\ &= \frac{(c^2-x^2-y^2)}{\sqrt{(c^2-x^2-y^2)}} = \sqrt{(c^2-x^2-y^2)} \end{aligned}$$

$$\Rightarrow \qquad z^2 = c^2-x^2-y^2$$

$$\Rightarrow \qquad x^2+y^2+z^2 = c^2.$$

• SUMMARY

- Standard form I : $f(p, \varepsilon) = 0$
  Its solution is $z = ax + by + c, \quad f(a, b) = 0.$
- Standard form II $f(z, p, q) = 0$
  To solve such D.E., put $X = x + ay.$
- Standard form III : $f_1(x, p) = f_2(x, q)$
  To solve such D.E., we put $c_1 = f_1(x, p) = f_2(y, q)$
- Standard form IV : $z = px + qy + f(p, q).$
  Its solution is $z = ax + by + f(a, b).$

• STUDENT ACTIVITY

1. Eliminate f and g from $y = f(x - at) + g(x + at)$

2. Solve $p^3 + z^3 = 27z.$

• TEST YOURSELF

Solve the following equations :

1. $\sqrt{p} + \sqrt{q} = 2x.$
2. $pe^y = qe^x.$
3. $pq = xy.$
4. $py = 2yx + \log q.$
5. $z(p^2 - q^2) = (x - y).$
6. Find the complete integral of $z = px + qy + p^2 + q^2.$
7. $z = px + qy - 2p - 3q.$
8. $z = px + qy - p^2 q.$
9. $z = px + qy + pq.$

ANSWERS

1. $z = \frac{1}{6}(a + 2x)^3 + a^2 y + b$
2. $z = ae^x + ae^y + b$
3. $z = \frac{1}{2a}(a^2x^2 + y^2 + 2ab)$
4. $z = \frac{1}{a}(ax^2 + a^2x + e^{ay} + a \,.\, b)$
5. $z^{3/2} = (x + a)^{3/2} + (y + a)^{3/2} + c$
6. $z = ax + by + a^2 + b^2$
7. $z = ax + by - 2a - 3b$
8. $z = ax + by - a^2 b$
9. $z = ax + by + ab$

OBJECTIVE EVALUATION

Fill in the Blanks :

1. The complete integral of the equation of the type $f(p, q) = 0$ is $z = ax + by + c$, where a and b are connected by the relation ....
2. The equations $f_1(x, y, z, p, q) = 0$ and $f_2(x, y, z, p, q) = 0$ are said to be compatible if $(f_1, f_2) =$
3. The equation of the type $f_1(x, p) = f_2(y, q)$ does not have any ...... solution.

True or False :

Write T for true and F for false :

1. A partial differential equation does not contain any partial derivative. *(T/F)*
2. The second order partial differential equation may also contain first order terms. *(T/F)*

Multiple Choice Questions (MCQ's) :

Choose the most appropriate one :

1. The equation of the envelope of the surfaces represented by the complete integral of the given PDE is called :
   (a) Particular integral (b) Singular integral
   (c) General solution (d) None of these.
2. The complete integral of $z = px + qy + p^2 + q^2$ is :
   (a) $z = ax + by$ (b) $z = a^2 + b^2$
   (c) $z = ax + by + a^2 + b^2$ (d) None of these.
3. The complete integral of $p = e^q$ is :
   (a) $a = e^b$ (b) $b = e^a$
   (c) $z = e.a$ (d) $z = ax + y \log a + c$.

ANSWERS

Fill in the Blanks :

1. $f(a, b) = 0$ 2. 0 3. singular.

True or False :

1. F 2. T

Multiple Choice Questions :

1. (b) 2. (c) 3. (d)

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UNIT 5

SOME METHODS FOR THE SOLUTION OF PARTIAL DIFFERENTIAL EQUATION

STRUCTURE

- Lagrange's Linear differential equation
- Geometric interpretation of Lagrange's differential equation
- Charpit's Method
  - Summary
  - Student Activity
  - Test Yourself

LEARNING OBJECTIVES

After going through this unit you will learn :

- What is the Lagrange's D.E. ?
- How to find its solution ?
- What is the Charpit's method ?
- How to find the solution P.D.E. by using Charpit's method.

5.1. LAGRANGE'S LINEAR DIFFERENTIAL EQUATION

The partial differential equation of the type $Pp + Qq = R$, where P, Q, R are the functions of x, y and z and $p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$. Then this partial differential of order one is called Lagrange's Linear Differential Equation.

Lagrange's Auxiliary Equations :

Let u and v be two functions of x, y, z which are related by the relation

$$f(u, v) = 0 \quad \ldots(1)$$

Differentiating (1) partially w.r.t. x and y, we get

$$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial x}\right) + \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial x}\right) = 0$$

or

$$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z} p\right) + \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}\cdot p\right) = 0 \quad \ldots(2)$$

and

$$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial y}\right) + \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial y}\right) = 0$$

or

$$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} q\right) + \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z}\cdot q\right) = 0 \quad \ldots(3)$$

Eliminating $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$ from (2) and (3), we get

From (2),

$$\frac{\partial f/\partial u}{\partial f/\partial v} = -\frac{\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z} p\right)}{\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z} p\right)} \quad \ldots(4)$$

From (3), $$\frac{\partial f/\partial u}{\partial f/\partial v} = -\frac{\left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z}q\right)}{\left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}q\right)} \quad ...(5)$$

From (4) and (5), we get

$$\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}p\right)\left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z}q\right) = \left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}q\right)\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}p\right)$$

Solving this equation, we get

$$\left(\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial z} - \frac{\partial v}{\partial y}\cdot\frac{\partial u}{\partial z}\right)p + \left(\frac{\partial v}{\partial x}\cdot\frac{\partial u}{\partial z} - \frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial z}\right)q = \left(\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial x}\right)$$

or $$\boldsymbol{Pp + Qq = R} \quad ...(6)$$

where $$\boldsymbol{P} = \frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial z} - \frac{\partial v}{\partial y}\cdot\frac{\partial u}{\partial z} = \frac{\partial (u, v)}{\partial (y, z)},$$ (Jacobian of u and v w.r.t. y and z)

$$\boldsymbol{Q} = \frac{\partial v}{\partial x}\cdot\frac{\partial u}{\partial z} - \frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial z} = \frac{\partial (u, v)}{\partial (z, x)}$$

and $$\boldsymbol{R} = \frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial x} = \frac{\partial (u, v)}{\partial (x, y)}$$

Thus $f(u, v) = 0$ is the general integral of the differential equation $Pp + Qq = R$. Now we shall determine the values of u and v. For this, let $u = a$ and $v = b$ be two equations, where a and b are arbitrary constants. That is

$$u(x, y, z) = a \text{ and } v(x, y, z) = b$$

This implies

$$du = 0 \text{ and } dv = 0$$

But $$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz$$

and $$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy + \frac{\partial v}{\partial z}dz$$

Thus, we obtained

$$\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz = 0 \quad ...(7)$$

and $$\frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy + \frac{\partial v}{\partial z}dz = 0 \quad ...(8)$$

Solving, (7) and (8) by cross multiplication method for dx, dy and dz, we get

$$\frac{dx}{\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial z} - \frac{\partial v}{\partial y}\cdot\frac{\partial u}{\partial z}} = \frac{dy}{\frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial z}} = \frac{dz}{\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial x}}$$

or $$\frac{dx}{\frac{\partial (u, v)}{\partial (y, z)}} = \frac{dy}{\frac{\partial (u, v)}{\partial (z, x)}} = \frac{dz}{\frac{\partial (u, v)}{\partial (x, y)}}$$

or $$\frac{\boldsymbol{dx}}{\boldsymbol{P}} = \frac{\boldsymbol{dy}}{\boldsymbol{Q}} = \frac{\boldsymbol{dz}}{\boldsymbol{R}} \quad ...(9)$$

Thus equations (9) are known as **Lagrange's auxiliary equations or Lagrange's subsidiary** equations.

• 5.2. GEOMETRICAL INTERPRETATION OF LAGRANGE'S LINEAR DIFFERENTIAL EQUATION

Lagrange's Linear differential equation is

$$Pp + Qq = R \quad ...(1)$$

where $p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$ and P, Q, R are the functions of x, y and z.

Equation (1) can be written as

$$Pp + Qq - R = 0$$

or $$Pp + Qq + R(-1) = 0 \qquad \ldots(2)$$

Lagrange's auxiliary equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \qquad \ldots(3)$$

These equations represent a family of curves and P, Q, R are the **direction ratio** of the tangent drawn at any point on the curves.

Since $f(u, v) = 0$ represents a surface through these curves. where $u = a$ (constant) and $v = b$ (constant) are the two particular integrals of the equation (3) and are the functions of x, y and z.

Further since, we know that the direction cosines of the normal to the surface $f(x, y, z) = 0$ at any point on it are proportional to

$$\frac{\partial f}{\partial x} : \frac{\partial f}{\partial y} : \frac{\partial f}{\partial z}$$

Divide by $\frac{\partial f}{\partial z}$, we get

$$\frac{\partial f/\partial x}{\partial f/\partial z} : \frac{\partial f/\partial y}{\partial f/\partial z} : 1 \qquad \ldots(4)$$

Since $p = \frac{\partial z}{\partial x} = -\frac{\partial f/\partial x}{\partial f/\partial z}$ and $q = \frac{\partial z}{\partial y} = -\frac{\partial f/\partial y}{\partial f/\partial z}$, then (4) becomes

$$-p : -q : 1$$

or $$p : q : -1$$

Thus equation (2) represents that the normal at any point on the surface is perpendicular to the tangent to the curve obtained by equation (3) through which this surface passes. Hence we can say that the equations (1) and (3) give the same equivalent surfaces.

SOLVED EXAMPLES

Example 1. *Solve the differential equation $yzp + zxq = xy$.*

Solution. Compare the given partial differential equation with

$$Pp + Qq = R$$

We get $P = yz$, $Q = zx$ and $R = xy$

Then the subsidiary equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

or $$\frac{dx}{yz} = \frac{dy}{zx} = \frac{dz}{xy} \qquad \ldots(1)$$

Taking the first two members of (1), we get

$$\frac{dx}{yz} = \frac{dy}{zx}$$

or $$xdx - ydy = 0.$$

Integrating, we get

$$x^2 - y^2 = c_1 \qquad \ldots(2)$$

Now taking second and third members of (1), we get

$$\frac{dy}{zx} = \frac{dz}{xy}$$

or $$ydy - zdz = 0.$$

Integrating, we get

$$y^2 - z^2 = c_2 \qquad \ldots(3)$$

Thus the general solution is

$$f(x^2 - y^2, y^2 - z^2) = 0.$$

Example 2. *Solve the partial differential equation $pz - qz = z^2 + (x + y)^2$.*

Solution. Compare the given partial differential equation with the standard partial differential equation

$$Pp + Qq = R$$

We get $P = z$, $Q = -z$, and $R = z^2 + (x+y)^2$

The subsidiary equations are given by

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\Rightarrow \quad \frac{dx}{z} = \frac{dy}{-z} = \frac{dz}{z^2 + (x+y)^2} \qquad \ldots(1)$$

Taking first and second ratio of (1), we get

$$\frac{dx}{z} = \frac{dy}{-z}$$

$$\Rightarrow \quad dx = -dy$$

$$\Rightarrow \quad dx + dy = 0$$

$$\Rightarrow \quad x + y = c_1 \qquad \text{(on integrating)}$$

Now taking first and third ratio of (1), we get

$$\frac{dx}{z} = \frac{dz}{z^2 + (x+y)^2}$$

or

$$dx = \frac{zdz}{z^2 + (x+y)^2}$$

or

$$dx = \frac{zdz}{z^2 + c_1^2} \qquad (\because \quad x + y = c_1)$$

On integrating, we get

$$2x = \log(z^2 + c_1^2) + \log c_2$$

or

$$e^{2x} = c_2 (z^2 + c_1^2)$$

$$e^{2x} = c_2 [z^2 + (x+y)^2]$$

$$c_2 = \frac{e^{2x}}{x^2 + y^2 + z^2 + 2xy}$$

Thus the general integral is given by

$$f\left(x + y, \frac{e^{2x}}{x^2 + y^2 + z^2 + 2xy}\right) = 0.$$

Example 3. *Solve* $xzp + yzq = xy$.

Solution. Compare this differential equation with Lagrange's linear differential equation

$$Pp + Qq = R.$$

We get

$$P = xz, \; Q = yz, \; R = xy.$$

Then, the Lagrange's subsidiary equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\Rightarrow \quad \frac{dx}{xz} = \frac{dy}{yz} = \frac{dz}{xy} \qquad \ldots(1)$$

Taking first and second ratio of (1), we get

$$\frac{dx}{xz} = \frac{dy}{yz}$$

$$\Rightarrow \quad \frac{dx}{x} = \frac{dy}{y}$$

$$\Rightarrow \quad \frac{dx}{x} - \frac{dy}{y} = 0.$$

On integrating, we get

$$\log x - \log y = \log c_1$$

or $$\frac{x}{y} = c_1.$$

Now taking second and third ratio of (1), we get

$$\frac{dy}{yz} = \frac{dz}{xy}$$

$$\Rightarrow \qquad \frac{dy}{z} = \frac{dz}{x}$$

$$\Rightarrow \qquad xdy = zdz$$

$$\Rightarrow \qquad c_1\, ydy = zdz \qquad (\because \; x = c_1\, y)$$

On integrating, we get

$$c_1\, y^2 - z^2 = c_2$$

or $$\left(\frac{x}{y}\right) y^2 - z^2 = c_2$$

or $$xy - z^2 = c_2.$$

Thus the general integral is

$$f\left(\frac{x}{y},\; xy - z^2\right) = 0.$$

Example 4. *Find the general solution of the following differential equation*

$$(mz - ny)\, p + (nx - lz)\, q = ly - mx.$$

Solution. Compare the given differential equation with Lagrange's differential equation $Pp + Qq = R$, we get

$$P = mz - ny,\; Q = nx - lz,\; R = ly - mx.$$

Then Lagrange's auxiliary equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\Rightarrow \qquad \frac{dx}{mz - xy} = \frac{dy}{nx - lz} = \frac{dz}{ly - mx} \qquad \ldots(1)$$

Taking the multipliers x, y, z, then (1) becomes

$$\frac{dx}{mz - xy} = \frac{dy}{nx - lz} = \frac{dz}{ly - mx} = \frac{xdx + ydy + zdz}{0}$$

$$\therefore \qquad xdx + ydy + zdz = 0.$$

Integrating, we get

$$x^2 + y^2 + z^2 = c_1.$$

Again taking the multipliers l, m, n, then (1) becomes

$$\frac{dz}{mz - ny} = \frac{dy}{nx - lz} = \frac{dz}{ly - mx} = \frac{ldx + ndy + ndz}{0}$$

$$\therefore \qquad ldx + mdy + ndz = 0.$$

Integrating, we get

$$lx + my + nz = c_2.$$

Thus the general solution is

$$f(x^2 + y^2 + z^2,\; lx + my + nz) = 0.$$

• TEST YOURSELF

Find the general integrals of the linear partial differential equations :

1. $\left(\dfrac{y - z}{yz}\right) p + \left(\dfrac{z - x}{zx}\right) q = \left(\dfrac{x - y}{xy}\right).$

2. $\dfrac{y^2 z}{x}\, p + zxq = y^2.$

3. $p + q = \dfrac{z}{a}.$

and $$dq = 0$$

$\Rightarrow$ $$q = b \text{ (constant)}.$$

Substituting these values of p and q into (1), we get

$$z = ax + by + a^2 + b^2.$$

This is required complete integral.

Example 2. *Find the complete integral of* $2zx - px^2 - 2qxy + pq = 0$.

Solution. Assume $\quad f \equiv 2zx - px^2 - 2qxy + pq = 0.$ ...(1)

Now finding partial derivatives of f with respect to x, y, z, p and q respectively.

$$\frac{\partial f}{\partial x} = 2z - 2px - 2qy,\ \frac{\partial f}{\partial y} = -2qx,\ \frac{\partial f}{\partial z} = 2x,\ \frac{\partial f}{\partial p} = -x^2 + q,\ \frac{\partial f}{\partial q} = -2xy + p.$$

Then the Charpit's auxiliary equation are

$$\frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}}$$

$$\Rightarrow \quad \frac{dx}{x^2 - q} = \frac{dy}{2xy - p} = \frac{dz}{px^2 - pq + 2xyq - pq} = \frac{dp}{2z - 2qy} = \frac{dq}{0} \qquad \text{...(2)}$$

From (2),

$$dq = 0.$$

Integrating, $q = a$ (constant).

Putting the value of $q = a$ into (1), we get

$$2zx - px^2 - 2axy + pa = 0$$

or $$p = \frac{2x(z - ay)}{x^2 - a}.$$

Now substituting these values of p and q into $dz = pdx + qdy$, we get

$$dz = \frac{2x(z - ay)}{x^2 - a}dx + ady$$

or $$dz - ady = \frac{2x(z - ay)}{x^2 - a}dx$$

or $$\frac{dz - ady}{z - ay} = \frac{2x\,dx}{x^2 - a}.$$

Integrating, we get

$$\log(z - ay) = \log(x^2 - a) + \log b$$

or $$z - ay = b(x^2 - a)$$

or $$z = ay + b(x^2 - a).$$

This is the required complete integral.

Example 3. *Solve* $\quad p = (z + qy)^2.$

Solution. Assuming $\quad f \equiv (z + qy)^2 - p = 0$ ...(1)

Now finding the partial derivatives of f w.r.t. x, y, z, p and q

$$\frac{\partial f}{\partial x} = 0,\ \frac{\partial f}{\partial y} = 2q(z + qy),\ \frac{\partial f}{\partial z} = 2(z + qy),\ \frac{\partial f}{\partial p} = -1,\ \frac{\partial f}{\partial q} = 2y(z + qy).$$

Then the Charpit's auxiliary equations are

$$\frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}}$$

$$\Rightarrow \quad \frac{dx}{1} = \frac{dy}{-2y(z + qy)} = \frac{dz}{p - 2qy(z + qy)} = \frac{dp}{2p(z + yq)} = \frac{dq}{4q(z + qy)} \qquad \text{...(2)}$$

Taking second and fourth ratio of (2), we get

$$\frac{dy}{-2y(z + qy)} = \frac{dp}{2p(z + yq)}$$

$$\Rightarrow \qquad \frac{dp}{p} + \frac{dy}{y} = 0.$$

Integrating, we get

$$\log p + \log y = \log a \quad \text{or} \quad py = a$$

or $$p = \frac{a}{y}.$$

Substitute the value of p into (1), we get

$$(z + qy)^2 = \frac{a}{y}$$

or $$(z + qy) = \sqrt{\frac{a}{y}}$$

or $$q = \frac{\sqrt{a}}{y^{3/2}} - \frac{z}{y}.$$

Now substituting the values of p and q into

$$dz = pdx + qdy$$

$$\therefore \qquad dz = \frac{a}{y}\,dx + \left(\frac{\sqrt{a}}{y^{3/2}} - \frac{z}{y}\right) dy$$

or $$ydz = adx + \sqrt{\frac{a}{y}}\,dy - zdy$$

or $$ydz + zdy = adx + \sqrt{\frac{a}{y}}\,dy$$

or $$d\,(yz) = adx + \sqrt{\frac{a}{y}}\,dy$$

Integrating, we get

$$yz = ax + 2\sqrt{ay} + b.$$

This is the required complete integral.

• SUMMARY

- Lagrange's D.E. $\quad Pp + Qq = R.$
- Lagrange's A.E. $\quad \dfrac{dx}{P} = \dfrac{dy}{Q} = \dfrac{dz}{R}$
- Charpitz's A.E. $\quad \dfrac{dx}{-\dfrac{\partial f}{\partial p}} = \dfrac{dy}{-\dfrac{\partial f}{\partial q}} = \dfrac{dz}{-p\dfrac{\partial f}{\partial p} - q\dfrac{\partial f}{\partial q}} = \dfrac{dp}{\dfrac{\partial f}{\partial x} + p\dfrac{\partial f}{\partial z}} = \dfrac{dq}{\dfrac{\partial f}{\partial y} + q\dfrac{\partial f}{\partial z}}$

• STUDENT ACTIVITY

1. Solve $p^2 + q^2 = x + y$.

2. Solve $yzp + zxq = xy$.

• TEST YOURSELF–2

Using Charpit's method, find the complete integral of the following differential equation :

1. $(p^2 + q^2)\, y = qz$. **2.** $px^5 - 4q^3x^2 + 6x^2z - 2 = 0$. **3.** $yzp^2 = q$.

4. $2\,(pq + py + qx) + x^2 + y^2 = 0$. **5.** $2z + p^2 + 2y^2 + qy = 0$.

6. $p^2 - y^2q + x^2 = y^2$. **7.** $z = pq$.

ANSWERS

1. $(ax + b)^2 + a^2y^2 = az^2$ **2.** $z = -\frac{2}{3}a^3\, e^{q/x^2} + \frac{1}{9} + \frac{1}{3x^2} + (ay + b)\, e^{3/x^2}$

3. $z^2 = \frac{(x+b)^2}{(a - y^2)}$

4. $2z = ax - x^2 + ay - y^2 + \frac{1}{2}(x - y)\sqrt{2\,(x-y)^2 + a^2} + \frac{a^2}{2\sqrt{2}} \log\,[\{\sqrt{2}\,(x - y) + \sqrt{2\,(x-y)^2 + a^2}\,] + b$

5. $y^2\,\{(x - a)^2 + y^2 + 2z\} = b$. **6.** $z = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) - \frac{a^2}{y} - y + b$.

7. $2\sqrt{z} = \sqrt{a}\,.\,x + \frac{1}{\sqrt{a}}\,y + b$.

OBJECTIVE EVALUATION

Fill in the Blanks :

1. The Lagrange's method can be used to solve ...... order PDE.
2. The general method to solve *PDE* is known as .... method.
3. The complete integral of $px + qy = pq$ is ....

True or False :

Write T for the true and F for false :

1. The complete mehod of $4z = pq$ is $az = (x + ay + b)^2$. *(T/F)*
2. The complete integral of $z\,pq = p + q$ is $z^2 = 2\,(a + 1)\,(x + y/a) + b$. *(T/F)*

Multiple Choice Questions (MCQ's) :

Choose the most appropriate one :

1. The complete integral of $f(p, q) = 0$ is :
   (a) $z = ax + b$ (b) $z = ax + by + c$ (c) $z = ax + f(a)\,.\,y + b$ (d) None of these.
2. The complete integral of $z = pq$ is :
   (a) $2\sqrt{z} = \sqrt{ax} + b$ (b) $2\sqrt{z} = \sqrt{ax} + \frac{1}{\sqrt{a}}\,y$
   (c) $z = \sqrt{ax} + y$ (d) $2\sqrt{z} = \sqrt{ax} + \frac{1}{\sqrt{a}}\,y + b$.
3. The complete integral of $q = 3y^2$ is :
   (a) $z = ax + b$ (b) $z = ax + y$
   (c) $z = ax + y^3 + b$ (d) None of these.

ANSWERS

Fill in the Blanks :

1. first 2. Charpit's 3. $az = \frac{1}{2}(y + ax)^2 + b$

True or False :

1. T 2. T.

Multiple Choice Questions :

1. (c) 2. (d) 3. (c)

□□□

UNIT

6

THE LAPLACE TRANSFORM

STRUCTURE

- Definitions
- Linearity property
- Existence of Laplace transform
- Laplace transforms of some elementary functions
- Some important theorems
- Laplace transforms of derivatives
  - Summary
  - Student Activity
  - Test Yourself

LEARNING OBJECTIVES

After going through this unit you will learn :

- What is Laplace transforms ?
- How to find Laplace transform of given functions using Laplace transforms ?

• 6.1. DEFINITIONS

Definition 1. An integral of the form

$$\int_{-\infty}^{\infty} k(p, t)\, F(t)\, dt$$

is defined as the integral transform of $F(t)$, provided it is convergent.

Differential Equations

It is denoted by $f(p)$ or $T\{F(t)\}$.

$$\therefore \qquad f(p) = T\{F(t)\} = \int_{-\infty}^{\infty} k(p, t)\, F(t)\, dt.$$

Definition 2. If $F(t)$ be a function of t defined for all values of t, then **Laplace transform** of $F(t)$, denoted by $L\{F(t)\}$ or $f(p)$ is defined by

$$L\{F(t)\} = f(p) = \int_{0}^{\infty} e^{-pt}\, F(t)\, dt \qquad ...(1)$$

Definition 3. A function $f(x)$ is said to be exponential order a as $x \to \infty$ if $\lim_{x \to \infty} e^{-ax} f(x) =$ a finite quantity.

i.e., for a given positive integer n if a real number M such that

$$|e^{-ax} f(x)| < M, \quad \forall\, x \geq n$$

which can be written as $f(x) = 0\,(e^{ax}),\ x \to \infty$.

Definition 4. A function $f(x)$ is called sectionally continuous (piecewise continuous) over the closed interval $x_1 \leq x \leq x_2$ if the closed interval can be divided into a finite number of subintervals $a \leq x \leq b$ such that

(i) $f(x)$ is continuous in the closed interval $[a, b]$

(ii) $\lim_{x \to a+0} f(x)$ and $\lim_{x \to b-0} f(x)$ both exist.

Definition 5. A function, which is sectionally (or piecewise) continuous over every finite interval in the range $t \geq 0$ and ∞ of exponential order as $t \to \infty$ is called a function of class A.

• 6.2. LINEARITY PROPERTY

Theorem. *The Laplace transformation is a linear transformation*

$$L\{a_1F_1(t) + a_2F_2(t)\} = a_1 L\{F_1(t)\} + a_2 L\{F_2(t)\}.$$

Proof. We know that

$$L\{f(t)\} = \int_0^\infty e^{-pt} f(t)\, dt.$$

Therefore,

$$L\{a_1 f_1(t) + a_2 f_2(t)\} = \int_0^\infty e^{-pt} [a_1 f_1(t) + a_2 f_2(t)]\, dt$$

$$= a_1 \int_0^\infty e^{-pt} f_1(t)\, dt + a_2 \int_0^\infty e^{-pt} f_2(t)\, dt$$

$$= a_1 L\{f_1(t)\} + a_2 L\{f_2(t)\}.$$

• 6.3. EXISTENCE OF LAPLACE TRANSFORM

Theorem. *If $F(t)$ is a function which is piecewise continuous on every finite interval in the range $t \geq 0$ and satisfies*

$$|F(t)| \leq Me^{at}$$

for all $t \geq 0$ and for some constant a and M, then the Laplace transform of $F(t)$ exists for all $p > a$.

Proof. We know that

$$L\{F(t)\} = \int_0^\infty e^{-pt} F(t)\, dt$$

$$= \int_0^{t_0} F(t)\, e^{-pt}\, dt + \int_{t_0}^\infty F(t)\, e^{-pt}\, dt \qquad ...(1)$$

Now $\int_0^{t_0} F(t)\, e^{-pt}\, dt$ exists since $F(t)$ is sectionally continuous on every finite interval $0 \leq t \leq t_0$

and

$$\left| \int_{t_0}^\infty F(t)\, e^{-pt}\, dt \right| \leq \int_{t_0}^\infty |F(t)\, e^{-pt}|\, dt$$

$$\leq \int_{t_0}^\infty e^{-pt} Me^{at}\, dt, \qquad (\because\ |F(t)| \leq Me^{at})$$

$$= \int_{t_0}^\infty e^{(a-p)t} M\, dt$$

$$= M\left[\frac{e^{-(p-a)t}}{-(p-a)}\right]_{t_0}^\infty$$

$$= \frac{M}{p-a} e^{-(p-a)t_0}, \quad \text{if } p > a$$

$$\Rightarrow \qquad \left| \int_{t_0}^\infty e^{-pt} f(t)\, dt \right| \leq \frac{M}{p-a} e^{-(p-a)t_0}, \quad \text{if } p > a.$$

Now $\dfrac{Me^{-(p-a)t_0}}{p-a}$ can be made small as we please by taking t_0 sufficiently large. Hence, from (1), we conclude that $L\{f(t)\}$ exists for all $p > a$.

• 6.4. LAPLACE TRANSFORMS OF SOME ELEMENTRY FUNCTIONS

(i) $F(t) = 1$.

Solution. We have $L\{F(t)\} = \int_0^\infty e^{-pt} f(t)\, dt$ ...(1)

Here $F(t) = 1$.

Therefore, from (1)

$$L\{1\} = \int_0^\infty e^{-pt} . 1\, dt = \left[-\frac{e^{-pt}}{p}\right]_0^\infty$$

$$= \frac{1}{p}, \quad p > 0$$

Hence $L\{1\} = \dfrac{1}{p}$.

(ii) $F(t) = t^n$.

Solution. We have $L\{F(t)\} = \int_0^\infty e^{-pt} F(t)\, dt$

$\Rightarrow$ $$L\{t^n\} = \int_0^\infty e^{-pt}\, t^n\, dt = \int_0^\infty e^{-pt} . t^{(n+1)-1} . dt$$

$$= \frac{\Gamma(n+1)}{p^{n+1}} \qquad \left[\because \int_0^\infty e^{-u} u^n\, du = \Gamma(n+1)\right]$$

$$= \frac{n!}{p^{n+1}}, \quad p > 0$$

Hence $L\{t^n\} = \dfrac{n!}{p^{n+1}}$.

(iii) $F(t) = t$.

Solution. We have $L\{t\} = \int_0^\infty e^{-pt} . t\, dt$

$$= \left[-\frac{1}{p} te^{-pt}\right]_0^\infty + \frac{1}{p}\int_0^\infty e^{-pt}\, dt$$

$$= \frac{1}{p^2}, \quad p > 0.$$

(iv) $F(t) = e^{at}$.

Solution. We have $L\{e^{at}\} = \int_0^\infty e^{-pt} e^{at}\, dt$

$$= \int_0^\infty e^{-(p-a)t}\, dt.$$

If $p \le a$, integral diverges. For $p > a$, the integral converges. Hence, for $p > a$.

$$L\{e^{at}\} = \int_0^\infty e^{-(p-a)t}\, dt$$

$$= \left[-\frac{e^{-(p-a)t}}{p-a} \right]_0^\infty = 0 + \frac{1}{p-a}$$

$$= \frac{1}{p-a}, \quad p > a.$$

(v) $F\{t\} = \sin at$.

Solution. $L\{\sin at\} = \int_0^\infty e^{-pt} \sin at \, dt$

$$= \left[\frac{e^{-pt}(-p \sin at - a \cos at)}{p^2 + a^2} \right]_0^\infty$$

$$= \frac{a}{p^2 + a^2}, \quad p > a$$

Hence $L\{\sin at\} = \dfrac{a}{p^2 + a^2}$.

(vi) $F\{t\} = \cos at$.

Solution. We know that

$$\int e^{ax} \cos bx \, dx = \frac{e^{ax}(a \cos bx + b \sin bx)}{a^2 + b^2}.$$

Therefore, we have

$$L\{\cos at\} = \int_0^\infty e^{-pt} \cos at \, at$$

$$= \left[\frac{e^{-pt}(-p \cos at + a \sin at)}{a^2 + p^2} \right]_0^\infty$$

$$= \frac{p}{p^2 + a^2}; \quad p > 0.$$

(vii) $F\{t\} = \sinh at$.

Solution. Consider

$$L\{\sinh at\} = L\left\{ \frac{e^{at} - e^{-at}}{2} \right\}$$

$$= \frac{1}{2} L[e^{at}] - \frac{1}{2} L[e^{-at}]$$

$$= \frac{1}{2} \cdot \frac{1}{p-a} - \frac{1}{2} \cdot \frac{1}{p+a}$$

$$= \frac{a}{p^2 - a^2}$$

Hence $L\{\sinh at\} = \dfrac{a}{p^2 - a^2}$.

(viii) $F\{t\} = \cosh at$.

Solution. Consider

$$L\{\cosh at\} = L\left[\frac{1}{2}(e^{at} + e^{-at}) \right].$$

$$= \frac{1}{2} L[e^{at}] + \frac{1}{2} L[e^{-at}]$$

$$= \frac{1}{2} \cdot \frac{1}{p-a} + \frac{1}{2} \cdot \frac{1}{p+a}, \quad p > a \text{ and } p > -a$$

$$= \frac{p}{p^2 - a^2}, \quad p > |a|$$

Hence, $$L\{\cosh at\} = \frac{p}{p^2 - a^2}.$$

Table of Laplace Transforms of Special Functions

	$F(t)$	$L\{F(t)\}$
1.	1	$\frac{1}{p},\ p > 0$
2.	$t^n,\ n \in \mathbf{Z}^+$	$\frac{n!}{p^{n+1}},\ p > 0$
3.	$t^a,\ a > -1$	$\frac{\Gamma(a+1)}{p^{a+1}},\ p > 0$
4.	e^{at}	$\frac{1}{p-a},\ p > a$
5.	$\sin at$	$\frac{a}{p^2+a^2},\ p > 0$
6.	$\cos at$	$\frac{p}{p^2+a^2},\ p > 0$
7.	$\sinh at$	$\frac{a}{p^2-a^2},\ p > \lvert a \rvert$
8.	$\cosh at$	$\frac{p}{p^2-a^2},\ p > \lvert a \rvert$

SOLVED EXAMPLES

Example 1. *Find the Laplace transform of the function* $F\{t\} = \frac{e^{at} - 1}{a}$.

Solution. We have

$$L\{F(t)\} = L\left[\frac{e^{at}-1}{a}\right] = L\left[\frac{1}{a}e^{at} - \frac{1}{a}\right]$$

$$= \frac{1}{a}L\{e^{at}\} - \frac{1}{a}L(1)$$

$$= \frac{1}{a}\left(\frac{1}{p-a}\right) - \frac{1}{a}\left(\frac{1}{p}\right)$$

$$= \frac{1}{p(p-a)}.$$

Example 2. *Find* $L\{(t^2+1)^2\}$.

Solution. $L\{(t^2+1)^2\} = L(t^4 + 2t^2 + 1)$

$$= L\{t^4\} + 2L\{t^2\} + L(1) \qquad \text{(By linearty property)}$$

$$= \frac{4!}{p^5} + 2 \cdot \frac{2!}{p^3} + \frac{1}{p} = \frac{24 + 4p^2 + p^4}{p^5},\ p > 0.$$

Example 3. *Find* $L\{F(t)\}$ *where* $F(t) = (\sin t - \cos t)^2$.

Solution. Consider

$$L\{(\sin t - \cos t)^2\} = L\{\sin^2 t + \cos^2 t - 2\sin t \cos t\}$$

$$= L\{1 - \sin 2t\}$$

$$= L\{1\} - L\{\sin 2t\}$$

$$= \frac{1}{p} - \frac{2}{p^2+2^2},\ p > 0$$

$$= \frac{p^2 - 2p + 4}{p(p^2+4)},\ p > 0.$$

Example 4. *Find* $L\{6\sin 2t - 5\cos 2t\}$.

Solution. $L\{6\sin 2t - 5\cos 2t\} = 6\,L\{\sin 2t\} - 5L\{\cos 2t\}$

$$= 6 \cdot \frac{2}{p^2+2^2} - 5 \cdot \frac{p}{p^2+2^2}, \quad p > 0$$

$$= \frac{12-5p}{p^2+4}, \quad p > 0.$$

Example 5. *Find* $L\{2e^{3t} - e^{-3t}\}$.

Solution. $L\{2e^{3t} - e^{-3t}\} = 2L\{e^{3t}\} - L\{e^{-3t}\}$

$$= 2 \cdot \frac{1}{p-3} - \frac{1}{p+3}, \quad p > 3 \text{ and } p > -3$$

$$= \frac{p+9}{p^2-9}, \quad p > |3|.$$

Example 6. *Find* $L\{F(t)\}$, *if* $F(t) = \begin{cases} e^t, & 0 < t \le 1 \\ 0, & t > 1. \end{cases}$

Solution. $L\{F(t)\} = \int_0^{\infty} e^{-pt} F(t)\,dt$

$$= \int_0^1 e^{-pt} \cdot e^t\,dt + \int_1^{\infty} e^{-pt} \cdot 0\,dt$$

$$= \int_0^1 e^{-(p-1)t} \cdot dt$$

$$= \left[-\frac{e^{-(p-1)t}}{p-1} \right]_0^1$$

$$= \frac{1}{(p-1)}\left[1 - e^{-(p-1)}\right], \quad p \ne 1.$$

• TEST YOURSELF 1

Find the Laplace transform of the following functions :

1. $\sin t \cos t$.
2. $4\cos^2 t$.
3. $\sin^2 at$.
4. $3\cosh 5t - 4\sinh 5t$.
5. $3t^4 - 2t^3 + 4e^{-3t} - 2\sin 5t + 3\cos 2t$.
6. $e^{-2t} - e^{-3t}$.
7. $\dfrac{e^{at}-1}{a}$.
8. $F(t) = \begin{cases} \sin t, & 0 < t < \pi \\ 0, & t > \pi. \end{cases}$

ANSWERS

1. $\dfrac{1}{p^2+4}, \ p > 0$
2. $\dfrac{4(p^2+8)}{p(p^2+16)}, \ p > 0$
3. $\dfrac{2a^2}{p(p^2+4a^2)}, \ p > 0$
4. $\dfrac{3p-20}{p^2-25}, \ p > 5$
5. $\dfrac{72}{p^5} - \dfrac{12}{p^4} + \dfrac{4}{p+3} - \dfrac{10}{p^2+25} + \dfrac{3p}{p^2+4}, \ p > 0$
6. $\dfrac{1}{p^2+5p+6}, \ p > -2$
7. $\dfrac{1}{p(p-a)}$
8. $\dfrac{e^{-p\pi}+1}{p^2+1}$

• 6.5. SOME IMPORTANT THEOREMS

Theorem 1. (First translation or shifting theorem). *If $f(p)$ is the Laplace transform of $F(t)$, then $f(p-a)$ is the Laplace transforms of $e^{at} F(t)$. i.e.,* *If* $L\{F(t)\} = f(p)$, *when* $p > a$,

$L\{e^{at} F(t)\} = f(p-a), \quad p > \alpha + a.$

Proof. We have, by definition of Laplace transform

$$L\{F(t)\} = f(p) = \int_0^\infty e^{-pt} F(t)\, dt.$$

Therefore, $$L\{e^{at} F(t)\} = \int_0^\infty e^{-pt} . e^{at} F(t)\, dt$$

$$= \int_0^\infty e^{-(p-a)t} . F(t)\, dt$$

$$= \int_0^\infty e^{-ut} F(t)\, dt, \text{ where } u = p - a > 0$$

$$= f(u) \qquad \text{(By definition)}$$

$$= f(p-a).$$

Theorem 2. (Second translation or Heaviside's shifting theorem)

If $$L\{F(t)\} = f(p) \text{ and } G(t) = \begin{cases} F(t-a), & t > a \\ 0, & t < a. \end{cases}$$

Then $$L\{G(t)\} = e^{-ap} f(p).$$

Proof. Let $L\{F(t)\} = f(p)$

and $$G(t) = \begin{cases} F(t-a), & \text{if } t > a \\ 0, & \text{if } t < a \end{cases}$$

Then $$L\{G(t)\} = \int_0^\infty e^{-pt} G(t)\, dt$$

$$= \int_0^a e^{-pt} G(t)\, dt + \int_a^\infty e^{-pt} G(t)\, dt$$

$$= \int_0^a e^{-pt} . 0\, dt + \int_a^\infty e^{-pt} F(t-a)\, dt$$

$$= 0 + \int_a^\infty e^{-pt} F(t-a)\, dt.$$

Let $t - a = u$, therefore $dt = du$.

If $t = a$, then $u = t - a, = a - a = 0.$

If $t = \infty$, then $u = \infty - a = \infty.$

Hence, $$L\{G(t)\} = \int_0^\infty e^{-p(u+a)} F(u)\, du$$

$$= e^{-pa} \int_0^\infty e^{-pu} F(u)\, du$$

$$= e^{-pa} f(p).$$

Theorem 3. (Change of scale property).

If $L\{F(t)\} = f(p)$, *then* $L\{F(at)\} = \dfrac{1}{a} f\left(\dfrac{p}{a}\right)$.

Proof. By definition

$$L\{F(at)\} = \int_0^\infty e^{-pt} F(at)\, dt$$

$$= \int_0^\infty e^{-pu/a} F(u) \frac{du}{a} \qquad \text{(where } at = u\text{)}$$

$$= \frac{1}{a} \int_0^\infty e^{-pu/a} F(u)\, du$$

$$= \frac{1}{a} \int_0^\infty e^{-st} F(t)\, dt, \text{ where } s = \frac{p}{a}$$

$$= \frac{1}{a} f(s) = \frac{1}{a} f\left(\frac{p}{a}\right).$$

SOLVED EXAMPLES

Example 1. *Find* $L\left\{\frac{e^{-at}\, t^{n-1}}{(n-1)\,!}\right\}$.

Solution. We have

$$L\left\{\frac{t^{n-1}}{(n-1)\,!}\right\} = \frac{1}{(n-1)\,!} \cdot \frac{(n-1)\,!}{p^n} = \frac{1}{p^n}.$$

Therefore, using first shifting theorem, we have

$$L\left\{e^{-at} \frac{t^{n-1}}{(n-1)\,!}\right\} = f(p+a) = \frac{1}{(p+a)^n}.$$

Example 2. *Find* $L\{e^t \cos^2 t\}$.

Solution. We have

$$L\{\cos^2 t\} = L\left\{\frac{1}{2}(1+\cos 2t)\right\} = \frac{1}{2}\{L\{1\} + L\{\cos 2t\}\}$$

$$= \frac{1}{2}\left\{\frac{1}{p} + \frac{p}{p^2+2^2}\right\}$$

$$= \frac{p^2+2}{p\,(p^2+4)} = f(p) \text{ (say)}.$$

Using first shifting theorem, we have

$$L\{e^t \cos^2 t\} = f(p-1) = \frac{(p-1)^2+2}{(p-1)\,\{(p-1)^2+4\}} = \frac{p^2-2p+3}{(p-1)(p^2-2p+3)}.$$

Example 3. *Find* $L\{e^{-t}(3 \sin 2t - 5 \cosh 2t)\}$.

Solution. We have

$$L\{3 \sin 2t - 5 \cosh 2t\} = 3 \,.\, \frac{2}{p^2+2^2} - \frac{5p}{p^2-2^2} = f(p) \text{ (say)}.$$

Using first shifting theorem, we have

$$L\{e^{-t}(3 \sin 2t - 5 \cosh 2t)\} = f(p+1)$$

$$= \frac{6}{(p+1)^2+4} - \frac{5\,(p+1)}{(p+1)^2-4}$$

$$= \frac{6}{p^2+2p+4} - \frac{5\,(p+1)}{p^2+2p-3}.$$

Example 4. *Find* $L\{F(t)\}$, *where*

$$F(t) = \begin{cases} \cos\left(t - \frac{2}{3}\pi\right) & , \quad t > \frac{2\pi}{3} \\ 0 & , \quad t < \frac{2\pi}{3} \end{cases}$$

Solution. Let $G(t) = \cos t$

Then $$F(t) = \begin{cases} G\left(t - \dfrac{2\pi}{3}\right), & t > \dfrac{2\pi}{3} \\ 0, & t < \dfrac{2\pi}{3} \end{cases}$$

We have $$L\{G(t)\} = L\{\cos t\} = \frac{p}{p^2+1} = f(p) \text{ (say)}$$

Using second shifting theorem, we have

$$L\{F(t)\} = e^{\left(-\frac{2\pi}{3}\right)\cdot p} \cdot f(p)$$
$$= e^{-2\pi p/3} \cdot \frac{p}{p^2+1}.$$

• TEST YOURSELF 2

1. Find $L\{t^3 e^{-3t}\}$.
2. Find $L\{e^{3t} \cos 5t\}$.
3. Find $L\{e^{-t} \sin^2 t\}$.
4. Find $L\{e^t \sin^2 t\}$.
5. Find $L\{e^{-4t} \cosh 2t\}$.
6. Find $L\{e^{-2t}(3\cos 6t - 5\sin 6t)\}$.

ANSWERS

1. $\dfrac{6}{(p+3)^4}$ 2. $\dfrac{p-3}{p^2-6p+34}$ 3. $\dfrac{2}{(p+1)(p^2+2p+5)}$

4. $\dfrac{2}{(p-1)(p^2-2p+5)}$ 5. $\dfrac{p+4}{p^2+8p+12}$ 6. $\dfrac{3p-24}{p^2+4p+40}$

• 6.6. LAPLACE TRANSFORMS OF DERIVATIVES

Theorem 1. *Let $F(t)$ be continuous for all $t \geq 0$ and be of exponential order as $t \to \infty$ and if $F'(t)$ is of class A, the Laplace transforms of derivatives $F'(t)$ exists when $p > a$ and*

$$L\{F'(t)\} = p\,L\{F(t)\} - F(0).$$

Proof. By definition, we have

$$L\{F'(t)\} = \int_0^\infty e^{-pt} F'(t)\,dt$$

$$= \left[e^{-pt} F(t)\right]_0^\infty + p\int_0^\infty e^{-pt} F(t)\,dt \qquad \text{[On integrating by parts]}$$

$$= -F(0) + p\,L\{F(t)\} \qquad \left[\because \lim_{t\to\infty} e^{-pt} F(t) = 0\right]$$

$$= pL\{F(t)\} - F(0).$$

REMARK

➤ Proceeding same as above, we get

$$L\{F''(t)\} = p\,L\{F'(t)\} - F'(0)$$
$$= p\,[p\,L\{F(t)\} - F(0)] - F'(0)$$
$$= p^2 L\{F(t)\} - p\,F(0) - F'(0)$$
$$= p^2 f(p) - p\,F(0) - F'(0).$$

Theorem 2. *If $F(t)$, $F'(t)$, ... $F^{n-1}(t)$ are continuous for $t \geq 0$ and be of exponential order as $t \to \infty$ and if $F^n(t)$ is of class A and if $L\{F(t)\} = f(p)$, then*

$$L\{F^n(t)\} = p^n f(p) - p^{n-1} F(0) - p^{n-2} F'(0) \ldots pF^{(n-2)}(0) - F^{(n-1)}(0)$$

$$= p^n f(p) - \sum_{r=0}^{n-1} p^{n-1-r} F^{(r)}(0).$$

Proof. Using above theorem, we have

$$L\{F'(t)\} = p\,L\{F(t)\} - F(0) \qquad ...(1)$$

and
$$L\{F''(t)\} = p^2 L\{F(t)\} - p\,F(0) - F'(0) \qquad ...(2)$$

Similarly, we can find

$$L\{F'''(t)\} = pL\{F''(t)\} - F''(0)$$
$$= p\,[p^2 L\{F(t)\} - p\,F(0) - F'(0)] - F''(0)$$
$$= p^3 L\{F(t)\} - p^2 F(0) - p\,F'(0) - F''(0).$$

Proceeding, similarly, we get

$$L\{F^n(t)\} = p^n L\{F(t)\} - p^{n-1} F(0) - p^{n-2} F'(0) - \ldots - F^{n-1}(0)$$
$$= p^n L\{F(t)\} - \sum_{r=0}^{n-1} p^{n-1-r} F^r(0).$$

Theorem 3. *If $F(t)$ is a function of class A and if $L\{F(t)\} = f(p)$, then*

$$L\{t\,.\,F(t)\} = -f'(p).$$

Proof. We know that

$$f(p) = L\{F(t)\} = \int_0^\infty e^{-pt} F(t)\,dt$$

Therefore
$$f'(p) = \frac{d}{dp}\int_0^\infty e^{-pt} F(t)\,dt$$

$$= \int_0^\infty \frac{\partial}{\partial p}\{e^{-pt} F(t)\}\,dt$$ (By Leibnitz rule of differentiation under the sign of integral)

$$= -\int_0^\infty t\,e^{-pt} F(t)\,dt$$

$$= -\int_0^\infty e^{-pt}\{t\,F(t)\}\,dt$$

$$= -L\{t\,F(t)\}$$

$\Rightarrow$
$$L\{t\,F(t)\} = -f'(p).$$

Theorem 4. *If $F(t)$ is a function of class A and if $L\{F(t)\} = f(p)$.*

Then
$$L\{t^n F(t)\} = (-1)^n \frac{d^n}{dp^n} f(p).$$

Proof. We shall prove this theorem by the Principle of Mathematical induction.

Step I. Using previous theorem, we have

$$L\{t\,F(t)\} = (-1)^1 \frac{d}{dp} f(p)$$

$\Rightarrow$ Theorem is true for $n = 1$.

Step II. Assume that the theorem is true for a particular value of n say k. Then, we have

$$L\{t^k F(t)\} = (-1)^k \frac{d^k}{dp^k} f(p)$$

$$\Rightarrow \int_0^\infty e^{-pt}\, t^k F(t)\,dt = (-1)^k \frac{d^k}{dp^k} f(p)$$

Step III. Differentiating both sides w.r.t. p, we have

$$\frac{d}{dp}\int_0^\infty e^{-pt}\, t^k\, F(t)\, dt = (-1)^k \frac{d^{k+1}}{dp^{k+1}} f(p).$$

Applying, Leibnitz's rule for differentiation under the sign of integration, we have

$$-\int_0^\infty e^{-pt}\, t^{k+1}\, F(t)\, dt = (-1)^{k+2} \frac{d^{k+1}}{dp^{k+1}} f(p)$$

$$\Rightarrow \qquad \int_0^\infty e^{-pt}\, \{t^{k+1} F(t)\}\, dt = (-1)^{k+1} \frac{d^{k+1}}{dp^{k+1}} f(p)$$

$$\Rightarrow \qquad L\,\{t^{k+1} F(t)\} = (-1)^{k+1} \frac{d^{k+1}}{dp^{k+1}} f(p)$$

$\Rightarrow$ Theorem is true for $n = k + 1$

Hence by the principle of mathematical induction, it is true for every positive integral value of n.

Theorem 5. (Laplace Transforms of Integrals). *If $F(t)$ is piecewise continuous and satisfies*

$$|\,F(t)\,| \le Me^{at}, \quad \forall\, t \ge 0$$

for some constant a and M, then

$$L\left\{\int_0^t F(x)\, dx\right\} = \frac{1}{p}\, L\,\{F(t)\}$$

Proof. Let $F(t)$ be piecewise continuous such that

$$|\,F(t)\,| \le Me^{at} \qquad \ldots(1)$$

for some constants a and M.

If (1) holds for some negative value of a, then it is also holds for positive value of a. Therefore, suppose that a is positive.

Let $$G(t) = \int_0^t F(x)\, dx.$$

Then $G(t)$ is continuous (∵ Integral of an integrable function is continuous)

Now, $$|\,G(t)\,| \le \int_0^t |\,F(x)\,|\, dx \le \int_0^t Me^{ax}\, dx$$

$$\Rightarrow \qquad |\,G(t)\,| \le \frac{M}{a}(e^{at} - 1), \qquad a > 0$$

Further $G'(t) = F(t)$, except for points at which $F(t)$ is discontinuous. Therefore, $G'(t)$ is piecewise continuous on each finite interval.

$$\therefore \qquad L\,\{G'(t)\} = pL\,\{G(t)\} - G(0)$$

$$= pL\,\{G(t)\} \qquad [\because\ G(0) = 0]$$

$$\Rightarrow \qquad L\,\{G(t)\} = \frac{1}{p}\, L\,\{G'(t)\}$$

$$\Rightarrow \qquad L\left\{\int_0^t F(x)\, dx\right\} = \frac{1}{p}\, L\,\{F(t)\}.$$

Theorem 6. (Division by t). *If $L\,\{F(t)\} = f(p)$, then*

$$L\left\{\frac{1}{t} F(t)\right\} = \int_p^\infty f(x)\, dx$$

provided $\lim\limits_{t \to 0} \left\{\frac{1}{t} F(t)\right\}$ *exists.*

Proof. Let $\qquad G(t) = \frac{1}{t} F(t)$ *i.e.*, $F(t) = t\,G(t)$

Therefore, $\quad L\{F(t)\} = L\{t\,G(t)\} = -\frac{d}{dp} L\{G(t)\}$

$$\Rightarrow \qquad f(p) = -\frac{d}{dp} L\{G(t)\}.$$

On integrating both sides with respect to p from p to ∞, we get

$$-\Big[L\{G(t)\}\Big]_p^\infty = \int_p^\infty f(p)\,dp$$

$$\Rightarrow \qquad -\lim_{p\to\infty} L\{G(t)\} + L\{G(t)\} = \int_p^\infty f(p)\,dp$$

$$\Rightarrow \qquad 0 + L\{G(t)\} = \int_p^\infty f(p)\,dp, \text{ by using } \lim_{p\to\infty} L\{G(t)\} = \lim_{p\to\infty} \int_0^\infty e^{-pt} G(t)\,dt = 0$$

$$\Rightarrow \qquad L\left\{\frac{1}{t} F(t)\right\} = \int_p^\infty f(x)\,dx.$$

SOLVED EXAMPLES

Example 1. *Find* $L\{t \cos at\}$.

Solution. We know that

$$L\{\cos at\} = \frac{p}{p^2 + a^2}, \quad p > 0.$$

Therefore, $\quad L\{t \cos at\} = -\frac{d}{dp} L\{\cos at\} = -\frac{d}{dp}\left(\frac{p}{p^2+a^2}\right)$

$$= \frac{p^2 - a^2}{(p^2 + a^2)^2}.$$

Example 2. *Find* $\quad L\{t^2 \sin at\}$.

Solution. We know that

$$L\{\sin at\} = \frac{a}{p^2 + a^2}.$$

Therefore, $\quad L\{t^2 \sin at\} = (-1)^2 \frac{d^2}{dp^2} L\{\sin at\} = \frac{d^2}{dp^2}\left\{\frac{a}{p^2+a^2}\right\}$

$$= \frac{d}{dp}\left\{\frac{-2ap}{(p^2+a^2)^2}\right\} = \frac{2a\,(3p^2 - a^2)}{(p^2+a^2)^3}, \quad p > 0.$$

Example 3. *Given* $L\{\sin \sqrt{t}\} = \frac{\sqrt{\pi}}{2p^{3/2}} e^{-1/4p}$, *show that*

$$L\left\{\frac{\cos \sqrt{t}}{\sqrt{t}}\right\} = \sqrt{\left(\frac{\pi}{p}\right)} \cdot e^{-1/4p}.$$

Solution. Let $\qquad F(t) = \sin \sqrt{t}$.

Then, we have $\qquad F'(t) = \cos \frac{\sqrt{t}}{2\sqrt{t}}$ and $F(0) = 0$.

Put all these values in

$$L\{F'(t)\} = pL\{F(t)\} - F(0)$$

we get

$$L\left\{\frac{\cos \sqrt{t}}{2\sqrt{t}}\right\} = pL\{\sin \sqrt{t}\}$$

$$= p\left[\frac{\sqrt{\pi}}{2p^{3/2}} e^{-1/4p}\right]$$

$$= \frac{1}{2}\sqrt{\left(\frac{\pi}{p}\right)}\, e^{-1/4p}.$$

Hence $$L\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = \sqrt{\left(\frac{\pi}{p}\right)}\,.\,e^{-1/tp}.$$

Example 4. *Show that* $L\left\{\frac{\sin t}{t}\right\} = \tan^{-1}\frac{1}{p}$ *and hence find* $L\left\{\frac{\sin at}{t}\right\}$. *Does the Laplace transform of* $\frac{\cos at}{t}$ *exists ?*

Solution. Let $F(t) = \sin t$

Then $$\lim_{t\to 0}\frac{F(t)}{t} = \lim_{t\to 0}\frac{\sin t}{t} = 1.$$

We know that

$$L\{\sin t\} = \frac{1}{p^2+1} = f(p) \quad \text{(say)}$$

Then, we have

$$L\left\{\frac{\sin t}{t}\right\} = \int_p^\infty f(x)\,dx = \int_p^\infty \frac{dx}{x^2+1} = \left(\tan^{-1}\right)_p^\infty$$

$$= \frac{\pi}{2} - \tan^{-1} p$$

$$= \cot^{-1} p = \tan^{-1}\left(\frac{1}{p}\right).$$

Now, $$L\left\{\frac{\sin at}{t}\right\} = aL\left\{\frac{\sin at}{at}\right\}$$

$$= a\,.\,\frac{1}{a}\tan^{-1}\left(\frac{1}{p/a}\right) \qquad \left[\because\ L\{f(at)\} = \frac{1}{a}f\left(\frac{p}{a}\right)\right]$$

$$= \tan^{-1}\left(\frac{a}{p}\right).$$

Also, since $$L\{\cos at\} = \frac{p}{p^2+a^2} = f(p) \quad \text{(say)}$$

Then $$L\left\{\frac{\cos at}{t}\right\} = \int_p^\infty \frac{x}{x^2+a^2}\,dx$$

$$= \left[\frac{1}{2}\log(x^2+a^2)\right]_p^\infty$$

$$= \frac{1}{2}\lim_{x\to\infty}\log(x^2+a^2) - \frac{1}{2}\log(p^2+a^2)$$

which does not exists, since $\lim_{x\to\infty}\log(x^2+a^2)$ is infinite.

Therefore, $L\left\{\frac{\cos at}{t}\right\}$ does not exist.

• SUMMARY

- Laplace Transform of $F(t)$

$$L\{F(t)\} = \int_0^\infty e^{-pt} F(t)\,dt$$

- $L\{1\} = \frac{1}{p}$, $L\{t^n\} = \frac{n!}{p^{n+1}}$
- $L\{e^{at}\} = \frac{1}{p-a}$, $p \neq a$
- $L\{\sin at\} = \frac{a}{p^2+a^2}$, $p > 0$, $L\{\cos at\} = \frac{p}{p^2+a^2}$, $p > a$
- $L\{\sinh at\} = \frac{a}{p^2-a^2}$, $p \neq \pm a$, $L\{\cosh at\} = \frac{p}{p^2-a^2}$, $p \neq \pm a$
- If $L\{F(t)\} = f(p)$, then

$$L\{e^{at} F(t)\} = f(p-a);\quad p > a$$

- If $L\{F(t)\} = f(p)$ and $G(t) = \begin{cases} F(t-a), & t > a \\ 0, & t < a \end{cases}$ then $L\{G(t)\} = e^{-ap} f(p)$.
- If $L\{F(t)\} = f(p)$, then $L\{F(at)\} = \frac{1}{a} f\left(\frac{p}{a}\right)$.
- $L\{F'(t)\} = pL\{F(t)\} - F(0)$.
- $L\{F''(t)\} = p^2 L\{F(t)\} - pF(0) - F'(0)$.
- If $L\{F(t)\} = f(p)$, then $L\{t^n F(t)\} = (-1)^n \frac{d^n}{dp^n}(f(p))$.
- If $|F(t)| \leq Me^{at}\ \forall\ t \geq 0$ and $F(t)$ is piecewise continuous, then

$$L\left\{\int_0^t F(t)\,dt\right\} = \frac{1}{p} L\{F(t)\}.$$

- If $L\{F(t)\} = f(p)$, then $L\left\{\frac{1}{t} F(t)\right\} = \int_p^\infty f(x)\,dx$.

• STUDENT ACTIVITY

1. Find $L\{e^t \cos^2 t\}$.

2. Find $L\{t^2 \sin at\}$.

• TEST YOURSELF–3

1. Show that $L\{-a \sin at\} = -\frac{a^2}{p^2+a^2}$.
2. Evaluate
   (a) $L\{t \cosh 3t\}$ (b) $L\{t \sinh at\}$.
3. Show that $L\{t^2 \cos at\} = \frac{2p(p^2-3a^2)}{(p^2+a^2)^3}$, $p > 0$.
4. Show that $L\{t^n e^{at}\} = \frac{n!}{(p-a)^{n+1}}$, $p > a$.
5. Show that $L\{t(3 \sin 2t - 2 \cos 2t)\} = \frac{8+12p-2p^2}{(p^2+4)^2}$.
6. Show that $L\{\sin \alpha t + t \cos \alpha t\} = \frac{(\alpha+1)p^2+(\alpha-1)\alpha^2}{(p^2+\alpha^2)^2}$.

ANSWERS

2. (a) $\frac{p^2+9}{(p^2-9)^2}$ (b) $\frac{2ap}{(p^2-a^2)^2}$.

OBJECTIVE EVALUATIONS

Fill in the blanks :

1. $L\{e^{at}\} = \ldots\ldots\ldots\ldots$
2. $L\{\sin at\} = \ldots\ldots\ldots\ldots$
3. $L\{t \cos at\} = \ldots\ldots\ldots\ldots$

True or False

1. If $L\{F(t)\} = f(t)$, then $L\{F(at)\} = \frac{1}{p} f\left(\frac{p}{a}\right)$. (T/F)
2. If $L\{F(t)\} = f(p)$, then $L\{e^{at} F(t)\} = f(p+a)$. (T/F)
3. $L\{F'(t)\} = pL\{F(t)\} - F(0)$. (T/F)

Multiple Choice Questions (MCQ's) :

1. $L(1)$ equals :
   (a) $\frac{1}{p}$ (b) $\frac{1}{p^2}$ (c) $\frac{1}{p-1}$ (d) $\frac{1}{p+1}$
2. $L\{t^2\}$ equals :
   (a) $\frac{1}{p^2}$ (b) $\frac{2}{p^3}$ (c) $\frac{1}{p^3}$ (d) $\frac{1}{p^4}$

ANSWERS

Fill in the Blanks :

1. $\frac{1}{p-a}$ 2. $\frac{a}{p^2+a^2}$ 3. $\frac{p^2-a^2}{(p^2+a^2)^2}$

True or False :

1. T 2. F 3. T

Multiple Choice Questions :

1. (a) 2. (b).

UNIT

7

THE INVERSE LAPLACE TRANSFORM

STRUCTURE

- Inverse Laplace transform
- Some Inverse Laplace transforms
- Important properties of inverse Laplace transforms
- Inverse Laplace transforms of derivatives
- Division by p
- Multiplication by p
- Inverse Laplace transform of integrals
- Convolution
  - Summary
  - Student Activity
  - Test Yourself

LEARNING OBJECTIVES

After going through this unit you will learn :

- What is inverse Laplace transform ?
- How to find the inverse Laplace transform of given functions
- What is convolution ?
- How to find the inverse Laplace using convolution.

7.1. INVERSE LAPLACE TRANSFORM

If the Laplace transform of a function $f(t)$ is $f(p)$ *i.e.*, if $L\{F(t)\} = f(p)$.

Then $F(t)$ is known as inverse Laplace transform of $f(p)$.

Symbolically. $F(t) = L^{-1}\{f(p)\}$.

Where L^{-1} is called the inverse Laplace transformation operator.

For example. If $L\{e^{-2t}\} = \dfrac{1}{p+2}$. Then we can write $L^{-1}\left(\dfrac{1}{p+2}\right) = e^{-2t}$.

Null Function :

A function $N(t)$ of t such that

$$\int_0^t N(t)\, dt = 0, \quad \forall\, t > 0 \text{ is called Null function.}$$

Uniqueness of Inverse Laplace Transforms : Learch Theorem :

Since, we know that the Laplace transform of a null function $N(t)$ is zero. Also, it is clearly that if $L\{F(t)\} = f(p)$, then also

$$L\{F(t) + N(t)\} = f(p).$$

It follows that we can have two different functions with same Laplace transfrom.

If we allow null functions, we see that the inverse Laplace transform is not unique. It is unique, however if we disallow null functions.

Learch's theorem. If we restrict ourselves to functions $F(t)$ which are sectionally continuous in every finite interval $0 \le t \le N$ and of exponential order for $t > N$, then the inverse Laplace transform of $f(p)$

i.e., $\quad L^{-1}\{f(p)\} = F(t)$, is unique.

7.2. SOME INVERSE LAPLACE TRANSFORMS

	$f(p)$	$L^{-1}\{f(p)\} = F(t)$
1.	$\frac{1}{p}$	1
2.	$\frac{1}{p^2}$	t
3.	$\frac{1}{p^{n+1}}, n = 0, 1, 2, \ldots$	$t^n/(n!)$
4.	$\frac{1}{p-a}$	e^{at}
5.	$\frac{1}{p^2+a^2}$	$\frac{\sin at}{a}$
6.	$\frac{p}{p^2+a^2}$	$\cos at$
7.	$\frac{1}{p^2-a^2}$	$\frac{\sinh at}{a}$
8.	$\frac{p}{p^2-a^2}$	$\cosh at$

7.3. IMPORTANT PROPERTIES OF INVERSE LAPLACE TRANSFORM

(i) Linearty property. *If C_1 and C_2 are any constants while $f_1(p)$ and $f_2(p)$ are the Laplace transform $F_1(t)$ and $F_2(t)$ respectively, then*

$$L^{-1}\{C_1 f_1(p) + C_2 f_2(p)\} = C_1 L^{-1}\{f_1(p)\} + C_2 L^{-1}\{f_2(p)\}.$$

Proof. We have

$$L\{C_1 F_1(t) + C_2 F_2(t)\} = C_1 L\{F_1(t)\} + C_2 L\{F_2(t)\}$$
$$= C_1 f_1(p) + C_2 f_2(p)$$

$$\Rightarrow \quad L^{-1}\{C_1 f_1(p) + C_2 f_2(p)\} = C_1 F_1(t) + C_2 F_2(t)$$
$$= C_1 L^{-1}\{f_1(p)\} + C_2 L^{-1}\{f_2(p)\}$$

(ii) First translation or shifting theorem.

If $L^{-1}\{f(p)\} = F(t)$ then

$$L^{-1}\{f(p-a)\} = e^{at} F(t) = e^{at} L^{-1}\{f(p)\}.$$

Proof. We have

$$f(p) = \int_0^\infty e^{-pt} F(t)\, dt$$

$$\Rightarrow \qquad f(p-a) = \int_0^\infty e^{-(p-a)t} F(t)\, dt$$

$$= \int_0^\infty e^{-pt} \{e^{at} F(t)\}\, dt$$

$$= L\{e^{at} F(t)\}.$$

Hence, $\quad L^{-1}\{f(p-a)\} = e^{at} F(t) = e^{at} L^{-1}\{f(p)\}.$

(iii) Second translation or shifting theorem.

If $L^{-1}\{f(p)\} = F(t)$ then $L^{-1}\{e^{-ap} f(p)\} = G(t)$ where

$$G(t) = \begin{cases} F(t-a) & t > a \\ 0 & , t < a. \end{cases}$$

Proof. We know that

$$f(p) = \int_0^\infty e^{-pt} F(t)\, dt.$$

Therefore, $$e^{-ap} f(p) = \int_0^\infty e^{-p(t+a)} F(t)\, dt$$

$$= \int_a^\infty e^{-px} F(x-a)\, dx \qquad \text{putting } t + a = x \Rightarrow dt = dx$$

$$= \int_0^a e^{-px} \cdot 0\, dx + \int_a^\infty e^{-px} F(x-a)\, dx$$

$$= \int_0^a e^{-pt} \cdot 0\, dt + \int_a^\infty e^{-pt} F(t-a)\, dt$$

$$= \int_0^\infty e^{-pt} G(t)\, dt = L\{G(t)\}$$

where $$G(t) = \begin{cases} F(t-a) & , t > a \\ 0 & , t < a. \end{cases}$$

shows $L^{-1}\{e^{ap} f(p)\} = G(t)$.

(iv) Change of scale property.

If $L^{-1}\{f(p)\} = F(t)$, *then* $L^{-1}\{f(ap)\} = \frac{1}{a} F\left(\frac{t}{a}\right)$.

Proof. We know that

$$f(p) = \int_0^\infty e^{-pt} F(t)\, dt$$

$$\Rightarrow \qquad f(ap) = \int_0^\infty e^{-apt} F(t)\, dt$$

Putting $at = x \Rightarrow dt = \frac{1}{a} dx$, we get

$$f(ap) = \frac{1}{a} \int_0^\infty e^{-px} F\left(\frac{x}{a}\right) dx$$

$$= \frac{1}{a} \int_0^\infty e^{-pt} F\left(\frac{t}{a}\right) dt \qquad \text{(By the property of definite integral)}$$

$$= \frac{1}{a} L\left\{F\left(\frac{t}{a}\right)\right\}$$

$$= L\left\{\frac{1}{a} F\left(\frac{t}{a}\right)\right\}.$$

Hence, $$L^{-1}\{f(ap)\} = \frac{1}{a} F\left(\frac{t}{a}\right).$$

SOLVED EXAMPLES

Example 1. *Find the inverse Laplace transforms of the following functions*

(*i*) $\dfrac{2p+1}{p(p+1)}$ (*ii*) $\dfrac{3p-8}{4p^2+25}$.

Solution. (i) We have

$$L^{-1}\left\{\frac{2p+1}{p(p+1)}\right\} = L^{-1}\left\{\frac{p+(p+1)}{p(p+1)}\right\}$$

$$= L^{-1}\left\{\frac{1}{p+1}\right\} + L^{-1}\left\{\frac{1}{p}\right\}$$

$$= e^{-t} + 1.$$

(ii) Here, we have

$$L^{-1}\left\{\frac{3p-8}{4p^2+25}\right\} = \frac{3}{4}L^{-1}\left\{\frac{p}{p^2+\left(\frac{5}{2}\right)^2}\right\} - 2L^{-1}\left\{\frac{1}{p^2+\left(\frac{5}{2}\right)^2}\right\}$$

$$= \frac{3}{4}\cos\left(\frac{5}{2}t\right) - 2\,.\,\frac{2}{5}\sin\left(\frac{5}{2}t\right)$$

$$= \frac{3}{4}\cos\left(\frac{5}{2}t\right) - \frac{4}{5}\sin\left(\frac{5}{2}t\right).$$

Example 2. *Find* $L^{-1}\left\{\frac{3p-2}{p^{5/2}} - \frac{7}{3p+2}\right\}$.

Solution. Here, we have

$$L^{-1}\left\{\frac{3p-2}{p^{5/2}} - \frac{7}{3p+2}\right\} = 3L^{-1}\left\{\frac{1}{p^{3/2}}\right\} - 2L^{-1}\left\{\frac{1}{p^{5/2}}\right\} - \frac{7}{3}L^{-1}\left\{\frac{1}{p+(2/3)}\right\}$$

$$= 3\frac{t^{1/2}}{\Gamma\left(\frac{3}{2}\right)} - 2\frac{t^{3/2}}{\Gamma\left(\frac{5}{2}\right)} - \frac{7}{3}e^{-\left(\frac{2}{3}\right)t}.$$

$$= 6\sqrt{\left(\frac{t}{\pi}\right)} - \frac{8}{3}t\sqrt{\left(\frac{t}{\pi}\right)} - \frac{7}{3}e^{-2t/3}.$$

Example 3. *Show that* $L^{-1}\left\{\frac{1}{p}\cos\frac{1}{p}\right\} = 1 - \frac{t^2}{(2\,!)^2} + \frac{t^4}{(4\,!)^2} - \frac{t^6}{(6\,!)^2} + \ldots$.

Solution. $$L^{-1}\left\{\frac{1}{p}\cos\frac{1}{p}\right\} = L^{-1}\left\{\frac{1}{p}\left(1 - \frac{(1/p)^2}{2\,!} + \frac{(1/p)^4}{4\,!} - \frac{(1/p)^6}{6\,!} + \ldots\right)\right\}$$

$$= L^{-1}\left\{\frac{1}{p}\right\} - \frac{1}{2\,!}L^{-1}\left\{\frac{1}{p^3}\right\} + \frac{1}{4\,!}L^{-1}\left\{\frac{1}{p^5}\right\} - \frac{1}{6\,!}L^{-1}\left\{\frac{1}{p^7}\right\} + \ldots$$

$$= 1 - \frac{t^2}{(2\,!)^2} + \frac{t^4}{(4\,!)^2} - \frac{t^6}{(6\,!)^2} + \ldots .$$

Example 4. *Evaluate* $L^{-1}\left\{\frac{1}{(p+2)(p-1)^2}\right\}$.

Solution. $$L^{-1}\left\{\frac{1}{(p+2)(p-1)^2}\right\} = L^{-1}\left\{\frac{1}{(p-1+3)(p-1)^2}\right\}$$

$$= e^t L^{-1}\left\{\frac{1}{p+3}\cdot\frac{1}{p^2}\right\}$$

$$= e^t L^{-1}\left\{\frac{1}{p^2}\left(\frac{1}{3} - \frac{1}{9}p + \frac{1}{9}\frac{p^2}{p+3}\right)\right\}$$

(Dividing 1 by $3+p$ till p^2 is a common factor in the remainder)

$$= e^t L^{-1}\left\{\frac{1}{3}\cdot\frac{1}{p^2}-\frac{1}{9}\cdot\frac{1}{p}+\frac{1}{9}\cdot\frac{1}{(p+3)}\right\}$$

$$= e^t\left(\frac{1}{3}t-\frac{1}{9}+\frac{1}{9}e^{-3t}\right)$$

$$= \frac{1}{9}[(3t-1)e^t+e^{-2t}].$$

Example 5. *Evaluate* $L^{-1}\left\{\frac{1}{(p+1)(p-2)}\right\}$.

Solution. Consider

$$L^{-1}\left\{\frac{1}{(p+1)(p-2)}\right\} = L^{-1}\left\{-\frac{1}{3}\cdot\frac{1}{p+1}+\frac{1}{3}\cdot\frac{1}{p-2}\right\}$$

$$= \frac{1}{3}(-e^{-t}+e^{2t}).$$

Example 6. *Evaluate* $L^{-1}\left\{\frac{p+5}{(p+2)(p^2+4)}\right\}$.

Solution. We have

$$L^{-1}\left\{\frac{p+5}{(p+2)(p^2+4)}\right\} = L^{-1}\left\{\frac{1}{8}\left(\frac{3}{p+2}-\frac{3p-14}{p^2+4}\right)\right\}$$

$$= \frac{1}{8}\left[3L^{-1}\left\{\frac{1}{p+2}\right\}-3L^{-1}\left\{\frac{p}{p^2+4}\right\}+14L^{-1}\left\{\frac{1}{p^2+4}\right\}\right]$$

$$= \frac{1}{8}(3e^{-2t}-3\cos 2t+7\sin 2t).$$

• TEST YOURSELF

1. Find the inverse Laplace transform of the following functions :

   (a) $\frac{1}{p^4}$ (b) $\frac{1}{p^2+4}$

   (c) $\frac{4}{p-2}$ (d) $\frac{1}{\sqrt{p}}$

   (e) $\frac{p}{p^2+2}+\frac{6p}{p^2-16}+\frac{3}{p-3}$ (f) $\frac{2p-5}{p^2-9}$

2. Find the inverse Laplace transform of the following functions :

   (a) $\frac{1}{p^2-6p+10}$ (b) $\frac{p+b}{(p+b)^2+a^2}$ (c) $\frac{3p+7}{p^2-2p-3}$

   (d) $\frac{1}{(p+a)^n}$ (e) $\frac{p}{(p+1)^5}$ (f) $\frac{p^2-2p+3}{(p-1)^2(p+1)}$.

ANSWERS

1. (a) $\frac{t^3}{6}$ (b) $\frac{1}{2}\sin 2t$ (c) $4e^{2t}$ (d) $\frac{1}{\sqrt{\pi t}}$

   (e) $\cos\sqrt{2}\,t+6\cosh 4t+3e^{3t}$ (f) $2\cosh 3t-\frac{5}{3}\sinh 3t$

2. (a) $e^{3t}\sin t$ (b) $e^{-bt}\cos at$ (c) $4e^{3t}-e^{-t}$

   (d) $e^{-at}\frac{t^{n-1}}{n-1)!}, n\in \mathbf{Z}^+$ (e) $e^{-t}(4t^3-t^4)/24$ (f) $\left(t-\frac{1}{2}\right)e^t+\frac{3}{2}e^{-t}$

• 7.4. INVERSE LAPLACE TRANSFORMS OF DERIVATIVES

Theorem. *If* $L^{-1}\{f(p)\}=F(t)$, *then* $L^{-1}\{f^{(n)}(p)\}=(-1)^n\cdot t^n\cdot F(t)$.

Proof. Since, we know that

$$L\{t^n F(t)\} = (-1)^n f^{(n)}(p).$$

Therefore,

$$t^n F(t) = L^{-1}\{(-1)^n f^{(n)}(p)]$$
$$= (-1)^n L^{-1}\{f^{(n)}(p)\}.$$

Hence, $\qquad L^{-1}\{f^{(n)}(p)\} = (-1)^n t^n F(t)$

• 7.5. DIVISION BY p

Theorem. *If* $L^{-1}\{f(p)\} = F(t)$, *then* $L^{-1}\left\{\frac{f(p)}{p}\right\} = \int_0^t F(u)\,du.$

Proof. Since we know that

$$\frac{f(p)}{p} = L\left\{\int_0^t F(u)\,du\right\}$$

$$\Rightarrow \qquad L^{-1}\left\{\frac{f(p)}{p}\right\} = \int_0^t F(u)\,du$$

• 7.6. MULTIPLICATION BY POWERS OF p

Theorem. *If* $L^{-1}\{f(p)\} = F(t)$ *and* $F(0) = 0$, *then* $L^{-1}\{pf(p)\} = F'(t)$.

Proof. We know that

$$L\{F'(t)\} = pL\{F(t)\} - F(0)$$
$$= pL[F(t)] \qquad [\because F(0) = 0]$$
$$= p\,f(p)$$

Hence, $\qquad L^{-1}\{pf(p)\} = F'(t).$

• 7.7. INVERSE LAPLACE TRANSFORMS OF INTEGRALS

Theorem. *If* $L^{-1}\{f(p)\} = F(t)$, *then*

$$L^{-1}\left[\int_p^\infty f(x)\,dx\right] = \frac{F(t)}{t}.$$

Proof. We know that

$$L\left\{\frac{1}{t}F(t)\right\} = \int_p^\infty f(x)\,dx$$

provided $\lim_{t\to 0}\left\{\frac{F(t)}{t}\right\}$ exists.

Hence, $\qquad L^{-1}\left\{\int_p^\infty f(x)\,dx\right\} = \frac{F(t)}{t}.$

SOLVED EXAMPLES

Example 1. *Find* $L^{-1}\left\{\frac{p}{(p^2+a^2)^2}\right\}$.

Solution. We have

$$L^{-1}\left\{\frac{p}{(p^2+a^2)^2}\right\} = L^{-1}\left\{-\frac{1}{2}\frac{d}{dp}\left(\frac{1}{p^2+a^2}\right)\right\}$$

$$= -\frac{1}{2}L^{-1}\left\{\frac{d}{dp}\left(\frac{1}{p^2+a^2}\right)\right\}$$

$$= -\frac{1}{2}t(-1)L^{-1}\left\{\frac{1}{p^2+a^2}\right\} = \frac{t}{2a}\sin at.$$

Example 2. *Evaluate* $L^{-1}\left\{\log\left(1-\frac{1}{p^2}\right)\right\}$.

Solution. Let us suppose

$$f(p) = \log\left(1-\frac{1}{p^2}\right)$$

$$= \log\left(\frac{p^2-1}{p^2}\right) = -2\log p + \log(p^2-1)$$

$$\Rightarrow \quad f'(p) = -2\left(\frac{1}{p}-\frac{p}{p^2-1}\right)$$

$$\Rightarrow \quad L^{-1}\{f'(p)\} = -2(1-\cosh t)$$

$$\Rightarrow \quad -tL^{-1}\{f(p)\} = -2(1-\cosh t)$$

$$\Rightarrow \quad L^{-1}\left\{\log\left(1-\frac{1}{p^2}\right)\right\} = \frac{2}{t}(1-\cosh t).$$

Example 3. *Evaluate*

(*i*) $L^{-1}\left\{\log\left(1+\frac{1}{p^2}\right)\right\}$.

(*ii*) $L^{-1}\left\{\frac{1}{p}\log\left(1+\frac{1}{p^2}\right)\right\}$.

Solution. (i) Let $\quad f(p) = \log\left(1+\frac{1}{p^2}\right) = -\log\left(\frac{p^2}{p^2+1}\right)$

$$= -2\log p + \log(p^2+1).$$

Therefore, $\quad f'(p) = -\frac{2}{p}+\frac{2p}{p^2+1}$

$$\Rightarrow \quad L^{-1}\{f'(p)\} = -2+2\cos t$$

$$\Rightarrow \quad -tL^{-1}\{f(p)\} = -2(1-\cos t)$$

Hence, $\quad L^{-1}\left\{\log\left(1+\frac{1}{p^2}\right)\right\} = \frac{2(1-\cos t)}{t}$

(ii) Since $\quad L^{-1}\left\{\log\left(1+\frac{1}{p^2}\right)\right\} = \frac{2(1-\cos t)}{t}.$

Therefore, $L^{-1}\left\{\frac{1}{p}\log\left(1+\frac{1}{p^2}\right)\right\} = L^{-1}\left\{\frac{1}{p}f(p)\right\} = \int_0^t F(x)\,dx$

$$= \int_0^t \frac{2}{x}(1-\cos x)\,dx.$$

• TEST YOURSELF–2

1. Evaluate the following inverse Laplace transforms :

(a) $L^{-1}\left\{\frac{p}{(p^2-a^2)^2}\right\}$ (b) $L^{-1}\left\{\frac{p}{(p^2-16)^2}\right\}$ (c) $L^{-1}\left\{\frac{1}{(p-a)^3}\right\}$

(d) $L^{-1}\left\{\frac{p+1}{(p^2+2p+2)^2}\right\}$ (e) $L^{-1}\left\{\frac{p^2}{(p^2+4)^2}\right\}$

2. Show that

(a) $L^{-1}\left\{\dfrac{1}{p^3(p+1)}\right\} = 1 - t + \dfrac{t^2}{2} - e^{-t}$

(b) $L^{-1}\left\{\dfrac{1}{p^3(p^2+1)}\right\} = \dfrac{t^2}{2} + \cos t - 1$

(c) $L^{-1}\left\{\log\dfrac{p+2}{p+1}\right\} = \dfrac{1}{t}(e^{-t} - e^{-2t})$

ANSWERS

1. (a) $\dfrac{t}{2a}\sinh at$ (b) $\dfrac{t}{8}\sinh 4t$ (c) $\dfrac{1}{2}t^2 e^{at}$

(d) $\dfrac{t}{2}e^{-t}\sin t$ (e) $\dfrac{1}{4}(\sin 2t + 2t\cos 2t)$.

• 7.8. CONVOLUTION

If $L^{-1}\{f(p)\} = F(t)$ and $L^{-1}\{g(p)\} = G(t)$, where $F(t)$ and $G(t)$ are two functions of class A. Then

$$L^{-1}\{f(p)\,.\,g(p)\} = \int_0^t F(u)\,G(t-u)\,du = F * G$$

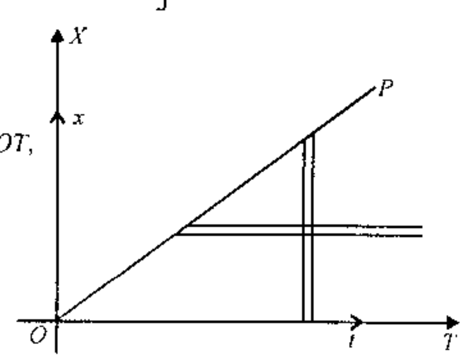
we call $F * G$ the convolution or falting of F and G.

Proof. Let $\int_0^t F(x)\,G(t-x)\,dx = H(t)$

Then
$$L\{H(t)\} = \int_0^\infty e^{-pt} H(t)\,dt$$
$$= \int_0^\infty e^{-pt}\left[\int_0^t F(x)\,G(t-x)\,dx\right]dt$$
$$= \int_0^\infty \left[\int_0^t e^{-pt} F(x)\,G(t-x)\,dx\right]dt$$

The integration being first with respect to x and then t.

The integration (1) is within the region lying below the line OP whose equation is $x = t$ and above OT, t being taken along OT and x along OX, with O is the origin the axes being perpendicular to each other. If the order of integration is changed, the strip will be taken parallel to OT, so that the limits of t are from x to ∞ and of x from 0 to ∞.



Therefore,

$$L\{H(t)\} = \int_0^\infty dx \int_x^\infty e^{-pt} F(x)\,G(t-x)\,dt$$

$$= \int_0^\infty e^{-px} F(x)\,dx \int_x^\infty e^{-p(t-x)}\,G(t-x)\,dt.$$

Putting $t - x = \theta \Rightarrow dt = d\theta$

$$\Rightarrow \qquad L\{H(t)\} = \int_0^{\infty} e^{-px} F(x) \left\{ \int_0^{\infty} e^{-p\theta} G(\theta)\, d\theta \right\} dx$$

$$= \int_0^{\infty} e^{-px} F(x)\, g(p)\, dx$$

$$= f(p)\, g(p)$$

$$\Rightarrow \quad L\left\{ \int_0^t F(x)\, G(t-x)\, dx \right\} = f(p)\, g(p)$$

$$\Rightarrow \qquad \int_0^t F(x)\, G(t-x)\, dx = L^{-1}\{f(p)\, g(p)\}$$

$$= F * G.$$

Properties of Convolution :

(1) $F * G$ is commutative *i.e.*, $F * G = G * F$

(2) $F * G$ is associative

(3) $F * G$ is distributive over addition.

SOLVED EXAMPLES

Example 1. *Using convolution theorem, evaluate* $L^{-1}\left\{\dfrac{1}{(p-1)(p+2)}\right\}$.

Solution. We have

$$L^{-1}\left\{\frac{1}{p+1}\right\} = e^t = F_1(t) \text{ (say)}$$

and

$$L^{-1}\left\{\frac{1}{p+2}\right\} = e^{-2t} = F_2(t) \text{ (say)}.$$

Using convolution theorem, we have

$$L^{-1}\left\{\frac{1}{p-1} \cdot \frac{1}{p+2}\right\} = F_1 * F_2 = \int_0^t F_1(x)\, F_2(t-x)\, dx$$

$$= \int_0^t e^x e^{-2(t-x)}\, dx \;= e^{-2t} \int_0^t e^{3x}\, dx = \frac{1}{3}(e^t - e^{-2t}).$$

Example 2. *Using convolution theorem, evaluate* $L^{-1}\left\{\dfrac{1}{(p^2+4)(p+2)}\right\}$.

Solution. We know that

$$L^{-1}\left\{\frac{1}{p^2+4}\right\} = \frac{1}{2}\sin 2t = F_1(t) \text{ (say)}$$

also,

$$L^{-1}\left\{\frac{1}{p+2}\right\} = e^{-2t} = F_2(t) \text{ (say)}.$$

Then by convolution theorem, we have

$$L^{-1}\left\{\frac{1}{(p^2+4)(p+2)}\right\} = F_1(t) * F_2(t) = \int_0^t F_1(x)\, F_2(t-x)\, dx$$

$$= \int_0^t \frac{1}{2}\sin 2x \,.\, e^{-2(t-x)} \,.\, dx$$

$$= \frac{1}{8}[e^{-2t} + \sin 2t - \cos 2t].$$

• SUMMARY

- **Inverse Laplace Transform :** If $L\{F(t)\} = f(p)$, then $L^{-1}\{f(p)\} = F(t)$.
- **Shifting theorem :** If $L^{-1}\{f(p)\} = F(t)$, then $L^{-1}\{f(p-a)\} = e^{at}L^{-1}\{f(p)\} = e^{at}F(t)$.
- **Second shifting thorem :** If $L^{-1}\{f(p)\} = F(t)$, then $L^{-1}\{e^{-ap}f(p)\} = G(t)$,
  where $G(t) = \begin{cases} F(t-a) & t > a \\ 0, & t < a \end{cases}$
- **Change of scale :** If $L^{-1}\{f(p)\} = F(t)$, then $L^{-1}\{f(ap)\} = \frac{1}{a}F\left(\frac{t}{a}\right)$.
- **Inverse Laplace of Derivative :** If $L^{-1}\{f(p)\} = F(t)$, then $L^{-1}\left\{\frac{d^n}{dp^n}(f(p))\right\} = (-1)^n t^n F(t)$.
- **Division by p :** If $L^{-1}\{f(p)\} = F(t)$, then $L^{-1}\left\{\frac{f(p)}{p}\right\} = \int_0^t F(u)\,du$.
- **Multiplication by p :** If $L^{-1}\{f(p)\} = F(t)$, then $L^{-1}\{p\,f(p)\} = F'(t)$.
- **Inverse Laplace of integrals :** If $L^{-1}\{f(p)\} = F(t)$, then $L^{-1}\left[\int_p^\infty f(x)\,dx\right] = \frac{F(t)}{t}$.
- **Convolution theorem :** If $L^{-1}\{f(p)\} = F(t)$ and $L^{-1}\{g(p)\} = G(t)$, then

$$L^{-1}\{f(p)\,g(p)\} = \int_0^t F(u)\,G(t-u)\,du = F * G.$$

• STUDENT ACTIVITY

1. If $L^{-1}\{f(p)\} = F(t)$, then show that $L^{-1}\{f(ap)\} = \frac{1}{a}F\left(\frac{t}{a}\right)$.

2. Evaluate $L^{-1}\left\{\log\left(1-\frac{1}{p^2}\right)\right\}$

• TEST YOURSELF–3

1. Use convolution theorem, show that

(a) $L^{-1}\left\{\dfrac{1}{(p+1)(p-2)}\right\}=\dfrac{1}{3}[e^{2t}-e^{-t}]$

(b) $L^{-1}\left\{\dfrac{p}{(p^2+a^2)^2}\right\}=\dfrac{1}{2a}t\sin at$

(c) $L^{-1}\left\{\dfrac{1}{p(p^2+4)^2}\right\}=\dfrac{1}{16}(1-t\sin 2t-\cos 2t)$

OBJECTIVE EVALUATION

Fill in the blanks

1. If $L^{-1}\{f(p)\}=F(t)$, then $L^{-1}\{f(p-a)\}=$ ....
2. $L^{-1}\left\{\dfrac{1}{p^2}\right\}=$ ....
3. $L^{-1}\left\{\dfrac{a}{p^2+a^2}\right\}=$ ....

True or False

1. $L^{-1}\left\{\dfrac{1}{p+a}\right\}=e^{at}$ (T/F)
2. If $L^{-1}\{f(p)\}=F(t)$, then $L^{-1}\{f(ap)\}=\dfrac{1}{a}F\left(\dfrac{t}{a}\right)$. (T/F)
3. $L^{-1}\left\{\dfrac{1}{p^4}\right)=\dfrac{t^3}{3\,!}$ (T/F)

Multiple Choic Questions (MCQ's) :

1. $L^{-1}\left\{\dfrac{p}{p^2-a^2}\right\}$ equal to :

   (a) $\cos at$ (b) $\sin at$ (c) $\dfrac{1}{a}\cos at$ (d) $\dfrac{1}{a}\sin at$

2. For $t>a$, if $L^{-1}\{f(p)\}=F(t)$, then $L^{-1}\{e^{ap}f(p)\}$ equals to :

   (a) $F\left(\dfrac{t}{a}\right)$ (b) $F(t-a)$ (c) $\dfrac{1}{a}F\left(\dfrac{t}{a}\right)$ (d) $F(at)$

ANSWERS

Fill in the blanks

1. $e^{at}F(t)$ 2. t 3. $\sin at$

True or False :

1. F 2. T 3. T

Multiple Choice Questions :

1. (a) 2. (b).

□□□

UNIT

8

APPLICATION OF LAPLACE TRANFORMS TO SOLUTIONS OF DIFFERENTIAL EQUATIONS

STRUCTURE

- Solution of Ordinary Differential Equations with constant coefficients
  - Test Yourself
- Solution of Partial Differential Equation using Laplace Transform
  - Summary
  - Student Activity
  - Test Yourself

LEARNING OBJECTIVES

After going through this unit you will learn :

- How to find the solution of Ordinary Differential Equation and Partial Differential Equation using Laplace transform

8.1. SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

Consider a linear differential equation with constant coefficients

$$\frac{d^n y}{dt^n} + A_1 \frac{d^{n-1} y}{dt^{n-1}} + \ldots + A_{n-1} \frac{dy}{dt} + A_n y = F(t) \qquad \ldots(1)$$

where t is the independent variable and $F(t)$ is a function of t.

Let $\quad y(0) = C_1, y'(0) = C_2, \ldots, y^{n-1}(0) = C_{n-1} \qquad \ldots(2)$

be the given initial or boundary conditions where $C_1, C_2, \ldots, C_{n-1}$ are constants. Now, taking the Laplace transform of both sides of (1) and using the conditions given by (2), we get an algebraic equation from which $\overline{y}(p) = L\{y(t)\}$ is determined. The required solution is then obtained by finding the inverse Laplace transform of $\overline{y}(p)$.

SOLVED EXAMPLES

Example 1. *Solve* $\frac{d^2y}{dt^2} + y = 0$ *under the condition that* $y = 1, \frac{dy}{dt} = 0$ *when* $t = 0$.

Solution. Here, the given equation is

$$\frac{d^2y}{dt^2} + y = 0. \qquad \ldots(1)$$

Taking the Laplace transform of both sides of the given differential equation, we get

$$L(y'') + L(y) = 0$$

$\Rightarrow \quad p^2 L(y) - py(0) - y'(0) + L\{y\} = 0$

$\Rightarrow \quad (p^2 + 1) L(y) - p \cdot 1 - 0 = 0 \qquad$ (using the given conditions)

$\Rightarrow \quad L(y) = \frac{p}{p^2+1}.$

Therefore,

$$y = L^{-1}\left\{\frac{p}{p^2+1}\right\} = \cos t.$$

Example 2. *Solve* $(D^2 + 1)\, y = 6 \cos 2t$ *if* $y = 3$, $Dy = 1$ *when* $t = 0$.

Solution. The given equation can be written as

$$y'' + y = 6 \cos 2t.$$

Taking the Laplace transform of both the sides of the given differential equation, we get

$$L(y'') + L(y) = 6L\{\cos(2t)\}$$

$$\Rightarrow \quad p^2 L\{y\} - py(0) - y'(0) + L\{y\} = 6\frac{p}{p^2 + 2^2}$$

$$\Rightarrow \quad (p^2 + 1)\, L\{y\} - 3p - 1 = \frac{6p}{p^2 + 4} \qquad \text{(Using the given conditions)}$$

$$\Rightarrow \quad L\{y\} = \frac{3p}{p^2 + 1} + \frac{1}{p^2 + 1} + \frac{6p}{(p^2 + 1)(p^2 + 4)}$$

$$= \frac{3p}{p^2 + 1} + \frac{1}{p^2 + 1} + \frac{2p\,[(p^2 + 4) - (p^2 + 1)]}{(p^2 + 1)(p^2 + 4)}$$

$$= \frac{3p}{p^2 + 1} + \frac{1}{p^2 + 1} + 2p\left\{\frac{1}{p^2 + 1} - \frac{1}{p^2 + 4}\right\}$$

$$= \frac{5p}{p^2 + 1} + \frac{1}{p^2 + 1} - \frac{2p}{p^2 + 4}.$$

Therefore,

$$y = 5L^{-1}\left\{\frac{p}{p^2 + 1}\right\} + L^{-1}\left\{\frac{1}{p^2 + 1}\right\} - 2L^{-1}\left\{\frac{p}{p^2 + 4}\right\}$$

$$\Rightarrow \quad y = 5\cos t + \sin t - 2\cos 2t.$$

Example 3. *Solve* $(D^2 + 9)\, y = \cos 2t$ *if* $y(0) = 1$, $y\left(\frac{\pi}{2}\right) = -1$.

Solution. The given equation can be written as

$$y'' + 9y = \cos 2t. \qquad \text{...(1)}$$

Taking the Laplace transform of both the sides of (1), we get

$$L\{y''\} + 9L\{y\} = L\{\cos 2t\}$$

$$\Rightarrow p^2 L\{y\} - py(0) - y'(0) + 9L\{y\} = \frac{p}{p^2 + 4}$$

$$\Rightarrow \quad (p^2 + 9)\, L\{y\} - p - C = \frac{p}{p^2 + 4}, \text{ where } C = y'(0)$$

$$\therefore \quad L\{y\} = \frac{p + C}{p^2 + 9} + \frac{p}{(p^2 + 9)(p^2 + 4)}$$

$$= \frac{p}{p^2 + 9} + \frac{C}{p^2 + 9} + \frac{p}{5(p^2 + 4)} - \frac{p}{5(p^2 + 9)}.$$

Therefore,

$$y = L^{-1}\left\{\frac{p}{p^2 + 9}\right\} + CL^{-1}\left\{\frac{1}{p^2 + 9}\right\} + \frac{1}{5}L^{-1}\left\{\frac{p}{p^2 + 4}\right\} - \frac{1}{5}L^{-1}\left\{\frac{p}{p^2 + 9}\right\}$$

$$= \cos 3t + \frac{1}{3}C\sin 3t + \frac{1}{5}\cos 2t - \frac{1}{5}\cos 3t$$

$$= \frac{4}{5}\cos 3t + \frac{1}{3}C\sin 3t + \frac{1}{5}\cos 2t \qquad \text{...(2)}$$

Now, since $y\left(\frac{\pi}{2}\right) = -1$, therefore, from (1), we have

$$-1 = \frac{4}{5}\cos\frac{3\pi}{2} + \frac{1}{3}C\sin\frac{3\pi}{2} + \frac{1}{5}\cos\pi.$$

On solving, we get $C = \frac{12}{5}$.

Put this value in (2), we get

$$y = \frac{4}{5}\cos 3t + \frac{4}{5}\sin 3t + \frac{1}{5}\cos 2t.$$

• TEST YOURSELF–1

1. Solve $\dfrac{dy}{dt}+y=1$ if $y=2$ when $t=0$.
2. Show that the general solution of the equation $(D^2+k^2)\,y=0$ is
$$y=C_1\cos kt+C_2\sin kt.$$
3. Solve $y''(t)+y(t)=t$ if $y'(0)=1,\ y(\pi)=0$.
4. Solve $(D^2-1)\,y=a\cosh nt$ if $y=Dy=0$, when $t=0$.
5. Solve $(D^2+m^2)\,x=a\cos nt,\ t>0$ where x, Dx equal to x_0 and x_1, when $t=0$, $n\neq m$.

ANSWERS

1. $y=e^t+1$ 3. $y=\pi\cos t+t$ 4. $y=\dfrac{a}{n^2-1}(\cosh nt-\cosh t)$

5. $x=x_0\cos mt+\dfrac{x_1}{m}\sin mt+\dfrac{a}{m^2-n^2}(\cos nt-\cos mt)$

• 8.2. SOLUTION OF PARTIAL DIFFERENTIAL EQUATIONS

The Laplace transforms is also useful in solving various partial differential equtions subject to the given boundary conditions.

Laplace Transforms of Some Partial Derivatives :

(1) $L\left\{\dfrac{\partial y}{\partial t}\right\}=p\bar{y}(x,p)-y(x,0)$

(2) $L\left\{\dfrac{\partial^2 y}{\partial t^2}\right\}=p^2\,\bar{y}(x,p)-py(x,0)-y_t(x,0)$

(3) $L\left\{\dfrac{\partial y}{\partial x}\right\}=\dfrac{d\bar{y}}{dx}$

(4) $L\left\{\dfrac{\partial^2 y}{\partial x^2}\right\}=\dfrac{d^2\bar{y}}{dx^2}$, where – (bar) denote the Laplace transform of that function.

SOLVED EXAMPLES

Example 1. *Solve* $\dfrac{\partial y}{\partial t}=2\dfrac{\partial^2 y}{\partial x^2}$, *where* $y(0,t)=0=y(5,t)$ *and* $y(x,0)=10\sin 4\pi x$.

Solution. Taking the Laplace transforms of both the sides of the given equation, we get
$$L\left\{\frac{\partial y}{\partial t}\right\}=2L\left\{\frac{\partial^2 y}{\partial x^2}\right\}$$
$$\Rightarrow\qquad p\,\bar{y}-y(x,0)=2\frac{d^2\bar{y}}{dx^2}$$
$$\Rightarrow\qquad \frac{d^2\bar{y}}{dx^2}-\frac{p}{2}\bar{y}=-5\sin 4\pi x. \qquad ...(1)$$
The general solution of (1) is given by
$$\bar{y}=C_1\,e^{\sqrt{(p/2)}\,.\,x}+C_2\,e^{-\sqrt{(p/2)}\,.\,x}-\frac{5\sin 4\pi x}{-(4\pi)^2-p/2}$$
$$\Rightarrow\qquad \bar{y}=C_1\,e^{\sqrt{(p/2)}\,.\,x}+C_2\,e^{-\sqrt{(p/2)}\,.\,x}+\frac{10}{32\pi^2+p}\,.\,\sin 4\pi x. \qquad ...(2)$$
Given that
$$y(0,t)=0=y(5,t).$$
Therefore,
$$\bar{y}(0,p)=0,\ \bar{y}(5,p)=0.$$
Put these values in (1), we get
$$0=C_1+C_2 \qquad ...(3)$$

and $$0 = C_1 e^{5\sqrt{p/2}} + C_2 e^{-5\sqrt{p/2}} + \frac{10}{32\pi^2 + p} \cdot \sin 20\pi$$

$$= C_1 e^{5\sqrt{(p/2)}} + C_2 e^{-5/\sqrt{(p/2)}} + 0. \quad \text{...(4)}$$

Solve (3) and (4), we get

$$C_1 = 0 = C_2.$$

Therefore, from (1), we have

$$\bar{y} = \frac{10}{32\pi^2 + p} \sin 4\pi x$$

$$\Rightarrow \quad y = L^{-1}\left\{\frac{10}{32\pi^2 + p} \cdot \sin 4\pi x\right\}$$

$$= 10e^{-32\pi^2 t} \cdot \sin 4\pi x.$$

Example 2. *Find the solution of* $\frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial x^2}$, $x > 0$, $t > 0$, *where* $y(0, t) = 1$, $y(x, 0) = 0$.

Solution. Taking the Laplace transforms of both the sides of the given equation, we get

$$L\left\{\frac{\partial y}{\partial t}\right\} = L\left\{\frac{\partial^2 y}{\partial x^2}\right\}$$

$$\Rightarrow \quad p\,\bar{y}(x, p) - y(x, 0) = \frac{\partial^2 \bar{y}}{\partial x^2}$$

$$\Rightarrow \quad \frac{d^2 \bar{y}}{dx^2} - p\,\bar{y} = 0. \quad \text{...(1)}$$

The general solution of (1) is given by

$$\bar{y} = C_1 e^{\sqrt{p}x} + C_2 e^{-\sqrt{p}x}.$$

By $y(x, t)$ must be bounded as $x \to \infty$.

Therefore, $\bar{y}(x, p) = L\{y(x, t)\}$ must also be bounded as $x \to \infty$

$$q \Rightarrow C_1 = 0$$

$$\Rightarrow \quad \bar{y} = C_2 e^{-\sqrt{p}x} \text{ if } \sqrt{p} > 0. \quad \text{...(2)}$$

But $\quad y(0, t) = 1.$

Therefore,

$$L\{y(0, t)\} = L\{1\}$$

$$\Rightarrow \quad \bar{y}(0, p) = \frac{1}{p}. \quad \text{...(3)}$$

From (2) and (3), we get $\quad C_2 = \frac{1}{p}$

$$\therefore \quad \bar{y} = \left(\frac{1}{p}\right) e^{-\sqrt{p}x}$$

$$\Rightarrow \quad y = L^{-1}\left\{\frac{1}{pe^{\sqrt{p}x}}\right\} = \text{erf}\left\{\frac{\sqrt{x}}{2\sqrt{t}}\right\}$$

• SUMMARY

- Consider

$$\frac{d^2 y}{dt^2} + a\frac{dy}{dt} + by = F(t) \quad \text{... (1)}$$

with $\quad y(0) = c_1, \; y'(0) = c_2 \quad \text{... (2)}$

Taking Laplace transform on both sides of (1) and using (2), we get an algebraic equation, from which $\bar{y}(p) = L\{y(t)\}$ is determined. The required solution of (1) is obtained by taking inverse Laplace of $\bar{y}(p)$.

• STUDENT ACTIVITY

1. Solve $\frac{d^2y}{dt^2} + y = 0$ with $y(0) = 1$, $y'(0) = 0$.

2. Solve $\frac{dy}{dt} + y = 1$ with $y(0) = 2$.

• TEST YOURSELF

1. *Solve* $\frac{\partial y}{\partial x} = 2\frac{\partial y}{\partial t} + y$, $y(x, 0) = 6e^{-3x}$ *which is bounded for* $x > 0$, $t > 0$.

2. Solve $\frac{\partial y}{\partial t} = 3\frac{\partial^2 y}{\partial x^2}$ where $y\left(\frac{\pi}{2}, t\right) = 0$, $\left(\frac{\partial y}{\partial x}\right)_{x=0} = 0$ and $y(x, 0) = 30\cos 5x$.

3. Solve $\frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial x^2}$, $y(x, 0) = 3\sin 2\pi x$, $y(0, t) = 0 = y(1, t)$, $0 < x < 1$, $t > 0$.

4. Solve
$$\frac{\partial y}{\partial t} = 2\frac{\partial^2 y}{\partial x^2}, \; y(0, t) = 0, \; y(5, t) = 0,$$
$$y(x, 0) = 10\sin 4\pi x - 5\sin 6\pi x.$$

OBJECTIVE EVALUATIONS

Fill in the blanks :

1. The general solution of $\frac{d^2y}{dt^2} + k^2y = 0$ is ....
2. $L\{y'(t)\} =$ ....

True or False

1. $L\{y''(t)\} = p^2L\{y(t)\} - py(0) - y'(0)$ *(T/F)*
2. Solution of $\frac{dy}{dt} + y = 1$ with $y(0) = 2$ is $e^t + 1$. *(T/F)*

Multiple Choice Questions (MCQ's)

1. Solution of $y''(t) + y(t) = t$ with $y(\pi) = 0$, $y'(0) = 1$ is :
   (a) $\pi\sin t + 1$ (b) $\pi\cos t + t$ (c) $\pi\sin t + t$ (d) $\pi\cos t - t$

ANSWERS

1. $y(x, t) = 6e^{-2t-3x}$ **2.** $y = 30e^{-75t}\cos 5x$

3. $y(x, t) = 3e^{-4\pi^2 t} . \sin 2\pi x$ **4.** $y(x, t) = -5e^{-72\pi^2 t} . \sin 6\pi x + 10 . e^{-32\pi^2 t} . \sin 4\pi x$

Fill in the blanks

1. $y = a\cos kt + b\sin kt$ **2.** $pL\{y(t)\} - y(0)$

True or False

1. T **2.** T

Multiple Choice Questions (MCQ's)

1. (b)

UNIT

9

FORCES IN THREE DIMENSIONS

STRUCTURE

- Equilibrium of forces in three dimensions
- Reduction of system of forces to a single force and a couple
- Wrench
- Poinsot's Central Axis
- Wrench and Screw
- Invariants
- Condition for a system of forces to be a single resultant force
- Equation of Central Axis
- Procedure for finding X, Y, Z and L, M, N
  - Summary
  - Student Activity
  - Test Yourself

LEARNING OBJECTIVES

After going through this unit you will learn :

- How to find the resultant of a system of forces acting on a particle
- What are the necessary and sufficient conditions of a rigid body to be in equilibrium
- What is poinsot's central axis and how to find its equation and the surface on which it lies.

• 9.1. EQUILIBRIUM OF FORCES IN THREE DIMENSIONS

1. *To find the resultant of any given system of forces acting at a particle.*

Let $\vec{F_1}, \vec{F_2}, \ldots, \vec{F_n}$ be the given system of forces acting at a particle which is at O. Let us choose three mutually perpendicular lines OX, OY and OZ through O as the axes of a co-ordinate system.

The resultant of the forces $\vec{F_1}, \vec{F_2}, \ldots, \vec{F_n}$ is obtained by the repeated application of the parallelogram law of forces. If $\vec{R}$ be the resultant of these forces, then we have

$$\vec{R} = \vec{F_1} + \vec{F_2} + \ldots + \vec{F_n}. \qquad \ldots(1)$$

Let X, Y, Z be the components of $\vec{R}$ along OX, OY and OZ respectively and let $\hat{i}$, $\hat{j}$ and $\hat{k}$ be the unit vectors along OX, OY and OZ respectively, then

$$\vec{R} = X\hat{i} + Y\hat{j} + Z\hat{k} \qquad \ldots(2)$$

$$\therefore \quad X = \hat{i} \cdot \vec{R} = \hat{i} \cdot (\vec{F_1} + \vec{F_2} + \ldots + \vec{F_n})$$

$$= \hat{i} \cdot \vec{F_1} + \hat{i} \cdot \vec{F_2} + \ldots + \hat{i} \cdot \vec{F_n}$$

and

$$Y = j \cdot \vec{R} = \hat{j} \cdot (\vec{F_1} + \vec{F_2} + \ldots + \vec{F_n})$$

$$= \hat{j} \cdot \vec{F_1} + \hat{j} \cdot \vec{F_2} + \ldots + \hat{j} \cdot \vec{F_n}$$

and

$$Z = \hat{k} \cdot \vec{R} = \hat{k} \cdot (\vec{F_1} + \vec{F_2} + \ldots \vec{F_n})$$

$$= \hat{k}\,.\,\vec{F}_1 + \hat{k}\,.\,\vec{F}_2 + \ldots + \hat{k}\,.\,\vec{F}.$$

Thus the resolved part of the resultant $\vec{R}$ along any axis is equal to the sum of resolved parts of $\vec{F}_1, \vec{F}_2, \ldots, \vec{F}_n$ along that axis :

If R be the magnitude of the resultant $\vec{R}$, then

$$R^2 = \vec{R}\,.\,\vec{R}$$
$$= (X\hat{i} + Y\hat{j} + Z\hat{k})\,.\,(X\hat{i} + Y\hat{j} + Z\hat{k})$$
$$= X^2 + Y^2 + Z^2$$
$$\Rightarrow \qquad R = \sqrt{X^2 + Y^2 + Z^2}.$$

Now dividing of both sides of (2) by R, we get

$$\hat{R} = \frac{\vec{R}}{R} = \left(\frac{X}{R}\right)\hat{i} + \left(\frac{Y}{R}\right)\hat{j} + \left(\frac{Z}{R}\right)\hat{k}$$

This is the unit vector along which the resultant $\vec{R}$ is acting. Hence $\frac{X}{R}, \frac{Y}{R}, \frac{Z}{K}$ are the direction cosines of the line of action of the resultant $\vec{R}$.

2. *The necessary and sufficient conditions of the particle under the action of a system of forces to be in equilibrium are that the algebraic sums of the resultant parts of the forces along any three mutually perpendicular directions vanish separately.*

Proof. Let $\vec{R}$ be the resultant of the system of forces acting on a particle at O and X, Y, Z be the algebraic sums of resolved parts of the forces along OX, OY and OZ axes respectively. Then

$$\vec{R} = X\hat{i} + Y\hat{j} + Z\hat{k} \qquad \ldots(1)$$

Conditions are necessary. Suppose the particle at O is in equilibrium, then the resultant $\vec{R}$ must be zero.

$$\therefore \qquad \vec{R} = \vec{O}, \ \vec{O} \text{ being the zero vector}$$
$$\Rightarrow \qquad X\hat{i} + Y\hat{j} + Z\hat{k} = \vec{O}$$
$$\Rightarrow \qquad X = 0,\ Y = 0,\ Z = 0.$$

Thus in a position of equilibrium of particle, the algebraic sums X, Y and Z along OX, OY and OZ respectively vanish separately.

Conditions are sufficient. Suppose the sums of the resolved parts of the forces X, Y and Z along OX, OY and OZ respectively vanish separately. Then

$$X = ,\ Y = 0,\ Z = 0.$$
$$\therefore \qquad \vec{R} = X\hat{i} + Y\hat{j} + Z\hat{k} \qquad \text{[using (1)]}$$
$$= O\hat{i} + O\hat{j} + O\hat{k}$$
$$= \vec{O}.$$

Thus the resultant $\vec{R}$ of all forces acting on a particle is zero. Hence the particle is in equilibrium.

• 9.2. REDUCTION OF A SYSTEM OF FORCES ACTING ON A RIGID BODY TO A SINGLE FORCES AND A COUPLE

(I) *When some forces act at different points on a rigid body, this system, of forces reduces to a single force and a couple whose axis passes through a point at which the single force acts.*

Let $\vec{F}_1, \vec{F}_2, \ldots, \vec{F}_n$ be the forces acting at the points $P_1, P_2, \ldots, P_n$ on a rigid body respectively. Let O be any arbitrary point treating as the origin of vectors and $\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_n$ be the position vectors of the points $P_1, P_2, \ldots, P_n$ with respect to the point O (Base point).

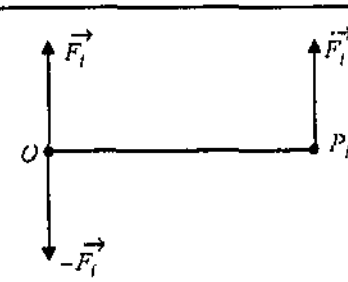


Fig. 1

Let us consider a force $\vec{F_i}$ acting at the point P_i with $\overrightarrow{OP_i} = \vec{r_i}$. Now apply two forces $\vec{F_i}$ and $-\vec{F_i}$ at O parallel to the forces $\vec{F_i}$ and at P_i in the opposite direction as shown in the adjoining fig.

On applying two and equal opposite forces at the same point, they will neutralise each other therefore there will be no extra effect on the body.

Thus the force $\vec{F_i}$ at P_i is equivalent to the single force $\vec{F_i}$ at P_i and two forces $\vec{F_i}$ and $-\vec{F_i}$ at O. Since the forces $\vec{F_i}$ at P_i and $-\vec{F_i}$ at O will form a couple of moment $\vec{r_i} \times \vec{F_i}$.

The force $\vec{F_i}$ acting at the point P_i of a rigid body is therefore equivalent to a single force $\vec{F_i}$ at O and a couple $\vec{G_i}$ of moment $\vec{r_i} \times \vec{F_i}$.

Similarly all the forces $\vec{F_1}, \vec{F_2}, \ldots, \vec{F_n}$ acting at the points $P_1, P_2, \ldots, P_n$ respectively are equivalent to the forces $\vec{F_1}, \vec{F_2}, \ldots, \vec{F_n}$ at O and the couples $\vec{G_1}, \vec{G_2}, \ldots, \vec{G_n}$ of moments $\vec{r_1} \times \vec{F_1}, \vec{r_2} \times \vec{F_2}, \ldots, \vec{r_n} \times \vec{F_n}$.

If $\vec{R}$ is the resultant of $\vec{F_1}, \vec{F_2}, \ldots, \vec{F_n}$ the n concurrent forces at O and $\vec{G}$ the moment of resultant of $\vec{G_1}, \vec{G_2}, \ldots, \vec{G_n}$, then we have

$$R = \vec{F_1} + \vec{F_2} + \ldots + \vec{F_n} = \sum_{i}^{n} \vec{F_i} \qquad \ldots(1)$$

and

$$\vec{G} = \vec{G_1} + \vec{G_2} + \ldots + \vec{G_n}$$

$$= \vec{r_1} \times \vec{F_1} + \vec{r_2} \times \vec{F_2} + \ldots + \vec{r_n} \times \vec{F_n}$$

$$= \sum_{i=1}^{n} \vec{r_i} \times \vec{F_i}. \qquad \ldots(2)$$

Hence the system of forces acting at the given points of a rigid body can be reduced to a single force $\vec{R}$ acting at O and a couple of moment $\vec{G}$, whose axis can be made to pass through the point O. since the couple is a free vector. The point O is also known as the **base point.**

Remark

- If L, M, N be the components of $\vec{G}$ about OX, OY and OZ respectively, then

$$\hat{G} = L\hat{i} + M\hat{j} + N\hat{k}$$

The unit vector along $\vec{G}$ is

$$\hat{G} = \frac{\vec{G}}{|\vec{G}|}$$

since $\qquad |\vec{G}| = \sqrt{L^2 + M^2 + N^2} = G$ (say)

$$\therefore \qquad \hat{G} = \frac{L}{G}\hat{i} + \frac{M}{G}\hat{j} + \frac{N}{G}\hat{k}$$

Hence, $\frac{L}{G}, \frac{M}{G}, \frac{N}{G}$ are the direction cosines of the axis of the couple G.

It has been observed from equation (1) that the single force $\vec{R}$ does not depend on the poisition of base point O. but from equation (2) it is obvious that the couple G depends on the points of base point.

We shall now discuss about the change in $\vec{G}$ when the position of the base point is changed.

(ii) *To find the change in couple when the base point is changed.*

Let O be the base point and suppose a system of forces $\vec{F_1}, \vec{F_2}, \ldots, \vec{F_n}$ acting at different points of a rigid body is reduced to a single force $\vec{R}$ and a couple $\vec{G}$ with reference to the base point O, then we have

$$\vec{R} = \sum_{i=1}^{n} \vec{F_i} \qquad \text{...(1)}$$

and $$\vec{G} = \sum_{i=1}^{n} \vec{r_i} \times \vec{F_i} \qquad \text{...(2)}$$

where $\vec{r_i}$ is the position vector of a point P_i at which a force $\vec{F_i}$ is acting.

Let us suppose that the base point O changes to another base point O' such that $\overrightarrow{OO'} = \vec{c}$.

Let $\vec{s_i}$ be the position vector the point P_i with respect to the base point O'. Then

$$\overrightarrow{O'P_i} = \vec{s_i}.$$

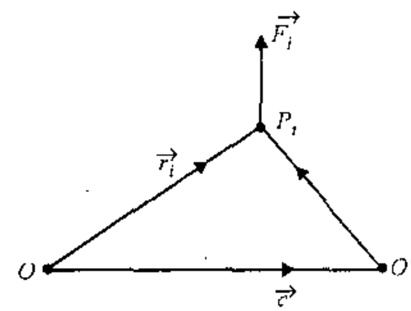


Fig. 2

Now in $\Delta OO'P_i$, we have

$$\overrightarrow{OO'} + \overrightarrow{O'P_i} = \overrightarrow{OP_i} \qquad \text{(By the property of addition of vectors)}$$

$$\Rightarrow \quad \vec{c} + \vec{s_i} = \vec{r_i}$$

$$\Rightarrow \quad \vec{s_i} = \vec{r_i} - \vec{c}. \qquad \text{...(3)}$$

Suppose a system of forces $\vec{F_1}, \vec{F_2}, \ldots, \vec{F_n}$ acting different points of a rigid body is reduced to a single force $\vec{R'}$ and a couple $\vec{G'}$ with reference to the base point O'.

Then we have

$$\vec{R'} = \vec{F_1} + \vec{F_2} + \ldots + \vec{F_n}$$

$$= \sum_{i=1}^{n} \vec{F_i} = \vec{R} \qquad \text{...(4)}$$

and
$$\vec{G'} = \sum_{i=1}^{n} \vec{s_i} \times \vec{F_i}$$

$$= \sum_{i=1}^{n} (\vec{r_i} - \vec{c}) \times \vec{F_i} \qquad \text{[using (3)]}$$

$$= \sum_{i=1}^{n} (\vec{r_i} \times \vec{F_i} - \vec{c} \times \vec{F_i})$$

$$= \sum_{i=1}^{n} \vec{r_i} \times \vec{F_i} - \sum_{i=1}^{n} \vec{c} \times \vec{F_i}$$

$$= \sum_{i=1}^{n} \vec{r_i} \times \vec{F_i} - \vec{c} \times \sum_{i=1}^{n} \vec{F_i} \qquad (\because \vec{c} \text{ is a constant vector})$$

$$\vec{G'} = \vec{G} - \vec{c} \times \vec{R} \qquad \text{...(5)}$$

Thus $$\vec{R'} = \vec{R} \text{ and } \vec{G'} = \vec{G} - \vec{c} \times \vec{R}.$$

Hence we get a conclusion that when the base point changes, the single force $\vec{R}$ remains the same but the couple $\vec{G}$ change to $\vec{G'}$ which is governed by the equation (5).

(iii) Conditions of equilibrium of a rigid body.

Theorem. *The necessary and sufficient conditions of a rigid body to be in equilibrium under the action of a system of forces acting at different points on it are that the sums of the resolved parts of the forces along any three mutually perpendicular axes and the sums of the moments of the forces about these axes must vanish separately.*

Proof. Suppose a rigid body is acted upon by a system of forces $\vec{F_1}, \vec{F_2}, \ldots, \vec{F_n}$ at the points $P_1, P_2, \ldots, P_n$ respectively.

Let $\vec{r_1}, \vec{r_2}, \ldots, \vec{r_n}$ be the position vectors of the points $P_1, P_2, \ldots, P_n$ with reference to the base point O. Then the system of forces reduces to a single force $\vec{R}$ and a couple $\vec{G}$ given by the equations:

$$\vec{R} = \sum_{i=1}^{n} \vec{F_i} \qquad \ldots(1)$$

and

$$\vec{G} = \sum_{i=1}^{n} \vec{r_i} \times \vec{F_i}. \qquad \ldots(2)$$

Now consider three mutually perpendicular axes OX, OY and OZ through O and let $\hat{i}, \hat{j}$ and $\hat{k}$ be the unit vectors along the axes OX, OY and OZ respectively.

Let (x_i, y_i, z_i) be the co-ordinates of a point P_i on a rigid body with reference to the axes OX, OY and OZ and let X_i, Y_i and Z_i be the components of a force $\vec{F_i}$ acting at P_i along OX, OY and OZ respectively.

Since $\vec{r_i}$ is the position vector of P_i so that

$$\vec{r_i} = x_i \hat{i} + y_i \hat{j} + z_i \hat{k}$$

and

$$\vec{F_i} = X_i \hat{i} + Y_i \hat{j} + Z_i \hat{k}.$$

Then from (1), we have

$$\vec{R} = \sum_{i=1}^{n} (X_i \hat{i} + Y_i \hat{j} + Z_i \hat{k}).$$

Also

$$\vec{R} = X \hat{i} + Y \hat{i} + Z \hat{k}.$$

$\therefore$

$$X = \sum_{i=1}^{n} X_i, \quad Y = \sum_{i=1}^{n} Y_i, \; Z = \sum_{i=1}^{n} Z_i. \qquad \ldots(3)$$

Here X, Y and Z are the sums of the components of the given forces along the axes OX, OY and OZ respectively.

Now equation (2) becomes :

$$\vec{G} = \sum_{i=1}^{n} [(x_i \hat{i} + y_i \hat{j} + z_i \hat{k}) \times (X_i \hat{i} + Y_i \hat{j} + Z_i \hat{k})]$$

$$= \sum_{i=1}^{n} [(y_i Z_i - z_i Y_i) \hat{i} + (z_i X_i - x_i Z_i) \hat{j} + (x_i Y_i - y_i X_i) \hat{k}].$$

If L, M and N be the components of $\vec{G}$ along OX, OY and OZ respectively, then

$$\vec{G} = L \hat{i} + M \hat{j} + N \hat{k}.$$

$$\therefore \quad L \hat{i} + M \hat{j} + N \hat{k} = \sum_{i=1}^{n} [(y_i Z_i - z_i Y_i) \hat{i} + (z_i X_i - x_i Z_i) \hat{j} + (x_i Y_i - y_i X_i) \hat{k}]$$

$$\Rightarrow \quad \left.\begin{aligned} L &= \sum_{i=1}^{n} (y_i Z_i - z_i Y_i) \\ M &= \sum_{i=1}^{n} (z_i X_i - x_i Z_i) \\ \text{and} \quad N &= \sum_{i=1}^{n} (x_i Y_i - y_i X_i) \end{aligned}\right\} . \qquad \ldots(4)$$

Equation (4) gives the sums of the components of couple about OX, OY and OZ respectively.

Conditions are Necessary :

Suppose the rigid body is in equilibrium, therefore, there is no movement in the body *i.e.*, there is neither the motion of translation nor the motion of rotation.

This implies, $\vec{R} = \vec{O}, \vec{G} = \vec{O}.$

Since $\vec{R} = X\hat{i} + Y\hat{j} + Z\hat{k}$ and $\vec{G} = L\hat{i} + M\hat{j} + N\hat{k}$

$\therefore$ $\vec{R} = \vec{O} \Rightarrow X\hat{i} + Y\hat{j} + Z\hat{k} = \vec{O} = (0, 0, 0)$

$\Rightarrow$ $X = 0, Y = 0, Z = 0$

and $\vec{G} = \vec{O} \Rightarrow L\hat{i} + M\hat{j} + N\hat{k} = \vec{O} = (0, 0, 0)$

$\Rightarrow$ $L = 0, M = 0, N = 0.$

Thus from (3) and (4), we get

$$\sum X_i = 0, \sum Y_i = 0, \sum Z_i = 0$$

and $\sum (y_i Z_i - z_i X_i) = 0, \sum (z_i X_i - x_i Z_i) = 0$

and $\sum (x_i Y_i - y_i X_i) = 0.$

Hence if a rigid is in equilibrium under the action of a system of forces, the sums of the components of all forces and couple vanish separately.

Condition are Sufficient :

Suppose the sums of components of the forces along the axes OX, OY and OZ vanish and *sums of the moments of the forces about* OX, OY and OZ vanish. Therefore,

$$X = 0, Y = 0, Z = 0$$

and $L = 0, M = 0, N = 0$

$\therefore$ $\vec{R} = X\hat{i} + Y\hat{j} + Z\hat{k} = \vec{O}$

and $\vec{G} = L\hat{i} + M\hat{j} + N\hat{k} = \vec{O}.$

Thus $\vec{R} = \vec{O}$ and $\vec{G} = \vec{O}$. Hence the rigid body is in equilibrium.

• 9.3. WRENCH

Definition. When a rigid body is acted upon by a system of forces at different points on the *body, then this system* can be reduced to a single force $\vec{R}$ acting at an arbitrary point O and *a* couple $\vec{G}$ whose axis passes through O. In case, when the line of action $\vec{R}$ is same to the axis of the couple $\vec{G}$, then $\vec{R}$ together with $\vec{G}$ form a **wrench** and common line of action of the single force $\vec{R}$ and the *axis of* $\vec{G}$ *is said to be the axis of the* **wrench**.

If R be the magnitude of $\vec{R}$, then R is called the **intensity** of the wrench. Also if $\vec{G} = p\vec{R}$, then p is called the **pitch** of the wrench.

Remark

➤ If $\vec{R}$ and $\vec{G}$ are parallel, then $\vec{R}$ and $\vec{G}$ form a wrench.

Theorem. *To show that any system of forces acting on a rigid body can be reduced to a single force together with a couple whose axis is along the direction of the force.*

Proof. It has already been proved that any system of forces acting on a rigid body can be reduced to a single force $\vec{R}$ and a couple $\vec{G}$ whose axis passes through O (base point) at which $\vec{R}$ acts.

Suppose a single force $\vec{R}$ *acts at* O *and along a line* OA *and a couple of moment* $\vec{G}$ about a line OB. Let $\angle AOB = \theta$.

Draw a line OC perpendicular to OA in the plane OAB and draw OD perpdendicular to the plane AOC.

The couple of moment G (magnitude of $\vec{G}$) acting about OB is equivalent to a couple of moment $G \cos \theta$ about OA and a couple $G \sin \theta$ about OC as shown in fig. 4.

Since the line OC is perpendicular to the plane AOD. Therefore the couple $G \sin \theta$ acts in the plane AOD and it can therefore be replaced by two equal unlike parallel forces in the plane AOD.

Let us choose one of these force R at O in the opposite direction to OA, therefore the other force must be equal to R acting at some point O' in OD along a line $O'A'$ (say) which is parallel to OA such that

$$R \cdot OO' = G \sin \theta$$

$$\Rightarrow \qquad OO' = \frac{G \sin \theta}{R}.$$

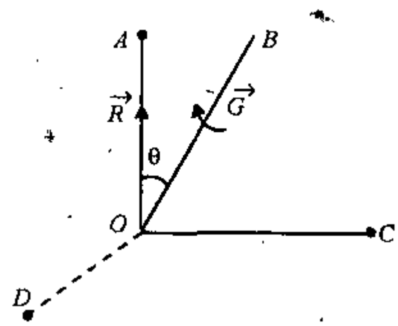


Fig. 3

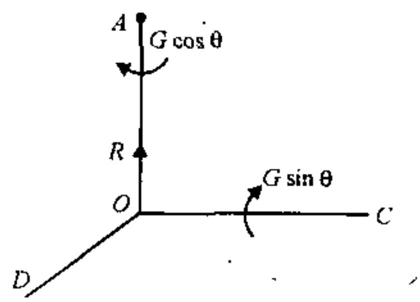


Fig. 4

Since the two equal forces of magnitude R are acting at O in the opposite direction, so they neutralise each other. Thus we obtain a force R at O' acting along $O'A'$ and a couple of moment $G \cos \theta$ about a line parallel to AO. Let us take a line $O'A'$ parallel to OA as shown in fig. 5.

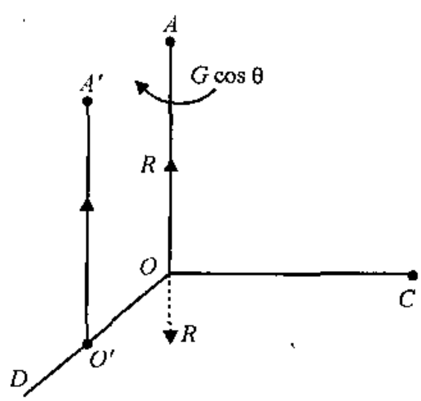


Fig. 5

Also, the axis of a couple can be transfered to any parallel axis, therefore we take the axis of $G \cos \theta$ as $O'A'$ as shown in fig. 6.

Hence a system of force acting on a rigid body can be reduced to a single force R and a couple of moment $G \cos \theta$ such that line of action of R and the axis of $G \cos \theta$ are the same. This same line is called *Poinsot's central axis*.

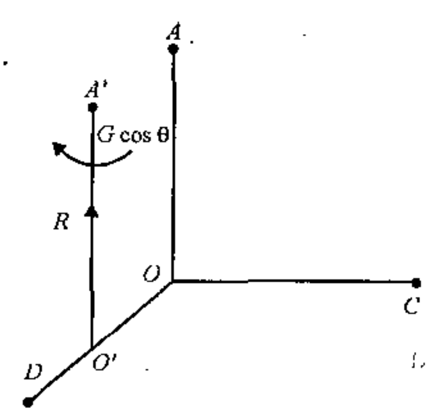


Fig. 6

• 9.4. POINSOT'S CENTRAL AXIS

1. Definition. A system of forces acting at different points of a rigid body can be reduced to a single force of magnitude R acting along a line and a single couple of moment $G \cos \theta$ about the same line. This same line is called **Poinsot's central axis.**

2. Properties of central axis.

(i) *Central axis for a system of forces acting on a rigid body is unique.*

Proof. Let, if possible for a given system of forces, there are two central axes. Let $O'A'$ and $O''A''$ be two central axes for a given system of forces, and p be the distance $O'O''$.

Therefore, the given system of forces is equivalent to a single force along $O'A'$ and a couple about a line $O'A'$ and also is equivalent to a force along $O''A''$ and a couple about $O''A''$. But the single force will be same in magnitude and direction, because the single force does not depend on the base point. Thus the line $O'A'$ is parallel to $O''A''$. Hence the wrench (R, G) about $O'A'$ is the same as the wrench (R, G') about a parallel line $O''A''$.

Since p is the distance between $O'A'$ and $O''A''$, so that the single force R along $O''A''$ is equivalent to R along $O'A'$ and a couple of moment $R \cdot p$ about an axis perpendicular to $O'A'$. Hence the wrench (R, G') is equivalent to R along $O'A'$, a couple G' about $O'A'$ and a couple $R \cdot p$ about an axis perpendicular to $O'A'$.

This implies that the system (R, G') is not same to the system (R, G). Which contradicts the hypothesis. Hence the two central axes $O'A'$ and $O''A''$ must be same. Consequently central axis is unique.

(ii) *The moment of the resultant couple about the central axis is less than the moment of the resultant couple corresponding to any point which is not on the cenral axis.*

Proof. Since the single force R is same for any base point O while the single couple G is not the same.

If O be any origin (not on the central axis) and G be the couple for O, and if its axis makes an angle θ with the single force R, then the couple for the central axis will $G \cos \theta$.

Since $\cos \theta \leq 1$, therefore $G \cos \theta \leq G$.

Hence the couple $G \cos \theta$ about the central axis is less that the couple G corresponding to any point O (not on the central axis).

• 9.5. WRENCH AND SCREW

(1) Wrench. A system of forces acting at different points on a rigid body can be reduced to a single force R acting at an arbitrary point O and a single couple G about an axis passing through O. If the axis of G makes an angle θ with the line of action of R, then $G \cos \theta$ is the magnitude of moment of couple about the central axis. If R is the single force and $K = G \cos \theta$ be the single couple whose axis coincides with the direction of R, *then R and K together constitute a wrench of the system of forces.*

The magnitude of single force R is called the intensity of the wrench and the ratio $\frac{K}{R}$, is called the pitch of the wrench. If p be the pitch, then $K = R \cdot p$. There are following cases depending on p (pitch).

(i) If $p = 0$, then the wrench (R, K) reduces to a single force R.

(ii) If $p = \infty$ (infinity), then the wrench (R, K) reduces to a couple K only.

(2) Screw. *The straight line along which the single force acts when considered together with the pitch is called a* **Screw.** *Therefore a Screw is a definite straight line associated with a definite pitch.*

• 9.6. INVARIANTS

(i) *Whatever origin or base point and axes are chosen, the quantities*

$$X^2 + Y^2 + Z^2 \text{ and } LX + MY + NZ$$

are invariable for any given system of forces acting on a rigid body

where $$X = \sum X_i,\ Y = \sum Y_i,\ Z = \sum Z_i$$

and $$L = \sum (y_i Z_i - z_i Y_i) \text{ etc.}$$

Proof. Let O be the origin and OX, OY and OZ are three mutually perpendicular axes, then a system of forces acting on a body can be reduced to a single force $\vec{R}$ and a couple $\vec{G}$. If $\hat{i}, \hat{j}$ and $\hat{k}$ be the unit vectors along the axes OX, OY and OZ respectively, then

$$\vec{R} = X\hat{i} + Y\hat{j} + Z\hat{k}$$

and $$\vec{G} = L\hat{i} + M\hat{j} + N\hat{k}.$$

Now if we consider other origin O' and $O'X'$, $O'Y'$ and $O'Z'$ as mutually perpendicular axes, then a system of forces reduces to a single force $\vec{R'}$ and a single couple $\vec{G'}$. If $\hat{i'}$, $\hat{j'}$, $\hat{k'}$ be the unit vectors along $O'X'$, $O'Y'$ and $O'Z'$ respectively, then

$$\vec{R'} = X'\hat{i'} + Y'\hat{j'} + Z'\hat{k'}$$

and $\quad \vec{G'} = L'\hat{i} + M'\hat{j} + N'\hat{k}'.$

We now actually prove that

$$X^2 + Y^2 + Z^2 = X'^2 + Y'^2 + Z'^2$$

and $\quad LX + MY + NZ = L'X' + M'Z' + N'Z'.$

Since single force R and R' does not depend on the position of base point, so that

$$\vec{R} = \vec{R'}$$

$$\Rightarrow \quad |\vec{R}| = |\vec{R'}|$$

$$\Rightarrow \quad \sqrt{X^2 + Y^2 + Z^2} = \sqrt{X'^2 + Y'^2 + Z'^2}$$

$$\Rightarrow \quad X^2 + Y^2 + Z^2 = X'^2 + Y'^2 + Z'^2.$$

On the other hand, the couple $\vec{G}$ depends on the position of base point. If $\overrightarrow{OO'} = \vec{c}$ (a constant vector), then

$$\vec{G'} = \vec{G} - \vec{c} \times \vec{R}$$

$$\Rightarrow \quad \vec{G'} \cdot \vec{R'} = (\vec{G} - \vec{c} \times \vec{R}) \cdot \vec{R'}$$

$$= (\vec{G} - \vec{c} \times \vec{R}) \cdot \vec{R} \qquad (\because \ \vec{R'} = \vec{R})$$

$$= \vec{G} \cdot \vec{R} - (\vec{c} \times \vec{R}) \cdot \vec{R}$$

$$= \vec{G} \cdot \vec{R} - 0 \qquad (\because \text{ Scalar triple product is always zero if two vectors are same})$$

$$\vec{G'} \cdot \vec{R'} = \vec{G} \cdot \vec{R}$$

$$\Rightarrow \quad L'X' + M'Y' + N'Z' = LX + MY + NZ$$

(ii) Pitch and intensity of wrench using invariants.

Suppose a system of forces acting on a rigid body reduces to a single force $\vec{R} = (X, Y, Z)$ and a couple $\vec{G} = (L, M, N)$.

If this system reduces to a wrench $(\vec{R'}, \vec{G'})$, then we have

$$\vec{R} = \vec{R'}$$

and $\quad \vec{G'} = p\vec{R'}.$

The magnitude of $\vec{R'} = \vec{R}$ is the intensity of wrench, so that the intensity of wrench

$$= |\vec{R'}|$$

$$= |\vec{R}|$$

$$= \sqrt{X^2 + Y^2 + Z^2} = R \text{ (say)} \qquad ...(1)$$

Also, $\quad \vec{G'} = p\vec{R'}$

$$\Rightarrow \quad \vec{G'} \cdot \vec{R'} = p\vec{R'} \cdot \vec{R'}$$

$$\Rightarrow \quad \vec{G} \cdot \vec{R} = p\vec{R} \cdot \vec{R} \qquad (\because \ \vec{G'} \cdot \vec{R'} = \vec{G} \cdot \vec{R})$$

$$\Rightarrow \quad \vec{G} \cdot \vec{R} = pR^2$$

$$\Rightarrow \quad LX + MY + NZ = pR^2$$

$$\Rightarrow \quad p = \frac{LX + MY + NZ}{R^2} = \frac{LX + MY + NZ}{X^2 + Y^2 + Z^2}. \qquad ...(2)$$

Equation (1) gives the intensity of wrench and equation (2) gives the pitch of wrench.

Remark

➤ If K be the couple of wrench, then $K = pR = \dfrac{LX + MY + NZ}{R}$.

• 9.7. CONDITION FOR A SYSTEM OF FORCES TO BE A SINGLE RESULTANT FORCE

Theorem. *The necessary and sufficient conditions for a system of forces to reduce to a single resultant force are*

$$LX + MY + NZ = 0 \;\; and \;\; X^2 + Y^2 + Z^2 \neq 0$$

where $\quad \vec{R} = (X, Y, Z) \;\; and \;\; \vec{G} = (L, M, N)$

Proof. Suppose a system of forces acting at different points on the rigid body reduces to a single force $\vec{R}$ (X, Y, Z) and a couple $\vec{G} = (L, M, N)$. The force $\vec{R}$ acts at O and the axis of $\vec{G}$ passes through O.

Condition is Necessary :

Let O' be other base point, and the system $(\vec{R}, \vec{G})$ reduces to $(\vec{R'}, \vec{G'})$, then

$$\vec{R'} = \vec{R}$$

and $$\vec{G'} = \vec{G} - \vec{c} \times \vec{R}, \text{ where } \overrightarrow{OO'} = \vec{c}.$$

If $(\vec{R'}, \vec{G'})$ reduces to a single force at O', then we must have

$$\vec{R'} \neq \vec{0} \;\; \text{and} \;\; \vec{G'} = \vec{0}$$

$$\Rightarrow \quad \vec{R} \neq \vec{0} \;\; \text{and} \;\; \vec{G'} \,.\, \vec{R'} = 0$$

$$\Rightarrow \quad \vec{R} \neq \vec{0} \;\; \text{and} \;\; \vec{G} \,.\, \vec{R} = 0$$

$$\therefore \quad \vec{G} \,.\, \vec{R} = 0$$

$$\Rightarrow \quad LX + MY + NZ = 0.$$

Condition are Sufficient :

Let us suppose that

$$LX + MY + NZ = 0.$$

Now take a point O' on the central axis and suppose the system reduces to $(\vec{R'}, \vec{G'})$ at O' which forms a wrench, therefore $\vec{G'}$ is parallel to $\vec{R'}$.

But $\quad LX + MY + NZ = 0$

$$\Rightarrow \quad \vec{G} \,.\, \vec{R} = 0$$

$$\Rightarrow \quad \vec{G'} \,.\, \vec{R'} = 0 \qquad (\because \vec{G} \,.\, \vec{R} \text{ is invariant})$$

Since $\vec{G'}$ is parallel to $\vec{R'}$; then $\vec{G'} \,.\, \vec{R'} = 0$ will be possible if $\vec{G} = O$, because $\vec{R} \neq \vec{O}$. Hence the system reduces to only $\vec{R'}$ at O' which is a single resultant force.

• 9.8. EQUATION OF CENTRAL AXIS

To find the equations of the central axis of the any given system of forces acting at different points on a rigid body.

Central axis. *A straight line which is the locus of the points referred to which as base point the system of forces reduces to a wrench, is called the central axis of the system of forces acting on a rigid body.*

Let O be the origin (base point) and OX, OY and OZ be three rectangular axes. Under this co-ordinate system, suppose a system of forces acting on a rigid body reduces to a single force $\vec{R} = (X, Y, Z)$ acting at O and a couple $\vec{G} = (L, M, N)$ about an axis passing through O.

Let $P(\alpha, \beta, \gamma)$ be any point on the central axis and $\vec{r}$ be its position vector with respect to O, then

$$\overrightarrow{OP} = \vec{r} = \alpha\,\hat{i} + \beta\,\hat{j} + \gamma\,\hat{k}.$$

Since P is on the central axis, so that the given system reduces to a wrench $(\vec{R}', \vec{G}')$ at P, then we have

$$\vec{R}' = \vec{R} \qquad ...(1)$$

and
$$\vec{G}' = \vec{G} - \vec{r} \times \vec{R} \qquad ...(2)$$

But $(\vec{R}', \vec{G}')$ is a wrench, so that

$$\vec{G}' = p\,\vec{R}', \; p \text{ being the pitch of wrench}$$

$$\Rightarrow \quad \vec{G} - \vec{r} \times \vec{R} = p\vec{R}' = p\vec{R} \qquad \text{[using (1) and (2)]}$$

$$\Rightarrow \quad (L\,\hat{i} + M\,\hat{j} + N\,\hat{k}) - (\alpha\,\hat{i} + \beta\,\hat{j} + \gamma\,\hat{k}) \times (X\,\hat{i} + Y\,\hat{j} + Z\,\hat{k}) = p\,(X\,\hat{i} + Y\,\hat{j} + Z\,\hat{k})$$

$$\Rightarrow \quad (L\,\hat{i} + M\,\hat{j} + N\,\hat{k}) - [\hat{i}\,(\beta Z - \gamma Y) + \hat{j}\,(\gamma X - \alpha Z) + \hat{k}\,(\alpha Y - \beta X)] = pX\,\hat{i} + pY\,\hat{j} + pZ\,\hat{k}$$

$$\Rightarrow \quad L - \beta Z + \gamma Z = pX,\; M - \gamma X + \alpha Z = pY,\; N - \alpha Y + \beta X = pZ$$

$$\Rightarrow \quad \frac{L - \beta Z + \gamma Y}{X} = \frac{M - \gamma X + \alpha Z}{Y} = \frac{N - \alpha Y + \beta X}{Z} = p.$$

Thus the locus of (α, β, γ) is

$$\frac{L - yZ + zY}{X} = \frac{M - zX + xZ}{Y} = \frac{N - xY + yX}{Z} = p = \frac{K}{R}. \qquad ...(4)$$

This is the required equation of the central axis.

Here the degree of x, y and z are all one, so that this line represents three planes whose ntersection is the above line. Hence the intersection of any two of these planes gives the equation of the central axis.

9.9. PROCEDURE FOR FINDING X, Y, Z; L, M, N

Suppose a system of forces of magnitudes $F_1, F_2, \ldots, F_n$ acting at different points of a rigid body. Let O be a base point and OX, OY and OZ be three mutually perpendicular axes.

Suppose F_1 is acting at a point (x_1, y_1, z_1) along a line

$$\frac{x - x_1}{l_1} = \frac{y - y_1}{m_1} = \frac{z - z_1}{n_1}$$

where l_1, m_1, n_1 are the direction cosines of a line. Then the components of F_1 along OX, OY and OZ can be determine as follows :

$X_1 = F_1 l_1,\; Y_1 = F_1 m_1,\; Z_1 = F_1 n_1.$

Similarly, for other forces we can find X_2, Y_2, Z_2 etc. Therefore, we find :

$$X = \sum X_1,\; Y = \sum Y_1 \text{ and } Z = \sum Z_1.$$

Also, we can find L, M and N as follows :

First we write the co-ordinates of points at which the forces are acting in the first row and then write the components of forces in the second row as shown below :

(i)
$$\begin{array}{cccccc} x_1 & y_1 & z_1; & x_2 & y_2 & z_2; \ldots \text{ etc.} \\ X_1 & Y_1 & Z_1; & X_2 & Y_2 & Z_2; \ldots \text{ etc.} \end{array}$$

Here we calculate L as follows :

$$L = (y_1 Z_1 - z_1 Y_1) + (y_2 Z_2 - z_2 Y_2) + \ldots$$

(ii)
$$\begin{array}{cccccc} x_1 & y_1 & z_1; & x_2 & y_2 & z_2; \ldots \text{ etc.} \\ X_1 & Y_1 & Z_1; & X_2 & Y_2 & Z_2; \ldots \text{ etc.} \end{array}$$

Here we calculate M as follows :

$$M = (z_1X_1 - x_1Z_1) + (z_2X_2 - x_2Z_2) + \dots$$

(iii)
$$\begin{array}{cccccccc} x_1 & y_1 & z_1; & x_2 & & y_2 & z_2; \dots \text{etc.} \\ X_1 & Y_1 & Z_1; & X_2 & & Y_2 & Z_2; \dots \text{etc.} \end{array}$$

Here we calculate N as follows :

$N = (x_1Y_1 - y_1X_1) + (x_2Y_2 - y_2X_2) + \dots$.

Remark

➤ If l_1, m_1, n_1 are not the direction cosines, then first make them direction cosines as follows :

$$\frac{l_1}{\sqrt{l_1^2 + m_1^2 + n_1^2}}, \frac{m_1}{\sqrt{l_1^2 + m_1^2 + n_1^2}}, \frac{n_1}{\sqrt{l_1^2 + m_1^2 + n_1^2}}.$$

SOLVED EXAMPLES

Example 1. *Three forces, each equal to P, act on a rigid body; one at point* $(a, 0, 0)$ *parallel to OY, the second at the point* $(0, b, 0)$ *parallel to OZ and the third at the point* $(0, 0, c)$ *parallel to OX axis, the axes being rectangular, find the resultant wrench in magnitude and position.*

Solution. First force P is acting at $(a, 0, 0)$ along a line paralle to OY axis, so that the direction cosines of the line are 0, 1, 0.

Then $\quad X_1 = 0 \,.\, P = 0,\ Y_1 = 1 \,.\, P,\ Z_1 = 0 \,.\, P = 0.$

Second force P is acting at $(0, b, 0)$ along a line parallel to OZ axis whose d.c.'s are 0, 0, 1 so that

$$X_2 = 0 \,.\, P = 0,\ Y_2 = 0 \,.\, P = 0,\ Z_2 = 1 \,.\, P = P.$$

And the third force P is acting at $(0, 0, c)$ along a line parallel to OX axis whose d.c.'s are 1, 0, 0, so that

$$X_3 = 1 \,.\, P = P,\ Y_3 = 0 \,.\, P = 0,\ Z_3 = 0 \,.\, P = 0.$$

$\therefore \quad X = X_1 + X_2 + X_3 = 0 + 0 + P = P$

$\quad Y = Y_1 + Y_2 + Y_3 = P + 0 + 0 = P$

and $\quad Z = Z_1 + Z_2 + Z_3 = 0 + P + 0 = P.$

Now we shall calculate L, M, N as follows :

Points at which the forces : $a,\ 0,\ 0; \quad 0,\ b,\ 0; \quad 0,\ 0,\ c$ are acting

Components of forces: $0,\ P,\ 0; \quad 0,\ 0,\ P; \quad P,\ 0,\ 0$

$\therefore \quad L = \sum (y_1Z_1 - z_1Y_1) = (0 - 0) + (bP - 0) + (0 - 0)$

$\quad = bP$

$\quad M = \sum (z_1X_1 - x_1Z_1) = (0 - 0) + (0 - 0) + (cP - 0)$

and $\quad N = \sum (x_1Y_1 - y_1X_1) = (aP - 0) + (0 - 0) + (0 - 0)$

$\quad = aP.$

If R be the force and K the couple of wrench, then

$$R = \sqrt{X^2 + Y^2 + Z^2} = \sqrt{P^2 + P^2 + P^2} = \sqrt{3} \,.\, P$$

and
$$K = \frac{LX + MY + NZ}{R}$$

$$= \frac{bP \,.\, P + cP \,.\, P + aP \,.\, P}{\sqrt{3} \,.\, P}$$

$$= \frac{P}{\sqrt{3}}(a + b + c).$$

Now the equation of central axis is

$$\frac{L - yZ + zY}{X} = \frac{M - zX + xZ}{Y} = \frac{N - xY + yX}{Z} = p$$

$$\Rightarrow \quad \frac{bP - yP + zP}{P} = \frac{cP - zP + xP}{P} = \frac{aP - xP + yP}{P} = p$$

$$\Rightarrow \quad b - y + z = c - z + x = a - x + y = p.$$

Since $\quad p = \frac{k}{R} = \frac{p}{\sqrt{3}}(a+b+c) \cdot \frac{1}{\sqrt{3}\,p}$

$$= \frac{1}{3}(a+b+c).$$

$$\therefore \quad b - y + z = c - z + x = a - x + y = \frac{1}{3}(a+b+c)$$

$$\Rightarrow \quad x + \frac{a+2b+3c}{3} = y + \frac{b+2c+3a}{3} = z + \frac{c+2a+3b}{3}$$

Thus the central axis is a straight line passing through the point

$$\left(-\frac{a+2b+3c}{3}, -\frac{b+2c+3a}{3}, -\frac{c+2a+3b}{3}\right)$$

and inclined at equal angles to the co-ordinate axes.

Example 2. *A force P acts along the axis of x and another force nP along a generator of the cylinder $x^2 + y^2 = a^2$. Show that the central axis lies on the cylinder*

$$n^2 (nx - z)^2 + (1+n)^2 y^2 = n^4 a^2.$$

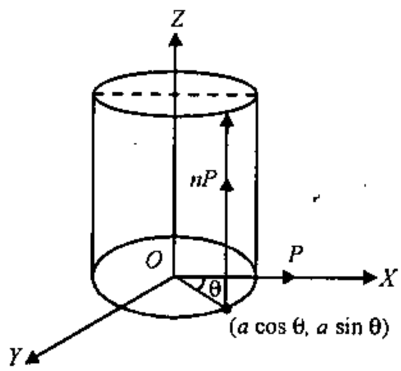


Fig. 7

Solution. Since the force P is acting along the x-axis whose equation is

$$\frac{x}{1} = \frac{y}{0} = \frac{x}{0}.$$

Thus P acting at (0, 0, 0) along a line whose d.c.'s are 1, 0, 0.

The axis of the cylinder $x^2 + y^2 = a^2$ is the axis of z, so that the generator of this cylinder is parallel to z-axis. Let $(a \cos\theta, a \sin\theta, 0)$ be any point on the cylinder. Thus the force nP is acting at the point $(a\cos\theta, a\sin\theta, 0)$ along a line whose d.c.'s are 0, 0, 1.

The components of P along OX, OY and OZ axes are respectively

$$X_1 = P \cdot 1 = P,\ Y_1 = P \cdot 0 = 0,\ Z_1 = P \cdot 0 = 0.$$

The components of nP along OX, OY and OZ axes are respectively

$$X_2 = nP \cdot 0 = 0,\ Y_2 = nP \cdot 0 = 0,\ Z_2 = nP \cdot 1 = nP.$$

Therefore, we get

$$X = X_1 + X_2 = P + 0 = P$$
$$Y = Y_1 + Y_2 = 0 + 0 = 0$$
$$Z = Z_1 + Z_2 = 0 + nP = nP.$$

Now we calculate L, M, N as follows :

Points of application	:	0	0	0 ;	$a\cos\theta$	$a\sin\theta$	0
Components of forces	:	P	0	0 ;	0	0	nP

Thus, $\quad L = (y_1 Z_1 - z_1 Y_1) + (y_2 Z_2 - z_2 Y_2)$

$$= (0 - 0) + (anP \sin\theta - 0)$$
$$= anP \sin\theta$$

$$M = (z_1 X_1 - x_1 Z_1) + (z_2 X_2 - x_2 Z_2)$$
$$= (0 - 0) + (0 - anP\cos\theta)$$
$$= -anP\cos\theta$$

and $\quad N = (x_1 Y_1 - y_1 X_1) + (x_2 Y_2 - y_2 X_2)$

$$= (0 - 0) + (0 - 0)$$
$$= 0.$$

The equation of the central axis is

$$\frac{L - yZ + zY}{X} = \frac{M - zX + xZ}{Y} = \frac{N - xY + yX}{Z}$$

$$\frac{anP\sin\theta - ynP}{P} = \frac{-anP\cos\theta - zP + xnP}{0} = \frac{0 - 0 + yP}{nP}$$

$$\frac{anP\sin\theta - nyP}{P} = \frac{y}{n}$$

$$n\,(an\sin\theta - ny) = y$$

$$an^2\sin\theta = y\,(1 + n^2)$$

$$\sin\theta = \frac{y\,(1 + n^2)}{an^2}$$

and
$$-anP\cos\theta - zP + xnP = 0$$

$$an\cos\theta = (xn - z)$$

$$\cos\theta = \frac{xn - z}{an} \qquad ...(2)$$

Squaring (1) and (2) and adding, we get

$$\sin^2\theta + \cos^2\theta = \frac{y^2\,(1+n^2)^2}{a^4n^2} + \frac{(xn - z)^2}{a^2n^2}$$

$$\Rightarrow \qquad 1 = \frac{y^2\,(1+n^2) + (xn - z)^2}{n^4a^2}$$

$$\Rightarrow \qquad y^2\,(1+n^2)^2 + n^2\,(xn - z)^2 = n^4a^2.$$

This is the required surface.

• SUMMARY

- The resultant of forces $\vec{F_1}, \vec{F_2}, \ldots\ldots, \vec{F_n}$ acting on a rigid body is given by

$$\vec{R} = \vec{F_1} + \vec{F_2} + \ldots\ldots + \vec{F_n} = \sum_{i=1}^{n} \vec{F_i}.$$

- If the forces $\vec{F_1}, \vec{F_2}, \ldots\ldots, \vec{F_n}$ acting at points wih position vectors $\vec{r_1}, \vec{r_2}, \ldots\ldots, \vec{r_n}$ on a rigid body, then the resultant moment $\vec{G}$ about $\vec{O}$ (origin) is given by

$$\vec{G} = \sum_{i=1}^{n} \vec{r_i} \times \vec{F_i}.$$

- If $\vec{R}$ and $\vec{G}$ are parallel, then $\vec{R}$ and $\vec{G}$ form a Wrench.
- Equation of cenral axis is

$$\frac{L - yZ + zY}{X} = \frac{M - zX + xZ}{Y} = \frac{N - xY + yX}{Z} = \frac{k}{R}.$$

• STUDENT ACTIVITY

1. The necessary and sufficient conditions of the particle under the action of a system of forces to be in equilibrium are that the algebraic sums of the resultant parts of the forces along any three mutually perpendicular directions vanish separately.

2. A force P acts along the axis of x and another force nP along a generator of the cylinder $x^2 + y^2 = a^2$. Show that the central axis lies on the cylinder

$$n^2 (nx - z)^2 + (1 + n)^2 y^2 = n^4 a^2.$$

TEST YOURSELF

1. Equal forces act along two perpendicular diagonals of opposite faces of a cube of side a. Show that they are equivalent to a single force R acting along a line through the centre of the cube, and a couple $\frac{1}{2} aR$ with the same line for axis.
2. Forces P, Q, R act along three non-intersecting edges of a cube. Find the central axis.
3. Equal forces act along the axes and along the straight line

$$\frac{x - \alpha}{l} = \frac{y - \beta}{m} = \frac{z - \gamma}{n}$$

find the equations of the central axis of the system.

4. Two forces P and Q act along the straight lines whose equations are $y = x \tan \alpha$, $z = c$ and $y = -x \tan \alpha$, $z = -c$ respectively. Show that their central axis lies on the straight line

$$y = x \cdot \frac{P - Q}{P + Q} \tan \alpha \quad \text{and} \quad \frac{z}{c} = \frac{P^2 - Q^2}{P^2 - 2PQ \cos 2\alpha + Q^2}$$

For all values of P and Q, prove that this line is a generator of the surface

$$(x^2 + y^2) z \sin 2\alpha = 2cxy.$$

ANSWERS

1. With respect to three coterminous edges as co-ordinate axes, the central axis is
$$\frac{zQ - yR - aQ}{P} = \frac{xR - zP - aR}{Q} = \frac{yP - xQ - aP}{R}$$
3. $$\frac{z(1 + m) - y(1 + n) + (\beta n - \gamma m)}{(1 + l)} = \frac{x(1 + n) - z(1 + l) + (\gamma l - \alpha n)}{(1 + m)} = \frac{y(1 + l) - x(1 + m) + (\alpha m - \beta l)}{(1 + n)}.$$

❑❑❑

UNIT 10

STABLE AND UNSTABLE EQUILIBRIUM

STRUCTURE

- Definitions
- Nature of Equilibrium Using Z-test
- Nature of Equilibrium of a body when resting on a fixed rough surface
  - Summary
  - Student Activity
  - Test Yourself

LEARNING OBJECTIVES

After going through this unit you will learn :
- What are different types of equilibrium ?
- How to find the condition that the given body is either in stable or unstable equilibrium.

10.1. DEFINITIONS

We thus now define all of three types of equilibriums :

(i) Stable equilibrium. *A body is said to be in* **stable equilibrium,** *if it is slightly displaced from its position of equilibrium, the forces acting on the body tend to move it back to its original position.*

(ii) Unstable equilibrium. *A body is said to be in* **unstable equilibrium,** *if it is slightly displaced from its original position, the forces acting on the body tend to move it still away from its position of equilibrium.*

(iii) Neutral equilibrium. *A body is said to be in a* **neutral equilibrium,** *if it is displaced from its position of equilibrium, the forces acting on it are in equilibrium in any new position of the body.*

10.2. NATURE OF EQUILIBRIUM USING z-TEST

Suppose a body or a system of bodies are in equilibrium under the influences of their weights only and supported by reactions with smooth fixed surfaces which do not present in the equation of virtual work.

If $w_1, w_2, \ldots$ be the weights of the different bodies and $z_1, z_2, \ldots$ the heights of their centre of gravity above some fixed plane, then the equation of virutal work is

$$-w_1\,\delta z_1 - w_2\,\delta z_2 \ldots = 0 \qquad \ldots(1)$$

If W be the total weight of the system and z be the height of its centre of gravity, then the equation (1) becomes :

$$-W\,\delta z = 0$$

$$\Rightarrow \qquad \delta z = 0$$

$$\Rightarrow \qquad \frac{dz}{d\theta}\,\delta\theta = 0$$

$$\Rightarrow \qquad \frac{dz}{d\theta} = 0.$$

Hence the necessary condition of a body to either be in stable or unstable equilibrium is that $\frac{dz}{d\theta} = 0.$

On solving $\frac{dz}{d\theta} = 0$ we are supposed to get $\theta = \alpha, \beta$ etc. which give the position of equilibruim.

Nature of equilibrium at $\theta = \alpha$:

Case I. Suppose $\frac{d^2z}{d\theta^2}$ is positive at $\theta = \alpha$, then z is minimum at $\theta = \alpha$, therefore, the height of the centre of gravity is minimum, so that for a small displacement, the height of the centre of gravity is incraesed and then on being set free the body will tend to come back to its original position of equilibrium. Hence in this case the body is in *stable* equilibrium.

Case II. Suppose $\frac{d^2z}{d\theta^2}$ is negative at $\theta = \alpha$, then z is maximum, therefore, the centre of gravity of the body will be lowered, during a small displacement and on being set free, the force of gravity will tend to keep the body away from its position of equilibrium.

Hence the body in this case is in *unstable* equilibrium.

Consequently, If $\frac{dz}{d\theta} = 0$ *gives the position of equilibrium, then the body will be stable or unstable at* $\theta = \alpha$ *according as* z *is minimum or maximum at* $\theta = \alpha$.

Remark

- If z be the depth of the centre of gravity of the combined body, then the body will be in unstable equilibrium if z is minimum and the body will be in stable equilibrium if z is maximum.

• 10.3. NATURE OF EQUILIBRIUM OF A BODY WHEN RESTING ON A FIXED ROUGH SURFACE

Theorem. *A body rests in equilibrium upon another, fixed body, the positions of the two bodies in contact have radii of curvatures* ρ_1 *and* ρ_2 *respectively, and the straight line joining their centres of gravity being vertical; if the first body being slightly displaced whose centre of gravity is at a height* h *above the point of contact, then the equilibrium is stable or unstable according as*

$$\frac{1}{h} > or \leq \frac{1}{\rho_1} + \frac{1}{\rho_2}. \text{ (without proof).}$$

Remarks :

- If both the body are spheres, then we will take ρ_1 and ρ_2 as their radii.
- If the lower body is a fixed plane, then we shall take $\rho_2 = \infty$.
- If the surface of contact of upper body is a plane, then we shall take $\rho_1 = \infty$.
- If the lower body at the point of contact is concave instead of convex, then ρ_2 is to be taken negative.

SOLVED EXAMPLES

Example 1. *A body consisting of a cone and a hemisphere on the same base rests on a rough horizontal table, the hemisphere being in contact with the table; show that the greatest height of the cone so that the equilibrium may be stable, is* $\sqrt{3}$ *times the radius of the hemisphere.*

Solution. Let G be the centre of gravity of the combined bodies and G_1 be the C.G. of the cone and G_2 the C.G. of hemisphere.

Let AB be the common base of hemisphere and the cone and COV the common axis which will be vertical in a position of equilibrium and C the point of contact of the hemisphere to the horizontal plane.

Let H be the height OV of the cone and r the radius OA (or OC) of the hemisphere. Then

$$OG_1 = \frac{H}{4}, \quad OG_2 = \frac{3r}{8}.$$

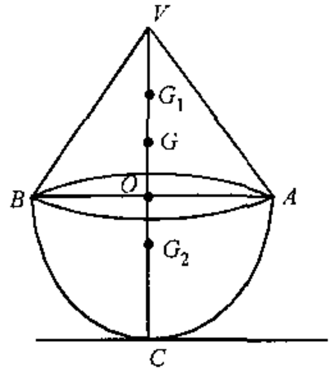


Fig. 6

Also h be the height of C.G. of combined body consisting of cone and a hemisphere above the point of contact C. Then

$$h = \frac{W_1 x_1 + W_2 x_2}{W_1 + W_2}.$$

Here $\quad W_1$ = weight of the cone

$$= \frac{1}{3}\pi r^2 H w, \text{ } w \text{ being the weight per unit volume}$$

and $\quad W_2$ = weight of hemisphere

$$= \frac{2}{3}\pi r^3 w$$

and

$$x_1 = OG_1 = \left(r + \frac{H}{4}\right)$$

$$x_2 = CG_2 = \frac{5}{8}r$$

$$\therefore \quad h = \frac{\frac{1}{3}\pi r^2 H w \cdot \left(r + \frac{H}{4}\right) + \frac{2}{3}\pi r^3 w \cdot \left(\frac{5}{8}r\right)}{\frac{1}{3}\pi r^2 H w + \frac{2}{3}\pi r^3 w}$$

$$= \frac{H\left(r + \frac{H}{4}\right) + \frac{5}{4}r^2}{(H + 2r)}$$

Now $\quad \rho_1$ = radius of curvature of the upper body at C which is hemisphere $= r$

and $\quad \rho_2$ = radius of curvature of the lower body at C which is a horizontal plane $= \infty$

$\therefore$ The equilibrium is stable if

$$\frac{1}{h} > \frac{1}{\rho_1} + \frac{1}{\rho_2}$$

$$\Rightarrow \quad \frac{1}{h} > \frac{1}{r} + \frac{1}{\infty}$$

$$\Rightarrow \quad \frac{1}{h} > \frac{1}{r}$$

$$\Rightarrow \quad h < r$$

$$\Rightarrow \quad \frac{H\left(r + \frac{H}{4}\right) + \frac{5}{4}r^2}{(H + 2r)} < r$$

$$\Rightarrow \quad Hr + \frac{H^2}{4} + \frac{5}{4}r^2 < r(H + 2r)$$

$$\Rightarrow \quad Hr + \frac{H^2}{4} + \frac{5}{4}r^2 < Hr + 2r^2$$

$$\Rightarrow \quad \frac{H^2}{4} < \frac{3}{4}r^2$$

$$\Rightarrow \quad H^2 < 3r^2$$

$$\Rightarrow \quad H < \sqrt{3}\,r.$$

Hence in the position of stable equilibrium the greatest height of the cone is $\sqrt{3}$ times the radius of hemisphere.

Example 2. *A hemisphere rests in equilibrium on a sphere of equal radius; show that the equilibrium is unstable when the curved, and stable when the flat surface of the hemisphere rets on sphere.*

Solution. (i) Let us consider the case when the curved surface rests on the sphere.

Let O and O' be the centres of sphere and hemisphere of same radius r (say) and C be the point of contact.

Since the C.G. of the sphere lies on the centre, so that C.G. of the lower body (sphere) is at O and let G be the C.G. of upper body.

In the position of equilibrium OCO' must be vertical.

Now ρ_1 = the radius of curvature at C of the upper body $= r$

and ρ_2 = the radius of curvature at C of lower body $= r$

$$\therefore \quad \frac{1}{\rho_1}+\frac{1}{\rho_2}=\frac{1}{r}+\frac{1}{r}=\frac{2}{r}$$

Also
$$h = CG$$
$$= CO' - O'G = r - \frac{3}{8}r \quad \left(\because \; O'G = \frac{3}{8}r\right)$$
$$= \frac{5}{8}r$$

$$\therefore \quad \frac{1}{h}=\frac{8}{5}r.$$

Obviously,
$$\frac{1}{h} < \frac{1}{\rho_1}+\frac{1}{\rho_2}.$$

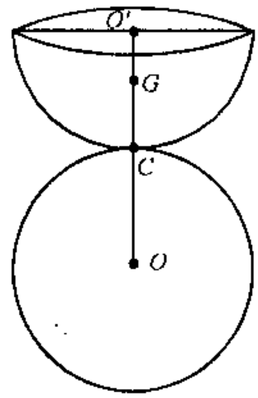


Fig. 7

Hence, in this case the equilibrium is unstable.

(ii) Now consider the case when the hemisphere rests on the sphere with flat surface in contact.

In this case the centre O' of the hemisphere is the contact point to the sphere. Therefore,

ρ_1 = the radius of curvature at O' of the upper body which is hemisphere whose flat part is in contact. $= \infty$

and ρ_2 = the radius of curvature at O' of the lower body which is a sphere $= r$

$$\therefore \quad \frac{1}{\rho_1}+\frac{1}{\rho_2}=\frac{1}{\infty}+\frac{1}{r}=\frac{1}{r}$$

Also
$$h = O'G = \frac{3}{8}r$$

$$\therefore \quad \frac{1}{h}=\frac{8}{3}r.$$

Obviously,
$$\frac{1}{h} > \frac{1}{\rho_1}+\frac{1}{\rho_2}.$$

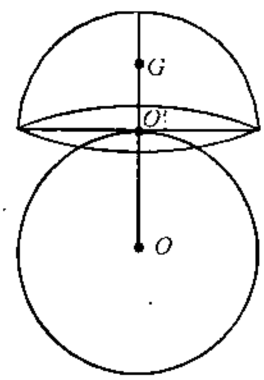


Fig. 8

Hence in this case the equilibrium is stable.

Example 3. *A uniform cubical box of edge a is placed on the top of a fixed sphere, the centre of the face of the cube being in contact with the highest point of the sphere. What is the least radius of the sphere for which the equilibrium will be stable ?*

Solution. Let O be the centre of the sphere over which a cubical box of edge a is placed. Let C be the point of contact and G be the centre of gravity of the cubical box and r be the radius of the sphere.

Therefore,
$$h = CG = \frac{a}{2}.$$

Now, ρ_1 = the radius of curvature at C of the upper body $= \infty$

and ρ_2 = the radius of curvature of the lower body $= r$

$$\therefore \quad \frac{1}{\rho_1}+\frac{1}{\rho_2}=\frac{1}{\infty}+\frac{1}{r}=\frac{1}{r}.$$

Since
$$h = CG = \frac{a}{2}.$$

Therefore, for stable equilibrium, we have

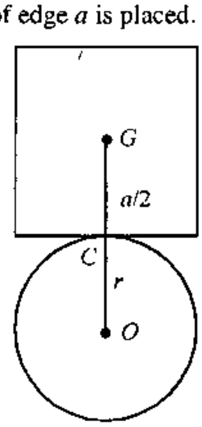


Fig. 9

$$\frac{1}{h} > \frac{1}{\rho_1} + \frac{1}{\rho_2}$$

$$\Rightarrow \qquad \frac{1}{a/2} > \frac{1}{r}$$

$$\Rightarrow \qquad r > \frac{a}{2}.$$

Hence, for the stable equilibrium, the least radius of the sphere must be $\frac{a}{2}$.

Example 4. *A uniform beam of length 2a rests with its ends on two smooth planes which intersect in a horizontal line. If the inclination of the planes to the horizontal are α and β with $\alpha > \beta$, show that the inclination θ of the beam with the horizontal in one of the equilibrium position is given by*

$$\tan \theta = \frac{1}{2} (\cot \beta - \cot \alpha)$$

and show that the beam is unstable in this position.

Solution. Let O be the intersection point of two inclined planes in the horizontal line and let AB be a uniform beam of length $2a$ rests on two inclined plane with A on one and B on other plane as shown in fig. 14.

Suppose the beam AB makes an angle θ with horizontal.

We have

$$\angle AOC = \beta, \ \angle BOD = \alpha.$$

Let G be the centre of gravity of the beam AB which is its middle point and let z be the height of G above the fixed horizontal plane COD.

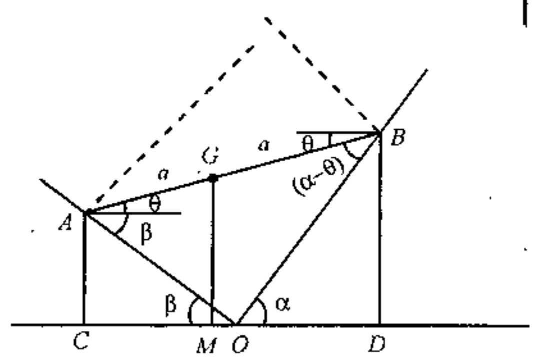


Fig. 14

$$\therefore \qquad z = GM = \frac{1}{2}(AC + BD)$$

$$= \frac{1}{2}(AO \sin \beta + OB \sin \alpha). \qquad \ldots(1)$$

Now in ΔAOB, we have

$$\frac{AB}{\sin\{\pi - (\alpha + \beta)\}} = \frac{AO}{\sin(\alpha - \beta)} = \frac{BO}{\sin(\beta + \theta)} \qquad \text{(By sine rule)}$$

$$\frac{2a}{\sin(\alpha + \beta)} = \frac{AO}{\sin(\alpha - \theta)} = \frac{BO}{\sin(\beta + \theta)}$$

$$\therefore \qquad AO = \frac{2a \sin(\alpha - \beta)}{\sin(\alpha + \beta)}, \ BO = \frac{2a \sin(\beta + \theta)}{\sin(\alpha + \beta)}.$$

Putting the values of AO and BO in (1), we get

$$z = \frac{1}{2}\left[\frac{2a \sin(\alpha - \theta)}{\sin(\alpha + \beta)} \sin \beta + \frac{2a \sin(\beta + \theta)}{\sin(\alpha + \beta)} \sin \alpha\right]$$

$$z = \frac{1}{\sin(\alpha + \beta)}[\sin(\alpha - \theta) \sin \beta + \sin(\beta + \theta) \sin \alpha].$$

Thus z is a function of θ.

$$\therefore \qquad \frac{dz}{d\theta} = \frac{a}{\sin(\alpha + \beta)}[-\cos(\alpha - \theta) \sin \beta + \cos(\beta + \theta) \sin \alpha] \qquad \ldots(2)$$

and

$$\frac{d^2z}{d\theta^2} = \frac{a}{\sin(\alpha + \beta)}[-\sin(\alpha - \theta) \sin \beta - \sin(\beta + \theta) \sin \alpha]. \qquad \ldots(3)$$

For the equilibrium position, we have

$$\frac{dz}{d\theta} = 0$$

$$\Rightarrow \qquad -\cos(\alpha - \theta) \sin \beta + \cos(\beta + \theta) \sin \alpha = 0$$

$\Rightarrow \quad \cos(\alpha - \theta)\sin\beta = \cos(\beta + \theta)\sin\alpha$

$\Rightarrow \quad (\cos\alpha\cos\theta + \sin\alpha\sin\theta)\sin\beta = (\cos\beta\cos\theta - \sin\beta\sin\theta)\sin\theta.$

Dividing by $\sin\alpha\sin\beta\sin\theta$,

$\Rightarrow \quad \cot\alpha\cot\theta + 1 = \cot\beta\cot\theta - 1$

$\Rightarrow \quad \cot\alpha + \tan\theta = \cot\beta - \tan\theta$

$\Rightarrow \quad 2\tan\theta = \cot\beta - \cot\alpha$

$$\Rightarrow \quad \tan\theta = \frac{1}{2}(\cot\beta - \cot\alpha). \qquad \ldots(4)$$

Equation (3) becomes :

$$\frac{d^2z}{d\theta^2} = \frac{a}{\sin(\alpha+\beta)}[-\sin\alpha\sin\beta\cos\theta + \cos\alpha\sin\beta\sin\theta$$
$$-\sin\alpha\sin\beta\cos\theta - \sin\alpha\cos\beta\sin\theta]$$
$$= \frac{a\sin\alpha\sin\beta\sin\theta}{\sin(\alpha+\beta)}[-2\cot\theta + \cot\theta - \cot\beta]$$
$$= \frac{-2a\sin\alpha\sin\beta\sin\theta}{\sin(\alpha+\beta)}\left[\frac{1}{2}(\cot\beta - \cot\alpha) + \cot\alpha\right]$$
$$= \frac{-2a\sin\alpha\sin\beta\sin\theta}{\sin(\alpha+\beta)}[\tan\theta + \cot\theta] \qquad \text{[using (4)]}$$
$$\frac{d^2z}{d\theta^2} = \frac{-2a\sin\alpha\sin\beta\cos\theta}{\sin(\alpha+\beta)}[1 + \tan^2\theta].$$

Since θ, α, β are all acute angles and $\alpha + \beta < \pi$, so that

$$\frac{d^2z}{d\theta^2} < 0.$$

$\therefore$ z is maximum when θ is governed by the relation

$$\tan\theta = \frac{1}{2}(\cot\beta - \cot\alpha)$$

Hence the beam is unstable if

$$\tan\theta = \frac{1}{2}(\cot\beta - \cot\alpha).$$

• SUMMARY

- **Nature of Equlibrium using z-test**

  (i) If z is the height of the centre of gravity of the combined body, then the body will be in stable or unstable equilibrium if z is minimum or maximum at $\theta = \alpha$, where $\left(\frac{dz}{d\theta}\right)_{\theta=\alpha} = 0$.

  (ii) If z is he depth of the centre of gravity of the combined body, then the body will be in stable or unstable equilibrium if z is maximum or minimum at $\theta = \alpha$, where $\left(\frac{dz}{d\theta}\right)_{\theta=\alpha} = 0$.

- If h be the height of C.G. of upper body (to be displaced) from the point of contant, and ρ_1 and ρ_2 be the radii of curvatures of above and lower bodies respectively, then body will be in stable or unstable equilibrum according as

$$\frac{1}{h} > \frac{1}{\rho_1} + \frac{1}{\rho_2} \quad \text{or} \quad \frac{1}{h} \leq \frac{1}{\rho_1} + \frac{1}{\rho_2}$$

• STUDENT ACTIVITY

1. A uniform cubical box of edge a is placed on the top of a fixed sphee, the centre of face o the cube being in contact with the highest point of the sphere. What is the least radius of the sphere for which the equilibrium will be stable ?

2. A uniform beam of length $2a$ rest with its ends on two smooth planes which intersect in a horizontal line. If the inclination of the planes to the horizontal are α and β with $\alpha > \beta$, show that the inclination θ of the beam with the horizontal in one of the equilibrium position is given by

$$\tan\theta = \frac{1}{2}(\cot\beta - \cot\alpha)$$

and show that the beam is unstable in this position.

• TEST YOURSELF

1. A solid sphere rest inside a fixed rough hemispherical bowl of twice its rdius. show that however large a weight is attached to the highest point of the sphere, the equilibrium is stable.
2. A heavy uniform rod rests with one end against a smooth vertical wall and with a point in its length resting on a smooth peg. Find the position of equilibrium and show that it is in unstable equilibrium.

ANSWERS

2. $\theta = \sin^{-1}\left(\frac{1}{a}\right)^{1/3}$, $2a$ = length of rod. b = distance of peg from vertical wall.

UNIT

11

KINEMATICS IN TWO DIMENSIONS

STRUCTURE

- Motion in a Straight Line
- Motion in a plane
- Angular Velocity and Acceleration
- Rate of change of a unit vector
- Relation between linear and angular velocities
- Radial and transverse velocities
- Radial and transverse accelerations
- Tangential and Normal Velocities
- Tangential and Normal accelerations
  - Summary
  - Student Activity
  - Test Yourself

LEARNING OBJECTIVES

After going through this unit you will learn :

- What is the motion of a particle in a straight line and in a plane ?

• 11.1. MOTION IN A STRAIGHT LINE : VELOCITY AND ACCELERATION

1. Velocity. Let a particle move along a straight line and the positions of the particle are determined from a fixed point on the line. Let this point be O and at any instant 't', the particle is at a point P, whose distance from O is x, *i.e.*, $OP = x$.

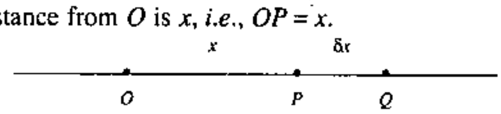


Fig. 1

Now at subsequent interval of time δt, the particle reaches to a point Q, whose distance from O is $x + \delta x$, *i.e.*, $OQ = x + \delta x$. Therefore, $PQ = \delta x$. Thus $PQ/\delta t = \delta x/\delta t$ *is known as the average velocity* of the particle during the time interval δt. As δt becomes smaller and smaller so that δx becomes smaller and smaller, the point $Q \to P$, then $\delta x/\delta t$ gives the rate of displacement of the particle, and thus $\delta x/\delta t$ gives the velocity v of the particle in the limit when $\delta t \to 0$. That is,

$$v = \lim_{\delta t \to 0} \frac{\delta x}{\delta t} = \frac{dx}{dt} = \dot{x}.$$

2. Acceleration. It is defined as the *rate of change of the velocity.*

Let v be the velocity of a moving particle at any time t and $v + \delta v$ be its velocity at time $t + \delta t$, then δv is the change in velocity in the interval δt. Thus the acceleration of the particle is given by

$$a = \lim_{\delta t \to 0} \frac{\delta v}{\delta t} = \frac{dv}{dt}.$$

Since, $$v = \frac{dx}{dt}$$

$\therefore$ $$a = \frac{d^2x}{dt^2} = \ddot{x}\ .$$

Remarks

- Velocity is a vector quantity, whose magnitude gives the speed.
- Acceleration is also a vector quantity.
- Negative acceleration is known as Retardation.

• 11.2. MOTION IN A PLANE : VELOCITY AND ACCELERATION

1. Velocity. When a particle moves in a plane, it traces a curve. Let O be the fixed point and OX, OY be the two fixed lines which are perpendicular and let A be another fixed point on the curve and P be the position of the particle at any time t on the curve. Let $AP = s$.

Let Q be the position of the particle on the curve at the time $t + \delta t$. Therefore, during δt, the displacement of the particle is the **chord PQ,** which is shown below :

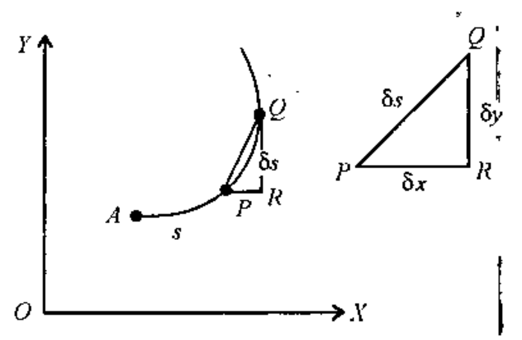


Fig. 2

Hence the velocity of the particle at the time t is given by

$$v = \lim_{\delta t \to 0} \frac{\text{chord } PQ}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{\text{chord } PQ}{\text{Arc } PQ} \cdot \frac{\text{Arc } PQ}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{\text{chord } PQ}{\delta s} \cdot \frac{\delta s}{\delta t} \qquad (\because \ \text{Arc } PQ = \delta s)$$

As $\delta t \to 0$ so that $\delta s \to 0$, then we have

$$v = \lim_{\delta s \to 0} \frac{\text{chord } PQ}{\delta s} \cdot \lim_{\delta t \to 0} \frac{\delta s}{\delta t}$$

$$= 1 \cdot \frac{ds}{dt} \qquad \left[\because \ \lim_{\delta s \to 0} \frac{\text{chord } PQ}{\delta s} = 1\right]$$

$$= \frac{ds}{dt} = \dot{s}.$$

Remark

- As $Q \to P$, then the chord PQ becomes the tangent at P and hence the direction of the velocity at P is along the tangent at P to the curve.

Components of the Velocity :

Let the co-ordinates of the points P and Q be respectively (x, y) and $(x + \delta x, y + \delta y)$. Thus the component of the displacement PQ are respectively $PR = \delta x$ parallel to OX and $QR = \delta y$ parallel to OY.

The component of the velocity parallel to OX is given by

$$v_x = \lim_{\delta t \to 0} \frac{PR}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{\delta x}{\delta t} = \frac{dx}{dt}$$

$$= \dot{x}$$

and the component of the velocity parallel to OY is,

$$v_y = \lim_{\delta t \to 0} \frac{QR}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{\delta y}{\delta t} = \frac{dy}{dt}$$

$$= \dot{y}$$

If v be the magnitude of the velocity moving in a plane, then

$$v^2 = v_x^2 + v_y^2$$

or $$v^2 = \dot{x}^2 + \dot{y}^2.$$

If ψ be the angle which gives the direction of motion, makes with x-axis *i.e.*, OX, then

$$\tan\psi = \frac{\dot{y}}{\dot{x}} = \frac{dy/dt}{dx/dt} = \frac{dy}{dx}.$$

2. Acceleration. *The rate of change of velocity is the acceleration.*

Let a be the acceleration of a moving particle in a plane, then we have

$$a = \frac{dv}{dt}$$

$$= \frac{d}{dt}\left(\frac{ds}{dt}\right) \qquad \left(\because \; v = \frac{ds}{dt}\right)$$

$$= \frac{d^2s}{dt^2}.$$

• 11.3. ANGULAR VELOCITY AND ACCELERATION

1. Angular velocity. *The rate of change of angular displacement is known as* **angular velocity.**

A particle is moving in a plane. Taking a fixed line OX as initial line with O as pole. Let P and Q be the positions of a moving particle at any time t and $t + \delta t$ respectively as shown in fig. 3.

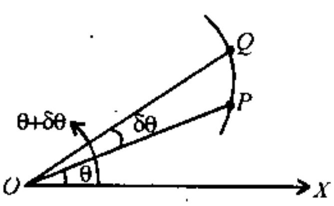


Fig. 3

And corresponding to P and Q, the angles $\angle POX = \theta$ and $\angle QOX = \theta + \delta\theta$ respectively. Therefore, the angular displacement of a moving particle during the interval δt is $\delta\theta$ and thus the average angular velocity of P about O is $\frac{\delta\theta}{\delta t}$.

As $\delta t \to 0$, $Q \to P$, then the **angular velocity** of the point P about O is

$$\lim_{\delta t \to 0} \frac{\delta\theta}{\delta t} = \frac{d\theta}{dt}$$

$$= \dot{\theta}.$$

Since $\dot{\theta}$ has direction as well as magnitude so that it is a vector quantity, which is perpendicular to the plane OPQ and the magnitude of this angular velocity vector is represented by ω. That is,

$$\omega = \frac{d\theta}{dt}.$$

2. Angular acceleration. *The rate of change of angular velocity is known as* **angular acceleration.**

Therefore, the angular acceleration is given by

$$= \frac{d}{dt}\left(\frac{d\theta}{dt}\right) = \frac{d^2\theta}{dt^2}$$

$$= \ddot{\theta}.$$

• 11.4. RATE OF CHANGE OF A UNIT VECTOR

Let $\overline{a}$ and $\overline{b}$ be two unit vectors lying in a plane, and let $\hat{i}$ and $\hat{j}$ be the unit vectors along X and Y axis respectively.

Let us suppose vector $\overline{a}$ makes an angle θ with the positive X-axis and the unit vector $\overline{b}$ is taken to be perpendicular to the unit vector $\overline{a}$, as shown in fig. 4.

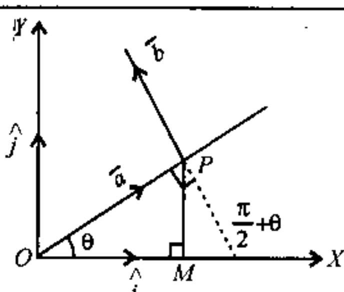


Fig. 4

In the fig. 4. Let $\overrightarrow{OP} = \overline{a}$, such that $OP = 1$ and $\angle POX = \theta$.

$\therefore$ In ΔOPM, we have

$$OM = OP\cos\theta = \cos\theta$$

and $\qquad MP = OP \sin\theta = \sin\theta$

$\therefore \qquad \overrightarrow{OM} = (\cos\theta)\,\hat{i}$ and $\overrightarrow{MP} = (\sin\theta)\,\hat{j}$

Then $\qquad \bar{a} = \overrightarrow{OM} + \overrightarrow{MP}$

$\therefore \qquad \bar{a} = (\cos\theta)\,\hat{i} + (\sin\theta)\,\hat{j}.$ ...(1)

Thus the unit vector $\bar{a}$ is obtained a function of θ, where θ is a function of t.
Similarly, the unit vector $\bar{b}$ is given by

$$\bar{b} = \cos\left(\theta + \frac{\pi}{2}\right)\hat{i} + \sin\left(\theta + \frac{\pi}{2}\right)\hat{j}$$

or $\qquad \bar{b} = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$

Differentiating (1) w.r.t. 't', we get

$$\frac{d\bar{a}}{dt} = (-\sin\theta\,\hat{i} + \cos\theta\,\hat{j})\frac{d\theta}{dt} \qquad \left(\because \frac{d\hat{i}}{dt} = \bar{0}, \frac{d\hat{j}}{dt} = \bar{0}\right)$$

$\therefore \qquad \dfrac{d\bar{a}}{dt} = \dfrac{d\theta}{dt}\bar{b}$ [using (2)]

Remark

➤ The unit vector $\bar{b}$ is perpendicular to $\bar{a}$ in the direction of θ increasing.

• 11.5. RELATION BETWEEN LINEAR AND ANGULAR VELOCITIES

Let $\bar{v}$ be the linear velocity vector of a moving particle at any point P which is along the tangent at P. Let OX and OY be the co-ordinate axes.

Also $\bar{e}_r$ and $\bar{e}_\theta$ be the unit vectors along the radius vector and perpendicular to the radius vector as shown in fig. 5.

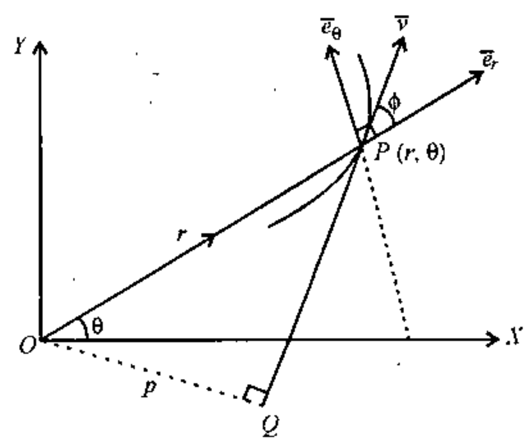


Fig. 5

$\therefore \qquad \dfrac{d\bar{e}_r}{dt} = \dfrac{d\theta}{dt}\bar{e}_\theta.$ ...(1)

Since $\qquad \bar{r} = OP\,\bar{e}_r = r\,\bar{e}_r$ $\qquad (\because OP = r)$

Now $\qquad \bar{v} = \dfrac{d\bar{r}}{dt}$

$$= \frac{d}{dt}(r\,\bar{e}_r)$$

$$= r\frac{d\bar{e}_r}{dt} + \frac{dr}{dt}\bar{e}_r$$

$$= r\frac{d\theta}{dt}\bar{e}_\theta + \frac{dr}{dt}\bar{e}_r \qquad \text{[using (1)]}$$

$$\bar{v} = \frac{dr}{dt}\bar{e}_r + r\frac{d\theta}{dt}\bar{e}_\theta \qquad ...(2)$$

Let ϕ be the angle between $\overline{v}$ and $\overline{e}_r$ and the components of $\overline{v}$ along $\overline{e}_r$ and $\overline{e}_\theta$ be v_r, v_θ respectively. Thus (2) becomes

$$\overline{v} = v_r\,\overline{e}_r + v_\theta\,\overline{e}_\theta$$

$$\therefore \quad v_\theta = \overline{v}\,.\,\overline{e}_\theta$$

$$= \left[\frac{dr}{dt}\overline{e}_r + \left(r\frac{d\theta}{dt}\right)\overline{e}_\theta\right].\,\overline{e}_\theta \qquad \text{[using (2)]}$$

$$= r\frac{d\theta}{dt} \qquad [\because\ \overline{e}_r\,.\,\overline{e}_\theta = 0 \text{ and } \overline{e}_\theta\,.\,\overline{e}_\theta = 1]$$

or $$\frac{d\theta}{dt} = \frac{v_\theta}{r} = \frac{v_\theta}{OP} \qquad (\because\ OP = r)$$

If ω is the angular velocity of a moving particle at P about O and $\angle POX = \theta$, then

$$\omega = \frac{d\theta}{dt}$$

$$\therefore \quad \boxed{\omega = \frac{v_\theta}{OP}}$$

or $$\omega = \frac{\text{component of velocity } \overline{v} \text{ at } P \text{ perpendicular to } OP}{OP}$$

Also $$\overline{v}\,.\,\overline{e}_\theta = v\cos(90^\circ - \phi) \qquad [\because \text{ Angle between } \overline{v} \text{ and } \overline{e}_\theta \text{ is } 90^\circ - \phi]$$

or $$v_\theta = v\sin\phi$$

or $$r\,\omega = v\sin\theta \qquad \left(\because\ v_\theta = \frac{\omega}{r}\right)$$

or $$\omega = \frac{v\sin\phi}{r}$$

or $$\boxed{\omega = \frac{vp}{r^2}} \qquad (\because\ p = r\sin\phi)$$

where p is length of the perpendicular drawn from O to the tangent at P.

Remarks

- If the particles P and Q are both in motion, then the angular velocity of Q relative to P is given by

$$= \frac{\text{the resolved part of the velocity } Q \text{ relative to } P \perp \text{ to } PQ}{PQ}.$$

• 11.6. RADIAL AND TRANSVERSE VELOCITIES

To find the components of the velocity in radial and Transverse direction.

Let a particle be moving in a plane and at any instant the particle be at P with velocity $\overline{v}$ along the tangent to the curve at P, as shown below :

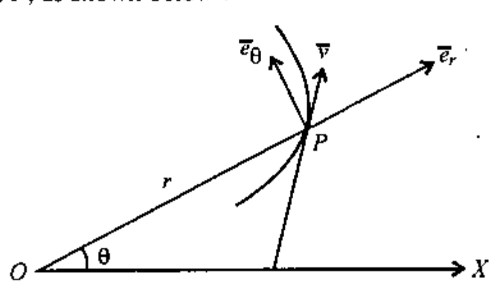


Fig. 6

Let $\overline{e}_r$ and $\overline{e}_\theta$ be the unit vectors along the radius vector $\overline{r}$ and perpendicular to the radius vector respectively,

$$\therefore \quad \overline{r} = r\,\overline{e}_r \qquad (\because\ \overrightarrow{OP} = \overline{r} \text{ and } OP = r)$$

Now
$$\bar{v} = \frac{d\bar{r}}{dt}$$
$$= \frac{d}{dt}(r\,\bar{e}_r)$$
$$= \frac{dr}{dt}\bar{e}_r + r\frac{d\bar{e}_r}{dt}$$
$$= \frac{dr}{dt}\bar{e}_r + r\frac{d\theta}{dt}\bar{e}_\theta \qquad \left[\because \frac{d\bar{e}_r}{dt} = \frac{d\theta}{dt}\bar{e}_\theta\right]$$

$\therefore$ Radial component of the velocity at $P = \frac{dr}{dt}$

and Transverse component of the velocity at $P = r\frac{d\theta}{dt}$

Hence $\boxed{\text{Radial velocity} = \frac{dr}{dt}}$

and $\boxed{\textbf{Transverse velocity} = r\frac{d\theta}{dt}}$

Since these two velocities are perpendicular to each other, therefore, the resultant velocity of the particle at P is given by,
$$v = \sqrt{\left(\frac{dr}{dt}\right)^2 + \left(r\frac{d\theta}{dt}\right)^2}.$$

Remarks

- Radial velocity $= \frac{dr}{dt}$ will be positive in the direction of r increasing.
- Transverse velocity $= r\frac{d\theta}{dt}$ will be postiive in the direction of θ increasing.

• 11.7. RADIAL AND TRANSVERSE ACCELERATION

To find the components of the acceleration along and perpendicular to the radius vector.

Let $\bar{a}$ be the acceleration vector of the moving particle at P, where the velocity vector be $\bar{v}$. Then
$$\bar{a} = \frac{d\bar{v}}{dt}$$
$$= \frac{d}{dt}\left(\frac{dr}{dt}\bar{e}_r + r\frac{d\theta}{dt}\bar{e}_\theta\right)$$
$$= \frac{d^2r}{dt^2}\bar{e}_r + \frac{dr}{dt}\frac{d\bar{e}_r}{dt} + \frac{d}{dt}\left(r\frac{d\theta}{dt}\right)\bar{e}_\theta + \left(r\frac{d\theta}{dt}\right)\frac{d\bar{e}_\theta}{dt}$$
$$= \frac{d^2r}{dt^2}\bar{e}_r + \frac{dr}{dt}\left(\frac{d\theta}{dt}\bar{e}_\theta\right) + \left(\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}\right)\bar{e}_\theta + \left(r\frac{d\theta}{dt}\right)\left(-\frac{d\theta}{dt}\bar{e}_r\right)$$
$$\left[\because \frac{d\bar{e}_r}{dt} = \frac{d\theta}{dt}\bar{e}_\theta \text{ and } \frac{d\bar{e}_\theta}{dt} = -\frac{d\theta}{dt}\bar{e}_r\right]$$
$$= \left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]\bar{e}_r + \left[2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}\right]\bar{e}_\theta$$
$$\therefore\ \bar{a} = \left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]\bar{e}_r + \left[\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)\right]\bar{e}_\theta.$$

Thus $\bar{a}$ is obtained as the linear combination of unit vectors $\bar{e}_r$ and $\bar{e}_\theta$. Therefore,

Radial acceleration $= \bar{a}\cdot\bar{e}_r$

or $$\text{R.A.} = \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2$$

and Transverse acceleration $= \bar{a} \cdot \bar{e}_\theta$

or $$\text{T.A.} = \frac{1}{r}\frac{d}{dt}\left(r^2 \frac{d\theta}{dt}\right)$$

Since Radial Acceleration (R. A.) and Transverse Acceleration (T. A.) are perpendicular to each other, then the resultant acceleration of the particle at P is given by

$$a = \sqrt{(\text{R.A.})^2 + (\text{T.A.})^2}$$

or $$a = \sqrt{\left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]^2 + \left[\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)\right]^2}.$$

Remarks

- R. A. will be taken positive in the direction of r increasing.
- T. A. will be taken positive in the direction of θ increasing.

SOLVED EXAMPLES

Example 1. *A particle describes a parabola with uniform speed, show that its angular velocity about the focus S, at any point P, varies inversely as* $(SP)^{3/2}$.

Solution. The equation of a parabola with S as pole is

$$p^2 = ar \qquad ...(1)$$

Since $v = \text{constant} = c$ *(say)*

$\therefore$ Angular velocity $= \dfrac{vp}{r^2}$

$$= \frac{c\sqrt{ar}}{r^2} \qquad (\because\ v = c \text{ and } p^2 = ar)$$

$$= \frac{c\sqrt{a}}{r^{3/2}}$$

$$= \frac{c\sqrt{a}}{(SP)^{3/2}} \qquad (\because\ SP = r)$$

Hence, the angular velocity varies inversely $(SP)^{3/2}$.

Example 2. *If the radial and transverse velocities of a particle are always proportional to each other, show that the path is an equiangular spiral.*

Solution. Here, radial velocity α transverse velocity

i.e., $$\frac{dr}{dt} = k\,r\frac{d\theta}{dt}$$

where k is some constant

or $$\frac{dr}{r} = k\,d\theta.$$

Integrating, we get

$$\log r = k\theta + c$$

where c is a constant of integration

or $$\log r = k\theta + \log A \qquad (\text{let } c = \log A)$$

or $$r = Ae^{k\theta}.$$

This is an equiangular spiral.

Example 3. *The velocities of a particle along and perpendicular to the radius vector are* λr *and* $\mu\theta$; *find the path and show that the accelerations along and perpendicular to the radius vector are*

$$\lambda^2 r - \frac{\mu^2\theta^2}{r} \text{ and } \mu\theta(\lambda + \mu/r).$$

Solution. Since the radial and transverse velocities are given as λr and $\mu\theta$, then

$$\frac{dr}{dt} = \lambda r \qquad \text{...(1)}$$

and
$$r\frac{d\theta}{dt} = \mu\theta. \qquad \text{...(2)}$$

Dividing (1) and (2), we get

$$\frac{dr}{r\,d\theta} = \frac{\lambda r}{\mu\theta}$$

or
$$\frac{\mu}{\lambda}\frac{dr}{r^2} = \frac{d\theta}{\theta}$$

Integrating, we get

$$-\frac{\mu}{\lambda r} = \log\theta + A$$

where A is a constant of integration and also taken to be $\log c$

$$\therefore \qquad -\frac{\mu}{\lambda r} = \log\theta + \log c$$

or
$$-\frac{\mu}{\lambda r} = \log(\theta c)$$

or
$$c\theta = e^{-\mu/\lambda r}$$

or
$$\theta = ae^{b/r}, \text{ where } a \text{ and } b \text{ are constant.}$$

This is the required equation of a path.

Now, radial acceleration
$$= \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2$$
$$= \frac{d}{dt}\left(\frac{dr}{dt}\right) - \frac{1}{r}\left(r\frac{d\theta}{dt}\right)^2$$
$$= \frac{d}{dt}(\lambda r) - \frac{1}{r}(\mu\theta)^2$$
$$= \lambda\frac{dr}{dt} - \frac{\mu^2\theta^2}{r}$$
$$= \lambda(\lambda r) - \frac{\mu^2\theta^2}{r}.$$
$$= \lambda^2 r - \frac{\mu^2\theta^2}{r}.$$

and transverse accelerati
$$= \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)$$
$$= \frac{1}{r}\frac{d}{dt}(r\,.\,\mu\theta) \qquad \left(\because\; r\frac{d\theta}{dt} = \mu\theta\right)$$
$$= \frac{1}{r}\left[\mu\theta\frac{dr}{dt} + \mu r\frac{d\theta}{dt}\right]$$
$$= \frac{1}{r}[\mu\theta\,\lambda r + \mu\,.\,\mu\theta]$$
$$= \mu\theta(\lambda + \mu/r).$$

• TEST YOURSELF

1. Prove that the angular velocity of a projectile about the focus of its path varies inversely as its distance from the focus.
2. A rod moves with its ends on rectangular axes OX, OY. If (x, y) be a point P on the rod and if the angular velocity ω of the rod is constant, show that components of acceleration of P along the axes are $-x\omega^2$ and $-y\omega^2$ and the resultant acceleration is $OP\,.\,\omega^2$ towards O.

3. If a point moves along a circle with constant speed, prove that its angular velocity about any point on the circle is half of that about the centre.
4. A straight line of constant length moves with its ends on two fixed rectangular axes OX, OY and P is the foot of the perpendicular from O on the straight line. Show that the velocity of P perpendicular to OP is $OP \cdot \frac{d\theta}{dt}$ and along OP is $2CP \cdot \frac{d\theta}{dt}$, where C is the middle point of the line and θ is the angle COX.
5. The line joining two points A, B is of constant length a and the velocities of A, B are in the directions which make angles α and β respectively with AB. Prove that the angular velocity of AB is $\frac{u \sin(\alpha - \beta)}{a \cos \beta}$, where u is the velocity of A.
6. A wheel rolls along a straight road with constant speed v. Show that the actual velocity of P is $v \cdot (AP/CP)$, where A is the point of contact of the wheel with the road and C is the centre of the wheel. Also find its direction. Find also the angular velocity of P relative to A.
7. A point P is moving along a fixed straight line AB with uniform velocity v. Show that its angular velocity about a point O is inversely proportional to OP^2.
8. Two points are moving with uniform velocities u, v in perpendicular lines OX and OY, the motions being towards O. If initially, their distances from the origin are a and b respectively, calculate the angular velocity of the line joining them at the end of t seconds, and show that it is greatest when

$$t = \frac{au + bv}{u^2 + v^2}.$$

• 11.8. TANGENTIAL AND NORMAL VELOCITIES

A particle is moving in a plane curve and at any time t the particle is at a point P on the curve, whose position vector is $\bar{r}$ with respect to some fixed point O. Let A be a fixed point on the curve such that $AP = s$.

Let $\bar{t}$ be the unit tangent vector along the tangent at P to the path and $\bar{n}$ be the unit normal vector in the direction of ψ increasing. Then we have

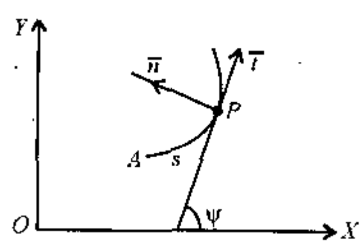


Fig. 9

$$\frac{d\bar{t}}{dt} = \frac{d\psi}{dt}\bar{n}. \qquad ...(1)$$

But we know that

$$\frac{d\bar{r}}{ds} = \bar{t}. \qquad ...(2)$$

Let $\bar{v}$ be the velocity of the moving particle at P, whose position vector is $\bar{r}$. Then,

$$\bar{v} = \frac{d\bar{r}}{dt} = \frac{d\bar{r}}{ds} \cdot \frac{ds}{dt}$$

$$\bar{v} = \frac{ds}{dt}\bar{t}$$

or
$$\bar{v} = \frac{ds}{dt}\bar{t} + 0 \cdot \bar{n}. \qquad ...(3)$$

Thus $\bar{v}$ is a linear combination of the unit vectors $\bar{t}$ and $\bar{n}$. Therefore, the tangential component of the velocity is $\frac{ds}{dt}$ in the direction of s increasing and the normal component of the velocity is zero. Hence, we obtain

$$\boxed{\text{Tangential velocity} = \frac{ds}{dt}}$$

and
$$\boxed{\text{Normal velocity} = \mathbf{0}}$$

Remarks

- Tangential velocity $= \frac{ds}{dt}$ is taken to positive in the direction of s increasing.
- The resultant velocity of a particle is always along the tangent to ts path *i.e.*, $v = \frac{ds}{dt}$.

11.9. TANGENTIAL AND NORMAL ACCELERATIONS

Let $\bar{a}$ be the acceleration of the particle at any point P, whose position vector is $\bar{r}$ and the velocity vector is $\bar{v}$. Then

$$\bar{a} = \frac{d\bar{v}}{dt}$$

$$= \frac{d}{dt}\left(\frac{ds}{dt}\bar{t}\right) \qquad \left(\because \; \bar{v} = \frac{ds}{dt}\bar{t}\right)$$

$$= \frac{d^2s}{dt^2}\bar{t} + \frac{ds}{dt}\cdot\frac{d\bar{t}}{dt}$$

$$= \frac{d^2s}{dt^2}\bar{t} + \frac{ds}{dt}\frac{d\psi}{dt}\bar{n} \qquad \left(\because \; \frac{d\bar{t}}{dt} = \frac{d\psi}{dt}\bar{n}\right)$$

$$= \frac{d^2s}{dt^2}\bar{t} + \left(\frac{ds}{dt}\right)^2\frac{d\psi}{ds}\bar{n}$$

$$\therefore \qquad \bar{a} = \frac{d^2s}{dt^2}\bar{t} + \frac{v^2}{\rho}\bar{n} \qquad \left[\because \; \frac{ds}{dt} = v \text{ and } \rho = \frac{ds}{d\psi}\right]$$

Thus $\bar{a}$ is obtained as the linear combination of the unit vectors $\bar{t}$ and $\bar{n}$. Therefore the coefficients of $\bar{t}$ and $\bar{n}$ give the tangential and Normal accelerations respectively.

Hence, $\boxed{\textit{Tangential acceleration} = \frac{d^2s}{dt^2}}$

and $\boxed{\text{Normal acceleration} = \frac{v^2}{\rho}}$

If a is the resultant acceleration, then

$$a = \sqrt{(\text{Tangential acceleration})^2 + (\text{Normal acceleration})^2}$$

i.e.,
$$a = \sqrt{\left(\frac{d^2s}{dt^2}\right)^2 + \left(\frac{v^2}{\rho}\right)^2}$$

Remarks

- Tangential acceleration $= \frac{d^2s}{dt^2}$, is taken to be positive *in the direction of s increasing.*
- Normal acceleration $= \frac{v^2}{\rho}$, is taken to be positive in the direction of inwards drawn normal.
- Other expressions of the tangential acceleration are $\frac{dv}{dt} = \frac{d}{dt}\left(\frac{ds}{dt}\right) = \frac{d^2s}{dt^2}$ and

$$\frac{d^2s}{dt^2} = \frac{dv}{dt} = \frac{dv}{ds}\cdot\frac{ds}{dt} = v\frac{dv}{ds}$$

- In normal acceleration $\frac{v^2}{\rho}$, ρ is a radius of curvature.

SOLVED EXAMPLES

Example 1. *A point describes a cycloid $s = 4a \sin \psi$ with uniform speed v. Find its acceleration at any point.*

Solution. The intrinsic equation of a cycloid is

$$s = 4a \sin \psi \qquad ...(1)$$

$$\therefore \qquad \rho = \frac{ds}{d\psi} = 4a \cos \psi.$$

Since particle moves on the cycloid with unifrom speed v, then

$$\text{Tangential acceleration} = \frac{d^2s}{dt^2} = \frac{dv}{dt} = 0$$

and
$$\text{Normal acceleration} = \frac{v^2}{\rho} = \frac{v^2}{4a \cos \psi}$$

$$\therefore \text{ The resultant acceleration} = \sqrt{\left(\frac{d^2s}{dt^2}\right)^2 + \left(\frac{v^2}{\rho}\right)^2}$$

$$= \sqrt{0 + \left(\frac{v^2}{4a \cos \psi}\right)^2}$$

$$= \frac{v^2}{4a \cos \psi}$$

$$= \frac{v^2}{4a\sqrt{1 - \sin^2 \psi}}$$

$$= \frac{v^2}{4a\sqrt{1 - \frac{s^2}{16a^2}}} \qquad \text{[using (1)]}$$

$$= \frac{v^2}{\sqrt{16a^2 - s^2}}.$$

Example 2. *Prove that the acceleration of a point moving in a curve with uniform speed is*

$$\rho\left(\frac{d\psi}{dt}\right)^2.$$

Solution. Since the particle is moving with unfirom speed, so that the tangential acceleration is zero. Now the normal acceleration is

$$\text{N.A.} = \frac{v^2}{\rho} = \left(\frac{ds}{dt}\right)^2 \frac{1}{\rho}$$

$$= \left(\frac{ds}{d\psi} \cdot \frac{d\psi}{dt}\right)^2 \cdot \frac{1}{\rho}$$

$$= \left(\frac{ds}{d\psi}\right)^2 \left(\frac{d\psi}{dt}\right)^2 \cdot \frac{1}{\rho}$$

$$= \rho^2 \left(\frac{d\psi}{dt}\right)^2 \cdot \frac{1}{\rho} \qquad \left(\because \frac{ds}{d\psi} = \rho\right)$$

$$= \rho\left(\frac{d\psi}{dt}\right)^2.$$

Hence the resultant acceleration $= \sqrt{(\text{T.A.})^2 + (\text{N.A.})^2}$

$$= \sqrt{0 + \left[\rho\left(\frac{d\psi}{dt}\right)^2\right]^2} = \rho\left(\frac{d\psi}{dt}\right)^2.$$

Example 3. *A particle is describing a plane curve. If the tangential and normal acceleration are each constant throughout the motion, prove that the angle ψ, through which the direction of motion turns in time t is given by*

$$\psi = A \log (1 + Bt).$$

Solution. Here, it is given that

$$\frac{d^2s}{dt^2} = \text{constant} = \lambda \text{ (say)} \qquad ...(1)$$

and
$$\frac{v^2}{\rho} = \text{constant} = \mu \text{ (say)}. \qquad ...(2)$$

From (1), we get on integrating,

$$\frac{ds}{dt} = \lambda t + a, \qquad ...(3)$$

where 'a' is a constant of integration.

From (2), we get

$$\frac{v^2}{\rho} = \frac{(ds/dt)^2}{(ds/d\psi)} = \mu$$

or
$$\frac{ds}{dt} \cdot \frac{d\psi}{dt} = \mu$$

or
$$(\lambda t + a) \frac{d\psi}{dt} = \mu \qquad \text{[using (3)]}$$

or
$$d\psi = \frac{\mu}{\lambda t + a} dt.$$

Integrating, we get

$$\psi = \frac{\mu}{\lambda} [\log (\lambda t + a) - \log a]$$

$$= \frac{\mu}{\lambda} \log \left[\frac{(\lambda t + a)}{a} \right]$$

or
$$\psi = A \log (1 + Bt),$$

where
$$A = \frac{\mu}{\lambda}, \ B = \frac{\lambda}{a}.$$

Example 4. *A point moves in a plane curve so that its tangential acceleration is constant and the magnitudes of the tangential velocity and normal acceleration are in a consatnt ratio; find the intrinsic equation of the curve.*

Solution. Here, it is given that

$$\frac{dv}{dt} = \lambda \text{ (constant)} \qquad ...(1)$$

and
$$\frac{v}{v^2/\rho} = \mu \text{ (constant)}$$

From (2), we get

$$\frac{\rho}{v} = \mu$$

or
$$\frac{ds/d\psi}{ds/dt} = \mu$$

or
$$\frac{dt}{d\psi} = \mu \qquad ...(3)$$

Multiplying (1) and (3), we get

$$\frac{dv}{d\psi} = \lambda\mu$$

or
$$dv = \lambda\mu \, d\psi.$$

Integrating, we get

$$v = \lambda\mu\psi + a \qquad ...(4)$$

where a is a constant.

Since $\rho = \mu v$

$\therefore$ $\rho = \mu(\lambda\mu\psi + a)$ [using (4)]

or $$\frac{ds}{d\psi} = \mu^2\lambda\psi + a\mu.$$

Integrating, we get

$$s = \frac{\mu^2\lambda}{2}\psi^2 + a\mu\psi + C$$

or $$s = A\psi^2 + B\psi + C$$

where $A = \frac{1}{2}\mu^2\lambda$, $B = a\mu$, C are constant.

Hence the intrinsic equation of the path is

$$s = A\psi^2 + B\psi + C.$$

• SUMMARY

- **Velocity and acceleration in a plane :**

  Velocity $v = \frac{ds}{dt}$, Acceleration $a = \frac{d^2s}{dt^2} = \frac{dv}{dt}$

- **Angular velocity and angular acceleration :**

  Angular velocity $= \frac{d\theta}{dt}$, Angular acceleration $= \frac{d^2\theta}{dt^2}$

- **Radial and Transverse velocities :**

  Radial velocity $= \frac{dr}{dt}$, Transverse velocity $= r\frac{d\theta}{dt}$

- **Radial and Transverse acceleration**

  Radial acceleration $= \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2$, Transverse acceleration $= \frac{1}{r}\cdot\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)$

- **Tangential and Normal velocities :**

  Tangential velocity $= \frac{ds}{dt}$, Normal velocity $= 0$

- **Tangential and Normal accelerations :**

  Tangential acceleration $= \frac{d^2s}{dt^2}$, Normal acceleration $= \frac{v^2}{\rho}$

• STUDENT ACTIVITY

1. If the radial and transverse velocities of a particle are always proportional to each other, show that the paths is an equiangular spiral.

2. A partial is describing a plane curve. If the tangential and normal accelerations are each constant throughout the motion, prove that the angle ψ, through which the direction of motion turns in time t is given by

$$\psi = A \log (1 + Bt)$$

• STUDENT ACTIVITY

1. A particle describes a curve (for which s and ψ vanish simultaneously) with uniform speed v. If the acceleration at any point s be $v^2c/(s^2 + c^2)$, find the intrinsic equation of the curve.
2. A particle moves in a plane in such a manner that its tangential and normal accelerations are always equal and its velocity varies as exp. $[\tan^{-1}(s/c)]$, s being the length of the arc of the curve measured from a fixed point on the curve. Find the path.
3. If the tangential and normal accelerations of a particle describing a plane curve be constant throughout, prove that the radius of curvature at any point t is given by $\rho = (at + b)^2$.

ANSWERS

1. $s = c \tan \psi$ 2. $s = c \tan \psi$

OBJECTIVE EVALUATION

Fill in the Blanks :

1. The rate of change of displacement is called ....
2. If $\bar{v} = \frac{dx}{dt}$, then the acceleration is ....
3. The magnitude of the velocity vector is ....
4. Negative of an acceleration is called ....
5. The rate of change of velocity is called ....

True or False :

Write T for true and F for false statements :

1. Velocity is a vector quantity. *(T/F)*
2. The magnitude of the velocity vector is called speed. *(T/F)*
3. If the acceleration a of a particle in a line is $\frac{d^2x}{dt^2}$, then its velocity is $x\frac{dx}{dt}$. *(T/F)*
4. If ω is the angular velocity of a particle, then $\omega = \frac{d\theta}{dt}$. *(T/F)*
5. If acceleration $= \frac{d^2x}{dt^2}$, then $-\frac{d^2x}{dt^2}$ = retardation. *(T/F)*

Multiple Choice Questions (MCQ's) :

Choose the most appropriate one :

1. The magnitude of a velocity vector is :
   (a) speed (b) velocity (c) acceleration (d) none of these.
2. If ω be a angular velocity of a particle, then its value is :
   (a) 0 (b) $d\theta/dt$ (c) $dt/d\theta$ (d) $d^2\theta/dt^2$.
3. If $\frac{d\hat{a}}{dt} = \omega\,\hat{b}$, then $\hat{a}\,.\,\hat{b}$ is :
   (a) 1 (b) ab (c) 0 (d) ω.

4. Radial acceleration of a particle is :

(a) $\frac{d^2r}{dt^2}$ (b) $\frac{d^2r}{dt^2}-\left(r\frac{d\theta}{dt}\right)^2$ (c) $\frac{d^2r}{dt^2}-\left(\frac{d\theta}{dt}\right)^2$ (d) $\frac{d^2r}{dt^2}-r\left(\frac{d\theta}{dt}\right)^2$.

ANSWERS

Fill in the Blanks :

1. Velocity **2.** $\frac{d^2x}{dt^2}$ **3.** speed **4.** Retardation **5.** Acceleration

True or False :

1. T **2.** T **3.** F **4.** T **5.** T

Multiple Choice Questions :

1. (a) **2.** (b) **3.** (c) **4.** (d)

❑❑❑

UNIT 12

RECTILINEAR MOTION (Variable Acceleration)

STRUCTURE

- Rectilinear Motion
- Velocity and acceleration in a straight line
- Motion under inverse square law
- Motion due to the attraction of the Earth
- Siple Harmonic Motion
- Some important definitions
- Geometrical representation of S.H.M.
  - ☐ Summary
  - ☐ Student Activity
  - ☐ Test Yourself

LEARNING OBJECTIVES

After going through this unit you will learn :

- What is rectilinear motion of a particle ?
- How to move a particle under inverse square law ?
- What is S.H.M. ?

12.1. RECTILINEAR MOTION

Definition. *When a particle moves in a straight line, its motion is known as* **Rectilinear motion.** *Whether the straight line is horizontal or vertical.*

12.2. VELOCITY AND ACCELERATION IN A STRAIGHT LINE

Velocity. Let OX be a straight line, where O is a fixed point on the line. Let us suppose a particle is moving along this line and at any instant t it is at a point P distant x from O.

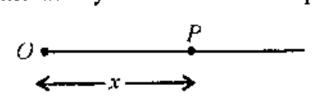


Fig. 1

Let $\overline{r}$ be the position vector P and $\hat{i}$ the unit vector along OX. Then

$$\overline{r} = x\,\hat{i} \qquad (\because \; OP = x)$$

$\therefore$ The velocity at P,

$$\overline{v} = \frac{d\overline{r}}{dt} = \frac{dx}{dt}\,\hat{i}.$$

Thus the direction of the velocity vector $\overline{v}$ is always along the line, in which the particle is moving. If v is the magnitude of the velocity $\overline{v}$, then

$$v = |\overline{v}| = \left|\frac{dx}{dt}\,\hat{i}\right| = \frac{dx}{dt}.$$

Also, if the particle is moving in the direction of x increasing, then $\frac{dx}{dt}$ will be positive, otherwise negative if moving in the direction of x decreasing.

Acceleration. *The rate of change of velocity is known as acceleration.* Let $\bar{a}$ be the acceleration of the particle at P, then

$$\bar{a} = \frac{d\bar{v}}{dt}$$

$$= \frac{d}{dt}\left(\frac{dx}{dt}\hat{i}\right)$$

$$= \frac{d^2x}{dt^2}\hat{i}.$$

Thus $\bar{a}$ is collinear with $\hat{i}$, therefore, the acceleration is also always along the line itself and the magnitude of the acceleration $\bar{a}$ is given by

$$|\bar{a}| = a = \frac{d^2x}{dt^2}.$$

It is positive in the direction of x increasing and negative in the direction of x decreasing.

Other Forms of the Acceleration :

If a particle is moving in a straight line and it is at a distance x from some fixed point O on the line at time t. Then the velocity and acceleration at this point P are

$$v = \frac{dx}{dt}$$

and $$a = \frac{d^2x}{dt^2}$$

$\therefore$ $$a = \frac{d}{dt}\left(\frac{dx}{dt}\right)$$

$$= \frac{d}{dt}(v) = \frac{dv}{dt}$$

and $$a = \frac{dv}{dt} = \frac{dv}{dx}\cdot\frac{dx}{dt}$$

$$= v\frac{dv}{dx}.$$

Hence $\frac{d^2x}{dt^2}, \frac{dv}{dt}$ and $v\frac{dv}{dx}$ three expressions of the acceleration and all will have positive sign in the direction of x increasing.

• 12.3. MOTION UNDER INVERSE SQUARE LAW

To discuss the motion of a particle when it moves in a straight line under an attraction towards a fixed point, which is inversely proportional to the square of the distance measured from the fixed point.

Let a particle be moving along a straight line OX, where O is a fixed point on the line and let the particle start from rest from a point A such that $OA = a$ towards the point O.

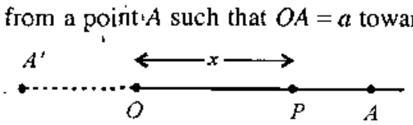


Fig. 2

Let P be the position of the particle at any time t whose distance from fixed point O is x *i.e.*, $OP = x$, and v be the velocity at P. Then the acceleration at P is equal to μ/x^2 towards O, where μ is a constant.

$\therefore$ The equation of motion of the particle at P is

$$\frac{d^2x}{dt^2} = -\frac{\mu}{x^2}. \qquad \text{...(1)}$$

[Here, negative sign is taken, because $\frac{d^2x}{dt^2}$ is positive in the direction of x increasing, while $\frac{\mu}{x^2}$ is towards O, in the direction of x decreasing so that $\frac{\mu}{x^2}$ is negative.]

Multiplying (1) by $2\frac{dx}{dt}$ and then integrating, we get

$$\left(\frac{dx}{dt}\right)^2 = \frac{2\mu}{x} + C \quad ...(2)$$

where C is a constant of integration.

Initially, when $x = a$, $\frac{dx}{dt} = 0$

$$\therefore \quad 0 = \frac{2\mu}{a} + C$$

or
$$C = -\frac{2\mu}{a}.$$

Putting the value of C in (2), we get

$$\left(\frac{dx}{dt}\right)^2 = \frac{2\mu}{x} - \frac{2\mu}{a} = 2\mu\left(\frac{1}{x} - \frac{1}{a}\right). \quad ...(3)$$

Equation (3) gives the velocity at P.
From (3), we get

$$\frac{dx}{dt} = -\sqrt{\frac{2\mu}{a}}\sqrt{\frac{a-x}{x}}$$

[Here negative sign is taken, because particle is moving in the direction of x decreasing]

$$\therefore \quad dt = -\sqrt{\frac{a}{2\mu}}\sqrt{\frac{x}{a-x}}\,dx$$

Integrating, we get

$$t = -\sqrt{\frac{a}{2\mu}}\int\sqrt{\frac{x}{a-x}}\,dx + D,$$

where D is a constant of integration.

Putting $x = a\cos^2\theta$, so that $dx = -2a\sin\theta\cos\theta\,d\theta$, then we get

$$t = \sqrt{\frac{a}{2\mu}}\int\sqrt{\frac{a\cos^2\theta}{a - a\cos^2\theta}}\,2a\sin\theta\cos\theta\,d\theta + D$$

$$= a\sqrt{\frac{a}{2\mu}}\int 2\cos^2\theta\,d\theta + D$$

$$= a\sqrt{\frac{a}{2\mu}}\int(1+\cos 2\theta)\,d\theta + D$$

$$= a\sqrt{\frac{a}{2\mu}}\left[\theta + \frac{\sin 2\theta}{2}\right] + D$$

$$= a\sqrt{\frac{a}{2\mu}}\,[\theta + \sin\theta\cos\theta] + D$$

$$= a\sqrt{\frac{a}{2\mu}}\,[\theta + \cos\theta\sqrt{1-\cos^2\theta}\,] + D.$$

Since $x = a\cos^2\theta$ *i.e.*, $\cos\theta = \sqrt{\frac{x}{a}}$ and $\theta = \cos^{-1}\sqrt{\frac{x}{a}}$, then

$$t = a\sqrt{\frac{a}{2\mu}}\left[\cos^{-1}\sqrt{\frac{x}{a}} + \sqrt{\frac{x}{a}}\sqrt{1-\frac{x}{a}}\right] + D.$$

Initially, when $t = 0$, $x = a$, then, we get

$$0 = a\sqrt{\frac{a}{2\mu}}\,[\cos^{-1} 1 + 0] + D$$

$$0 = a\sqrt{\frac{a}{2\mu}}\,[0 + 0] + D$$

or $$D = 0.$$

$$\therefore \qquad t = a\sqrt{\frac{a}{2\mu}}\left[\cos^{-1}\sqrt{\frac{x}{a}} + \sqrt{\frac{x}{a}}\sqrt{1-\frac{x}{a}}\right]. \qquad ...(4)$$

This equation (4) gives the time at the point P at a distance x from O (*i.e.*, the centre of force). If we put $x = 0$ in (3), we get the infinite velocity at O and, therefore the particle moves to the left of O with the acceleration always directed towards O and thus the velocity is continuously decreasing. The particle will come to instantaneous rest at A' such that $OA' = OA = a$ and then the particle retraces its path. Hence the particle will oscillate about O between A and A'.

Let t_1 be the time taken by the particle to reach from the point A to O (the centre of the froce). Then put $x = 0$ in (4), we get

$$t_1 = a\sqrt{\frac{a}{2\mu}}\,[\cos^{-1} 0 + 0]$$

$$= a\sqrt{\frac{a}{2\mu}}\left(\frac{\pi}{2}\right)$$

$$= \frac{\pi}{2}\sqrt{\frac{a^3}{2\mu}}.$$

Now, the time of one complete oscillation $= 4 \times t_1$

$$= 4\cdot\frac{\pi}{2}\sqrt{\frac{a^3}{2\mu}}$$

$$= 2\pi\sqrt{\frac{a^3}{2\mu}}.$$

• 12.4. MOTION DUE TO THE ATTRACTION OF THE EARTH

1. *Earth attracts every body outside its surface with a force (gravitational force), which is always proportional to* $\frac{1}{(\textbf{distance})^2}$, *where the distance is measured from the centre of earth. Thus the attraction of the earth follows the* **inverse square law.**

2. *On the other hand, when a body moves inside the earth, it is experienced a force, which is always directly proportional to the* **distance,** *towards the centre where the distance is measured.*

3. *At the surface of the earth, the acceleration of a body is taken to be g (acceleration due to gravity).*

SOLVED EXAMPLES

Example 1. *If h be the height due to the velocity v at the earth's surface, supposing its attraction constant and H the corresponding height when the variation of gravity is taken into account, prove that*

$$\frac{1}{h} = \frac{1}{H} + \frac{1}{r}$$

where r is the earth's radius.

Solution. Since a particle attains a height h outside the earth due to the velocity v at the earth's surface under constant attraction. Then we have

$$v^2 = 2gh \qquad ...(1)$$

$$(\because \; v^2 = u^2 - 2gh)$$

Now when the particle moves under the variation of gravity. Let P be the position of the particle at any time t at a distance x measured from the centre of the earth in the vertically upwards motion and v be the velocity with which the particle projected. Then the acceleration of the particle at P is μ/x^2, which is directed towards the centre of the earth.

$\therefore$ The equation of motion is

$$\frac{d^2x}{dt^2} = -\frac{\mu}{x^2}. \quad ...(2)$$

Since, we have $\frac{d^2x}{dt^2} = -g$ at the surface.

$\therefore$ When $x = r$ (radius of the earth).

$$\frac{d^2x}{dt^2} = -g.$$

Then from (2), we get

$$\frac{\mu}{r^2} = g$$

$\Rightarrow$ $$\mu = r^2 g.$$

Now (2) becomes,

$$\frac{d^2x}{dt^2} = -\frac{r^2 g}{x^2}. \quad ...(3)$$

Multiplying (3) by $2\frac{dx}{dt}$ and then integrating, we get

$$\left(\frac{dx}{dt}\right)^2 = \frac{2r^2 g}{x} + A, \text{ where } A \text{ is a constant.}$$

Initially, at the earth surface, $x = r$ and $\frac{dx}{dt} = v$, then

$$v^2 = \frac{2r^2 g}{r} + A$$

or $$A = v^2 - 2rg$$

$\therefore$ $$\left(\frac{dx}{dt}\right)^2 = \frac{2r^2 g}{x} + v^2 - 2rg. \quad ...(4)$$

In this motion, suppose the particle reaches at the maximum height H. That is, at the height H above the earth $\frac{dx}{dt} = 0$ and $x = r + H$, then from (4), we get

$$0 = \frac{2r^2 g}{r+H} + v^2 - 2rg$$

or $$0 = \frac{2r^2 g}{r+H} + 2gh - 2rg \quad (\because\ v^2 = 2gh)$$

or $$0 = 2r^2 g + 2gh\,(r+H) - 2r^2 g - 2rgH$$

or $$0 = h\,(r+H) - rH$$

or $$\frac{1}{h} = \frac{1}{H} + \frac{1}{r}.$$

Hence proved.

Example 2. *A particle is projected vertically upwards from the surface earth with a velocity just sufficient to carry it to the infinity. Prove that the time it takes to reach a height h is*

$$\frac{1}{3}\sqrt{\frac{2a}{g}}\left[\left(1+\frac{h}{a}\right)^{3/2} - 1\right]$$

where a is the radius of the earth.

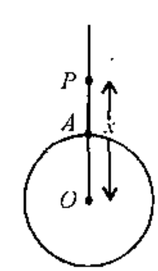


Fig. 3

Solution. Let v be the velocity of a particle with which it is projected vertically upwards from the earth's surface and it is just sufficient to carry the particle to the infinity. Let P be the position of the particle at any time during the upwards motion, whose distance from the centre of the earth is x. Then the acceleration of the particle at P is $\frac{\mu}{x^2}$ directed towards O.

$\therefore$ The equation of the motion of the particle is

$$\frac{d^2x}{dt^2} = -\frac{\mu}{x^2}. \quad \text{...(1)}$$

(Here negative sign is taken, because $\frac{\mu}{x^2}$ is measured in the direction of x decreasing).

Since the acceleration at the surface of the earth is g so that, when $x = a$ (radius of the earth) $\frac{d^2x}{dt^2} = -g$, then from (1), we get

$$-\frac{\mu}{a^2} = -g$$

$$\Rightarrow \qquad \mu = a^2 g.$$

Thus the equation (1) becomes

$$\frac{d^2x}{dt^2} = -\frac{a^2 g}{x^2}. \quad \text{...(2)}$$

Multiplying (2) by $2\frac{dx}{dt}$ and then integrating, we get

$$\left(\frac{dx}{dt}\right)^2 = \frac{2a^2 g}{x} + A, \text{ where } A \text{ is a constant.}$$

Initially, when $x \to \infty$, $\frac{dx}{dt} = 0$, we have

$$0 = 0 + A$$

or $$A = 0$$

$\therefore$ $$\left(\frac{dx}{dt}\right)^2 = \frac{2a^2 g}{x}$$

or $$\frac{dx}{dt} = \sqrt{2a^2 g}\,\frac{1}{\sqrt{x}}. \quad \text{...(3)}$$

(Here, $\frac{dx}{dt}$ is taken to be positive, because the particle is moving in the direction of x increasing).

Separating the variables in (3), we get

$$dt = \frac{1}{\sqrt{2a^2 g}}\sqrt{x}\,dx.$$

Integrating from $x = a$ to $x = h + a$, we get

$$t = \frac{1}{\sqrt{2a^2 g}}\int_a^{h+a}\sqrt{x}\,dx$$

$$= \frac{1}{\sqrt{2a^2 g}}\left[\frac{2}{3}x^{3/2}\right]_a^{h+a}$$

$$= \frac{1}{\sqrt{2a^2 g}}\cdot\frac{2}{3}[(h+a)^{3/2} - a^{3/2}]$$

$$t = \frac{1}{3}\sqrt{\frac{2a}{g}}\left[\left(1 + \frac{h}{a}\right)^{3/2} - 1\right].$$ **Hence proved.**

• TEST YOURSELF

1. Discuss the motion of a particle under inverse square law.
2. If the earth's attraction vary inversely as the square of the distance from its centre and g be its magnitude at the earth's surface, the time of falling from a height h above the surface to the surface is

$$\sqrt{\frac{(a+h)}{2g}}\left[\sqrt{\frac{h}{a}}+\frac{(a+h)}{a}\sin^{-1}\sqrt{\frac{h}{a+h}}\right]$$

where a is the radius of the earth.

• 12.5. SIMPLE HARMONIC MOTION

Definition : *A particle moves in a straight line in such a way that its acceleration is always directed towards a fixed point on the line, which is directly proportional to the distance measured from the fixed point, then the motion of the particle is called* **Simple Harmonic Motion.**

To investigate the Simple Harmonic Motion :

Let O be the fixed point on a straight line $A'OA$, which is taken as the centre of the force. Suppose a particle starts its motion from rest from a point A on the line towards O.

Let P be the position of the particle at any time t such that $OP = x$. Then the acceleration of the particle at P is μx towards O.

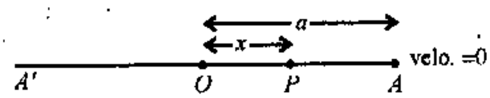


Fig. 4

$\therefore$ The equation of motion of the particle at P is

$$\frac{d^2x}{dt^2} = -\mu x. \qquad ...(1)$$

(Here negative sign is taken because the acceleration is measured in the direction of x decreasing).

Multiplying (1) by $2\frac{dx}{dt}$ and then integrating, we get

$$\left(\frac{dx}{dt}\right)^2 = -\mu x^2 + C, \text{ where } C \text{ is a constant.}$$

Initially, at A, $x = a$ and $\frac{dx}{dt} = 0$, then

$$0 = -\mu a^2 + C$$

or

$$C = \mu a^2$$

$\therefore$

$$\left(\frac{dx}{dt}\right)^2 = \mu(a^2 - x^2) \qquad ...(2)$$

This equation (2) gives the velocity at any time t. Let v be the velocity at the point P, then

$$\boxed{v^2 = \mu(a^2 - x^2)} \qquad ...(3)$$

Now from (2), we get

$$\frac{dx}{dt} = -\sqrt{\mu}\sqrt{(a^2 - x^2)}. \qquad ...(4)$$

(Here negative sign is taken because particle is moving in the direction of x decreasing).

Separating the variable in (4), we get

$$dt = -\frac{1}{\sqrt{\mu}}\frac{dx}{\sqrt{a^2 - x^2}}$$

or

$$\sqrt{\mu}\, dt = -\frac{dx}{\sqrt{a^2 - x^2}}.$$

Integrating, we have

$$t\sqrt{\mu} = \cos^{-1}\left(\frac{x}{a}\right) + D, \text{ where } D \text{ is a constant.}$$

Initially at A, $x = a$ and $t = 0$, then

$$0 = \cos^{-1}(1) + D.$$

$\therefore$ $D = 0$

Thus, $\cos^{-1}\left(\frac{x}{a}\right) = t\sqrt{\mu}$

or $$\boxed{x = a\cos(t\sqrt{\mu})} \quad ...(5)$$

when the particle reaches at O *i.e.*, $x = 0$, then the equation (4) gives the velocity $-a\sqrt{\mu}$. The particle thus passes through O and goes to the left of O, where acceleration changes to retardation and therefore the velocity of the particle continuously decreases. Ultimately the particle comes to rest instantaneously at A' such that $OA = OA'$. It then retraces its path and passes through O, and again is instantaneously at rest at A. Hence the particle oscillates about O between A and A'.

Let t_1 be the time taken by the particle to cover the distance from A to O *i.e.*, $x = 0$, then from (5), we get

$$t_1 = \frac{1}{\sqrt{\mu}}\cos^{-1} 0.$$

$\therefore$ $$t_1 = \frac{\pi}{2\sqrt{\mu}}.$$

Now the time of a complete oscillation $= 4t_1$

$$= \frac{2\pi}{\sqrt{\mu}}.$$

This time of a complete oscillation is called the **periodic time.**

• 12.6. SOME IMPORTANT DEFINITIONS

Definition (Periodic time) : *During a simple harmonic motion of a particle, the time taken by the particle to make a complete oscillation, is called* **Periodic time.** *If T is the time period of S.H.M., then*

$$T = \frac{2\pi}{\sqrt{\mu}}.$$

Definition (Amplitude) : *The maximum displacement of a particle during a Simple Harmonic Motion on either side of the centre of force is called an* **amplitude.**

Definition (Frequency) : *The number of complete oscillations in one second is called the frequency of Simple Harmonic Motion.*

Since T is the time period for one complete oscillations, therefore the number of complete oscillations in one second is $\frac{1}{T}$. Further since,

$$T = \frac{2\pi}{\sqrt{\mu}}$$

$\therefore$ $$\text{Frequency} = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\mu}.$$

Definition (Phase and Epoch) : The equation of motion of a particle in S.H.M. is

$$\frac{d^2x}{dt^2} = -\mu x$$

$\therefore$ $$\frac{d^2x}{dt^2} + \mu x = 0.$$

The solution of this differential equation is

$$x = a\cos(\sqrt{\mu}\, t + \phi).$$

The constant ϕ is called the **starting phase** or the **epoch** of the motion and the angle $(\sqrt{\mu}t + \phi)$ is called the **argument** of the motion, whilst the **phase** of the motion at any time t is the time that has elapsed since the particle passed through its maximum distance in the positive direction.

Suppose x is maximum at the time t_0, then

$$\sqrt{\mu}\, t_0 + \phi = 0.$$

Hence the phase at time $t = t - t_0$

$$= t + \frac{\phi}{\sqrt{\mu}}$$

$$= \frac{\sqrt{\mu}t + \phi}{\sqrt{\mu}}.$$

Remarks

- Maximum velocity of the particle in a S.H.M. is $\sqrt{\mu}a$, where a is the amplitude.
- Maximum acceleration at the extreme points is μa.

• 12.7. GEOMETRICAL REPRESENTATION OF S.H.M.

Suppose a particle moves round the circumference of a circle with uniform angular velocity ω.

Let AOA' be the fixed diameter of the circle and let P be the position of the particle at any time t such that angular displacement of P from A is θ, then

$$\omega = \frac{\theta}{t}.$$

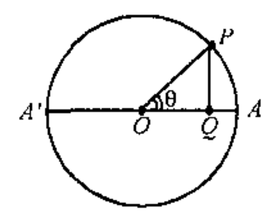


Fig. 5

Draw a perpendicular from P to AOA', whose foot is Q. Let $OQ = x$, then

$$x = a\cos\theta \qquad [\because\ OP = a \text{ (radius)}]$$

or
$$x = a\cos\omega t. \qquad ...(1)$$

Differentiate (1) w.r.t. 't', we get

$$\frac{dx}{dt} = -a\omega\sin\omega t. \qquad ...(2)$$

Again differentiating, we have

$$\frac{d^2x}{dt^2} = -a\omega^2\cos\omega t$$

$$\therefore \qquad \frac{d^2x}{dt^2} = -\omega^2 x. \qquad ...(3)$$

Thus the equation (3) represents that the acceleration of the point Q is directly proportional to the displacement from O and directed towards O. Therefore we get a conclusion that as the particle moves round the circumference of a circle, the foot Q oscillates on AA' about O and the equation (2) represents the velocity of Q at any time. From (1) we see that the amplitude of this S.H.M. is a, because the maximum value of x is obtained as a.

The time period of Q = The time taken by P to turn through an angle

2π with uniform angular velocity, $= \frac{2\pi}{\omega}$.

Hence, we can say that if a particle describes a circle with uniform velocity, then the foot of the perpendicular from its any position on any diameter executes **Simple Harmonic Motion.**

SOLVED EXAMPLES

Example 1. *A particle is moving with S.H.M. and while making an excursion from one position of rest to the other, its distances from the middle point of its path at three consecutive seconds are observed to be x_1, x_2, x_3. Prove that the time of a complete revolution is*

$$2\pi/\cos^{-1}\left(\frac{x_1 + x_3}{2x_2}\right).$$

Solution. Since we have

$$x = a \cos (\sqrt{\mu}\, t). \qquad ...(1)$$

Now x_1, x_2, x_3 are the displacement from the middle point of the path in three consecutive seconds, then

$$x_1 = a \cos \sqrt{\mu}\, t$$

$$x_2 = a \cos \sqrt{\mu}\, (t + 1)$$

and $$x_3 = a \cos \sqrt{\mu}\, (t + 2) \qquad \text{[using (1)]}$$

$\therefore$ $$x_1 + x_3 = a\, [\cos \sqrt{\mu}\, t + \cos \sqrt{\mu}\, (t + 2)]$$

$$= 2a \cos \sqrt{\mu}\, (t + 1) \,.\, \cos \sqrt{\mu}$$

$$= 2x_2 \,.\, \cos \sqrt{\mu}$$

$\therefore$ $$\sqrt{\mu} = \cos^{-1} \left(\frac{x_1 + x_3}{2x_2} \right).$$

The periodic time $$T = \frac{2\pi}{\sqrt{\mu}}$$

$$= 2\pi / \cos^{-1} \left(\frac{x_1 + x_3}{2x_2} \right).$$

Example 2. *In a S.H.M. of amplitude a and period T prove that :*

$$\int_0^T v^2\, dt = \frac{2\pi^2 a^2}{T}.$$

Solution. Since in a S.H.M., we have

$$x = a \cos \sqrt{\mu}\, t$$

$\therefore$ $$v = \frac{dx}{dt} = -\, a \sqrt{\mu} \sin \sqrt{\mu}\, t$$

and $$T = \frac{2\pi}{\sqrt{\mu}}$$

Now, $$\int_0^T v^2\, dt = a^2 \mu \int_0^T \sin^2 \sqrt{\mu}\, t\, dt$$

$$= a^2 \mu \int_0^T \sin^2 \frac{2\pi t}{T}\, dt$$

$$= a^2 \mu \int_0^{2\pi} (\sin^2)\, y\, \frac{T}{2\pi}\, dy \quad \text{put } y = \frac{2\pi t}{T}$$

$$= \frac{a^2 \mu T}{2\pi} \int_0^{2\pi} \sin^2 y\, dy$$

$$= \frac{a^2 \mu T}{2\pi} \left[\frac{1}{2} \int_0^{2\pi} (1 - \cos 2y)\, dy \right]$$

$$= \frac{a^2 \mu T}{2\pi} \cdot \frac{1}{2} \left[y - \frac{\sin 2y}{2} \right]_0^{2\pi}$$

$$= \frac{a^2 \mu T}{2\pi} \cdot \frac{1}{2}\, [2\pi]$$

$$= \frac{a^2 \mu T}{2}$$

$$= \frac{2\pi^2 a^2}{T} \qquad \left(\because\ \mu = \frac{4\pi^2}{T^2} \right)$$

• SUMMARY

- **Motion in a straight line :**

$$\text{Velocity} = \frac{dx}{dt}$$

$$\text{Acceleration} = \frac{d^2x}{dt^2}$$

- **Motion under inverse square law**

$$\text{Acceleration} = -\frac{\mu}{x^2}$$

- **S.H.M.**

$$\text{Acceleration} = -\mu x$$

• STUDENT ACTIVITY

1. If h be the height due to the velocity v at the earth's surface supposing its atttraction constant and H the corresponding height when the variation of gravity is taken into account, prove that

$$\frac{1}{h} = \frac{1}{H} + \frac{1}{r}$$

where r is the earth's radius.

2. In a S.H.M. of amplitude a and period T prove that

$$\int_0^T v^2 dt = \frac{2\pi^2 a^2}{T}$$

• TEST YOURSELF

1. A horizontal shelf is moved up and down with S.H.M. of period 1/2 sec. What is the amplitude admissible in order that a weight placed on the shelf may not be jerked off ?
2. A particle starts from rest under an acceleration k^2x directed towards a fixed point after time t another particle starts from the same position under the same acceleration. Show that the particles will collide at time $\frac{\pi}{k} + \frac{t}{2}$ after the start of the first particle provided $t < \frac{2\pi}{k}$.
3. Define a S.H.M. show that S.H.M. is periodic and its period is independent of the amplitude.
4. Show that if the displacement of a particle in a straight line is expressed by the equation $x = a \cos nt + b \sin nt$, it describes a S.H.M. whose amplitude is $\sqrt{(a^2 + b^2)}$ and period is $\frac{2\pi}{n}$.

5. A point moving in a straight line with S.H.M. has velocities v_1 and v_2 when its distances from the centre of force are x_1 and x_2. Show that the period of motion is

$$2\pi\sqrt{\left(\frac{x_1^2 - x_2^2}{v_2^2 - v_1^2}\right)}.$$

ANSWERS

1. $g/16\pi^2$.

OBJECTIVE EVALUATION

Fill in the Blanks :

1. The expressions $\frac{d^2x}{dt^2}, \frac{dv}{dt}$ and $v\frac{dv}{dx}$ are of ....
2. Earth attracts every body outside its surface with an acceleration which follows the law of ..... .
3. Inside the earth's surface, the acceleration is proportional to ....
4. In S.H.M. the acceleration is always towards .... and proportional to ....

True or False :

Write T for true and F for false statements :

1. Outside the earth's surface, the particle follows inverse square law. *(T/F)*
2. In S.H.M. the acceleration of the particle is always towards the centre of motion. *(T/F)*
3. If μ is the intensity of a force under which a particle is executing S.H.M., then its time period is $\frac{2\pi}{\mu}$. *(T/F)*
4. In S.H.M., the maximum velocity is obtained at the centre of motion. *(T/F)*

Multiple Choice Quesitons (MCQ's) :

Choose the most appropriate one :

1. Inside the earth's surface, the acceleration of the particle is proportional to :
   (a) (distance) (b) 1/(distance) (c) $(\text{distance})^2$ (d) none of these.
2. Outside the earth's surface, the acceleration of the particle is proportional to :
   (a) distance (b) 1/distance (c) $1/(\text{distance})^2$ (d) $(\text{distance})^2$.
3. Maximum velocity of the particle in S.H.M. is :
   (a) μa^2 (b) $\sqrt{\mu}a$ (c) μa (d) μ/a.
4. Maximum value of acceleration in S.H.M. is :
   (a) μa (b) $\sqrt{\mu}a$ (c) μa^2 (d) $\mu^2 a$.

ANSWERS

Fill in the Blanks :

1. Acceleration 2. Inverse square 3. Distance 4. Centre of motion, distance

True or False :

1. T 2. T 3. F 4. T

Multiple Choice Questions :

1. (a) 2. (c) 3. (b) 4. (a)

❑❑❑

UNIT 13

MOMENTS OF INERTIA

STRUCTURE

- Some simple cases of Moment of Inertia
- Parallel and Perpendicular axes Theorems
  - Summary
  - Student Activity
  - Test Yourself

LEARNING OBJECTIVES

After going through this unit you will learn :

- What is moment of inertia ?
- How to find the moment of inertia of the given body about the given line or axes.

13.1. SOME SIMPLE CASES OF MOMENT OF INERTIA

(1) Moment of Inertia of Uniform Rod of Length $2a$:

(a) *To find the moment of inertia of uniform rod of length $2a$ and mass M about a line through one end perpendicular to the rod.*

Let M be the mass of a uniform rod AB of length $2a$, then the mass per unit of length of the rod is $\frac{M}{2a}$.

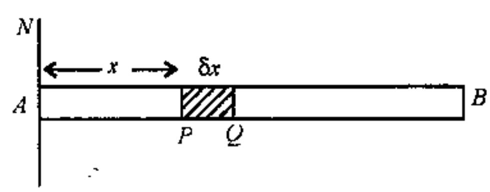


Fig. 1

Let us consider an element PQ of legnth δx distant x apart from an end A. Let NA be a line through A and perpendicular to AB.

$\therefore$ Mass of an element $PQ = \frac{M}{2a}\,\delta x.$

The moment of inertia of this element about the line NA is

$$\frac{M}{2a}\,\delta x \,.\, x^2.$$

Thus the moment of inertia of the whole rod about NA is

$$\int_{x=0}^{x=2a} \frac{M}{2a} x^2\, dx = \frac{M}{2a}\int_{x=0}^{x=2a} x^2\, dx$$

$$= \frac{M}{2a}\left[\frac{x^3}{3}\right]_0^{2a}$$

$$= \frac{4}{3} Ma^2.$$

(b) *To find the moment of inertia of a uniform rod of length $2a$ about a line through the middle point and perpendicular to it.*

Let M be the mass of the rod AB of length $2a$ and OL the line through the middle point O (say) of the rod.

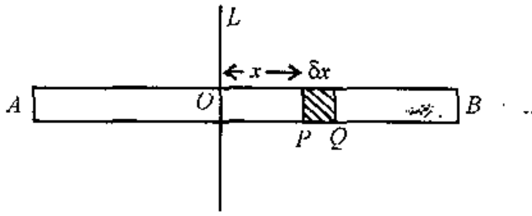


Fig. 2

Let us consider an element PQ of width δx at a distance x from the line OL. Then the mass of this element is $\frac{M}{2a}\delta x$.

The moment of inertia of this element PQ about OL

$$= \frac{M}{2a}\delta x \, . \, x^2.$$

Thus the moment of inertia of the rod about OL is

$$\int_{x=-a}^{x=a} \frac{M}{2a} x^2 \, dx \qquad (\because x \text{ takes the values from } - OA \text{ to } OB)$$

$$= \frac{M}{2a}\int_{-a}^{a} x^2 \, dx$$

$$= \frac{M}{2a}\left[\frac{x^3}{3}\right]_{-a}^{a}$$

$$= \frac{M}{2a}\left[\frac{a^3}{3} + \frac{a^3}{3}\right]$$

$$= \frac{1}{3} \boldsymbol{Ma^2}.$$

(2) Moment of Inertia of a Rectangular Lamina :

(a) *To find the moment of inertia of a rectangular lamina about a line through the centre and parallel to a side.*

Let $ABCD$ be a rectangular lamina of side $AB = 2a$ and $AD = 2b$ and let M be the mass of this lamina. Then the mass per unit of area is $\frac{M}{4ab}$.

Let OL be a line through O and parallel to AB about which the moment of inertia is to be required.

Let us consider an elementary strip of breadth δx and of length $2b$ at the distance x from O and parallel to AD.

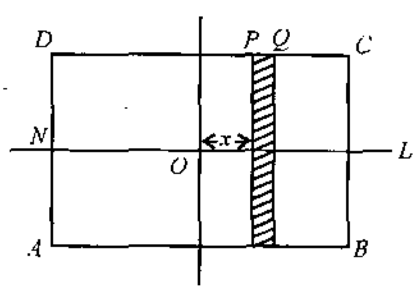


Fig. 3

$\therefore$ The mass of this elementary strip $= \frac{M}{4ab}(\delta x \, . \, 2b)$

$$= \frac{M}{2a}\delta x.$$

The moment of inertia of this strip about $LN = \frac{M}{2a}\delta x\left(\frac{b^2}{3}\right)$.

Thus the moment of inertia of the rectangular lamina about LN

$$= \int_{x=-a}^{x=a} \frac{M}{2a}\left(\frac{b^2}{3}\right) dx$$

$$= \frac{Mb^2}{6a}\int_{-a}^{-a} dx$$

$$= \frac{Mb^2}{6a}[a+a]$$
$$= \frac{1}{3}Mb^2.$$

Hence the moment of inertia of the rectangular lamina about a line through the centre and parallel to the side $2a$ is $\frac{1}{3}Mb^2$. Similarly the moment of inertia of the rectangular lamina about a line through the centre and parallel to the side $2b$ is $\frac{1}{3}Ma^2$.

(b) *To find the moment of inertia of a rectangular lamina about a line through the centre and perpendicular to the plane of lamina.*

Let OL be the line through the centre O of the lamina $ABCD$ and perpendicular to the lamina.

Let us consider an element $PQRS$ of area $\delta x\, \delta y$ at a distance $\sqrt{x^2+y^2}$ from O.

$\therefore$ The mass of this element $PQRS = \frac{M}{4ab}\delta x\, \delta y$.

The moment of inertia of this element about $OL = \frac{M}{4ab}\delta x\,.\,\delta y\,(\sqrt{x^2+y^2})^2$.

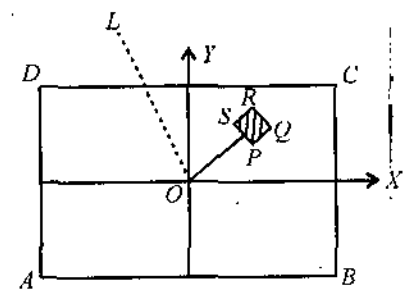


Fig. 4

Thus the moment of inertia of the lamina about OL

$$= \int_{x=-a}^{x=a}\int_{y=-b}^{y=b}\frac{M}{4ab}(x^2+y^2)\,dx\,dy$$
$$= \frac{M}{4ab}\,.\,4\int_0^a\int_0^b(x^2+y^2)\,dx\,dy$$
$$= \frac{M}{ab}\int_0^a\left[x^2y+\frac{y^3}{3}\right]_0^b dx$$
$$= \frac{M}{ab}\int_0^a\left(bx^2+\frac{b^3}{3}\right)dx$$
$$= \frac{M}{ab}\left[b\frac{x^3}{3}+\frac{b^3}{3}x\right]_0^a$$
$$= \frac{M}{ab}\left[\frac{a^3b}{3}+\frac{ab^3}{3}\right]$$
$$= \frac{1}{3}M(a^2+b^2).$$

(3) Moment of Inertia of a Rectangular Parallelopiped :

To find the moment of inertia of a rectangular parallelopiped.

Let $2a, 2b, 2c$ be the lengths of the sides of a rectangular parallelopiped. Take the centre of the parallelopiped as origin O and OX, OY and OZ parallel to the sides as mutually perpendicular axes.

Conceive the rectangular parallelopiped as made up of a very large number of thin parallel rectangular lamina (slices) all perpendicular to OX and consider one of such elementary slice $PQRS$ of width δx at the distance x from O. Let ρ be the mass per unit volume of the parallelopiped.

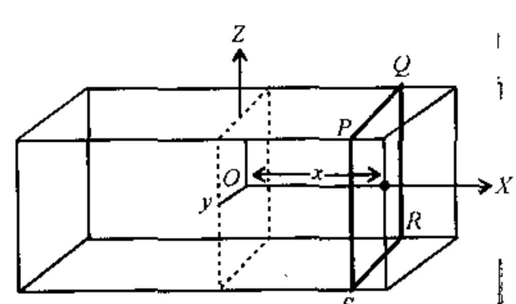


Fig. 5

$\therefore$ The mass of the element $PQRS = 2b\,.\,2c\,.\,\delta x\rho$

The moment of inertia of this element about OX

$$= \frac{(2b \cdot 2c \cdot \delta x \rho)}{3} (b^2 + c^2) \qquad \text{[see § 4.2 (2) (b)]}$$

Thus the moment of inertia of the rectangular parallelopiped about OX

$$= \int_{x=-a}^{x=a} \frac{2b \cdot 2c \cdot \rho}{3} (b^2 + c^2)\, dx$$

$$= \frac{4bc\rho}{3} (b^2 + c^2) \int_{-a}^{a} dx$$

$$= \frac{4bc\rho}{3} (b^2 + c^2) [x]_{-a}^{a}$$

$$= \frac{4bc\rho}{3} (b^2 + c^2) [a + a]$$

$$= \frac{8abc\rho}{3} (b^2 + c^2)$$

$$= \frac{M}{3} (b^2 + c^2) \qquad (\because M = 8abc\rho)$$

Hence the moment of inertia of the rectangular parallelopiped about a line through the centre and parallel to the side $2a$ is $\frac{M}{3}(b^2 + c^2)$.

Similarly M.I. of the parallelopiped about the lines through the centre and parallel to the side $2b$ and $2c$ are respectively, $\frac{M}{3}(a^2 + c^2)$ and $\frac{M}{3}(a^2 + b^2)$.

(4) Moment of Inertia of a Circular Ring :

(a) *To find the moment of inertia of a circular ring about its diameter.*

Let AB be the diameter of a circular ring of radius a with centre O as origin and OX as x-axis.

Let us consider an elementary arc $PQ = a\,\delta\theta$, then the mass of this element is $\rho\, a\delta\theta$.

The perpendicular distance of this element from $OX = PN = a \sin\theta$.

$\therefore$ The moment of inertia of this element about

$$OX = \rho\, a\delta\theta\, (a \sin\theta)^2.$$

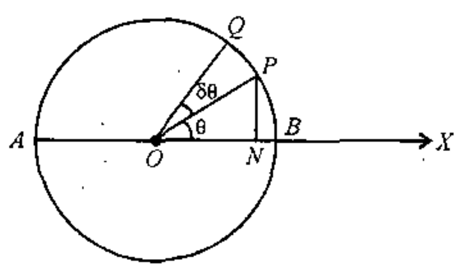


Fig. 6

Thus the moment of inertia of the circular ring about OX

$$= \int_{\theta=0}^{2\pi} \rho a\, (a \sin\theta)^2\, d\theta$$

$$= \rho a^3 \int_0^{2\pi} \sin^2\theta\, d\theta$$

$$= 4\rho a^3 \int_0^{\pi/2} \sin^2\theta\, d\theta$$

$$= 4\rho a^3 \left[\frac{1}{2} \cdot \frac{\pi}{2}\right]$$

$$= \pi \rho a^3$$

$$= \frac{M}{2} a^2. \qquad \left(\because \rho = \frac{M}{2\pi a}\right)$$

(b) *To find the moment of inertia of the circular ring about a line through the centre and perpendicular to the plane of the ring.*

Let OL be a line through the centre O of a circular ring and perpendicular to the plane of the ring.

Consider an elementary arc $PQ = a\,\delta\theta$. Then the mass of this element is $\rho\, a\delta\theta$ and the perpendicular distance of this element from the line OL is a.

$\therefore$ The moment of inertia of this element about OL

$$= \rho\, a\delta\theta \,.\, (a)^2.$$

Thus the moment of inertia of the circular ring about OL

$$= \int_0^{2\pi} \rho\, a^3\, d\theta$$

$$= \rho\, a^3 \int_0^{2\pi} d\theta$$

$$= \rho\, a^3 \Big[\theta\Big]_0^{2\pi}$$

$$= 2\pi\, \rho a^3$$

$$= Ma^2 \qquad \left(\because\ \rho = \frac{M}{2\pi a}\right)$$

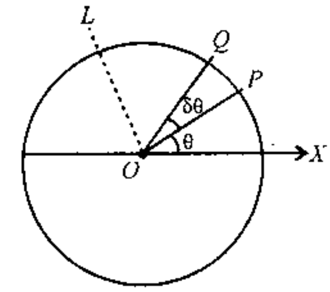


Fig. 7

(5) Moment of Inertia of a Circular Disc :

(a) *To find the moment of inertia of circular disc about the diameter.*

Let AB be the diameter of a circular disc of radius a with the centre O as origin and OX as x-axis.

Let ρ be the mass per unit area of the disc. Then we have $\rho = \dfrac{M}{\pi a^2}$. Let us consider two circles of radius r and $r + \delta r$ with centre O and form a circular ring. The area of this circular ring is $2\pi r\,\delta r$ and thus its mass is $2\pi\, r\delta r\, \rho$. Suppose the disc is made up of a very large number of such circular rings.

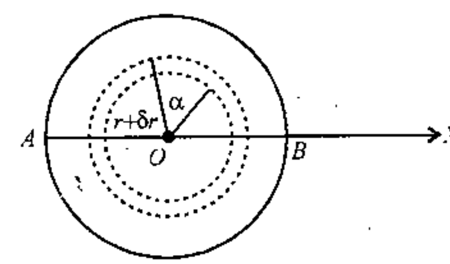


Fig. 8

$\therefore$ The moment of this circular ring about

$$OX = \frac{(2\pi r\, \rho\delta r)}{2}\, r^2$$

$$= \pi\, \rho r^3 \delta r$$

Thus the moment of the circular disc about OX

$$= \int_{r=0}^{a} \pi\, \rho r^3\, dr$$

$$= \pi\, \rho \int_{r=0}^{r=a} r^3\, dr$$

$$= \pi\rho \left[\frac{r^4}{4}\right]_0^a$$

$$= \frac{\pi\rho}{4} a^4 = \frac{Ma^2}{4} \qquad \left(\because\ \rho = \frac{M}{\pi a^2}\right)$$

Hence the moment of inertia of the circular disc of radius a about its diameter is $\dfrac{1}{4}\boldsymbol{Ma^2}$.

(b) *To find the moment of inertia of a circular disc about the line through the centre and perpendicular to the plane of the disc.*

Let OL be a line through the centre O of the circular disc and perpendicular to its plane.

Let us consider an element $PQRS$ of area $r\,\delta r\,\delta\theta$ at the distance r from the line OL. Then the mass of this element is $\rho\, r\delta r\,\delta\theta$.

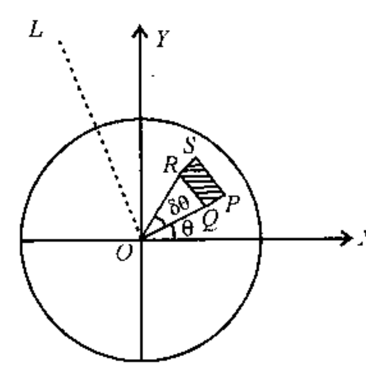


Fig. 9

$\therefore$ The moment of inertia of this element about $OL = \rho r\, \delta r\, \delta\theta\, (r)^2$.

Thus the moment of inertia of circular disc about OL

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{a} \rho\, r^3\, dr\, d\theta$$

$$= \rho \int_{\theta=0}^{2\pi} \left[\frac{r^4}{4}\right]_0^a d\theta$$

$$= \frac{\rho\, a^4}{4} \int_0^{2\pi} d\theta$$

$$= \frac{\rho\, a^4}{4} \left[\theta\right]_0^{2\pi} = \frac{\rho a^4}{4}\,(2\pi)$$

$$= \frac{\rho\, a^4 \pi}{2}$$

$$= \frac{Ma^2}{2} \qquad \left(\because\ \rho = \frac{M}{\pi a^2}\right)$$

Hence the moment of inertia of a circular disc of radius a about the line through the centre and perpendicular to its plane is $\frac{Ma^2}{2}$.

(6) Moment of Inertia of an Elliptic Disc :

(a) *To find the moment of inertia of an elliptic disc of axes* $2a$, *and* $2b$ *about its major axis.*

Let OX and OY be the major and minor axes of an elliptic disc, where O is the centre of it. Let ρ be the mass per unit area of the disc.

Suppose the elliptic disc is made up of a very large number of slices all perpendicular to OX and consider an elementary such slice PQ of width δx parallel to OY with the co-ordinates of P as $(a \cos\theta, b \sin\theta)$.

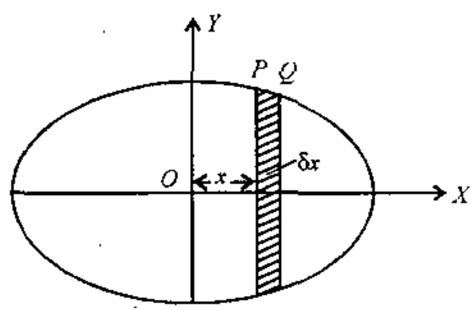


Fig. 10

$\therefore\quad \delta x = \delta\,(a \cos\theta) = -\,a \sin\theta\, d\theta$

and length of the slice $PQ = 2b \sin\theta$.

$\therefore$ The mass of this elementary slice $PQ = \rho\,(2b \sin\theta)\, \delta x$

$$= \rho\,(2b \sin\theta)\,(-\,a \sin\theta\, \delta\theta)$$

$$= -\,2ab\, \rho \sin^2\theta\, \delta\theta$$

The moment of inertia of this element about OX is,

$$= \frac{(-\,2ab\, \rho \sin^2\theta\, \delta\theta)}{3}\, y^2$$

$$= -\frac{2}{3}\, ab\, \rho \sin^2\theta\, (b \sin\theta)^2\, \delta\theta$$

$$= -\frac{2}{3}\, ab^3\, \rho \sin^4\theta\, \delta\theta$$

Thus the moment of inertia of the elliptic disc about OX

$$= \int_{\theta=0}^{\theta=\pi} \frac{2}{3}\, ab^3\, \rho \sin^4\theta\, d\theta \qquad \text{(ignore the negative sign)}$$

$$= \frac{2}{3}\, ab^3\, \rho \int_0^{\pi} \sin^4\theta\, d\theta$$

$$= \frac{4}{3}\, ab^3\, \rho \int_0^{\pi/2} \sin^4\theta\, d\theta$$

$$= \frac{4}{3} ab^3 \rho \left[\frac{(4-1)(4-3)}{4.2} \cdot \frac{\pi}{2} \right]$$

$$= \frac{1}{4} ab^3 \rho\pi$$

$$= \frac{1}{4} ab^3 \pi \left[\frac{M}{\pi ab} \right] \qquad \left(\because \ \rho = \frac{M}{\pi ab} \right)$$

$$= \frac{1}{4} Mb^2.$$

Hence the moment of inertia of an elliptic disc about the major axis is $\frac{1}{4}Mb^2$.

Similarly the moment of inertia of the elliptic disc about minor axis 2b is $\frac{1}{4}Ma^2$.

(b) *To find the moment of inertia of an elliptic disc about the line through the centre and perpendicular to its plane.*

Let OL be a line through the centre and perpendicular to the plane of an elliptic disc.

Let us consider an element $PQRS$ of area $\delta x\, \delta y$ at the distance $\sqrt{x^2 + y^2}$ from the line OL.

The mass of this element $PQRS = \rho\, \delta x\, \delta y$. Therefore, the moment of inertia of this element about OL is,

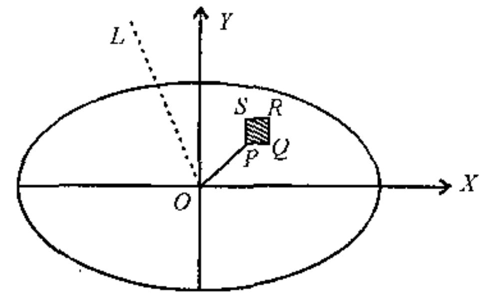


Fig. 11

$$= \rho\, \delta x\, \delta y\, (x^2 + y^2)$$

Thus the moment of inertia of whole elliptic disc about OL

$$= \int_{x=-a}^{a} \int_{y=-b}^{b} \rho\, (x^2 + y^2)\, dx\, dy$$

$$= \rho \int_{-a}^{a} \left[x^2 y + \frac{y^3}{3} \right]_{-b}^{b} dx$$

$$= 2\rho \int_{-a}^{a} \left(x^2 b + \frac{b^3}{3} \right) dx$$

$$= 2\rho \left[\frac{x^3 b}{3} + \frac{b^3 x}{3} \right]_{-a}^{a}$$

$$= 4\rho \left[\frac{a^3 b + b^3 a}{3} \right]$$

$$= \frac{4}{3} \rho\, ab\, (a^2 + b^2)$$

$$= \frac{4}{3\pi} M\, (a^2 + b^2) \qquad \left(\because \ \rho = \frac{M}{\pi ab} \right)$$

Hence the moment of inertia of an elliptic disc about the line through the centre and perpendicular to its plane is $\frac{4}{3\pi} M (a^2 + b^2)$.

(7) Moment of Inertia of a Hollow Sphere :

Hollow Sphere. When a semi-circular arc is revolved about its bounding diameter, the surface thus generated is called **hollow sphere.**

To find the moment of inertia of a hollow sphere about its diameter.

Let AOB be the diameter of a hollow sphere of radius a and ρ be the mass per unit surface area of the sphere.

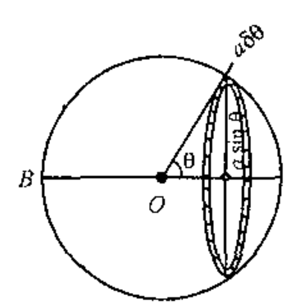


Fig.12

Consider an elementary arc $a\,\delta\theta$, which when revolved about the diameter AB, a circular ring of radius $a \sin\theta$ is generated. Therefore, the mass of this elementary ring $= \rho\,(2\pi a \sin\theta)\,.\,(a\,\delta\theta)$.

$\therefore$ The moment of inertia of this elementary ring about AB

$$= \rho\,(2\pi a \sin\theta)\,(a\delta\theta)\,.\,(a \sin\theta)^2 \qquad \text{(see § 4.2 (4) (b))}$$

Thus the moment of inertia of hollow sphere about AOB

$$= \int_{\theta=0}^{\pi} \rho\,(2\pi a \sin\theta)\,(a\,d\theta)\,.\,(a \sin\theta)^2$$

$$= 2\pi a^4 \rho \int_0^{\pi} \sin^3\theta\, d\theta$$

$$= 4\pi a^4 \rho \int_0^{\pi/2} \sin^3\theta\, d\theta$$

$$= 4\pi a^4 \rho \left[\frac{(3-1)}{3}\,.\,1\right]$$

$$= \frac{8}{3}\pi a^4 \rho$$

$$= \frac{2}{3} M a^2 \qquad \left(\because\ \rho = \frac{M}{4\pi a^2}\right)$$

Hence the moment of inertia of a hollow sphere about its diameter is $\frac{2}{3} Ma^2$.

(8) Moment of Inertia of a Solid Sphere :

Solid sphere. *When a semi-circular area is revolved about its diameter, the solid thus generated is called* **solid sphere.**

To find the moment of inertia of a solid sphere about its diameter.

Let AOB be the diameter of a solid sphere of radius a and ρ be the mass per unit volume of the solid sphere.

Let us consider an elementary area $PQRS = r\delta\theta\,\delta r$ at the distance r from the centre O. When this elementary area is revolved about the diameter AOB, a ring of cross-section $r\delta\theta\delta r$ and radius $r \sin\theta$ thus generated.

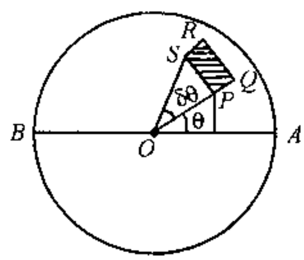


Fig. 13

$\therefore$ The mass of this elementary ring

$$= \rho\,(2\pi r \sin\theta)\,.\,r\delta\theta\,\delta r$$

The moment of inertia of this elementary ring about AOB

$$= \rho\,(2\pi r \sin\theta)\,(r\delta\theta\,\delta r)\,.\,(r \sin\theta)^2$$

Thus the moment of inertia of whole solid sphere about AOB

$$= \int_{\theta=0}^{\pi} \int_{r=0}^{r=a} \rho\,(2\pi r \sin\theta)\,(r^2 \sin^2\theta)\, r\, dr\, d\theta$$

$$= 2\pi\rho \int_0^{\pi} \int_0^{a} r^4 \sin^3\theta\, dr\, d\theta$$

$$= 2\pi\rho \int_0^{\pi} \left[\frac{r^5}{5}\right]_0^a \sin^3\theta\, d\theta$$

$$= \frac{2\pi\rho a^5}{5} \int_0^{\pi} \sin^3\theta\, d\theta$$

$$= 2\,.\,\frac{2}{5}\pi\rho\, a^5 \int_0^{\pi/2} \sin^3\theta\, d\theta$$

$$= \frac{4}{5}\pi\rho a^5 \left[\frac{2}{3}\right]$$

$$= \frac{8}{15}\pi\rho a^5$$

$$= \frac{2}{5}Ma^2 \qquad \left(\because \; \rho = \frac{M}{\frac{4}{3}\pi a^3}\right)$$

Hence the moment of inertia of a solid sphere of radius a and mass M about its diameter is $\frac{2}{5}Ma^2$.

• 13.2. THE THEOREMS OF PARALLEL AND PERPENDICULAR AXES

(1) Theorem of Parallel axis. *If the moments and products of inertia about any line or lines through the centre of gravity of a body, are given, to find the moments and products of inertia about parallel line or lines.*

Let $G(\bar{x}, \bar{y}, \bar{z})$ be the centre of gravity of a rigid body and let GX', GY', GZ' be axes taken through G parallel to OX, OY and OZ through O. Let (x', y', z') be the new co-ordinates of P with respect to the axes GX', GY' and GZ' while the co-ordinates of P with respect to OX, OY and OZ is (x, y, z), so that

$$x = \bar{x} + x',\; y = \bar{y} + y',\; z = \bar{z} + z'.$$

∴ The moment of inertia of the body about OX

$$A = \Sigma m\,(y^2 + z^2)$$
$$= \Sigma m\,\{(\bar{y} + y')^2 + (\bar{z} + z')^2\}$$
$$= \Sigma m\,\{\bar{y}^2 + \bar{z}^2 + y'^2 + z'^2 + 2\bar{y}y' + 2\bar{z}z'\}$$
$$= \Sigma m\,(\bar{y}^2 + \bar{z}^2) + \Sigma m\,(y'^2 + z'^2) + 2\bar{y}\,\Sigma my' + 2\bar{z}\,\Sigma mz'$$
$$= \Sigma m\,(\bar{y}^2 + \bar{z}^2) + \Sigma m\,(y'^2 + z'^2)$$

($\because \Sigma my' = 0 = \Sigma mz'$, from the centroid property)

$$A = M(\bar{y}^2 + \bar{z}^2) + A'$$

where $M = \Sigma m$, the total mass of the body; $A' = \Sigma m\,(y'^2 + z'^2)$, the moment of inertia about the parallel X'-axis through G.

or $$A = A' + Mh^2 \qquad \ldots(1)$$

where $h = \sqrt{\bar{y}^2 + \bar{z}^2}$, the distance of the centre of gravity from X-axis through O. Thus equation (1) is the *parallel axes theorem for moment of inertia.*

(2) Theorem of Perpendicular Axis for a Lamina distribution. *If the moments and products of inertia of a plane lamina about two perpendicular axes in the plane of lamina are given; to find the moments and products of inertial about any other axis through the intersection of two perpendicular axes.*

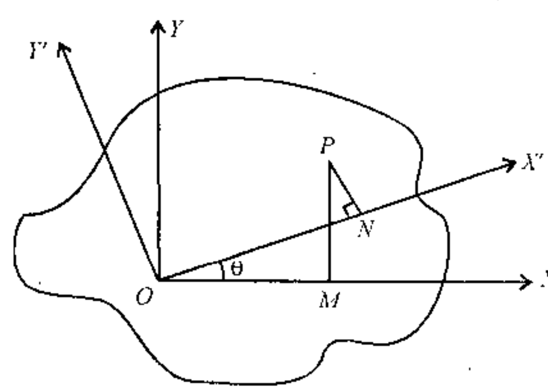


Fig. 15

Let A and B be the moments of inertia and F be the product of inertia about the axes OX and OY in the plane. Let us consider an elementary mass m of a rigid body at $P(x, y)$ with respect to axes OX and OY, then we have

$A = \Sigma my^2$, $B = \Sigma mx^2$ and $F = \Sigma mxy$.

If (x', y') be the co-ordinates of a point P with respect to new system of co-ordinate axes OX' and OY' such that $\angle XOX' = \theta$.

Then we have,

$$x = x'\cos\theta - y'\sin\theta$$

and $$y = x'\sin\theta + y'\cos\theta.$$

∴ $$x' = x\cos\theta + y\sin\theta$$

and $$y' = -x\sin\theta + y\cos\theta$$

Thus the moment of inertia of the rigid body about OX' is

$$\begin{aligned} A' &= \Sigma m y'^2 \\ &= \Sigma m\,(-x\sin\theta + y\cos\theta)^2 \\ &= \Sigma m\,(x^2\sin^2\theta + y^2\cos^2\theta - 2xy\sin\theta\cos\theta) \\ &= \sin^2\theta\,\Sigma mx^2 + \cos^2\theta\,\Sigma my^2 - 2\sin\theta\cos\theta\,\Sigma mxy \\ &= B\sin^2\theta + A\cos^2\theta - F\sin 2\theta. \end{aligned}$$

$\therefore$ $$\boldsymbol{A' = A\cos^2\theta + B\sin^2\theta - F\sin 2\theta.}$$

Remarks

- If A and B be the moments of inertia about any two perpendicular lines in a plane, then the moment of inertia about a line through the point of intersection of the perpendicular lines and perpendicular to the plane is

$$\begin{aligned} \Sigma m\,(x^2 + y^2) &= \Sigma mx^2 + \Sigma my^2 \\ &= \boldsymbol{A + B.} \end{aligned}$$

SOLVED EXAMPLES

Example 1. *Find the moment of inertia of a hollow sphere about a diameter, its external and internal radii being b and a.*

Solution. Let us consider a spherical shell of radius x such that $a < x < b$. Let δx be the width of this shell and ρ be the mass per unit volume of the hollow sphere.

$\therefore$ Mass of this spherical shell

$$= 4\pi\rho x^2 \,.\, \delta x.$$

The moment of inertia of this shell about the diameter

$$= \frac{2}{3}(4\pi\rho x^2 \,.\, \delta x)\, x^2 \qquad \left(\because \text{ M.I.} = \frac{2}{3}Ma^2\right)$$

Thus the moment of inertia of the given hollow sphere about a diameter

$$\begin{aligned} &= \int_{x=a}^{b} \frac{2}{3}\, 4\pi\rho\, x^4\, dx \\ &= \frac{8}{3}\pi\rho \int_a^b x^4\, dx \\ &= \frac{8}{3}\pi\rho \left[\frac{x^5}{5}\right]_a^b \\ &= \frac{8}{3}\pi\rho\, \frac{(b^5 - a^5)}{5} \\ &= \frac{2}{5} M \frac{(b^5 - a^5)}{(b^3 - a^3)} \qquad \left(\because\ M = \frac{4}{3}\pi\,(b^3 - a^3)\,\rho\right) \end{aligned}$$

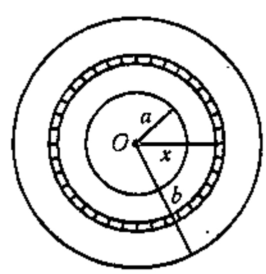


Fig. 17

• SUMMARY

- **Moment of inertia of a rod of length $2a$ and mass M**

  (i) About a line perpendicular to the rod through its centre $= \frac{1}{3}Ma^2$

  (ii) About a line perpendicular to the rod through its one end $= \frac{4}{3}Ma^2$.

- **M.I. of a rectangular lamina of sides $2a$, $2b$ and mass M**

  (i) M.I. about a line through its centre and parallel to side $2a = \frac{1}{3}Mb^2$.

  (ii) M.I. about a line through its centre and parallel to side $2b = \frac{1}{3}Ma^2$.

(iii) M.I. about a line through its centre and perpendicular to its plane $= \frac{1}{3} M (a^2 + b^2)$.

- M.I. of rectangulr parallelopiped of sides $2a, 2b, 2c$ and mass M

(i) M.I. of about a line through its centre and parallel to edge $2a = \frac{1}{3} M (b^2 + c^2)$

(ii) M.I. of about a line through its centre and parallel to edge $2b = \frac{1}{3} M (a^2 + c^2)$

(iii) M.I. of about a line through its centre and parallel to edge $2c = \frac{1}{3} (M (a^2 + b^2)$.

- **M.I. of circular ring of radius a, $2b$, $2c$ and mass M**

(i) M.I. about a diameter $= \frac{1}{2} Ma^2$

(ii) M.I. about a line through its centre and perpendicular to its plane $= Ma^2$

- **M.I. of circular disc of radius a and mass M**

(i) M.I. about a diameter $= \frac{1}{4} Ma^2$

(ii) M.I. about a line through its centre and perpendicular to its plane $= \frac{1}{2} Ma^2$.

- **M.I. of elliptic disc of axes $2a$, $2b$ and mass M**

(i) M.I. about the major axis $2a = \frac{1}{4} Mb^2$

(ii) M.I. about the minor axis $2b = \frac{1}{4} Ma^2$

(iii) M.I. about a line through its centre and perpendicular to its plane $= \frac{4}{3\pi} M (a^2 + b^2)$.

- **M.I. of a hollow sphere of radius a and mass M**

(i) M.I. about the diameter $= \frac{2}{3} Ma^2$.

(ii) M.I. about its tangent $= \frac{5}{3} Ma^2$.

- **M.I. of a solid sphere of radius a and mass M**

(i) M.I. about the diameter $= \frac{2}{5} Ma^2$

(ii) M.I. about its tangent $= \frac{7}{5} Ma^2$

• STUDENT ACTIVITY

1. Find M.I. of a uniform rod of length $2a$ and mass M about a line through its one end and perpendicular to the rod.

2. Find M.I. of a solid sphere of radius a and mass M about its diameter.

• TEST YOURSELF

1. Find the moment of inertia of a circular area about a line in its own plane whose perpendicular distance from its centre is c.
2. Find the moment of inertia of an isosceles triangle about a perpendicular from the vertex upon the opposite side.
3. Find the moment of inertia of the arc of circle about
   (i) the diameter bisecting the arc
   (ii) an axis through the centre, perpendicular to its plane
   (iii) an axis through its middle point perpendicular to its plane.

OBJECTIVE EVALUATION

Fill in the Blanks :

1. The moment of inertia of a uniform rod of length $2a$ about a line through its middle point and perpendicular to it is ....
2. M.I. of a circular ring of radius a and mass M about its diameter is ....
3. M.I. of a circular disc of radius a and mass M about a line through its centre and perpendicular to its plane is ....

True or False :

1. The moment of inertia of uniform rod of length $2a$ and mass M about a line through one end is $\frac{4}{3}Ma^2$. *(T/F)*
2. If M be the mass of the rigid body and I its moment of inertia about an axis, then its radius of gyration about its axis is given by $\sqrt{I/M}$. *(T/F)*
3. M.I. of a circular ring of radius a and mass M about a line through its centre and perpendicular to its plane is Ma^2. *(T/F)*
4. M.I. of a circular disc of radius a and mass M about its diameter is $\frac{1}{4}Ma^2$. *(T/F)*

Multiple Choice Questions (MCQ's) :

Choose the most appropriate one :

1. M.I. of a thin uniform rod of length $2a$ and mass M about an axis through one end and perpendicular to it is :
   (a) $\frac{4}{3}Ma^2$ (b) $\frac{1}{4}Ma^2$ (c) $\frac{1}{2}Ma^2$ (d) Ma^2.
2. M.I. of a rectangular plate of sides $2a$ and $2b$ and mass M about a line through its centre parallel to the side $2a$ is :
   (a) $\frac{2}{3}Mb^2$ (b) $\frac{1}{3}Ma^2$ (c) $\frac{1}{3}Mb^2$ (d) $\frac{2}{3}Ma^2$.
3. M.I. of a circular ring of radius a and mass M about its diameter is :
   (a) Ma^2 (b) $\frac{1}{4}Ma^2$ (c) $\frac{1}{3}Ma^2$ (d) $\frac{1}{2}Ma^2$.

ANSWERS

1. $\frac{1}{2}M(a^2+2c^2)$ **2.** $\frac{1}{24}Ma^2$, a is length of opposite side

3. (i) $\frac{Ma^2}{2\alpha}(\alpha - \sin\alpha\cos\alpha)$ as $M = 2\alpha a\rho$ (ii) Ma^2 (iii) $\frac{2Ma^2}{\alpha}(\alpha - \sin\alpha)$

Fill in the Blanks :

1. $\frac{1}{3}Ma^2$ **2.** $\frac{1}{2}Ma^2$ **3.** $\frac{1}{2}Ma^2$

True or False :

1. T **2.** T **3.** T. **4.** T

Multiple Choice Questions :

1. (a) **2.** (c) **3.** (d)

UNIT 14

D'ALEMBERT'S PRINCIPLE

STRUCTURE

LEARNING OBJECTIVES

After going through this unit you will learn :

- What is D'Alembert's principle ?
- How to apply D'Alembert's principle to solve the given questions ?

• 14.1. IMPRESSED AND EFFECTIVE FORCES

Impressed forces. *The external forces acting on a rigid body are called* **impressed forces.** For examples, Gravitational force and Magnetic force, and weight of the body etc.

Effective forces. *When a rigid body is in motion then the effective force on the body is defined as the product of its mass and its acceleration.*

If m denotes the mass of a moving particle and (x, y, z) be the co-ordinates of the particle at any time t, then the components of the effective force on the particle are $m\frac{d^2x}{dt^2}$, $m\frac{d^2y}{dt^2}$ and $m\frac{d^2z}{dt^2}$ parallel to x, y and z-axes respectively.

Remark

➤ $-m\frac{d^2x}{dt}$, $-m\frac{d^2y}{dt^2}$, and $-m\frac{d^2z}{dt^2}$ are the components of reversed effective force.

• 14.2. D'ALEMBERT'S PRINCIPLE

Statement. *The reversed effective forces acting on each particle of the moving rigid body and the impressed forces on body are in equilibrium.*

Proof. Let a rigid body be in motion and $\overline{r}$ be the position vector of a particle of mass m at any time t, then $\frac{d^2\overline{r}}{dt^2}$ is the acceleration of the particle. Suppose $\overline{F}$ and $\overline{R}$ be the external and internal forces acting on it, then the equation of motion of the paricle is

$$m\frac{d^2\overline{r}}{dt^2} = \overline{F} + \overline{R} \qquad ...(1)$$

[By Newton's second law of motion]

or
$$\left(-m\frac{d^2\overline{r}}{dt^2}\right) + \overline{F} + \overline{R} = \overline{0}.$$

This equation shows that the three forces $-m\frac{d^2\overline{r}}{dt}$, $\overline{F}$ and $\overline{R}$ are in equilibrium.

Now applying the same hypothesis to each particle of the rigid body, the forces

$$\Sigma\left(-m\frac{d^2\bar{r}}{dt^2}\right),\ \Sigma\bar{F}\ \text{and}\ \Sigma\bar{R}$$

are in equilibrium, where Σ runs over each particle of the rigid body.

But the internal forces acting on the rigid body form pairs of equal and opposite forces, therefore,

$$\Sigma\bar{R} = \bar{0}.$$

Hence the forces $\Sigma\left(-m\frac{d^2\bar{r}}{dt^2}\right)$ and $\Sigma\bar{F}$ are in equilibrium,

$$\therefore \qquad \Sigma\bar{F} + \Sigma\left(-m\frac{d^2\bar{r}}{dt^2}\right) = \bar{0}.$$

Hence the reversed effective forces acting on each particle of the rigid body and the impressed (External) forces on the body are in equilibrium.

Remark

- D'Alembert's principle reduces the dynamic problem to the static problem.

• 14.3. GENERAL EQUATIONS OF A MOTION OF A RIGID BODY

To deduce the general equations of motion of a rigid body by D'Alembert's principle.

Let a rigid body be in motion and $\bar{r}$ be the position vector of a particle of mass m at any time t and $\bar{F}$ be the external force acting on it, then by D'Alembert's principle, we have

$$\Sigma\left(-m\frac{d^2\bar{r}}{dt^2}\right) + \Sigma\bar{F} = \bar{0}$$

or

$$\Sigma m\frac{d^2\bar{r}}{dt^2} = \Sigma\bar{F}. \qquad ...(1)$$

Taking vector product with $\bar{r}$ of both sides of (1), we get

$$\Sigma\bar{r}\times m\frac{d^2\bar{r}}{dt^2} = \Sigma\bar{r}\times\bar{F}. \qquad ...(2)$$

Hence the equaions (1) and (2) give the general equations of motion of a rigid body.

Cartesian Form of General Equations :

Let $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $\bar{F} = X\hat{i} + Y\hat{j} + Z\hat{k}$, so that

$$\frac{d^2\bar{r}}{dt^2} = \frac{d^2x}{dt^2}\hat{i} + \frac{d^2y}{dt^2}\hat{j} + \frac{d^2z}{dt^2}\hat{k}$$

and

$$\bar{r}\times\bar{F} = (x\hat{i} + y\hat{j} + z\hat{k})\times(X\hat{i} + Y\hat{j} + Z\hat{k})$$

$$= (yZ - zY)\hat{i} + (zX - xZ)\hat{j} + (xY - yX)\hat{k}$$

Then from (1) and (2), we get after equating the coefficients of $\hat{i}, \hat{j}$ and $\hat{k}$,

$$\left.\begin{aligned} \Sigma m\frac{d^2x}{dt^2} &= \Sigma X \\ \Sigma m\frac{d^2y}{dt^2} &= \Sigma Y \\ \Sigma m\frac{d^2z}{dt^2} &= \Sigma Z \end{aligned}\right\} \qquad ...(3)$$

and

$$\left.\begin{aligned} \Sigma m\left(y\frac{d^2z}{dt^2} - z\frac{d^2y}{dt^2}\right) &= \Sigma(yZ - zY) \\ \Sigma m\left(z\frac{d^2x}{dt^2} - x\frac{d^2z}{dt^2}\right) &= \Sigma(zX - xZ) \\ \Sigma m\left(x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2}\right) &= \Sigma(xY - yX) \end{aligned}\right\} \qquad ...(4)$$

These equations (3) and (4) give the general equations of motion of a rigid body in cartesian form.

• 14.4. CENTROID OF A RIGID BODY AND ITS LINEAR MOMENTUM

Centroid of a rigid body. Let $\overline{r}$ be the position vector of any particle of mass m of a rigid body at any instant with respect to a fixed point O, then the centroid of a body is defined as the position vector

$$\overline{r}_1 = \frac{\Sigma m\overline{r}}{\Sigma m}. \qquad \ldots(1)$$

If $\Sigma m = M$, then

$$\overline{r}_1 = \frac{\Sigma m\overline{r}}{M}$$

and if $\overline{r}_1$ $(\overline{x}, \overline{y}, \overline{z})$ and $\overline{r}$ is (x, y, z), then we have

$$\overline{x} = \frac{\Sigma mx}{M}, \overline{y} = \frac{\Sigma my}{M}, \overline{z} = \frac{\Sigma mz}{M}.$$

Thus $(\overline{x}, \overline{y}, \overline{z})$ gives the co-ordinates of the centroid of a rigid body.

Linear momentum of a rigid body. If $\overline{v}$ be the velocity of a particle of mass m at the point (x, y, z) and $\overline{V}$ be the velocity vector of the centroid of the body whose position vector is $\overline{r}_1$.

Now we have, $\qquad \overline{r}_1 = \frac{\Sigma m\overline{r}}{M}.$

Differentiating this vector equation w.r.t. 't', we get

$$\overline{V} = \frac{d\,\overline{r}_1}{dt} = \frac{1}{M}\frac{d}{dt}(\Sigma m\overline{r})$$

$$= \frac{1}{M}\Sigma\frac{d\,\overline{r}}{dt}$$

$$= \frac{1}{M}\Sigma m\overline{v} \qquad \left(\because \ \overline{v} = \frac{d\,\overline{r}}{dt}\right)$$

$$\therefore \qquad \overline{V} = \frac{\Sigma m\overline{v}}{M}. \qquad \ldots(2)$$

Since $\qquad \overline{V} = \left(\frac{d\,\overline{x}}{dt}, \frac{d\,\overline{y}}{dt}, \frac{d\,\overline{z}}{dt}\right)$ and $\overline{v} = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right)$,

then (2) becomes

$$\frac{d\,\overline{x}}{dt} = \frac{1}{M}\Sigma\left(m\frac{dx}{dt}\right), \ \frac{d\,\overline{y}}{dt} = \frac{1}{M}\Sigma\left(m\frac{dy}{dt}\right), \frac{d\,\overline{z}}{dt} = \frac{1}{M}\Sigma\left(m\frac{dz}{dy}\right).$$

Thus the equation (2) gives the velocity of the centroid of a rigid body.

Remark

➤ $\overline{V} = \frac{1}{M}\Sigma m\,\overline{v}$ shows that the linear momentum of a rigid body in a given direction is equal to the product of whole mass of the body and the velocity of its centroid.

SOLVED EXAMPLES

Example 1. *A rough uniform board, of mass m and length 2a, rests on a smooth horizontal plane and a man of mass M walks on it from one end to the other. Find the distance through which the board moves in this time.*

Solution. When a man moves from one end to the other end on a rough uniform board, the only external forces are

(i) weight of the board mg acting vertically downwards,

(ii) the weight of the man Mg also acting vertically downwards.

Fig. 2

Thus there is no external force along horizontal plane, then by D'Alembert's principle, we have that during complete motion C.G. of the board will remain at rest.

Let AB be the position of a rough uniform board of mass m and length $2a$ rests on a smooth horizontal plane, when the man of mass M is at A.

Then the distance of the centre of gravity G from A is

$$AG = \frac{M \times o + a \times m}{M + m} = \frac{am}{M + m}.$$

Now, when the man reaches at the other end of the board, the position of the board becomes $A'B'$; suppose the board slips through a distance $AA' = x$ backwards during the motion of man from A to B.

Then in this position the distance of C.G. of the system from A is

$$AG = \frac{M(2a - x) + m(a - x)}{M + m}$$

$$= \frac{2aM + am - x(M + m)}{M + m}.$$

But in both cases AG must be same, then we have

$$\frac{ma}{M + m} = \frac{2aM + ma - x(M + m)}{M + m}$$

or $$ma = 2aM + ma - x(M + m)$$

or $$x = \frac{2aM}{m + M}.$$

This gives the required distance that moved by the board.

Example 2. *A rod of length 2a, is suspended by a string of length l, attached to one end; if the string and rod revolve about the vertical with uniform angular velocity, and their inclinations to the vertical be* θ *and* ϕ *respectively, show that*

$$\frac{3l}{a} = \frac{(4 \tan\theta - 3 \tan\phi) \sin\phi}{(\tan\phi - \tan\theta) \sin\theta}.$$

Solution. Let a rod AB of length $2a$ be suspended by a string OA of length l and the whole system revolves about the vertical line with uniform angular velocity ω (say). The string and the rod make the angles θ and ϕ with the vertical respectively.

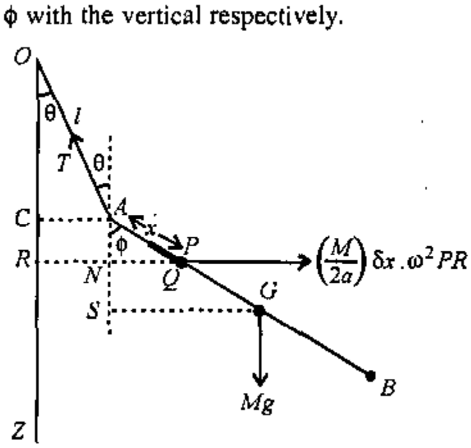


Fig. 3

Let us consider an element PQ of width δx at a distance x from A, then the mass ot this element PQ is

$$\left(\frac{M}{2a}\right)\delta x.$$

Now, this element PQ describes a circle of radius PR in the horizontal plane, when the rod revolves about the vertical line OZ with angular velocity ω, then the reversed effective force on the element PQ is

$$\left(\frac{M}{2a}\delta x\right).PR\omega^2 \text{ along } RP$$

$$= \left(\frac{M}{2a}\delta x\right).(l\sin\theta + x\sin\phi)\,\omega^2 \qquad (\because\ PR = l\sin\theta + x\sin\phi)$$

The external forces acting on the rod are

(i) Tension T at A along AO, and

(ii) The weight Mg of the rod acting at C.G. of the rod vertically downwards.

Then, resolving the forces along horizontal and vertical, we get

$$T \sin \theta = \Sigma \frac{M}{2a} \delta x \omega^2 (l \sin \theta + x \sin \phi) \quad ...(1)$$

and $$T \cos \theta = Mg \quad ...(2)$$

From (1), we have

$$T \sin \theta = \frac{M}{2a} \omega^2 \int_0^{2a} (l \sin \theta + x \sin \phi)\, dx$$

($\because$ The rod is distributed uniformly into a large number of elements like PQ)

$$= \frac{M}{2a} \omega^2 \left[lx \sin \theta + \frac{x^2}{2} \sin \phi \right]_0^{2a}$$

$$= \frac{M}{2a} \omega^2 [2al \sin \theta + 2a^2 \sin \phi]$$

$$= M\omega^2 (l \sin \theta + a \sin \phi) \quad ...(3)$$

Now taking the moments of the forces at A, we get

$$\Sigma \frac{M}{2a} \delta x \omega^2 (l \sin \theta + x \sin \phi) \,.\, AN - Mg \,.\, SG = 0$$

or $$\frac{M\omega^2}{2a} \int_0^{2a} (l \sin \theta + x \sin \phi)\, x \cos \phi \, dx - Mga \sin \phi = 0$$

($\because$ $AN = x \cos \phi$, $SG = a \sin \phi$)

or $$Mga \sin \phi = \frac{M\omega^2}{2a} \cos \phi \left[\frac{lx^2}{2} \sin \theta + \frac{x^3}{3} \sin \phi \right]_0^{2a}$$

$$= \frac{M\omega^2}{2a} \cos \phi \left[2a^2 l \sin \theta + \frac{8a^3}{3} \sin \phi \right]$$

$$= M\omega^2 \cos \phi \left[al \sin \theta + \frac{4a^2}{3} \sin \phi \right]$$

or $$g \sin \phi = \frac{1}{3} \omega^2 \cos \phi \,(3l \sin \theta + 4a \sin \phi)$$

or $$g \tan \phi = \frac{1}{3} \omega^2 (3l \sin \theta + 4a \sin \phi). \quad ...(4)$$

Dividing (3) by (2), we get

$$\tan \theta = \frac{\omega^2}{g} (l \sin \theta + a \sin \phi). \quad ...(5)$$

Eliminating ω^2 and g between (4) and (5), we get

$$\tan \phi = \frac{1}{3} \frac{\tan \theta \,(3l \sin \theta + 4a \sin \phi)}{(l \sin \theta + a \sin \phi)}$$

or $$3 \tan \phi \,(l \sin \theta + a \sin \phi) = \tan \theta \,(3l \sin \theta + 4a \sin \phi)$$

or $$3l (\tan \phi \sin \theta - \tan \theta \sin \theta) = a (\tan \theta \sin \phi - 3 \tan \phi \sin \phi)$$

or $$\frac{3l}{a} = \frac{(\tan \theta - 3 \tan \phi) \sin \phi}{(\tan \phi - \tan \theta) \sin \theta}.$$ **Hence proved.**

Example 3. *A rod of length* $2a$, *revolves with uniform angular velocity* ω *about a vertical axis through a smooth joint at one extremity of the rod so that it describes a cone of semi-vertical angle* α, *show that*

$$\omega^2 = \frac{3}{4} \frac{g}{a \cos \alpha}.$$

Also prove that the direction of reaction at the hinge makes with vertical an angle $\tan^{-1}\left(\frac{3}{4}\tan\alpha\right)$.

Solution. Let a rod AB of length $2a$ and mass M (say) revolves with uniform angular velocity ω about a vertical axis through A.

Let us consider an element PQ of width δx at a distance x from A. Then the mass of this element PQ is $\left(\frac{M}{2a}\right)\delta x$.

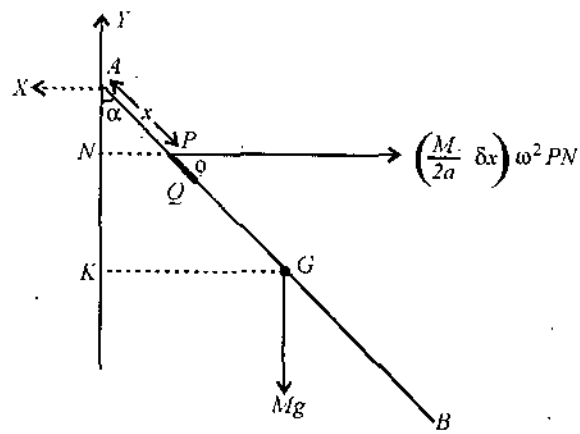


Fig. 4

As the rod AB revolves about the vertical axis, then this element PQ describes a circle in a horizontal plane of radius $PN = x\sin\alpha$. Then the reversed effective force on this element is

$$\left(\frac{M}{2a}\delta x\right)\omega^2 . PN \qquad \text{along } NP$$

$$= \left(\frac{M}{2a}\delta x\right)\omega^2 (x\sin\alpha) \qquad \text{along } NP.$$

The external forces acting on the rod are the weight Mg of the rod vertically downwards and the reaction at A.

Taking the moments of force about A, we have

$$Mg \,.\, GK = \Sigma\left(\frac{M}{2a}\delta x\right).\, \omega^2 x\sin\alpha \,.\, AN$$

or

$$Mg \,.\, a\sin\alpha = \frac{M}{2a}\omega^2 \sin\alpha\cos\alpha \int_0^{2a} x^2\,dx \qquad (\because\ GK = a\sin\alpha,\ AN = x\cos\alpha)$$

$$= \frac{M}{2a}\omega^2\sin\alpha\cos\alpha\left[\frac{x^3}{3}\right]_0^{2a}$$

$$= \frac{M}{2a}\omega^2\sin\alpha\cos\alpha\left[\frac{8a^3}{3}\right]$$

$$= \frac{4}{3}Ma^2\omega^2\sin\alpha\cos\alpha$$

$\therefore$

$$g = \frac{4}{3}a\omega^2\cos\alpha$$

or

$$\omega^2 = \frac{3}{4}\frac{g}{a\cos\alpha}.$$

This proved the first result.

Further, let X and Y be the components of the reaction at A, then we have

$$X = \Sigma\frac{M}{2a}\delta x\,\omega^2 x\sin\alpha$$

$$= \int_0^{2a} \frac{M}{2a}\omega^2 x \sin\alpha \, dx$$

$$= \frac{M}{2a}\omega^2 \sin\alpha \int_0^{2a} x\, dx$$

$$= \frac{M}{2a}\omega^2 \sin\alpha \left[\frac{x^2}{2}\right]_0^{2a}$$

$$= \frac{M}{2a}\omega^2 \sin\alpha \left[\frac{4a^2}{2}\right]$$

$$= Ma\omega^2 \sin\alpha$$

and $$Y = Mg.$$

If θ be the angle that the direction of reaction makes with the vertical, then

$$\tan\theta = \frac{X}{Y}$$

$$= \frac{Ma\omega^2 \sin\alpha}{Mg}$$

$$= \frac{a}{g}\omega^2 \sin\alpha$$

$$= \frac{3}{4}\tan\alpha \qquad \left(\because \ \omega^2 = \frac{3}{4}\frac{g}{a\cos\alpha}\right)$$

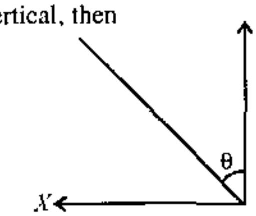


Fig. 5

$\therefore$ $$\theta = \tan^{-1}\left(\frac{3}{4}\tan\alpha\right).$$

Hence proved the second result.

• SUMMARY

- **Effective force :** E.F. = mass × acceleration = $m\dfrac{d^2\vec{r}}{dt^2}$
- **Reversed effective force :** R.E.F. = $-m\dfrac{d^2\vec{r}}{dt^2}$
- **D'Alembert's Principle :** The reversed effective forces acting on each particle of the moving rigid body and impressed forces on the body are in equilibrium.

  i.e., $$\Sigma\vec{F} + \Sigma\left(-m\frac{d^2\vec{r}}{dt^2}\right) = \vec{0}$$

• STUDENT ACTIVITY

1. State and prove D'Alembert's Principle.

2. A rough uniform board, of mass m and length $2a$, rests on a smooth horizontal plane nd man of mass M walks on it from one end to the other. Find the distance through which the board moves in this time.

• TEST YOURSELF

1. A rod revolving on a smooth horizontal plane about one end, which is fixed, breaks into two parts; what is the subsequent motion of the two parts ?
2. Find the motion of the rod OPQ, with two masses M and M' attached to it at P and Q respectively, when it moves round the vertical as a conical pendulum with uniform angular velocity, then angle θ which the rod makes with the vertical being constant.
3. A uniform rod OA, of length $2a$, free to turn about its end O, revolves with uniform angular velocity ω about the vertical OZ through O, and is inclined at a constant angle α to OZ, show that the value of α is eiher zero or $\cos^{-1}\left(\frac{3g}{4a\omega^2}\right)$.
4. A plank of mass M is initially at rest along a line of greatest slope of a smooth plane inclined at an angle α to the horizon and a man of mass M', starting from the upper end, walks down the plank so that it does not move; show that he gets to the other end in time

$$\sqrt{\left\{\frac{2M'a}{(M+M')\,g\sin\alpha}\right\}}$$

where a is the length of the plank.

ANSWERS

1. Rod OA revolving about fixed point O, the part AB with its C.G. C will fly off in a tangent line at C to the circle with O as centre and OC as radius and will also continue to rotate about C and the part OB will continue to rotate about O with the same angular velocity.

OBJECTIVE EVALUATION

Fill in the Blanks :

1. D'Alembert's principle reduces the dynamical problem into ....
2. $m\frac{d^2x}{dt^2}, m\frac{d^2y}{dt^2}, m\frac{d^2z}{dt^2}$ are the components of .... on the particle of m at any time t parallel to the co-ordinate axes.
3. $\Sigma(-mf)$ is called ....

True or False :

Write T for true and F for false statements :

1. D'Alembert's principle says that the reversed effective forces on the body is in equilibrium with the impressed forces acting on the body. *(T/F)*
2. $\Sigma\left(-m\frac{d^2\vec{r}}{dt^2}\right)$ is the expression of effective force. *(T/F)*
3. The impulse of the force is the time integral of the force. *(T/F)*

Multiple Choice Questions :

Choose the most appropriate one :

1. According to the D'Alembert's principle, $\Sigma\left(-m\frac{d^2\bar{r}}{dt^2}\right) = ?$

   (a) $\Sigma\bar{F}$ (b) $\Sigma(-\bar{F})$ (c) 0 (d) *none of these.*

2. If $\bar{R}$ be the internal force acting between two particles of a rigid body, then $\Sigma\bar{R} = ?$

   (a) 1 (b) -1 (c) 0 (d) none of these.

3. If $\bar{r}$ be the position vector of a particle at P wth respect to the its centre of gravity and M be its mass, then $\Sigma M\bar{r}$ is :

   (a) 0 (b) 1 (c) -1 (d) none of these.

ANSWERS

Fill in the Blanks :

1. Statics problem **2.** effective force **3.** reversed effective force

True or False :

1. T **2.** F **3.** T.

Multiple Choice Questions :

1. (b) **2.** (c) **3.** (a).