Contents

Syllabus

CALCULUS & GEOMETRY
SC–102

CHAPTER I

Successive differentiation, Leibnitz theorem, Maclaurin and Taylor series expansions. Indeterminate form, Tangent and Normal (Cartesian curve), curvatures, Asymptotes singular points and curve tracing (only cartesian curve).

CHAPTER II

Length of curves, Area of Cartesian curves, Volumes of revolution and surfaces of revolution.

CHAPTER III

Definition of differential equation, Order and degree of differential equation. Differential equation to first order and first degree. Exact differential equations, First order higher degree differential equations, Clairaut's form and singular solution. Linear differential equation with constant coefficient. Homogeneous linear ordinary differential equations. Linear differential equation of second order Wronskian.

CHAPTER IV

Geometry (2-dim) : Coordinate system, General equation of second degree, System of conics, Confocal conics.

Geometry (3-dim) : Coordinate system, Direction cosines and ratios, The plane, The straight line, Sphere, Cone Cylinder.

UNIT 1

SUCCESSIVE DIFFERENTIATION

STRUCTURE

- Introduction
- Successive Differentiation of Standard Forms
  - Test Yourself–1
- Leibnitz's Theorem
  - Test Yourself–2
  - Summary
  - Student Activity
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LEARNING OBJECTIVES

After going through this unit you will learn :

- How to differentiate the given functions upto finite number of times
- Leibnitz's rule which is applicable for the product of two or more functions

• 1.1. INTRODUCTION

Let $y = f(x)$ be a function, then the differential coefficient of $f(x)$ denoted by $f'(x)$ is defined as follows

$$f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x} = \frac{dy}{dx}$$

If the limit exists (*i.e.*, limit is finite and unique), then $f'(x)$ is called *first differential coefficient of $f(x)$ with respect to x*. Similarly, if $f(x)$ is differentiable twice, it is denoted by $f''(x)$, if it is differentiable thrice, it is denoted by $f'''(x)$, *i.e.*,

$$f''(x) = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$$

$$f'''(x) = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3}.$$

If $y = f(x)$ be a function of x, then we adopt the following notations.

$$y_1 = f'(x) = \frac{dy}{dx} = Df(x) = \frac{d}{dx}(f(x)),\quad y_2 = f''(x) = \frac{d^2y}{dx^2} = D^2f(x) = \frac{d^2}{dx^2}(f(x))$$

$$y_3 = f'''(x) = \frac{d^3y}{dx^3} = D^3f(x) = \frac{d^3}{dx^3}(f(x))$$

Similarly, $$y_n = f^n(x) = \frac{d^ny}{dx^n} = D^nf(x) = \frac{d^n}{dx^n}(f(x))$$

This process of finding the differential coefficients of a function is called **successive differentiation.**

• 1.2. SUCCESSIVE DIFFERENTIATION OF STANDARD FORMS

(i) If $y = f(x) = x^n$ then $D^n(x^n) = n!$.

(ii) If $y = f(x) = x^m$, then $D^n(x^m) = \dfrac{m!}{(m-n)!}x^{m-n}$ $(m > n)$

(iii) If $y = f(x) = \dfrac{1}{(ax+b)}$, then $D^n\left[\dfrac{1}{ax+b}\right] = \dfrac{(-1)^n a^n n!}{(ax+b))^{m+n}}$

(iv) If $y = f(x) = \frac{1}{(ax+b)^m}$, then $D^n\left[\frac{1}{ax+b}\right] = \frac{(-1)^n a^n n!}{(m-1)!(ax+b))^{m+n}}$

(v) If $y = f(x) = \sin(ax+b)$, then $D^n = [\sin(ax+b)] = a^n \sin\left(\frac{n\pi}{2} + ax + b\right)$

(vi) If $y = f(x) = \cos(ax+b)$, then $D^n = [\cos(ax+b)] = a^n \cos\left(\frac{n\pi}{2} + ax + b\right)$

(vii) If $y = f(x) = e^{ax+b}$, then $D^n = (e^{ax+b}) = a^n e^{ax+b}$.

(viii) If $y = f(x) = \log(ax+b)$, then $D^n = [\log(ax+b)] = (-1)^{n-1}\frac{a^n(n-1)!}{(ax+b)^n}$

(ix) $y = f(x) = e^{ax}\sin(bx+c)$.

Here, we have

$$y_1 = f'(x) = ae^{ax}\cdot\sin(bx+c) + be^{ax}\cos(bx+c)$$
$$= e^{ax}[a\sin(bx+c) + b\cos(bx+c)]$$

Put $\quad a = r\cos\theta,\ b = r\sin\theta \Rightarrow r^2 = a^2 + b^2$

and $\quad \tan\theta = b/a$ *i.e.*, $\theta = \tan^{-1} b/a$.

Therefore, $\quad y_1 = f'(x) = r\cdot e^{ax}\sin(bx+c+\theta)$

$$= (a^2+b^2)^{1/2}\cdot e^{ax}\sin\left(bx+c+\tan^{-1}\frac{b}{a}\right).$$

Similarly, $\quad y_2 = f''(x) = (a^2+b^2)^{1/2}(a^2+b^2)^{1/2}\cdot e^{ax}\sin(bx+c+\tan^{-1}b/a+\tan^{-1}b/a)$

$$= (a^2+b^2)^{2/2}\cdot e^{ax}\sin(bx+c+2\tan^{-1}b/a)$$
$$y_3 = f'''(x) = (a^2+b^2)^{3/2}e^{ax}\sin(bx+c+3\tan^{-1}b/a)$$

Similarly, $\quad y_n = f^n(x) = (a^2+b^2)^{n/2}e^{ax}\sin(bx+c+n\tan^{-1}b/a)$

$$\Rightarrow \quad y_n = \frac{d^n}{dx^n}[e^{ax}\sin(bx+c)] = (a^2+b^2)^{n/2}e^{ax}\sin(bx+c+n\tan^{-1}b/a)$$

(x) $y = f(x) = e^{ax}\cos(bx+c)$.

Similarly, we may obtain

$$y_n = \frac{d^n}{dx^n}[e^{ax}\cos(bx+c)] = (a^2+b^2)^{n/2}\cdot e^{ax}\cos(bx+c+n\tan^{-1}b/a)$$

SOLVED EXAMPLES

Example 1. *Find the n^{th} differential coefficient of log $(ax+x^2)$.*

Solution. Let $\quad y = \log(ax+x^2) = \log[x(a+x)] = \log x + \log(a+x)$

Differentiating n times, we get

$$y_n = \frac{d^n}{dx^n}(\log x) + \frac{d^n}{dx^n}\log(a+x)$$
$$= \frac{(-1)^{n-1}(n-1)!\,up15}{x^n} + \frac{(-1)^{n-1}(n-1)!}{(x+a)^n}$$
$$= (-1)^{n-1}\cdot(n-1)!\left[\frac{1}{x^n} + \frac{1}{(x+a)^n}\right].$$

Example 2. *Find the n^{th} differential coeffcieints of*

(*i*) e^{ax} *sin bx cos cx.*

(*ii*) $e^{2x}\sin^3 x$.

Solution. (i) Let $\quad y = e^{ax}\sin bx\cos cx$

$$= \frac{1}{2}e^{ax}[2\sin bx\cos cx]$$
$$= \frac{1}{2}e^{ax}[\sin(bx+cx) + \sin(bx-cx)]$$
$$= \frac{1}{2}[e^{ax}\sin(b+c)x + e^{ax}\sin(b-c)x] \quad \text{...(1)}$$

Since, we know that

$$\frac{d^n}{dx^n}[e^{ax}\sin(bx+c)] = (a^2+b^2)^{n/2}e^{ax}\sin(bx+c+n.\tan^{-1}b/a)$$

Therefore, by differentiating (1) n times, we get

$$\frac{d^n}{dx^n}[y] = y_n = \frac{1}{2}[\{a^2+(b+c)^2\}^{n/2} e^{ax} \sin\{(b+c)x$$

$$+ n\tan^{-1}(b+c)/a\} + \{a^2+(b-c)^2\}^{n/2} e^{ax} \sin\{(b-c)x + n\tan^{-1}(b-c)/a\}].$$

(ii) Let $y = e^{2x}\sin^3 x$.

Now using the result

$$\sin 3x = 3\sin x - 4\sin^3 x.$$

We have $\quad 4\sin^3 x = 3\sin x - \sin 3x$

$\Rightarrow \quad \sin^3 x = \frac{1}{4}(3\sin x - \sin 3x).$

Therefore, $\quad y = \frac{1}{4}e^{2x}[3\sin x - \sin 3x] = \frac{3}{4}e^{2x}\sin x - \frac{1}{4}e^{2x}\sin 3x.$

Now, differentiating n times, we get

$$y_n = \frac{3}{4}[(2^2+1^2)^{1/2}]^n e^{2x}\sin[x + n\tan^{-1}1/2]$$

$$-\frac{1}{4}[(2^2+3^2)^{1/2}]^n e^{2x}\sin[2x + n\tan^{-1}3/2].$$

Example 3. *If $y = \sin mx + \cos mx$, prove that $y_n = m^n[1+(-1)^n \sin 2mx]^{1/n}$.*

Solution. We know that

$$\frac{d^n}{dx^n}[\sin(ax+b)] = a^n \sin\left(n\cdot\frac{\pi}{2} + ax + b\right)$$

and

$$\frac{d^n}{dx^n}[\cos(ax+b)] = a^n \cos\left(n\cdot\frac{\pi}{2} + ax + b\right)$$

Therefore, $\quad y_n = \frac{d^n}{dx^n}(\sin mx) + \frac{d^n}{dx^n}(\cos mx)$

$$= m^n \sin\left(mx + n\frac{\pi}{2}\right) + m^n \cos\left(mx + n\frac{\pi}{2}\right)$$

$$= m^n\left[\left\{\sin\left(mx + n\frac{\pi}{2}\right) + \cos\left(mx + n\frac{\pi}{2}\right)\right\}^2\right]^{1/2}$$

$$= m^n\left[1 + 2\sin\left(mx + n\frac{\pi}{2}\right)\cdot\cos\left(mx + n\frac{\pi}{2}\right)\right]^{1/2}$$

$$= m^n[1+\sin(2mx+n\pi)]^{1/2} = m^n[1\pm\sin 2mx]^{1/2}$$

$$= m^n[1+(-1)^n\sin 2mx]^{1/2}.$$

Example 4. *Find the n^{th} differenial coefficient of $\log[(ax+b)(cx+d)]$.*

Solution. Let $y = \log[(ax+b)(cx+d)] = \log(ax+b) + \log(cx+d)$.

We know that $\quad D^n \log(ax+b) = (-1)^{n-1}(n-1)!\,a^n(ax+b)^{-n}$.

$\therefore \quad y_n = (-1)^{n-1}(n-1)!\,a^n(ax+b)^{-n} + (-1)^{n-1}(n-1)!\,c^n(cx+d)^{-n}$

$$= (-1)^{n-1}(n-1)!\left[\frac{a^n}{(ax+b)^n} + \frac{c^n}{(cx+d)^n}\right].$$

Example 5. *Find the n^{th} derivative if $y = \cos^4 x$.*

Let $\quad y = \cos^4 x = (\cos^2 x)^2 = [1/2(1+\cos 2x)]^2$

$= 1/4(1 + 2\cos 2x + \cos^2 2x)$

$= 1/4[1 + 2\cos 2x + 1/2(1+\cos 4x)]$

$= 1/4[3/2 + 2\cos 2x + 1/2\cos 4x]$

$= 3/8 + 1/2\sin 2x + 1/8\cos 4x.$

Now $\quad D^n\cos(ax+b) = a^n\cos(ax+b+n\pi/2).$

$\therefore \quad y_n = 0 + \frac{1}{2}\cdot 2^n\cos\left(2x + \frac{1}{2}n\pi\right) + \frac{1}{8}\cdot 4^n\cos\left(4x + \frac{1}{2}n\pi\right)$

$$= 2^{n-1}\cdot\cos\left(2x + \frac{1}{2}n\pi\right) + 2^{2n-3}\cos\left(4x + \frac{1}{2}n\pi\right).$$

• TEST YOURSELF–1

1. Find the n^{th} derivatives of
   (i) $\sin^3 x$ (ii) $\cos x \cos 2x \cos 3x$
   (iii) $e^{ax} \cos^2 x \sin x$ (iv) $\sin ax \cos bx$
   (v) $\sin^2 x \sin 2x$.
2. Show that the value of the n^{th} differential coefficients of $\frac{x^3}{x^2-1}$ for $x = 0$, is zero if n is even and is $-n!$, if n is odd and greater than 1.
3. (i) If $x = a(t - \sin t)$ and $y = a(1 + \cos t)$, prove that $\frac{d^2y}{dx^2} = \frac{1}{4a} \operatorname{cosec}^4\left(\frac{t}{2}\right)$.

   (ii) If $x = a(\cos\theta + \theta\sin\theta)$, $y = a(\sin\theta - \theta\cos\theta)$, find $\frac{d^2y}{dx^2}$.

ANSWERS

1. (i) $y_n = \frac{3}{4}\sin\left(x + \frac{n\pi}{2}\right) - \frac{1}{4} \cdot 3^n \sin\left(3x + \frac{n\pi}{2}\right)$

   (ii) $y_n = \frac{1}{4}\left\{6^n \cos\left(6x + \frac{1}{2}n\pi\right) + 4^n \cos\left(4x + \frac{n\pi}{2}\right) + 2^n \cos\left(2x + \frac{n\pi}{2}\right)\right\}$

   (iii) $y_n = \frac{1}{4}[(a^2+1)^{n/2} e^{ax} \sin\{x + n\tan^{-1} 1/a\} + (a^2+9)^{n/2} e^{ax} \sin(3x + n\tan^{-1} 3/a)]$

   (iv) $y_n = \frac{1}{2}\left[(a+b)^n \sin\left\{(a+b)x + \frac{1}{2}n\pi\right\} + (a-b)^n \sin\left\{(a-b)x + \frac{1}{2}n\pi\right\}\right]$

   (v) $y_n = 2^{n-1}\sin\left(2x + \frac{1}{2}n\pi\right) - 4^{n-1}\sin\left(4x + \frac{1}{2}n\pi\right)$

3. (ii) $y_2 = \frac{1}{a} \cdot \frac{\sec^3\theta}{\theta}$.

• 1.3. LEIBNITZ'S THEOREM

This theorem help us to find the n^{th} *differential coefficient of the product of two functions* in terms of the successive derivatives of the functions.

Statement. *If u, v be two functions of x, having derivative of n^{th} order, then*

$$D^n(uv) = u_n v + {}^nC_1 u_{n-1} v_1 + {}^nC_2 u_{n-2} v_2 + \ldots + {}^nC_r u_{n-r} v_r + \ldots + {}^nC_n u v_n$$

where suffixes of u and v denote differentiations w.r.t. x.

SOLVED EXAMPLES

Example 1. *Find the n^{th} derivative of $x^2 \sin x$.*

Solution. Let $u = \sin x$ and $v = x^2$.

Then,
$$u_n = \sin\left[x + \frac{n\pi}{2}\right]$$
$$u_{n-1} = \sin\left(x + (n-1)\frac{\pi}{2}\right)$$
$$u_{n-2} = \sin\left[x + (n-2)\frac{\pi}{2}\right].$$

Also
$$v_1 = 2x$$
$$v_2 = 2$$
$$v_3 = 0.$$

Now, by Leibnitz theorem, we have

$$\frac{d^n}{dx^n}(uv) = u_n \cdot v + {}^nC_1 u_{n-1} \cdot v_1 + {}^nC_2 u_{n-2} \cdot v_2$$

$$\Rightarrow \frac{d^n}{dx^n}(x^2 \sin x) = \sin\left(x + \frac{n\pi}{2}\right)x^2 + {}^nC_1 \sin\left[x + (n-1)\frac{\pi}{2}\right]2x + {}^nC_2 \sin\left[x + (n-2)\frac{\pi}{2}\right]2$$

Q-3

End

$$= x^2 \sin\left(x + \frac{n\pi}{2}\right) + 2nx \sin\left[x + (n-1)\frac{\pi}{2}\right] + n(n-1)\sin\left[x + (n-2)\frac{\pi}{2}\right].$$

Example 2. *If $y = a\cos(\log x) + b\sin(\log x)$, show that*

$x^2y_2 + xy_1 + y = 0$ *and* $x^2y_{n+2} + (2n+1)xy_{n+1} + (n^2+1)y_n = 0.$

Solution. Here, we have

$$y = a\cos(\log x) + b\sin(\log x). \qquad \text{...(1)}$$

Differentiating (1) with respect to x, we have

$$y_1 = -\frac{a}{x}\sin(\log x) + \frac{b}{x}\cos(\log x)$$

$$\Rightarrow \qquad xy_1 = -a\sin(\log x) + b\cos(\log x).$$

Again, differentiating w.r.t. x, we get

$$xy_2 + y_1 = -\frac{a}{x}\cos(\log x) - \frac{b}{x}\sin(\log x).$$

$$\Rightarrow \qquad x^2y_2 + xy_1 = -a\cos(\log x) - b\sin(\log x) = -y$$

$$\Rightarrow \qquad x^2y_2 + xy_2 + y = 0. \qquad \textbf{Proved} \quad \text{...(2)}$$

Now, differentiating (2) both n times by Leibnitz theorem

$$\Rightarrow \quad D^n(x^2y_2) + D^n(xy_1) + D^n(y) = 0$$

$$\Rightarrow \quad (D^ny_2)x^2 + {}^nC_1(D^{n-1}y_2)(Dx^2) + {}^nC_2(D^{n-2}y_2)(D^2x^2)$$
$$+ (D^ny_1)x + {}^nC_1(D^{n-1}y_1)(Dx) + D^ny = 0$$

$$\Rightarrow \quad x^2y_{n+2} + 2nxy_{n+1} + \frac{n(n-1)}{2}2y_n + xy_{n+1} + ny_n + y_n = 0$$

$$\Rightarrow \quad x^2y_{n+2} + (2n+1)xy_{n+1} + (n^2+1)y_n = 0. \qquad \textbf{Proved.}$$

• SUMMARY

- $y = x^m \Rightarrow y_n = \dfrac{n!}{(m-n)!}x^{m-n} \ (m > n)$
- $\dfrac{d^n}{dx^n}(\sin(ax+b)) = a^n \sin\left(\dfrac{n\pi}{2} + ax + b\right)$
- $\dfrac{d^n}{dx^n}(\cos(ax+b)) = a^n \cos\left(\dfrac{n\pi}{2} + ax + b\right)$
- $\dfrac{d^n}{dx^n}(\log(ax+b)) = (-1)^{n-1}\dfrac{a^n(n-1)!}{(ax+b)^n}$
- $\dfrac{d^n}{dx^n}(e^{ax}\sin(bx+c)) = (a^2+b^2)^{n/2}e^{ax}\sin\left(bx + c + n\tan^{-1}\left(\dfrac{b}{a}\right)\right)$
- $\dfrac{d^n}{dx^n}(e^{ax}\cos(bx+c)) = (a^2+b^2)^{n/2}e^{ax}\cos\left(bx + c + n\tan^{-1}\left(\dfrac{b}{a}\right)\right)$
- Liebnitz's Theorem : If u, v be two functions of x having n^{th} order derivatives, then
  $$D^n(uv) = u_nv + {}^nc_1u_{n-1}v_1 + {}^nc_2u_{n-2}v_2 + \ldots\ldots + {}^nc_nuv_n$$
- $(y_n)_0$ = value of $y_n(x)$ at $x = 0$.

• STUDENT ACTIVITY

1. Find the n^{th} derivative of $\sin ax \cos bx$.

2. If $y = [x + \sqrt{1+x^2}]^m$, then find $(y_n)_0$.

• TEST YOURSELF–2

1. Use Leibnitz's theorem, to find y_n in the following cases :
   (i) $x^3 e^{ax}$ (ii) $x^2 e^x$ (iii) $x^3 \sin ax$
   (iv) $x^3 \log x$ (v) $x^2 e^x \cos x$ (vi) $e^x \log x$ (vii) $x^n \log x$
2. If $I_n = \dfrac{d^n}{dx^n}(x^n \log x)$, prove that $I_n = nI_{n-1} + (n-1)\,!$ and hence show that
$$I_n = n\,!\left(\log x + 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}\right).$$
3. If $y = [x + \sqrt{(1+x^2)}]^m$, prove that
$$(1+x^2)\, y_{n+2} + (2n+1)\, xy_{n+1} + (n^2 - m^2)\, y_n = 0.$$
4. If $y^{1/m} + y^{-1/m} = 2x$, prove that
$$(x^2-1)\, y_{n+2} + (2n+1)\, xy_{n+1} + (n^2 - m^2)\, y_n = 0.$$
5. If $y = \cos(\log x)$, prove that
$x^2 y_{n+2} + (2n+1)\, xy_{n+1} + (n^2+1)\, y_n = 0.$

ANSWERS

1. (i) $e^{ax} a^{n-3} [a^3x^3 + 3n\, a^2x^2 + 3n(n-1)\, ax + n(n-1)(n-2)]$
   (ii) $e^x [x^2 + 2nx + n(n-1)]$
   (iii) $a^{n-3}\left[a^3x^3 \sin\left(ax + \frac{n\pi}{2}\right) + 3na^2x^2 \sin\left(ax + (n-1)\frac{\pi}{2}\right) + 3n(n-1)\, ax \sin\left\{ax + (n-2)\frac{\pi}{2}\right\}\right.$
   $\left. + n(n-1)(n-2) \sin\left(ax + (n-3)\frac{\pi}{2}\right)\right].$
   (iv) $\dfrac{(-1)^{n-1}\, n\,!}{x^{n-3}}\left[\dfrac{1}{n} - \dfrac{3}{n-1} + \dfrac{3}{n-2} - \dfrac{1}{n-3}\right]$
   (v) $e^x\left[2^{n/2} x^2 \cos\left(x + \frac{n\pi}{4}\right) + 2^{(n-1)/2}\, 2nx \cos\left(x + (n-1)\frac{\pi}{4}\right)\right.$
   $\left. + 2^{(n-2)/2}\, n(n-1) \cos\left(x + (n-2)\frac{\pi}{4}\right)\right]$

(vi) $e^x [\log x + {}^nC_1 x^{-1} - {}^nC_2 x^{-2} + {}^nC_3 2! x^{-3} - \ldots + {}^nC_n (-1)^{n-1} (n-1)! x^{-n}]$

(vii) $y_{n+1} = \frac{n!}{x}$

OBJECTIVE EVALUATION

➤ FILL IN THE BLANKS :

1. $D^n (\log x)$ is equal to ....
2. To find the n^{th} derivative of the product of two functions we use .... theorem.
3. If $y = \sin (ax + b)$, then $D^n \sin (ax + b) =$ ....
4. If $y = (ax + b)^{-1}$, then $D^n (ax + b)^{-1} =$ ....

➤ TRUE OR FALSE :

Write 'T' for True and 'F' for False :

1. To find the n^{th} derivative of the product of two functions we use Leibnitz's theorem. *(T/F)*
2. If we observe that one of the two functions is such that all its differential coefficients after a certain steps, become zero, then we should take this function as second function. *(T/F)*
3. If $y = a \cos (\log x) + b \sin (\log x)$, then $x^2 y_2 + xy_1 = y$. *(T/F)*

➤ MULTIPLE CHOICE QUESTIONS :

Choose the most appropriate one :

1. $D^n (e^{ax+b})$ is equal to :

   (a) $a^n e^{ax}$ (b) e^{ax+b}

   (c) $a^n b^n e^{ax+b}$ (d) $a^n e^{ax+b}$

2. $D^n \log x$ is equal to :

   (a) $\frac{(n-1)!}{x^n}$ (b) $\frac{(-1)^n (n-1)!}{x^n}$ (c) $\frac{(-1)^{n-1} (n-1)!}{x^n}$ (d) $\frac{(-1)^n n!}{x^n}$

ANSWERS

Fill in the Blanks :

1. $\frac{(-1)^{n-1} (n-1)!}{x^n}$ 2. Leibnitz's 3. $a^n \sin\left(ax + b + \frac{n\pi}{2}\right)$

4. $(-1)^n \cdot n! \, a^n (ax+b)^{-n-1}$

True or False :

1. T 2. T 3. F

Multiple Choice Questions :

1. (c) 2. (c)

□□□

UNIT 2

EXPANSIONS

STRUCTURE

- Taylor's Theorem
- Maclaurin's Theorem
- Failure of Taylor's and Maclaurin's Theorem
  - Summary
  - Student Activity
  - Test Yourself

LEARNING OBJECTIVES

After going through this unit you will learn :

- Some standard results like Taylor's and Maclaurin's Theorem
- Expansion of given function in a power series of its variable.

• 2.1. TAYLOR'S THEOREM

Statement. *Let $f(x)$ be a function of x which can be expanded in powers of x and let the expansion be differentiable term by term any number of times, then*

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2\,!} f''(a) + \ldots + \frac{h^n}{n\,!} f^n(a) + \ldots$$

(i) Writing $\boldsymbol{x}$ **for** $\boldsymbol{a}$ in (2), we have

$$f(x+h) = f(x) = hf'(x) + \frac{h^2}{2\,!} f''(x) + \ldots + \frac{h^n}{n\,!} f^n(x) + \ldots$$

(ii) Putting $\boldsymbol{a+h=b}$ or $\boldsymbol{h=b-a}$, in (2), we get

$$f(b) = f(a) + (b-a) f'(a) + \frac{(b-a)^2}{2\,!} f''(a) + \ldots + \frac{(b-a)^n}{n\,!} f^n(a) + \ldots$$

(iii) Changing $a+h$ to x *i.e.*, $\boldsymbol{h}$ **to** $\boldsymbol{x-a}$ in (2), we have

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2\,!} f''(a) + \ldots + \frac{(x-a)^n}{n\,!} f^n(a) + \ldots$$

• 2.2. MACLAURIN'S THEOREM

Let $f(x)$ be a function of x which possesses continuous derivatives of all orders in the interval $[0, x]$ *and can be expanded as an infinite series in x, then*

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2\,!} f''(0) + \ldots + \frac{x^n}{n\,!} f^n(0) + \ldots$$

SOLVED EXAMPLES

Example 1. *Expand*

(i) e^x *(ii)* $(1+x)^n$ *(iii) sin x.*

(iv) log $(1+x)$. *(v)* a^x.

Solution. (i) Let $f(x) = e^x \Rightarrow f(0) = e^0 = 1, f'(x) = e^x \Rightarrow f'(0) = 1\ f^n(0) = 1.$

Put all these values in Maclaurin's series

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2\,!} f''(0) + \frac{x^3}{3\,!} f'''(0) + \ldots$$

We get $$\boldsymbol{e^x = 1 + x + \frac{x^2}{2\,!} + \frac{x^3}{3\,!} + \frac{x^4}{4\,!} + \ldots + \frac{x^n}{n\,!} + \ldots}.$$

This is known as ***exponential series.***

(ii) Here, we have

$$f(x) = (1+x)^n$$

$\Rightarrow$ $$f(0) = 1$$

$$f'(x) = n(1+x)^{n-1} \Rightarrow f'(0) = n$$

$$f''(x) = n(n-1)(1+x)^{n-2} \Rightarrow f''(0) = n(n-1)$$

....

$$f^m(x) = n(n-1)(n-2)\ldots(n-m+1)(1+x)^{n-m}$$

$\Rightarrow$ $$f^m(0) = n(n-1)\ldots(n-m+1).$$

Put all these values in Maclaurin's series

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} + f''(0) + \ldots$$

We get $$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots + \frac{n(n-1)\ldots(n-m+1)}{m!}x^m + \ldots .$$

This is known as ***Binomial series.***

(iii) Here, we have

$$f(x) = \sin x \Rightarrow f(0) = 0$$

$$f'(x) = \cos x \Rightarrow f'(0) = 1$$

$$f''(x) = -\sin x \Rightarrow f''(0) = 0$$

$$f'''(x) = -\cos x \Rightarrow f''(0) = -1$$

....

$$f^n(x) = \sin\left(x + \frac{n\pi}{2}\right)$$

$\Rightarrow$ $$f^n(0) = \begin{cases} 0, & \text{when } n = 2m \\ (-1)^m & \text{when } n = 2m+1. \end{cases}$$

Putting all these values in Maclaurin's series, we get

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots + (-1)^m \frac{x^{2m+1}}{(2m+1)!} + \ldots .$$

(iv) Here, $f(x) = \log(1+x) \Rightarrow f(0) = 0$

$$f^n(x) = \frac{(-1)^{n-1}(n-1)!}{(x+1)^n} \Rightarrow f^n(0) = (-1)^{n-1}(n-1)!\ n = 1, 2, 3, \ldots .$$

Put all these values in Maclaurin's series, we get

$$\log(1+x) = 0 + x - \frac{x^2}{2!}\cdot 1! + \frac{x^3}{3!}\cdot 2! - \frac{x^4}{4!}\cdot 3! + \ldots + \frac{x^n}{n!}(-1)^{n-1}(n-1)! + \ldots$$

$\Rightarrow$ $$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots + (-1)^{n-1}\frac{x^n}{n!} + \ldots$$

This is known as logarihmic series.

(v) Here, we have

$$f(x) = a^x \quad \Rightarrow \quad f(0) = a^0 = 1$$

$$f'(x) = a^x \log a \quad \Rightarrow \quad f'(0) = \log a$$

$$f''(x) = a^x (\log a)^2 \quad \Rightarrow \quad f''(0) = (\log a)^2$$

$$f'''(x) = a^x (\log a)^3 \quad \Rightarrow \quad f'''(0) = (\log a)^3$$

....

Putting all these values in Maclaurin's series, we get

$$a^x = 1 + x\log a + \frac{x^2}{2!}(\log a)^2 + \frac{x^3}{3!}(\log a)^3 + \ldots .$$

Example 2. *Expand log sin $(x+h)$ in power of h by Taylor's theorem.*

Solution. Let $f(x+h) = \log\sin(x+h)$

$\Rightarrow$ $$\left.\begin{aligned} f(x) &= \log\sin x \\ f'(x) &= \frac{1}{\sin x}\cdot\cos x = \cot x \\ f''(x) &= -\operatorname{cosec}^2 x \\ f'''(x) &= 2\operatorname{cosec} x\operatorname{cosec} x\cot x = 2\operatorname{cosec}^2 x\cot x \\ &\ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots \end{aligned}\right\} \quad \ldots(1)$$

Now by Taylor's theorem, we have

$$f(x+h)=f(x)+hf'(x)+\frac{h^2}{2\,!}f''(x)z+\frac{h^3}{3\,!}f'''(x)+\ldots \qquad \ldots(2)$$

Putting all the values from (1) in (2), we get

$$\mathbf{\log\sin(x+h)=\log\sin x+h\cot x-\frac{h^2}{2}\operatorname{cosec}^2 x+\frac{h^3}{3}\operatorname{cosec}^2 x\cot x+\ldots\,.}$$

Example 3. *Expand* $\sin x$ *in powers of* $\left(x-\frac{\pi}{2}\right)$ *by using Taylor's series.*

Solution. Let $f(x)=\sin x$.

We may write $f(x)=f\left[\frac{\pi}{2}+\left(x-\frac{\pi}{2}\right)\right]$

Now, expanding $f\left[\frac{\pi}{2}+\left(x-\frac{\pi}{2}\right)\right]$ by Taylor's theorem in powers of $\left(x-\frac{\pi}{2}\right)$, we get

$$f(x)=f\left[\frac{1}{2}\pi+\left(x-\frac{\pi}{2}\right)\right]$$

$$=f\left(\frac{\pi}{2}\right)+\left(x-\frac{\pi}{2}\right)f'\left(\frac{\pi}{2}\right)+\frac{1}{2\,!}\left(x-\frac{\pi}{2}\right)^2 f''\left(\frac{\pi}{2}\right)+\frac{1}{3\,!}\left(x-\frac{\pi}{2}\right)^3 f'''\left(\frac{\pi}{2}\right)+\ldots \qquad \ldots(1)$$

Now,

$$f(x)=\sin x\Rightarrow f\left(\frac{\pi}{2}\right)=\sin\frac{\pi}{2}=1$$

$$f'(x)=\cos x\Rightarrow f'\left(\frac{\pi}{2}\right)=\cos\frac{\pi}{2}=0$$

$$f''(x)=-\sin x\Rightarrow f''\left(\frac{\pi}{2}\right)=-\sin\frac{\pi}{2}=-1$$

$$f'''(x)=-\cos x\Rightarrow f'''\left(\frac{\pi}{2}\right)=-\cos\frac{\pi}{2}=0$$

$$f^{iv}(x)=\sin x\Rightarrow f^{iv}\left(\frac{\pi}{2}\right)=\sin\frac{\pi}{2}=1$$

....

Putting all these values in (1), we get

$$\mathbf{\sin x=1-\frac{1}{2\,!}\left(x-\frac{\pi}{2}\right)^2+\frac{1}{4\,!}\left(x-\frac{\pi}{2}\right)^4-\ldots\,.}$$

• SUMMARY

- **Taylor's Theorem :** Let $f(x)$ be a function of x which can be expanded in powers of x and that the expansion be differentiable term by term any number of times, then :

$$f(a+x)=f(a)+xf'(a)+\frac{x^2}{2\,!}f''(a)+\ldots\ldots+\frac{x^n}{n\,!}f^n(a)+\ldots\ldots$$

Other form of Taylor's theorem

$$f(x)=f(a)+(x-a)f'(a)+\frac{(x-a)^2}{2\,!}f''(a)+\ldots\ldots$$

- **Maclaurin's Theorem :** Let $f(x)$ be a function of x which possesses continuous derivatives of all orders in the interval $[0, x]$ and can be expanded as an infinite series in x, then

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2\,!}f''(0)+\ldots\ldots+\frac{x^n}{n\,!}f^n(0)+\ldots\ldots$$

• STUDENT ACTIVITY

1. Expand $\sin x$ in powers of $\left(x-\frac{\pi}{2}\right)$ by using Taylor's series.

2. Apply Maclaurin's theorem to prove that

$$\log \sec x = \frac{1}{2}x^2 + \frac{1}{2}x^2 + \frac{1}{12}x^4 + \frac{1}{45}x^6 + \ldots\ldots$$

• TEST YOURSELF

1. Expand the following function by Maclaurin's theorem
   (i) $\sec x$ (ii) $e^x \sec x$ (iii) $\log(1 + \sin x)$
2. Apply Maclaurin's theorem to prove that $\log \sec x = \frac{1}{2}x^2 + \frac{1}{12}x^4 + \frac{1}{45}x^6 + \ldots$.
3. If $y = \sin^{-1} x = a_0 + a_1 x + a_2 x^2 + \ldots$ Prove that $(n+1)(n+2)\, a_{n+2} = n^2 a_n$.
4. Expand the following
   (i) $\sin\left(\frac{\pi}{4} + \theta\right)$ in powers of θ. (ii) $2x^3 + 7x^2 + x - 1$ in powers of $x - 2$.
   (iii) $\sin^{-1}(x + h)$ in power of x. (iv) $\log \sin x$ in power of $(x - a)$.

ANSWERS

1. (i) $1 + \frac{x^2}{2!} + \frac{5x^4}{4!} + \frac{31x^6}{6!} + \ldots$ (ii) $1 + x + \frac{2x^2}{2!} + \frac{4x^3}{3!} + \ldots$
   (iii) $x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{12} + \frac{x^5}{24} - \ldots$
4. (i) $\frac{1}{\sqrt{2}}\left(1 + \theta - \frac{\theta^2}{2!} - \frac{\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{\theta^5}{5!} \ldots\right)$
   (ii) $45 + 53(x-2) + 19(x-2)^2 + 2(x-2)^3 + \ldots$
   (iii) $\sin^{-1} h + x(1-h^2)^{-1/2} + \frac{x^2}{2!} h(1-h^2)^{-3/2} + \frac{x^3}{3!}[(1-h^2)^{-5/2}(1+2h^2)] + \ldots$
   (iv) $\log \sin a + (x-a)\cot a - \frac{(x-a)^2}{2!}\operatorname{cosec}^2 a + \frac{(x-a)^3}{3!} 2\operatorname{cosec}^2 a \cot a + \ldots$

OBJECTIVE EVALUATION

➤ FILL IN THE BLANKS :

1. If $y = \tan x$ then $y_5(0)$ is ....
2. If $y = e^{\sin x}$ then $y_3(0)$ is ....
3. By Maclaurin's theorem

$$y = e^x \sin x = x + x^2 + \frac{2}{3\,!}x^3 + \ldots + \frac{2^{n/2}\sin(n\pi/4)}{n\,!}x^n + \ldots$$

Then $y_3(0) = \ldots\ldots\ldots$.

➤ TRUE OR FALSE :

Write 'T' for True and 'F' for False :

1. We get the Maclaurin's series by putting $a = 0$, $h = x$ in Taylor's series. (T/F)

2. If $f(x) = a^x$, then $f^n(0)$ is equal to $(\log a)^n$. (T/F)

3. If $y = \log \sec x$ then $y_4(0)$ is 3. (T/F)

4. Second term in the expansion of $\log(1 + \tan x)$ is $-\frac{1}{2}x^2$. (T/F)

➤ MULTIPLE CHOICE QUESTIONS :

Choose the most appropriate one :

1. If $f(x) = a^x$ then $f^n(0)$ is equal to :

(a) 0 (b) 1

(c) $\frac{(\log a)^n}{n}$ (d) $(\log a)^n$.

2. If $y = \tan x$ then $y_5(0)$ is :

(a) 4 (b) 8 (c) 12 (d) 16.

3. If $y = \log \sec x$ then $y_4(0)$ is :

(a) 0 (b) 1 (c) 2 (d) 3.

4. Expansion of $e^x \sec x$ is equal to :

(a) $1 + x + \frac{2x^2}{2\,!} + \ldots$ (b) $1 + x + \frac{x^2}{2\,!} + \ldots$

(c) $1 - x + \frac{x^2}{2\,!}$ (d) $1 - x + \frac{2x^2}{2\,!} + \ldots$.

ANSWERS

Fill in the Blanks :

1. 16 **2.** 0 **3.** 2

True or False :

1. T **2.** T **3.** F **4.** T

Multiple Choice Questions :

1. (a) **2.** (d) **3.** (c) **4.** (a)

UNIT 3

INDETERMINATE FORMS

STRUCTURE

- Indeterminate Forms
- L'Hospital Rule for the indeterminate form 0/0
- L'Hospital Rule for the Indeterminate form ∞/∞
  - Test Yourself–1
- The Indeterminate form $0 \times \infty$
- The Indeterminate form $\infty - \infty$
- The Indeterminate forms 0°, 1^∞, ∞^0
  - Student Activity
  - Summary
  - Test Yourself–2

LEARNING OBJECTIVES

After going through this unit you will learn :

- About all determinate and indeterminate forms witnessed while evaluating the limit of the given functions.

• 3.1. INDETERMINATE FORMS

When a function involves the independent variablé in such a manner that for a certain assigned value of that variable, its value cannot be found by simply substituting that value of the variable, the function is said to take an **indeterminate form.**

The most common cases occuring is that of *a fraction whose numerator and denominator both vanish for the value of the variable involved.*

As $f(x) \to 0$ and $g(x) \to 0$ when $x \to a$, then the quotient $\dfrac{f(x)}{g(x)}$ is said to have attained the indeterminate form $\dfrac{0}{0}$.

Similarly if $\lim\limits_{x \to a} f(x) = \infty$ and $\lim\limits_{x \to a} g(x) = \infty$, then the fraction $\dfrac{f(x)}{g(x)}$ is said to have attained the indeterminate form $\dfrac{\infty}{\infty}$.

The other important indeterminate forms are $0 \times \infty$, $\infty - \infty$, 0°, 1^∞ and ∞^0.

• 3.2. L'HOSPITAL RULE FOR THE INDETERMINATE FORM 0/0

If $$\lim_{x \to a} f(x) = 0 = \lim_{x \to a} g(x)$$

then $$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} \text{ provided}$$

provided $$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} \text{ exists.}$$

• 3.3. L'HOSPITAL RULE FOR THE INDETERMINATE FROM ∞/∞

If $\lim\limits_{x \to a} f(x) = \infty$ *and* $\lim\limits_{x \to a} g(x) = \infty$, *then*

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

provided $\lim\limits_{x \to a} \dfrac{f'(x)}{g'(x)}$ *exists.*

Solved Examples

Example 1. *Find* $\lim\limits_{x \to 0} \dfrac{e^x - e^{\sin x}}{x - \sin x}$.

Solution :

$$\lim_{x \to 0} \frac{e^x - e^{\sin x}}{x - \sin x}. \qquad \left| \frac{0}{0} \text{ form} \right.$$

$$= \lim_{x \to 0} \frac{e^x - e^{\sin x} . \cos x}{1 - \cos x} \qquad \left| \text{again } \frac{0}{0} \text{ form} \right.$$

$$= \lim_{x \to 0} \frac{e^x - [\cos x . e^{\sin x} . \cos x + e^{\sin x} (-\sin x)]}{\sin x}$$

$$= \lim_{x \to 0} \frac{e^x - e^{\sin x} [\cos^2 x - \sin x]}{\sin x} \qquad \left| \text{again } \frac{0}{0} \text{ form} \right.$$

$$= \lim_{x \to 0} \frac{e^x - e^{\sin x} [2 \cos x (-\sin x) - \cos x] - [(\cos^2 x - \sin x) e^{\sin x} \cos x]}{\cos x}$$

$$= \lim_{x \to 0} \frac{e^x - e^{\sin x} [-\sin 2x - \cos x + \cos^3 x - \sin x \cos x]}{\cos x}$$

$$= \frac{1 - 1(-1 + 1)}{1} = \frac{1}{1}$$

$$= 1.$$

Example 2. *Find* $\lim\limits_{x \to 0} \dfrac{x \cos x - \log(1 + x)}{x^2}$.

Solution : We have $\lim\limits_{x \to 0} \dfrac{x \cos x - \log(1 + x)}{x^2}$

$$= \lim_{x \to 0} \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\right) - \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots\right) \qquad \left| \frac{0}{0} \text{ form} \right.$$

$$= \lim_{x \to 0} \left(\frac{\frac{x^2}{2} - \frac{5}{6} x^3 + \ldots}{x^2} \right)$$

$$= \lim_{x \to 0} \left(\frac{1}{2} - \frac{5}{6} x + \text{terms containing } x \right)$$

$$= \frac{1}{2}.$$

Example 3. *Find* $\lim\limits_{x \to 0} \dfrac{\cosh x - \cos x}{x \sin x}$.

Solution : We have $\lim\limits_{x \to 0} \dfrac{\cosh x - \cos x}{x \sin x}$ $\qquad \left| \dfrac{0}{0} \text{ form} \right.$

$$= \lim_{x \to 0} \left[\left(\frac{\cosh x - \cos x}{x^2} \right) \left(\frac{x}{\sin x} \right) \right]$$

$$= \lim_{x \to 0} \frac{\cosh x - \cos x}{x^2} \qquad \left| \frac{0}{0} \text{ form} \right.$$

$$= \lim_{x \to 0} \frac{\sinh x + \sin x}{2x} \qquad \left| \frac{0}{0} \text{ form} \right.$$

$$= \lim_{x \to 0} \frac{\cosh x + \cos x}{2} = \frac{1 + 1}{2} = 1.$$

• TEST YOURSELF–1

1. Find the following limits :

(i) $\lim_{x \to 0} \frac{x - \sin x}{x^3}$ (ii) $\lim_{x \to 0} \frac{1 - \cos x}{x^2}$

(iii) $\lim_{x \to 0} \frac{a^x - b^x}{x}$ (iv) $\lim_{x \to 1} \frac{\log x}{x - 1}$

(v) $\lim_{x \to 0} \frac{(1+x)^n - 1}{x}$ (vi) $\lim_{x \to 0} \frac{xe^x - \log(1+x)}{x^2}$

(vii) $\lim_{x \to 0} \frac{\tan x - x}{x^2 \tan x}$ (viii) $\lim_{x \to a} \frac{a^x - x^a}{x^x - a^a}$

2. Find $\lim_{x \to 0} \frac{\sin x \sin^{-1} x - x^2}{x^6}$.

3. Find $\lim_{x \to 0} \frac{(1+x)^{1/x} - e + \frac{1}{2} ex}{x^2}$.

ANSWERS

1. (i) $\frac{1}{6}$ (ii) $\frac{1}{2}$ (iii) $\log \frac{a}{b}$ (iv) 1 (v) n (vi) $\frac{3}{2}$ (vii) $\frac{1}{3}$

(viii) $\frac{\log a - 1}{\log a + 1}$ 2. $\frac{1}{18}$ 3. $\frac{11e}{24}$.

3.4. THE INDETERMINATE FORM $0 \times \infty$.

To find $\lim_{x \to a} [f(x) \cdot g(x)]$, ***when*** $\lim_{x \to a} f(x) = 0$ ***and*** $\lim_{x \to a} g(x) = \infty$.

To determine this limit, the produce may be transformed into the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$, using any one of the following relations

$$f(x)\ g(x) = \frac{f(x)}{\frac{1}{g(x)}} \quad \text{or} \quad f(x) \cdot g(x) = \frac{g(x)}{\frac{1}{f(x)}}$$

and then apply previous method.

Solved Examples

Example 1. *Evaluate* $\lim_{x \to 0+} (x \log x)$.

Solution : $\lim_{x \to 0+} (x \log x) = \lim_{x \to 0+} \frac{\log x}{1/x}$ $\quad \left| \frac{0}{0} \right.$ form

$$= \lim_{x \to 0+} \frac{1/x}{-1/x^2} = \lim_{x \to 0+} (-x) = 0.$$

Example 2. *Evaluate* $\lim_{x \to 0} x \log \sin x$.

Solution : $\lim_{x \to 0} x \log \sin x.$ $\quad |$ from $0 \times \infty$

$$= \lim_{x \to 0} \left(\frac{\log \sin x}{1/x} \right)$$

$$= \lim_{x \to 0} \frac{(1/\sin x) \cdot \cos x}{-1/x^2} \quad \left| \text{form} \frac{\infty}{\infty} \right.$$

$$= \lim_{x \to 0} \frac{-x^2 \cos x}{\sin x} \quad \left| \text{form} \frac{\infty}{\infty} \right.$$

$$= \lim_{x \to 0} \frac{x^2 \sin x - 2x \cos x}{\cos x} \quad \left| \text{form} \frac{0}{0} \right.$$

$$= 0.$$

3.5. THE INDETERMINATE FORM $\infty - \infty$

To determine $\lim_{x \to a} [f(x) - g(x)]$, when $\lim_{x \to a} f(x) = \infty = \lim_{x \to \infty} g(x)$

Here, this can be reduced to the form $\frac{0}{0}$ by the relation

$$f(x) - g(x) = \left\{ \frac{\left[\frac{1}{g(x)} - \frac{1}{f(x)}\right]}{\frac{1}{f(x) \cdot g(x)}} \right\}$$

and they aply previus method.

Working Procedure.

1. Change all trigonometric-ratio into $\sin x$ *and* $\cos x$ *(if T-ratio ae present).*

2. Take L.C.M.

Now the indeterminate form is reduced into $\frac{0}{0}$ *form.*

Solved Examples

Example 1. Evaluate $\lim_{x \to 0} \left(\frac{1}{x^2} - \frac{1}{\sin^2 x} \right)$.

Solution : $\lim_{x \to 0} \left(\frac{1}{x^2} - \frac{1}{\sin^2 x} \right)$. | form $\infty - \infty$

$$= \lim_{x \to 0} \frac{\sin^2 x - x^2}{x^2 \sin^2 x} \quad \Big| \text{ form } \frac{0}{0}$$

$$= \lim_{x \to 0} \frac{\left(x - \frac{x^3}{3!} + \ldots \right)^2 - x^2}{x^2 \left(x - \frac{x^3}{3!} + \ldots \right)^2}$$

$$= \lim_{x \to 0} \frac{-\frac{2x^4}{3!} + \text{terms containing higher powers of } x}{x^4 + \text{terms containing higher powers of } x}$$

$$= \lim_{x \to 0} \frac{-\frac{2}{3!} \text{ terms containing } x \text{ in the numerator}}{1 + \text{terms containing } x \text{ in the numerator}}$$

$$= -\frac{2}{3!} = -\frac{1}{3}.$$

Example 2. *Evaluate* $\lim_{x \to \pi/2} (\sec x - \tan x)$.

Solution : We have $\lim_{x \to \pi/2} (\sec x - \tan x)$ | form $\infty - \infty$

$$= \lim_{x \to \pi/2} \left(\frac{1}{\cos x} - \frac{\sin x}{\cos x} \right) \quad \Big| \text{ form } \frac{0}{0}$$

$$= \lim_{x \to \pi/2} \left(\frac{1 - \sin x}{\cos x} \right)$$

$$= \lim_{x \to \pi/2} \frac{-\cos x}{-\sin x} = \lim_{x \to \pi/2} \cot x = \mathbf{0}.$$

3.6. THE INDETERMINATE FORMS 0°, 1^∞, ∞°

To determine $\lim_{x \to a} [f(x)]^{g(x)}$ *when the limit is of the form 0—,* 1^∞, ∞°.

Let $\quad y = [f(x)]^{g(x)}$

Taking logs; $\quad \log y = g(x) \log f(x)$.

The R.H.S. assumes the indeterminate form $0 \times \infty$ in each of these above cases. The limit can, therefore, be dtermined by the method used in the article (4).

Suppose

$$\lim_{x \to a} [f(x) \log f(x)] = l \text{ (say)}$$

$$\Rightarrow \quad \lim_{x \to a} \log y = l$$

$$\Rightarrow \quad \lim_{x \to a} \left[\lim_{x \to a} y \right] = l$$

$$\Rightarrow \quad \lim_{x \to a} y = e^l$$

$$\Rightarrow \quad \lim_{x \to a} [f(x)]^{g(x)} = e^l.$$

Working Procedure

1. *Let the given limit = y.*

2. *Take logs onboth sides to get the forms* $0, \infty$ *and proceed by the method of the type* $0 \times \infty$.

Solved Examples

Example 1. *Find* $\lim_{x \to 0} \left(\frac{\tan x}{x} \right)^{1/x^2}$.

Solution : Let
$$y = \left(\frac{\tan x}{x} \right)^{1/x^2} \qquad \Big| \text{ form } 1^\infty, \text{ for } x = 0$$

$$\Rightarrow \quad \log y = \frac{1}{\frac{1}{x^2}} \log \frac{\tan x}{x}$$

$$\Rightarrow \quad \lim_{x \to 0} \log y = \lim_{x \to 0} \frac{1}{x^2} \log \frac{\tan x}{x}$$

$$= \lim_{x \to 0} \frac{\log \frac{\tan x}{x}}{x^2} \qquad \Big| \text{ form } \frac{0}{0}$$

$$= \lim_{x \to 0} \frac{\dfrac{1}{\left(\frac{\tan x}{x} \right) \left[\frac{x \sec^2 x - \tan x}{x^2} \right]}}{2x}$$

$$= \lim_{x \to 0} \frac{x \sec^2 - \tan x}{2x^3} \qquad \Big| \because \lim_{x \to 0} \frac{\tan x}{x} = 1$$

$$= \lim_{x \to 0} \frac{x \,.\, 2 \sec x \sec x \tan x + \sec^2 x - \sec^2 x}{6x^2}$$

$$= \lim_{x \to 0} \frac{2x \tan x \sec^2 x}{6x^2} = \lim_{x \to 0} \frac{\tan x \sec^2 x}{3x}$$

$$= \lim_{x \to 0} \left(\frac{1}{3} \cdot \frac{\tan x}{x} \cdot \sec^2 x \right) = \frac{1}{3} \times 1 \times \sec^2 x = \frac{1}{3}$$

$$\therefore \quad \lim_{x \to 0} y = e^{1/3} \quad \lim_{x \to 0} \left(\frac{\tan x}{x} \right)^{1/x^2} = {}^{1/3}.$$

Example 2. *Evaluate* $\lim_{x \to 0} \left(\frac{\sin x}{x} \right)^{1/x^2}$.

Solution : Let $y = \lim_{x \to 0} \left(\frac{\sin x}{x} \right)^{1/x^2}$

$$\therefore \qquad \log y = \lim_{x\to 0}\left(\frac{1}{x^2}\log\frac{\sin x}{x}\right) = \lim_{x\to 0}\frac{1}{x^2}\left\{\frac{x-\frac{x^3}{3!}+\frac{x^5}{5!}-\ldots}{x}\right\}$$

$$= \lim_{x\to 0}\frac{1}{x^2}\log\left(1-\frac{x^2}{3!}+\frac{x^4}{4!}+\ldots\right)$$

$$= \lim_{x\to 0}\frac{1}{x^2}\left[1-\left(\frac{x^2}{6}-\frac{x^4}{120}+\ldots\right)\right]$$

$$= \lim_{x\to 0}\frac{1}{x^2}\log(1-z) \qquad \text{where } z=\frac{x^2}{6}-\frac{x^4}{120}+\ldots$$

$$= \lim_{x\to 0}\frac{1}{x^2}\left(-z-\frac{z^2}{2}-\ldots\right)$$

$$= \lim_{x\to 0}\frac{1}{x^2}\left[-\left(\frac{x^2}{6}-\frac{x^4}{120}+\ldots\right)-\frac{1}{2}\left(\frac{x^2}{6}-\frac{x^4}{120}+\ldots\right)^2-\ldots\right]$$

$$= \lim_{x\to 0}\frac{1}{x^2}\left[-\frac{x^2}{6}+\left(\frac{x^4}{120}-\frac{x^4}{72}\right)+\ldots\right]$$

$$= \lim_{x\to 0}\frac{1}{x^2}\left[-\frac{x^2}{6}-\frac{x^4}{180}+\ldots\right]$$

$$= \lim_{x\to 0}\left[\frac{1}{6}-\frac{x^2}{180}+\ldots\right]$$

$$= -\frac{\mathbf{1}}{\mathbf{6}}.$$

• SUMMARY

- **Indeterminate Forms :** Some indeterminate forms are

$$\frac{0}{0},\ \frac{\infty}{\infty},\ 0\times\infty,\ \infty-\infty,\ 0^\circ,\ 1^\infty \text{ and } \infty^\circ.$$

- If $\lim_{x\to 0} f(x)=\infty=\lim_{x\to 0} g(x)$, then $\lim_{x\to 0}\frac{f(x)}{g(x)}=\lim_{x\to 0}\frac{f'(x)}{g'(x)}$, $g(a)\neq 0$.
- If $\lim_{x\to 0} f(x)=\infty=\lim_{x\to 0} g(x)$, then $\lim_{x\to 0}\frac{f(x)}{g(x)}=\lim_{x\to 0}\frac{f'(x)}{g'(x)}$ provided $\lim_{x\to 0}\frac{f'(x)}{g'(x)}$ exists.
- If $\lim_{x\to a} f(x)=0$ and $\lim_{x\to a} g(x)=\infty$, then $\lim_{x\to a} f(x)\,g(x)=\lim_{x\to 0}\frac{f(x)}{1/g(x)}$.
- If $\lim_{x\to a}[f(x)-g(x)]=\lim_{x\to a}\left[\frac{\frac{1}{g(x)}-\frac{1}{f(x)}}{\frac{1}{f(x)\,g(x)}}\right]$
- If the $\lim_{x\to 0}[f(x)]^{g(x)}$ is of the form 0°, 1^∞ and ∞°, then $\log y=\lim_{x\to a} g(x)\log(f(x)=l$ (say).

$\Rightarrow \quad y=e^l.$

• STUDENT ACTIVITY

1. Find $\lim_{x\to\pi/2}\frac{\log(x-\pi/2)}{\tan x}$.

2. Evaluate $\lim\limits_{x \to 0} \left(\dfrac{\sin x}{x} \right)^{1/x^2}$.

• TEST YOURSELF–2

1. Evaluate the following limits :

   (i) $\lim\limits_{x \to 0} x \log \tan x$ (ii) $\lim\limits_{x \to 0} \tan\left(\dfrac{\pi}{2} - x\right)$

   (iii) $\lim\limits_{x \to \infty} 2^x \sin \dfrac{a}{2^x}$

2. Evaluate the following limits

   (i) $\lim\limits_{x \to 0} \left[\dfrac{1}{x} - \dfrac{1}{x^2} \log(1 + x) \right]$ (ii) $\lim\limits_{x \to 2} \left(\dfrac{1}{x - 2} - \dfrac{1}{\log(x - 1)} \right)$

   (iii) $\lim\limits_{x \to 0} \left[\dfrac{1}{x^2} - \operatorname{cosec}^2 x \right]$ (iv) $\lim\limits_{x \to 0} \left(\dfrac{1}{x^2} - \cot^2 x \right)$

3. Evaluate the following limits :

   (i) $\lim\limits_{x \to 0} \left(\dfrac{1}{x} \right)^{\tan x}$ (ii) $\lim\limits_{x \to \infty} \left(\dfrac{\pi}{2} - \tan^{-1} x \right)^{1/x}$

   (iii) $\lim\limits_{x \to 0} \left(\dfrac{\tan x}{x} \right)^{1/x}$

OBJECTIVE EVALUATION

Fill in the Blanks :

1. $\lim\limits_{x \to 1} \dfrac{\log x}{x - 1}$ is .... 2. $\lim\limits_{x \to 0} \dfrac{\sin ax}{\sin bx}$ is ....

3. $\lim\limits_{x \to \infty} \dfrac{x^2 + 2x}{5 - 3x^2}$ is .... 4. $\lim\limits_{x \to 0} \dfrac{\tan x}{x}$ is ....

True or False :

1. The indeterminate form $\dfrac{\infty}{\infty}$ can be converted into the form $\dfrac{0}{0}$. *(T/F)*

2. 1^{∞} is not an indeterminate form. *(T/F)*

Multiple Choice Questions :

1. $\lim_{x \to 0} \frac{\tan x}{x}$ is :

(a) 0 (b) ∞ (c) 1 (d) -1

2. $\lim_{x \to 0} (1 + nx)^{1/x}$ is :

(a) 1 (b) e^{-n} (c) e^2 (d) e^n

ANSWERS

1. (i) 0 (ii) ∞ (iii) a (iv)

2. (i) $\frac{1}{2}$ (ii) $-\frac{1}{2}$ (iii) 1 (iv) $\frac{2}{3}$

3. (i) 1 (ii) 1 (iii) 1

Fill in the Blanks :

1. 1 **2.** $\frac{a}{b}$ **3.** $-\frac{1}{3}$ **4.** 1

True or False

1. T **2.** T **3.** F

Multiple Choice Questions :

1. (c) **2.** (d)

UNIT

4

TANGENTS AND NORMALS

STRUCTURE

LEARNING OBJECTIVES

After going through this unit you will learn :

Determine the tangents and normals to the given curve at the given points.

How to determine Angle between two Curves.

• 4.1. TANGENT

Let P be any point on a curve $y = f(x)$ and Q any other point on it such that Q is very close to P. The point Q may be taken on either side of P.

As Q tends to P, secant line PQ, in general tends to a definite straight line TP passing through P. *This straight line TP is known as the tangent to the curve* $y = f(x)$ *at the point P.*

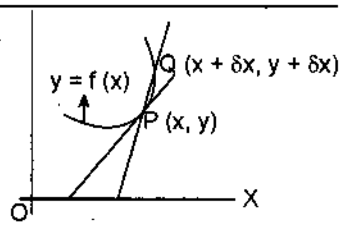


Fig. 4.1

• 4.2. EQUATION OF THE TANGENT

Let $y = f(x)$ be a given curve. Let P be any given point (x, y) on this curve and Q be any other point $(x + \delta x, y + \delta y)$ on it such that Q is very closed to P. Let (X, Y) be an arbitrary point on the secant line PQ, then the equation of the secant line PQ is

$$Y - y = \left(\frac{y + \delta y - y}{x + \delta x - x}\right)(X - x)$$

or $$Y - y = \frac{\delta y}{\delta x}(X - x) \qquad \ldots (1)$$

As Q tends to P, $\delta x \to 0$ and PQ tends to the tangent at P.

$\therefore$ Equation (1) tends to an equation

$$Y - y = \frac{dy}{dx}(X - x) \qquad \left[\because \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}\right]$$

Hence, the equation of the tangent to the curv at $P(x, y)$ is given by

$$Y - y = \frac{dy}{dx}(X - x)$$

REMARKS

If we are to find the tangent to the curve $y = f(x)$ at (x_1, y_1), we first find $\frac{dy}{dx}$ at (x_1, y_1). Then the equation of the tangent to the curve at (x_1, y_1) is given by

$$y - y_1 = \left(\frac{dy}{dx}\right)_{(x_1, y_1)} (x - x_1)$$

or
$$\frac{y - y_1}{x - x_1} = \left(\frac{dy}{dx}\right)_{(x_1, y_1)}$$

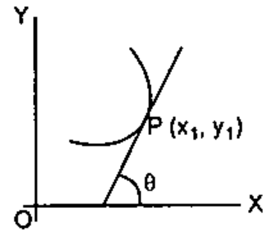


Fig. 4.2

If θ be the angle which the positive direction of the tangent at P makes with the positive direction of the x-axis, then

$$\tan\theta = \left(\frac{dy}{dx}\right)_{(x_1, y_1)}$$

If $\theta = 0$, then $\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = 0$, which means, the tangent is parallel to x-axis.

If $\theta = 90°$, then $\left(\frac{dx}{dy}\right)_{(x_1, y_1)} = 0$, which means, the tangent is perpendicular to x-axis or parallel to y-axis.

Two curves $y = f(x)$ and $y = g(x)$ are at right angle, if $m_1 \times m_2 = -1$ where m_1 = slope of the tangent to $y = f(x)$ at common point and m_2 = slope of tangent to $y = g(x)$ at common point.

Two curves $y = f(x)$ and $y = g(x)$ touch each other if they have the same tangent at the common point.

If θ be the angle between the curves $y = f(x)$ and $y = g(x)$, then

$$\tan\theta = \left|\frac{m_1 - m_2}{1 - m_1 m_2}\right|.$$

Result 1. The tangent to a curve $y = f(x)$ at a point $P(x_1, y_1)$ is parallel to x-axis if and only if

$$\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = 0.$$

Result 2. The tangent to a curve $y = f(x)$ at a point $P(x_1, y_1)$ is parallel to y-axis if and only if

$$\left(\frac{dx}{dy}\right)_{(x_1, y_1)} = 0.$$

• 4.3. NORMAL

The normal to a curve $y = f(x)$ at any point P on it, is the straight line passing through P and perpendicular to the tangent to the curve at P.

• 4.4. EQUATION OF A NORMAL

Let $y = f(x)$ be a given curve and $P(x_1, y_1)$ be any point on it.

Since, the normal to $y = f(x)$ at $P(x_1, y_1)$ is perpendicular to the tangent at P.

So, $$\text{slope of normal} = \frac{-1}{\left(\frac{dy}{dx}\right)_{(x_1, y_1)}}$$

∴ The equation of a normal at $P(x_1, y_1)$ is given by

$$y - y_1 = \frac{-1}{\left(\frac{dy}{dx}\right)_{(x_1, y_1)}} (x - x_1)$$

or
$$\frac{y - y_1}{x - x_1} = \frac{-1}{\left(\frac{dy}{dx}\right)_{(x_1, y_1)}}$$

or
$$(x - x_1) + \left(\frac{dy}{dx}\right)_{(x_1, y_1)} (y - y_1) = 0$$

Solved Examples

Example 1. Find the slope of the tangent of the following curves :

(i) $y^2 = 4ax$ at $(a^2, 2a)$ (ii) $y = x^3 - x$ at $(2, 6)$

Solution : (i) We have $y^2 = 4ax$

$$2y\frac{dy}{dx} = 4a$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2a}{y}.$$

So, the slope of the tangent at $(a^2, 2a) = \left(\frac{dy}{dx}\right)_{(a^2, 2a)}$

$$= \left(\frac{2a}{y}\right)_{(a, 2a)} = \frac{2a}{2a} = 1.$$

(ii) We have $y = x^3 - x$

$$\therefore \quad \frac{dy}{dx} = 3x^2 - 1.$$

So, the slope of the tangent at $(2, 6) = \left(\frac{dy}{dx}\right)_{(2, 6)}$

$$= (3x^2 - 1)_{(2, 6)}$$

$$= 3\,(2)^2 - 1 = 11.$$

Example 2. Find the equation of the tangent and the normal to the curve $y = x^3 - 2x + 7$ at (1, 6).

Solution : We have $y = x^3 - 2x + 7$

$$\therefore \quad \frac{dy}{dx} = 3x^2 - 2$$

So, $\left(\frac{dy}{dx}\right)_{(1, 6)} = 3\,(1)^2 - 2 = 3 - 2 = 1.$

The equation of the tangent to the given curve at (1, 6) is

$$y - 6 = \left(\frac{dy}{dx}\right)_{(1, 6)} (x - 1)$$

or $\quad y - 6 = 1\,(x - 1)$

or $\quad x - y + 5 = 0.$

The equation of the normal to the given curve at (1, 6) is

$$y - 6 = -\left(\frac{dy}{dx}\right)_{(1, 6)} (x - 1)$$

or $\quad y - 6 = \frac{-1}{1}(x - 1)$

or $\quad x + y - 7 = 0.$

Example 3. Find the equation of the tangent to the curve $x^2 + 3y = 3$, which is parallel to the line $y - 4x + 5 = 0$.

Solution : We have

$$x^2 + 3y = 3$$

$$\Rightarrow \quad 2x + 3\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{2x}{3}.$$

Let (x_1, y_1) be a point of contact.

So, $\quad x_1^2 + 3y_1 = 3 \quad \ldots (1)$

Now the slope of the tangent at (x_1, y_1) to the given curve is

$$\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = -\frac{2x_1}{3}$$

and, the slope of the line $y - 4x + 5 = 0$ is 4.

Since the tangent is parallel to the given line.

$$\therefore \qquad -\frac{2x_1}{3} = 4$$

or $$x_1 = -6$$

Putting $x_1 = -6$ in (1), we get $y_1 = -11$.

$\therefore$ Point of contact is $(-6, -11)$.

Thus, the equation of the tangent at $(-6, -11)$ is

$$y + 11 = 4(x + 6)$$

or $$4x - y + 13 = 0.$$

Example 4. Find the angles of intersection of the curves $y = 4 - x^2$ and $y = x^2$.

Solution : We have

$$\therefore \qquad y = 4 - x^2 \qquad \ldots (1)$$

and $$y = x^2 \qquad \ldots (2)$$

Solving eqns. (1) and (2), we get

$$y = 4 - x^2 \qquad \text{[From eq. (1)]}$$

$$\Rightarrow \qquad x^2 = 4 - x^2 \qquad \text{[using eq. (2)]}$$

$$\Rightarrow \qquad 2x^2 = 4$$

$$\Rightarrow \qquad x^2 = 2$$

$$\Rightarrow \qquad x = \pm\sqrt{2}$$

Putting $x = \pm\sqrt{2}$ in eq. (2), we get $y = 2$.

Thus the common points are $(\sqrt{2}, 2)$ and $(-\sqrt{2}, 2)$.

Now from eq. (1), we get

$$\frac{dy}{dx} = -2x \qquad \ldots (3)$$

So, m_1 = slope of the tangent to eq. (1) at $(\sqrt{2}, 2)$.

$$= \left(\frac{dy}{dx}\right)_{(\sqrt{2}, 2)} = -2\sqrt{2} \qquad \text{[using eq. (3)]}$$

From eq. (2), we get

$$\frac{dy}{dx} = 2x \qquad \ldots (4)$$

And, m_2 = sope of the tangent to the curve eq. (2) at $(\sqrt{2}, 2)$

$$= \left(\frac{dy}{dx}\right)_{(\sqrt{2}, 2)} = 2\sqrt{2} \qquad \text{[using eq. (4)]}$$

If θ be the angle between the curves at $(\sqrt{2}, 2)$ then,

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

$$= \left|\frac{-2\sqrt{2} - 2\sqrt{2}}{1 - 8}\right|$$

$$= \left|\frac{-4\sqrt{2}}{-7}\right| = \frac{4\sqrt{2}}{7}$$

$$\therefore \qquad \theta = \tan^{-1}\left(\frac{4\sqrt{2}}{7}\right).$$

• SUMMARY

- If $y = f(x)$, then the slope of the tangent to $y = f(x)$ at (x_1, y_1) is $m = \left(\frac{dy}{dx}\right)_{(x_1, y_1)}$.
- Equation of the tangent to the curve $y = f(x)$ at the point (x_1, y_1) is
$$y - y_1 = \left(\frac{dy}{dx}\right)_{(x_1, y_1)}(x - x_1).$$
- If $y = f(x)$, then the slope of the normal to the cuve $y = f(x)$ at (x_1, y_1) is given by
$$m_1 = -\left(\frac{dx}{dy}\right)_{(x_1, y_1)}$$
- Equation of the normal to the curve $y = f(x)$ at the point (x_1, y_1) is
$$y - y_1 = -\left(\frac{dx}{dy}\right)_{(x_1, y_1)}(x - x_1)$$

• STUDENT ACTIVITY

1. Find the equation of the tangent to the curve $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point $(a \cos \theta, b \sin \theta)$.

2. Find the angles of intersection of the curve $y = x - x^2$ and $y = x^2$.

• TEST YOURSELF

1. Find the slope of the tangent to the following curve :
   (i) $y = (2x^2 + 3\sin x)$ at $x = 0$ (ii) $y = (\sin 2x + \cot x + 2)^2$ at $x = \pi/2$
   (iii) $y = \sin^2 x$ at $x = \pi/4$
2. Find the equation of the tangent and the normal to the given curves at indicated point :
   (a) $y^2 = 4ax$ at $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$ (b) $xy = c^2$ at (c, c)
3. (i) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at $(a \cos \theta, b \sin \theta)$ (ii) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at $(a \sec \theta, b \tan \theta)$.
4. $y^2 = 4ax$ at $(at^2, 2at)$

5. Find the equation of the normal to the curve $y = (\sin 2x + \cot x + 2)^2$ at $x = \pi/2$.

6. Show that the tangents to the curve $y = 2x^3 - 4$ at the points $x = 2$ and $x = -2$ are parallel.

7. Show that the curves $xy = a^2$ and $x^2 + y^2 = 2a^2$ touch each other.

8. Find the angles of intersection of the curves $x^2 - y^2 = a^2$ and $x^2 + y^2 = a^2\sqrt{2}$.

ANSWERS

1. (i) 3 (ii) -12 (iii) 1

2. (i) $m^2x - my + a = 0,\ m^2x + m^3y - 2am^2 - a = 0$ (ii) $x + y - 2c = (\ , y = x$

3. (i) $bx \cos\theta + ay \sin\theta = ab,\ ax \sec\theta - by \cot\theta = a^2 + b^2$

(ii) $bx \sec\theta - ay \tan\theta = ab,\ ax\, ax \cos\theta + by \cot\theta = a^2 + b^2$

4. $x - ty + at^2 = 0,\ tx + y = at^3 + 2at$

5. $24y - 2x + \pi - 96 = 0$ **8.** $\pi/4$.

UNIT 5

CURVATURE

STRUCTURE

LEARNING OBJECTIVES

After going through this unit you will learn :
- How to calculate the radius of curvature of the given curve at a given point

• 5.1. DEFINITION OF CURVATURE

Let P, Q be two negihbouring points on a curve AB. Also, let $AP = s$, arc $AQ = s + \delta s$ and arc $PQ = \delta s$.

Let the tangent to the curve at points P and Q makes angle ψ and $\psi + \delta\psi$ respectively with a fixed line say X-axis, then

(*i*) *The angle $\delta\psi$ through which the tangent turns as its points of contact travels along the arc PQ is called the* ***total bending or total curvature*** *of arc PQ.*

(*ii*) *The ratio $\frac{\delta\psi}{\delta s}$ is called the* ***mean*** *or* ***average curvature*** *of arc PQ.*

(*iii*) *The limiting value of the mean curvature when Q tends to P is called the* ***curvature*** *of the curve at the point P. Therefore, the curvature K at point P is*

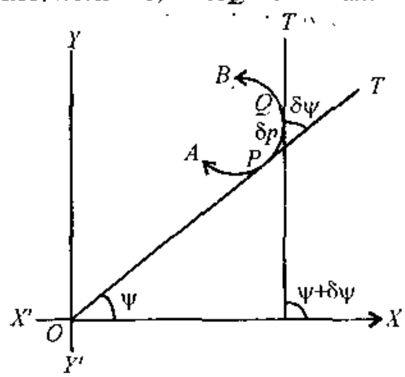


Fig. 2

$$\lim_{Q \to P} \frac{\delta\psi}{\delta s} = \lim_{\delta s \to 0} \frac{\delta\psi}{\delta s} = \frac{d\psi}{ds}.$$

(*iv*) *The reciprocal of the curvature of the given curve at P, (provided this curvature is not equal to zero), is called the* ***radius of curvature*** *of the curve at P. This is denoted by* ρ

$$\rho = \frac{1}{K} = \frac{ds}{d\psi}.$$

• 5.2. FORMULA FOR RADIUS OF CURVATURE (CARTESIAN FORM)

Let $y = f(x)$ be the equation of curve. Then the radius of curvature P is given by

$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}.$$

• 5.3. RADIUS OF CURVATURE AT THE ORIGIN

Let the curve $y = f(x)$ *passes through the origin.* Then, we may use the following methdos, to find the radius of curvature.

(i) Method of Direct Substitution. Since $y = f(x)$ be given. *Calculate the values of* $\frac{dy}{dx}$ *and* $\frac{d^2y}{dx^2}$ *at origin and then use the following formula*

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{d^2y/dx^2}.$$

(ii) Method of Expansion. Let $y = f(x)$ be the equation of curve. Since, it passes through the origin, therefore $f(0) = 0$. In this case we use the following formula :

$$\rho = \frac{(1+p_1^2)^{3/2}}{p_2}.$$

where $p_1 = \frac{dy}{dx}$ and $p_2 = \frac{d^2y}{dx^2}$

(iii) Newton's Method. ***If a curve passes through the origin, and axis of x is the tangent at the origin, then radius of curvature ρ at origin***

$$= \lim_{\substack{x \to 0 \\ y \to 0}} \frac{x^2}{2y}.$$

REMARK

➤ If a curve passes through the origin and axis of y is the tangent, then radius of curvature at the origin is given by $= \lim\limits_{\substack{x \to 0 \\ y \to 0}} \frac{y^2}{2x}$.

SOLVED EXAMPLES

Example 1. *Find the curvature of the curve $x^3 + y^3 = 3axy$ at the point $(3a/2, 3a/2)$.*

Solution. Here, we have the equation of the curve is

$$x^3 + y^3 = 3axy. \qquad \text{...(i)}$$

Differentiating w.r.t. x, we get

$$3x^2 + 3y^2\frac{dy}{dx} = 3ay + 3ax\frac{dy}{dx}$$

$$\Rightarrow \quad x^2 + y^2\frac{dy}{dx} = ay + ax\frac{dy}{dx} \qquad \text{...(ii)}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{x^2 - ay}{ax - y^2}$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{\text{at}\left(\frac{3}{2}a,\frac{3}{2}a\right)} = -1$$

From (ii), we have

$$2x + 2y\left(\frac{dy}{dx}\right)^2 + y^2\frac{d^2y}{dx^2} = a\frac{dy}{dx} + a\frac{dy}{dx} + ax\frac{d^2y}{dx^2}$$

$$\Rightarrow \quad (ax - y^2)\frac{d^2y}{dx^2} = 2x + 2y\left(\frac{dy}{dx}\right)^2 - 2a\frac{dy}{dx}. \qquad \text{...(iii)}$$

Putting $x = \frac{3a}{2}$, $y = \frac{3a}{2}$ and $\left(\frac{dy}{dx}\right)_{\left(\frac{3a}{2},\frac{3a}{2}\right)} = -1$, we get

$$\left[\frac{d^2y}{dx^2}\right]_{\left(\frac{3a}{2},\frac{3a}{2}\right)} = -\frac{32}{3}\cdot\frac{1}{a}$$

Hence, the radius of curvature ρ at $\left(\frac{3a}{2},\frac{3a}{2}\right)$, is given by

$$\rho = \left[\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}\right]_{\text{at}\left(\frac{3a}{2},\frac{3a}{2}\right)} = \frac{(1+1)^{3/2}}{-\frac{32}{3}\cdot\frac{1}{a}} = -\frac{3a}{8\sqrt{2}}.$$

Therefore, the curvature $= \frac{1}{\rho} = \frac{8\sqrt{2}}{3a}$. (By ignoring the negative sign)

Example 2. *Apply Newton's formula to find the radius of curvature at the origin for the curve*

$$x^3 - 2x^2y + 3xy^2 - 4y^3 + 5x^2 - 6xy + 7y^2 - 8y = 0.$$

Solution. Since, the curve passes through the origin. Equating to zero, the lowest degree terms, we may find $y = 0$

$\Rightarrow x$ axis is the tangent at the origin.

Therefore, by Newton's formula, ρ at $(0, 0) = \lim\limits_{\substack{x \to 0 \\ y \to 0}} \frac{x^2}{2y}$.

Dividing the equation of the curve by $2y$, we get

$$x \cdot \frac{x^2}{2y} - x^2 + \frac{3}{2}xy - 2y^2 + 5 \cdot \frac{x^2}{2y} - 3x + \frac{7}{2}y - 4 = 0.$$

Taking lim $x \to 0$ and $y \to 0$, we get

$$5 \lim_{\substack{x \to 0 \\ y \to 0}} \frac{x^2}{2y} - 4 = 0 \Rightarrow 5\rho - 4 = 0 \Rightarrow \rho = \frac{4}{5}.$$

• SUMMARY

- If $y = f(x)$, then the radius of curvature is given by

$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$

- Curvature $= \frac{1}{\rho}$.
- If a curve passes through the origin and x-axis is the tangent at the origin, then the radius of curvaure ρ at the origin is

$$\rho = \lim_{(x, y) \to (0, 0)} \left(\frac{x^2}{2y}\right)$$

- If a curve passes through the origin and y-axis is the tangent at the origin, then the radius of curvature at the oriin is given by

$$\rho = \lim_{(x, y) \to (0, 0)} \left(\frac{y^2}{2x}\right)$$

• STUDENT ACTIVITY

1. Find the curvature of the curve $x^3 + y^3 = 3axy$ at the point $\left(\frac{3a}{2}, \frac{3a}{2}\right)$.

2. Find the radius of curvature at the origin to the curve

$$x^3 - 2x^2y + 3xy^2 - 4y^3 + 5x^2 - 6xy + 7y^2 - 8y = 0.$$

- **TEST YOURSELF**

1. Find the radius of curvature of the following curves :
   (i) $x^{1/2} + y^{1/2} = a^{1/2}$. (ii) $a^2 y = x^3 - a^3$.
   (iii) $x^{2/3} + y^{2/3} = a^{2/3}$. (iv) $x^m + y^m = 1$.
   (v) $\sqrt{x} + \sqrt{y} = 1$ at $\left(\frac{1}{4}, \frac{1}{4}\right)$. (vi) $s = 4a \sin \psi$ at (s, ψ).
   (vii) $ay^2 = x^3$.
   (viii) $y = e^x$ at the point where it cuts the y-axis.
   (ix) $x^{2/3} + y^{2/3} = a^{2/3}$ at $(a \cos^3 \theta, a \sin^3 \theta)$.
   (x) $y = 4 \sin x - \sin 2x$ at $x = \frac{\pi}{2}$.
2. Find the radius of curvature at the origin of the following curves :
   (i) $x^3 + y^3 = 3axy$ (ii) $y = x^3 + 5x^2 + 6x$
   (iii) $5x^3 + 7y^3 + 4x^2y + xy^2 + 2x^2 + 3xy + y^2 + 4x = 0$
   (iv) $a(y^2 - x^2) = x^3$

ANSWERS

1. (i) $\frac{2(x+y)^{3/2}}{a^{1/2}}$ (ii) $\frac{(a^4 + 9x^4)^{3/2}}{6a^4 x}$ (iii) $3a^{1/3} x^{1/3} y^{1/3}$
   (iv) $\frac{(x^{2m-2} + y^{2m-2})^{3/2}}{(1-m) x^{m-2} y^{m-2}}$ (v) $\frac{1}{\sqrt{2}}$ (vii) $\frac{1}{6a} (4a + 9x)^{3/2} x^{1/2}$
   (viii) $\sqrt{8}$ (ix) $3a \sin \theta \cos \theta$ (x) $\frac{5\sqrt{5}}{4}$.
2. (i) $\frac{3a}{2}$ (ii) $\frac{37\sqrt{37}}{10}$ (iii) $\rho = 2$ (ignoring negative sign) (iv) $2a\sqrt{2}$

OBJECTIVE EVALUATION

➤ FILL IN THE BLANKS :

1. The curvature of the curve at any point P is defined as the .... of the radius of curvature at P.
2. For a curve $y = f(x)$, the radius of curvature $\rho =$ ....

➤ TRUE OR FALSE :

Write 'T' for True and 'F' for False :

1. The curvature of the curve at any point P is defined as the reciprocal of the radius of curvature of P. *(T/F)*
2. If y axis is the tangent to the given curve at the origin, then radius of curvature at the origin is equal to $\lim_{x \to 0} \frac{y^2}{2x}$. *(T/F)*
3. The curvature of the circle and circle of curvature, both are the same. *(T/F)*

➤ MULTIPLE CHOICE QUESTIONS :

Choose the most appropriate one :

1. The radius of curvature of the curve $y = e^x$ at the point where it crosses the y-axis is :
   (a) 2 (b) $\sqrt{2}$ (c) $2\sqrt{2}$ (d) 1.
2. For the curve $xy = a^2$ the radius of curvature at (2, 2) is :
   (a) 4 (b) 16 (c) 10 (d) None of these.

ANSWERS

Fill in the Blanks :

1. reciprocal **2.** $\dfrac{(1+y_1^2)^{3/2}}{y_2}$

True or False :

1. T **2.** T **3.** F

Multiple Choice Questions :

1. (c) **2.** (d)

❑❑❑

UNIT 6

ASYMPTOTES

STRUCTURE

- Asymptote
- Determination of Asymptotes
- Asymptotes of General Equation
- Number of Asymptotes of a Curve
- Asymptotes Parallel to Co-Ordinates Axes
  - Test Yourself–1
  - Test Yourself–2
  - Summary
  - Student Activity
  - Test Yourself–3

LEARNING OBJECTIVES

After going through this unit you will learn :

- How to determine the asymptotes of the given curves, which are helpful to draw the given curve

6.1. ASYMPTOTE

In calculus, there are some curves whose branches seem to go to infinity. It is not necessary that there always exists a definite straight line for all such curves which seems to touch the branch of the curves at infinity but more or less there are some certain curves for which this type of definite straight line exists, this straight line is therefore known as **asymptote.**

Definition. A definite straight line whose distance from branch of the curve continuously decreases as we move away from the origin along the branch of the curve and seems to touch the branch at infnity, provided the distance of this line from origin should be finite initially, is called an **asymptote** of the curve.

More than one asymptote of a curve. Suppose in the equation of a curve, two or more than two values of y exists for every value of x, then we obtain different branches of the curve corresponding to these distinct values of y. If each branch have its own separate asymptote, then we can say that a curve may have more than one asymptote.

6.2. DETERMINATION OF ASYMPTOTES

Let us consider a curve

$$f(x, y) = 0 \qquad ...(1)$$

and also consider that there are no asymptotes parallel to y-axis. Thus we shall take the equation which is not parallel to y-axis, in the form of

$$y = mx + c. \qquad ...(2)$$

Let us take a point $P(x, y)$ on the curve (1), therefore this point as tends to infinity along the straight line (2), x must tend to infinity. Now find the tangent to the curve $f(x, y) = 0$ at the point $P(x, y)$.

$\therefore$ The equation of tangent at $P(x, y)$ is

$$Y - y = \frac{dy}{dx}(X - x) \quad \text{or} \quad Y = \frac{dy}{dx}X + \left(y - x\frac{dy}{dx}\right). \qquad ...(3)$$

The equation (3) is of the form $y = mx + c$ so in order to exist the asymptote of the curve there must both $\frac{dy}{dx}$ and $\left(y - x\frac{dy}{dx}\right)$ tend to finite limits as x tends to infinity. Therefore, if the equation

(3) tends to the straight line given in (2) as x tends to infinity, then the line (2) will be an asymptote of the curve $f(x, y) = 0$ and also we have

$$m = \lim_{x \to \infty} \frac{dy}{dx} \qquad \text{(slope of the asymptote)}$$

and
$$c = \lim_{x \to \infty} \left(y - x\frac{dy}{dx} \right).$$

Since c is finite, then we have

$$\lim_{x \to \infty} \left(\frac{y - x\frac{dy}{dx}}{x} \right) = \lim_{x \to \infty} \frac{c}{x} = 0 \quad \text{or} \quad \lim_{x \to \infty} \left(\frac{y}{x} - \frac{dy}{dx} \right) = 0$$

or
$$\lim_{x \to \infty} \left(\frac{y}{x} \right) = \lim_{x \to \infty} \frac{dy}{dx} \quad \text{or} \quad \lim_{x \to \infty} \frac{y}{x} = m.$$

Also
$$c = \lim_{x \to \infty} \left(y - x\frac{dy}{dx} \right)$$

$$c = \lim_{x \to \infty} (y - mx).$$

Hence, if $y = mx + c$ is an asymptote to the curve $f(x, y) = 0$, then we obtain

$$m = \lim_{x \to \infty} \frac{dy}{dx} = \lim_{x \to \infty} \frac{y}{x}$$

and
$$c = \lim_{x \to \infty} (y - mx).$$

• 6.3. ASYMPTOTES OF GENERAL EQUATION

Let $f(x, y) = 0$ a general equations of a curve of degree n. To find the asymptotes $y = m + c$, we proceed as follows :

(i) Put $y = m$ and $x = 1$ in n^{th} degree terms (highest degree) of $f(x, y) = 0$ and we obtain an expression $\phi_n(m)$.

(ii) Find the values of m by solving equatin $\phi_n(m) = 0$.

We have two cases :

Case I : If all the values of m are distinct, then we calculate the values of c by using the formula

$$c\phi'_n(m) + \phi_{n-1}(n) = 0$$

Case II : If r values of m are identical, then we calculate the values of c for these identical values of m by using the formula

$$\frac{c^r}{r!}\phi_n^r(m) + \frac{c^{r-1}}{(r-1)!}\phi_{n-1}^{r-1}(n) + \ldots\ldots + \frac{c}{1!}\phi'_{n-r+1}(m) + \phi_{n-r}(m) = 0.$$

• 6.4. NUMBER OF ASYMPTOTES OF A CURVE

Suppose the degree of an algebraic curve is n, then *we find a polynomial $\phi_n(m)$ by putting $y = m$ and $x = 1$ in the n^{th} degree terms of the curve.* Thus the equation $\phi_n(m) = 0$ is of degree n in m and which gives almost n values of m real as well as imaginary. These n values of m are nothing but the slopes of the asymptotes, which are not parallel to y axis. If there are some asymptotes, parallel to y-axis, then the degree of $\phi_n(m)$ will be smaller than n by the same number of parallel asymptotes. Suppose all the roots of $\phi_n(m) = 0$ are distinct and real, then to each value of m we obtain one value of c. Hence, we obtain n asymptotes. In case, there some roots say r (out of n) of $\phi_n(m) = 0$ are same, then we can find the values of c for these same roots by the following equation

$$\frac{c^r}{r!}\phi_n^r(m) + \frac{c^{r-1}}{r-1!}\phi_{n-1}^{r-1}(m) + \ldots + \phi_{n-r}(m) = 0.$$

This equation in c is of degree r so we get r distinct values of c for the same roots. hence, again we obtain n asymptotes. Therefore we can say that the *total number of asymptotes of a curve are equal to the degree of the curve.* These asymptotes are real as well as imaginary but we have required only real asymptotes so we ignore all the imaginary asymptotes.

• 6.5. ASYMPTOTES PARALLEL TO CO-ORDINATES AXES

(a) Asymptotes parallel to x-axis. Let the general equation of an algebraic curve in decreasing powers of x be

$$x^n \phi(y) + x^{n-1}\phi_1(y) + x^{n-2}\phi_2(y) + \ldots = 0 \qquad \ldots(1)$$

where $\phi(y)$, $\phi_1(y)$, $\phi_2(y)$ ... are the function of y only.

To find the asymptotes parallel to x-axis we equate the coefficient of highest power of x in the given curve to 0.

(b) Asymptotes parallel to y-axis. Similarly, we may obtain the asymptotes parallel to y-axis by taking the coefficient of highest power of y in the equation of the curve equal to zero.

REMARK

- If the coefficient of highest power of x or y or both are constant, then no asymptotes parallel to either x or y or both axis exists respectively.

SOLVED EXAMPLES

Example 1. *Find the asymptotes of the curve* $x^3 + y^3 - 3axy = 0$.

Solution. Obviously, the degree of the curve is 3, so it will have 3 asymptotes real as well as imaginary. Here the coefficient of highest degree in x and y are constant so no asymptotes parallel to co-ordinate axis exist. Let

$$y = mx + c \qquad \ldots(1)$$

be the asymptote of the curve.

So putting $y = m$ and $x = 1$ in the highest degree terms of the curve, we get

$$\phi_3(m) = 1 + m^3.$$

Solving the equation $\phi_3(m) = 0$ *i.e.*, $1 + m^3 = 0$

or $\qquad (1 + m)(m^2 - m + 1) = 0$ or $m = -1$

is only real root and other two roots are imaginary so ignore them.

Next, putting $y = m$ and $x = 1$ is second degree terms in the equation of the curve (1), we get

$$\phi_2(m) = -3am.$$

Now we find value of c by the following equation

$$c\,\phi_n'(m) + \phi_{n-1}(m) = 0 \text{ or } c\,\phi_3'(m) + \phi_2(m) = 0$$

or $\qquad c\,[3m^2] + (-3am) = 0 \qquad [\because\ \phi_3(m) = 1 + m^3,\ \therefore\ \phi_3'(m) = 3m^2]$

when $m = -1$, then

$$c\,[3(-1)^2] + [-3a(-1)] = 0$$

$$3c + 3a = 0 \text{ or } c = -a.$$

Hence, the asymptote is $y = -x - a$ or $x + y + a = 0$.

Example 2. *Find all the asymptotes of the curve* $x^3 + x^2y - xy^2 - y^3 - 3x - y - 1 = 0$.

Solution. The degree of the curve is 3 so it has 3 asymptotes which are real as well as imaginary. Since the coefficients of highest degree *i.e.*, 3rd degree of x and y are constant so there are no asymptotes parallel to co-ordinate axes. Thus there are **oblique asymptotes** of the form $y = mx + c$.

Now putting $y = m$ and $x = 1$ in the third degree terms of the curve, we get

$$\phi_3(m) = 1 + m - m^2 - m^3.$$

Solving the equation $\phi_3(m) = 0$ *i.e.*, $1 + m - m^2 - m^3 = 0$, we get

$$(1 + m)(1 - m^2) = 0 \text{ or } m = -1, -1, 1.$$

Determination of c. For $m = 1$, we use the following equation

$$c\,\phi_n'(m) + \phi_{n-1}(m) = 0 \text{ or } c\,\phi_3'(m) + \phi_2(m) = 0.$$

Putting $y = m$ and $x = 1$ in the second degree terms of the equation, we get

$$\phi_2(m) = 0.$$

From (1), we get

$$c\,[1-2m-3m^2]+0=0$$

at $m=1$

$$c\,(1-2-3)+0=0 \text{ or } -4c=0 \text{ or } c=0.$$

Thus one of the asymptote is $y=x$.

Determination of c for $m=-1, -1$. Since two out of three roots of the equation $\phi_3(m)=0$ are same, then we use the following formula to determine c

$$\frac{c^2}{2\,!}\phi_3''(m)+\frac{c}{1\,!}\phi_2'(m)+\phi_1(m)=0. \qquad \ldots(2)$$

Putting $y=m$ and $x=1$ in the first degree terms of the equation we obtain $\phi_1(m)=-3-m$.

From (3), we have

$$\frac{c^2}{2\,!}(-2-6m)+\frac{c}{1\,!}\cdot 0+(-3-m)=0$$

at $m=-1$

$$\frac{c^2}{2}(-2+6)-3+1=0 \text{ or } 2c^2-2=0 \text{ or } c=\pm 1.$$

Thus other two asymptotes are $y=-x+1,\ y=-x-1$.

Hence, all the asymptotes of the given curve are $y=x,\ x+y-1=0,\ x+y+1=0$.

• TEST YOURSELF–1

Find all the asymptotes of the following curves :

1. $a^2/x^2-b^2/y^2=1$.
2. $a^2/x^2+b^2/y^2=1$.
3. $y^2(a^2-x^2)=x^4$.
4. $x^2y^2=a^2(x^2+y^2)$.
5. $x^2y^2-x^2y-xy^2-y+1=0$.
6. $3x^3+2x^2y-7xy^2+2y^3+14xy+7y^2+4x+5y=0$.
7. $2x^3-x^2y-2xy^2+y^3-4x^2+8xy-4x+1=0$.
8. $x^3+2x^2y+xy^2-x^2-xy+2=0$.
9. $y^3-5xy^2+8x^2y-4x^3-3y^2+9xy-6x^2+2y-2x+1=0$.
10. $y^3-x^2y-2xy^2+2x^3-7xy+3y^2+2x^2+2x+2y+1=0$.

ANSWERS

1. $x=\pm a$ **2.** $x=\pm a,\ y=\pm b$

3. $x=\pm a$ **4.** $x=\pm a;\ y=\pm a$

5. $y=0;\ y=1;\ x=0;\ x=1$ **6.** $x+2y=1,\ 2x-2y=7,\ 7x-6y=15$

7. $x+y-2=0;\ x-y+2=0;\ 2x-y-4=0$

8. $x=0;\ x+y=0;\ x+y-1=0$ **9.** $x-y=0;\ 2x-y+2=0;\ 2x-y+1=0$

10. $x-y-1=0;\ x+y+2=0;\ 2x-y=0$

• SUMMARY

- If $y=mx+c$ is an asymptote to the curve $f(x,y)=0$, then

$$m=\lim_{x\to\infty}\frac{dy}{dx}=\lim_{x\to\infty}\left(\frac{y}{x}\right)$$

and

$$c=\lim_{x\to\infty}(y-mx)$$

- The asymptotes parallel to x-axis are obtained by putting the coefficients of highest power of x in the quation of the given curve $f(x,y)=0$ to zero.
- The asymptotes parallel to y-axis are obtained by putting the coefficients of highest power of y in the equation of the given curve $f(x,y)=0$ to zero.

- If $y = mx + c$ is in asymptote to the given curve $f(x, y) = 0$, then
To dtermine m : Put $y = m$ and $x = 1$ in the highest degree terms say n of the curve so we get $\phi_n(m)$. The roots of $\phi_n(m) = 0$ gives the values of m.
To determine c : For r identical values of m we use the following formula to find c

$$\frac{c^r}{r!}\phi_n^r(m) + \frac{c^{r-1}}{(r-1)!}\phi_{n-1}^{r-1}(m) + \ldots + \phi_{n-r}(m) = 0$$

• STUDENT ACTIVITY

1. Find the asymptotes of the curve $x^3 + y^3 - 3axy = 0$.

2. Find all the asymptotes of the curve

$$y^3 - xy^2 - x^2y + x^3 + x^2 - y^2 - 1 = 0.$$

• TEST YOURSELF–2

OBJECTIVE EVALUATION

➤ FILL IN THE BLANKS :

1. If $y = mx + c$ is an asymptote of the curve $f(x, y) = 0$, then $m =$ .... and $c =$ ....
2. The equation $\phi_n(m) = 0$ gives the .... of the the asymptotes.
3. If one or more values of m obtained from $\phi_n(m) = 0$ are such that $\phi_n'(m) = 0$ and $\phi_{n-1}(m)$, then the asymptotes ....

➤ TRUE OR FALSE :

Write 'T' for True and 'F' for False :

1. The line $y = mx + c$ is an asymptote of the curve

$$y = mx + c + \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \ldots .$$ *(T/F)*

2. The polynomial $\phi_n(m)$ is obtained by putting $y = m$ and $x = m$ in the n^{th} degree terms of the curve. *(T/F)*
3. If $y = mx + c$ is an asymptote of the curve $f(x, y) = 0$ then

$$m = \lim_{x \to \infty}\left(\frac{y}{x}\right).$$ *(T/F)*

➤ MULTIPLE CHOICE QUESTIONS :

Choose the most appropriate one :

1. If $y = mx + c$ is an asymptote of the curve $f(x, y) = 0$, then $\lim\limits_{x \to \infty}(y/x)$ equals :
(a) c (b) m
(c) $-m$ (d) $-c$.

2. If $y = mx + c$ is an asymptote of the curve $f(x, y) = 0$, then $\lim\limits_{x \to \infty,\, y/x \to m} (y - mx)$ equals :

(a) m (b) $-c$

(c) c (d) $-m$.

ANSWERS

Fill in the Blanks :

1. $\lim\limits_{x \to \infty} y/x$, $\lim\limits_{x \to \infty,\, y/x \to m} (y - mx)$ **2.** Slopes **3.** Will not exist

True or False :

1. T **2.** F **3.** T

Multiple Choice Questions :

1. (b) **2.** (c)

❑❑❑

UNIT

7

SINGULAR POINTS AND CURVE TRACING

STRUCTURE

- Concave and Convex Curves
- Point of Inflaexion
- Formula for Finding the Point of Inflexion
  - Test Yourself–1
- Multiple Points and Singular Points
- Types of Double Points
- Tangents at Origin
- Position and Nature of Double Points
  - Test Yourself–2
- Curve Tracing
  - Test Yourself–3
  - Summary
  - Student Activity
  - Test Yourself–4

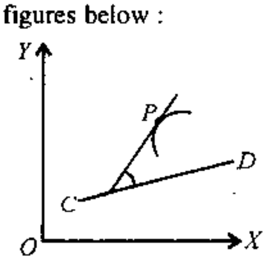
LEARNING OBJECTIVES

After going through this unit you will learn :

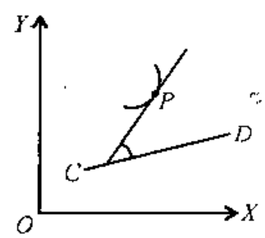
- About the singular points on the curve.
- How to trace the given curve.

• 7.1. CONCAVE AND CONVEX CURVES

Definition. If P is any point on a curve and CD is any given line which does not passes through this point P. Then the curve is said to be concave at P with respect to the line CD if the small arc of the curve containing P lies entirely within the acute angle between the tangent at P to the curve and the line CD and the curve is said to be convex at P if the arc of the curve containing P lies wholly outside the acute angle between that tangent at P and the line CD. Which are shown in figures below :



Concave



Convex

Fig. 1

• 7.2. POINT OF INFLEXION

Definition. A point P on the curve is said to be the point of inflexion, if the curve in one side of P is concave and other side of P is convex with respect to the line CD which does not passes through the point P as shown in fig. 2.

Inflexion tangent. The tangent at the point of inflexion of a curve is said to be inflexion tangent. In the fig. 2 the line PQ is the inflexion tangent.

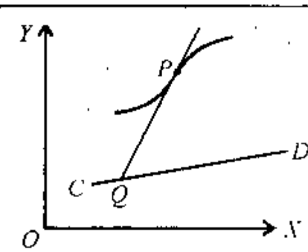


Fig. 2

• 7.3. FORMULA FOR FINDING THE POINT OF INFLEXION

We can have a point of inflexion at P, if $\dfrac{d^2y}{dx^2} = 0$ ***but*** $\dfrac{d^3y}{dx^3} \neq 0.$

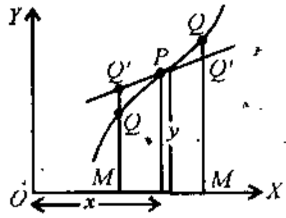


Fig. 3

SOLVED EXAMPLES

Example 1. *Find the points of inflexion of the curve* $x = (\log y)^3$.

Solution. Since the equation of the curve is

$$x = (\log y)^3. \quad \text{...(1)}$$

Differentiating (1) with respect to 'y', we get

$$\frac{dx}{dy} = 3\,(\log y)^2 \cdot \frac{1}{y}.$$

Again differentiating w.r.t. y

$$\frac{d^2x}{dy^2} = 3\left[\frac{2\log y}{y^2} - \frac{(\log y)^2}{y^2}\right]. \quad \text{...(2)}$$

Again differentiating w.r.t. 'y', we get

$$\frac{d^3x}{dy^3} = 3\left[\frac{2}{y^3} - \frac{4\log y}{y^3} - \frac{2\log y}{y^3} + \frac{2\,(\log y)^2}{y^3}\right]. \quad \text{...(3)}$$

For the point of inflexion, we have

$$\frac{d^2x}{dy^2} = 0.$$

$$\therefore \qquad 3\left[\frac{2\log y - (\log y)^2}{y^2}\right] = 0 \text{ or } 3\,(\log y)\,(2 - \log y) = ($$

or $\qquad \log y = 0,\ \log y = 2 \text{ or } y = 1, y = e^2.$

From (3) it is obvious that at $y = 1$, $y = e^2$, $\dfrac{d^3x}{dy^3} \neq 0$.

Hence, the points of inflexion are $(0, 1)$ $(8, e^2)$.

Example 2. *Find the points of inflexion of the curve* $y^2 = x\,(x + 1)^2$.

Solution. Since the equation of the curve can be written as

$$y = (x + 1)\sqrt{x}. \quad \text{...(1)}$$

Differentiating (1) w.r.t. 'x', we get

$$\frac{dy}{dx} = \frac{3}{2} \cdot x^{1/2} + \frac{1}{2\sqrt{x}}.$$

Again differentiating w.r.t. 'x'

$$\frac{d^2y}{dx^2} = \frac{3}{4\sqrt{x}} - \frac{1}{4x^{3/2}} \quad \text{...(2)}$$

and again differentiating w.r.t. 'x', we get

$$\frac{d^3y}{dx^3} = -\frac{3}{8x^{3/2}} + \frac{3}{8x^{5/2}}. \quad \text{...(3)}$$

For the point of inflexion, we have

$$\frac{d^2y}{dx^2} = 0.$$

$$\therefore \quad \frac{3}{4\sqrt{x}} - \frac{1}{4x\sqrt{x}} = 0 \text{ or } \left(3 - \frac{1}{x}\right) = 0 \text{ or } x = 1/3.$$

From (3) it is obvious that at $x = 1/3$ $\dfrac{d^3y}{dx^3} \neq 0$.

Thus, the point of inflexion are given by $(1/3, \pm 4/3\sqrt{3})$.

• TEST YOURSELF–1

1. Find the points of inflexion of the curve $x = \log (y/x)$.
2. Find the points of inflexion of the curve $y(a^2 + x^2) = x^3$.
3. Find the points of inflexion of the curve $y = (x-1)^4 (x-2)^3$.
4. Find the points of inflexion of the curve $xy = a^2 \log (y/a)$.

ANSWERS

1. $(-2, -2/e^2)$. 2. $(0, 0), \left(\sqrt{3}a, \frac{3\sqrt{3}}{4}a\right), \left(-\sqrt{3}a, \frac{-3\sqrt{3}}{4}a\right)$.

3. Point of inflexion at $x = 2$, $(11 \pm \sqrt{2})/7$. 4. $\left(\frac{3}{2}ae^{-3/2}, ae^{3/2}\right)$.

• 7.4. MULTIPLE POINTS AND SINGULAR POINTS

Definition. A point on the curve is said to be **multiple points** if through this point more than one branches of a curve passes.

Definition. A point on the curve is called a **double point** if through it two branches of the curve passes.

Definition. If three branches of the curve passes through a point, then this point is called **triple point**.

Definition. If n branches passes through a point on the curve, then this point is called a **multiple point of n^{th}** order.

Definition. The point of inflexion and multiple points are also called the **singular points.**

Or

An unusual point on the curve is basically called a **singular point.**

• 7.5. TYPES OF DOUBLE POINTS

(i) Node. A double point on a curve is said to be a **node,** if through this double point two branches of the curve passes which are real and having two different tangents at that point. (Fig. 4)

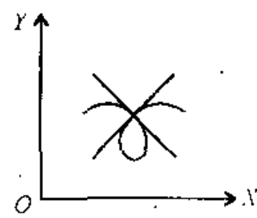


Fig. 4

(ii) Cusp. A double point on a curve is called a **cusp** if through this double point two real branches of the curve passes and have real coincident tangents at that point. (Fig. 5)

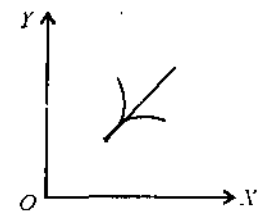


Fig. 5

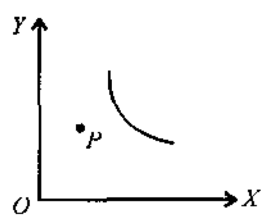


Fig. 6

(iii) Conjugate point. A point P on the curve is said to be **conjugate point** if there are no real points on the curve in the neighbourhood of that point and having no real tangent at that point. (Fig. 6)

• 7.6. TANGENTS AT ORIGIN

The nature of a double point depends on the tangents so we find the tangent or tangents there. ***If a curve passes through the origin, then the equation of the tangent or tangents at the origin are obtained by equating to zero the lowest degree terms in the equation of the curve.***

• 7.7. POSITION AND NATURE OF DOUBLE POINTS

Thus *the necessary and sufficient condition for any point of the curve $f(x, y) = 0$ to be a multiple point are that*

$$\frac{\partial f}{\partial x} = 0 = \frac{\partial f}{\partial y}.$$

The double point will be node, cusp or conjugate point according as

$$\left(\frac{\partial^2 f}{\partial x\, \partial y}\right)^2 >, = \text{ or } < \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2}.$$

REMARK

➤ If $\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial x\, \partial y}, \frac{\partial^2 f}{\partial y^2}$ are all zero, then the point $P(x, y)$ will be a multiple point of order greater that two.

SOLVED EXAMPLES

Example 1. *Show that the origin is a node on the curve $x^3 + y^3 - 3axy = 0$.*

Solution. The tangent at the origin are obtained by equating to zero the lowest degree terms *i.e.*, second degree term in the given equation of the curve.

$\therefore$ $\quad -3axy = 0$ or $x = 0, y = 0$.

Thus at the origin there are two real and distinct tangents. Hence (0, 0) is a node.

Example 2. *Find the double point of the curve $(x - 2)^2 = y(y - 1)^2$.*

Solution. Let $f(x, y) \equiv (x - 2)^2 - y(y - 1)^2 = 0$. ...(1)

Differentiating (1) partially w.r.t. x and y, we get

$$\frac{\partial f}{\partial x} = 2(x - 2) \quad ...(2)$$

and

$$\frac{\partial f}{\partial y} = -(y - 1)^2 - 2y(y - 1). \quad ...(3)$$

Since the necessary and sufficient condition for a double points are

$$\frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 0, \quad \therefore\ 2(x - 2) = 0 \quad ...(4)$$

$$-(y - 1)^2 - 2y(y - 1) = 0. \quad ...(5)$$

Now solving $f(x, y) = 0$, $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$ simultaneously.

$\therefore$ From (4), we get $x = 2$ and from (5), we get

$$-(y - 1)(y - 1 + 2y) = 0$$

or

$$-(y - 1)(3y - 1) = 0 \text{ or } y = 1, y = 1/3.$$

$\therefore$ Possible double points are (2, 1) and (2, 1/3).

But (2, 1/3) does not satisfy $f(x, y) = 0$. Hence only double point is (2, 1).

Example 3. *Examine the nature of the origin on the following curve :*

$$y^2 = a^2x^2 + bx^3 + cxy^2.$$

Solution. The given curve is

$$f(x, y) \equiv y^2 - a^2x^2 - bx^3 - cxy^2 = 0. \quad ...(1)$$

Equating to zero the lowest degree terms in the equtaion of curve (1), we get

$$y^2 - a^2x^2 = 0 \text{ or } y = \pm ax.$$

Thus we have obtained two real and distinct tangent at (0, 0). Hence (0, 0) is a node.

• TEST YOURSELF–2

1. Find the equations of the tangents at the origin to the following curves :
   (a) $(x^2 + y^2)(2a - x) = b^2x$ (b) $a^4y^2 = x^4(x^2 - a^2)$
   (c) $x^4 + 3x^3y + 2xy - y^2 = 0$ (d) $x^3 + y^3 = 3axy$.
2. Examine the nature of the origin on the curve $(2x + y)^2 - 6xy(2x + y) - 7x^3 = 0$..

ANSWERS

1. (a) $x=0$ (b) $y=0, y=0$ (c) $y=0, 2x-y=0$ (d) $x=0, y=0$
2. Origin is a single cusp of first species

• 7.8. CURVE TRACING

Cartesian Form of Equations :

To trace any curve of cartesian form we should apply following process :

(a) Symmetricity. In order to find the symmetry of the curve we should apply following rules :

(i) If the **powers of** y in the equation of the curve are all **even,** then curve is symmetrical about **x-axis.**

(ii) If the **powers of** x in the equation of the curve are all **even,** then the curve is symmetrical about **y-axis.**

(iii) If the **powers of x as well as y** in the equation of the curve are all **even,** the curve is symmetrical about **both axes.**

(iv) If the equation of curve remains **unchanged** when x is replaced by $-x$ and y is replaced by $-y$, then the cure is symmetrical in **opposite quadrants.**

(v) If the equation of the curve remains unchanged when x and y are interchanged, then the curve is symmetrical about the **line $y=x$.**

(b) Nature of the origin on the curve. If the curve passes through the origin, then *find the tangent at* (0, 0) *by equating to zero the lowest degree terms of the curve.* If we obtain two tangent at the origin, then origin will be a double point and then find the nature of this double point.

(c) Intersection of curve with co-ordinate axes. We should check whether the curve cuts the co-ordinate axes or not, for this *put $y=0$ in the equation of the curve and find the values of x, then we get the points at which the curve cuts the x-axis. Similarly if the curve cuts the y-axis, then put $x=0$ in the equation of the curve and obtain the points on the y-axis.* Hence in this way we obtain the points of intersection of the curve with co-ordinate axis. Thereafter we should find the tangents at these points of intersection. For this first we shift the origin at these points and then obtain the tangent at these new origin by equating to zero the lowest degree terms in the new equation of the curve. On the other hand *the value of dy/dx at these points of intersection can also be used to find the slope of the tangent at that point.*

(d) Nature of y or x in the curve. We should now solve the equation of the curve either for y or for x whichever is convenient. Suppose we solve for y and see that nature of y as x increases from 0 to $+\infty$. Similarly see the nature of y as x decreases from 0 to $-\infty$ and finally collect those values of x for which $y=0$ or $y \to \infty$ or $-\infty$.

REMARK

➤ If the curve is symmetrical about x-axis in opposite quadrants then there is no need to take the values of x of both positive and negative. We can take only positive values of x to see the variation in y.

(e) Regions in which curve does not exist. In order to find the regions where the curve does not exist we should solve the equation of curve for one variable in terms of the other. Therefore, the curve will not exist for those values of one variable which make the other variable imaginary.

(f) Asymptotes. Next, we should find all the asymptotes of the curve because the branches of the curve approach to the asymptotes if they exist.

(g) Sign of dy/dx. Next, we should find the value of dy/dx from the equation of the curve and find the points on the curve at which $dy/dx=0$ or $dy/dx=\infty$. Therefore at these points we obtain the nature of tangents. suppose in any region $a<x<b$, dy/dx remains positive throughout, then in this region y increase continuously as x increases. On the other hand if dy/dx remains negative, they y decreases continuously as x increases.

(h) Special points. If necessary, we should find the some special point on the curve.

(i) Points of inflexion. If necessary, we should find the point of inflexion to know the position of the curves at that point.

Now taking all above considerations in mind, draw an approximate shape of the curve.

SOLVED EXAMPLES

Example 1. *Trace the curve* $y^2(2a-x)=x^3$.

Solution. (i) Obviously the given curve is symmetrical about x-axis.

(ii) The curve passes through the origin and the tangents at the origin are obtained by equating to zero the lowest degree terms *i.e.*, $2ay^2$ in the equation of the curve.

$$\therefore \qquad 2ay^2 = 0 \quad \text{or} \quad y = 0, y = 0.$$

Thus at the origin we obtained two coincident tangents $y = 0$, $y = 0$ *i.e.*, x-axis. Therefore $(0, 0)$ is a cusp.

(iii) From the equation of the curve it is obvious that the curve does not cut the co-ordinate axes.

(iv) Now solving the equation of the curve for y, we get

$$y^2 = x^3/(2a - x)$$

when $x = 0$, $y^2 = 0$ and when $x = 2a$, thus $y^2 \to \infty$ thus $x = 2a$ is an asymptotes of the curve.

It is observed that y increases as x increases from 0 to $2a$.

(v) When x lies between 0 and $2a$, y^2 will be positive and the curve will exist in this region. When $x > 2a$, y^2 will be negative so the curve will not exist beyond the line $x = 2a$. When $x < 0$, again y^2 will be negative and thus the curve will also not exist for $x < 0$. Hence we can say that the curve only exists in the region $0 < x < 2a$.

(vi) In order to find the asymptotes, putting $y = m$ and $x = 1$ in the third degree terms in the equation of the curve, we get

$$\phi_3(m) = m^2 + 1.$$

Therefore the equation $\phi_3(m) = 0$ gives both its roots imaginary so ignore them. Consequently $x = 2a$ is only the asymptote of the curve.

(vii) Differentiating the equation of the curve

$$y = \frac{x^{3/2}}{\sqrt{(2a - x)}}$$

we get
$$\frac{dy}{dx} = \frac{(3a - x)\, x^{1/2}}{(2a - x)^{3/2}}.$$

In the region $0 < x < 2a$, $\dfrac{dy}{dx}$ will be positive, so therefore in this region y increase continuously as x increases.

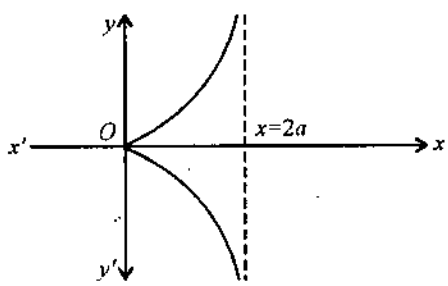


Fig. 7

Now taking all above points of consideration in the mind and draw the curve whose shape is shown in fig. 7.

Example 2. *Trace the curve* $y^2(1 - x^2) = x^2(1 + x^2)$.

Solution. (i) In the equation of the curve the powers of both x and y are all even so the curve is symmetrical about both axes.

(ii) The curve passes through the origin. The tangents at the origin are obtained by equating to zero the lowest degree terms in the equation of the curve.

$$\therefore \qquad y^2 - x^2 = 0 \quad \text{or} \quad y = \pm x.$$

Thus there are two real and distinct tangent at the origin so $(0, 0)$ is a node.

(iii) From the equation of the curve it is clear that curve does not cut any co-ordinate axes.

(iv) Solving the equation of the curve for y, we get

$$y^2 = \frac{x(1 + x^2)}{(1 - x^2)}.$$

When $x = 0$, $y^2 = 0$ and when $x = \pm 1$, $y^2 \to \infty$ so $x = \pm 1$ are two asymptotes parallel to y-axis.

(v) When $-1 < x < 1$, y^2 is positive, so the curve exists in this region. When $x > 1$, y^2 will be negaive thus curve will not exist beyond the line $x = 1$. Also when $x < -1$, y^2 will be negative so that curve will not exist for $x < -1$.

(vi) In order to find the asymptotes, putting $y = m$ and $x = 1$ in the fourth degree terms of the curve, we get $\phi_n(m) = m^2 + 1$.

Solving $\phi_n(m) = 0$, we get both values of m imaginary so ignore them. Consequently $x = \pm 1$ are only two real asymptotes.

(vii) Since, we have

$$y = x\sqrt{\left(\frac{1+x^2}{1-x^2}\right)}.$$

$$\therefore \quad \frac{dy}{dx} = \frac{\sqrt{1-x^2}\left(\sqrt{1+x^2}+\dfrac{x^2}{\sqrt{1+x^2}}\right) - x\sqrt{(1+x^2)}\left[\dfrac{-x}{\sqrt{1-x^2}}\right]}{(1-x^2)}$$

$$= \frac{(1-x^2)(1+x^2+x^2) - x(1+x^2)(-x)}{(1-x^2)^{3/2}(1+x^2)^{1/2}}$$

$$= \frac{2x^2+1-x^4}{(1-x^2)^{3/2}(1+x^2)^{1/2}}.$$

When $-1 < x < 0$ $\frac{dy}{dx}$ is negative this means that when x decreases from -1 to 0, y decreases. When $0 < x < 1$ $\frac{dy}{dx}$ is positive this implies that when x increases from 0 to 1, y increases.

Now taking all the above facts in mind and draw the shape of the curve. We get

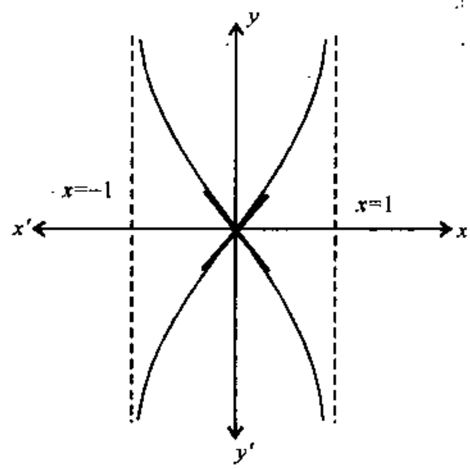


Fig. 8

• SUMMARY

- We can find a point of inflexion to the curve $y = f(x)$, if $\frac{d^2y}{dx^2} = 0$ and $\frac{d^3y}{dx^3} \neq 0$.
- For multiple points in $f(x, y) = 0$ we must have $\frac{\partial f}{\partial x} = 0 = \frac{\partial f}{\partial y}$.
- Multiple point (a, b) is node if $\left(\frac{\partial^2 f}{\partial x\,\partial y}\right)_{(a,b)} > \left(\frac{\partial^2 f}{\partial x^2}\right)_{(a,b)}\left(\frac{\partial^2 f}{\partial y^2}\right)_{(a,b)}$.
- Multiple point (a, b) is cusp if $\left(\frac{\partial^2 f}{\partial x\,\partial y}\right)_{(a,b)} = \left(\frac{\partial^2 f}{\partial x^2}\right)_{(a,b)}\left(\frac{\partial^2 f}{\partial y^2}\right)_{(a,b)}$.
- Multiple point (a, b) is a conjugate point if $\left(\frac{\partial^2 f}{\partial x\,\partial y}\right)_{(a,b)} < \left(\frac{\partial^2 f}{\partial x^2}\right)_{(a,b)}\left(\frac{\partial^2 f}{\partial y^2}\right)_{(a,b)}$.
- To trace any curve $f(x, y) = 0$, we must remember the following points :
  (i) Symmetricity
  (ii) Nature of origin (if $f(x, y) = 0$ passes through the origin).
  (iii) Points of intersection of $f(x, y) = 0$ with coordinate axes.
  (iv) Nature of y or x in $f(x, y) = 0$.
  (v) Regions in which curve does not exist
  (vi) Asymptotes
  (vii) Sign of $\frac{dy}{dx}$.
  (viii) Points of inflexion (if they exist).

• STUDENT ACTIVITY

1. Show that the origin is a node on the curve $x^3 + y^3 - 3axy = 0$.

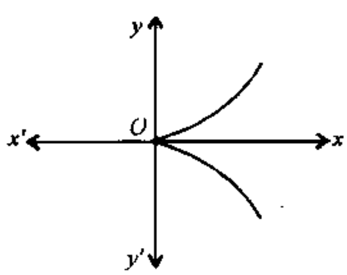
2. Trace the curve $ay^2 = x^2 (a - x)$.

• TEST YOURSELF–3

Trace the following curves :

1. $ay^2 = x^3$.
2. $a^2y = x^3$.
3. $y = x(x^2 - 1)$.
4. $xy^2 = 4a^2(2a - x)$.
5. $y^2(a + x) = x^2(a - x)$.
6. $x^2(x^2 - 4a^2) = y^2(x^2 - a^2)$.
7. $x^3 + y^3 = x$.
8. $x^2y^2 = a^2(x^2 + y^2)$.

ANSWERS



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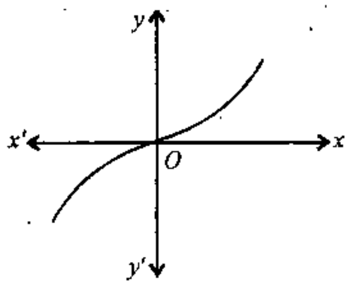

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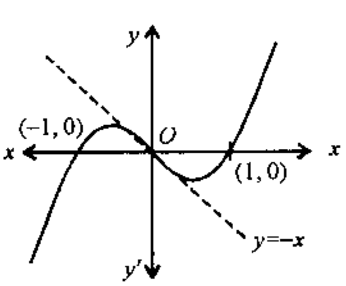

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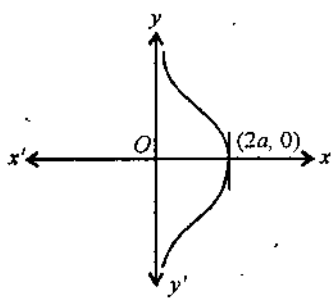

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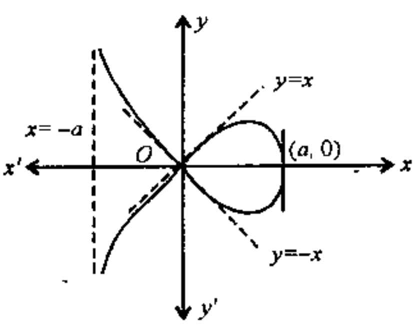

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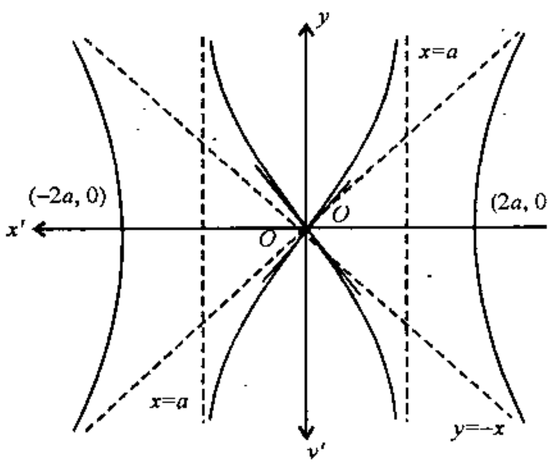

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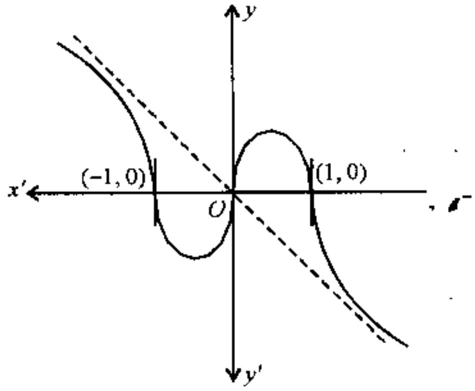

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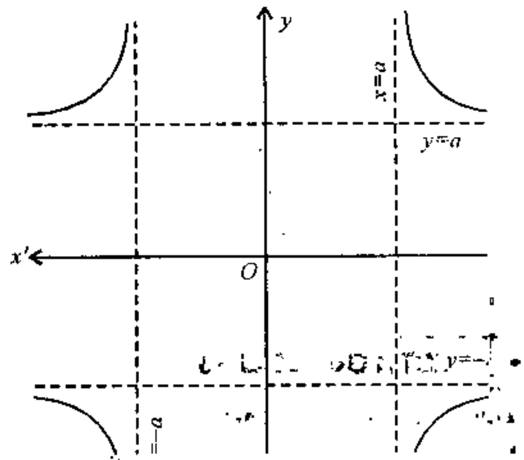

8.

OBJECTIVE EVALUATION

➤ FILL IN THE BLANKS :

1. A point $P(x, y)$ on the curve $y = f(x)$ will be a point of inflexion if $\frac{d^2y}{dx^2} = 0$ but $\frac{d^3y}{dx^3}$ is not equal to ....
2. A double point is a node if at this point there are .... distinct tangents.
3. A double point is a cusp if at this point there are two .... tangents.

➤ TRUE OR FALSE :

Write 'T' for True and 'F' for False :

1. For the point of inflexion of the curve $y = f(x)$, $\frac{d^2y}{dx^2} = 0$ but $\frac{d^2y}{dx^3} \neq 0$. *(T/F)*
2. If we obtained two real and distinct tangents at the double point, then double point is a cusp. *(T/F)*

➤ MULTIPLE CHOICE QUESTIONS :

Choose the most appropriate one :

1. At the point of inflexion of the curve $x = f(y)$, $\frac{d^2x}{dy^2} = 0$ and $\frac{d^3x}{dy^3}$ is not equal to :
   (a) 1 (b) 0 (c) − 1 (d) None of these.
2. The points of inflexion of the curve $y^2 = (x - a)^2 (x - b)$ lie on the line :
   (a) $3x + a = 4b$ (b) $3x + a = b$ (c) $3x + b = 4a$ (d) $3x = a$.

ANSWERS

Fill in the Blanks :

1. Zero 2. Two 3. Coincident

True or False :

1. T 2. F 3. T 4. T

Multiple Choice Questions :

1. (b) 2. (a)

❑❑❑

UNIT

8

RECTIFICATION OF CURVES

STRUCTURE

- Formulae for Finding the Lengths of the Curves
  - Summary
  - Student Activity
  - Test Yourself

LEARNING OBJECTIVES

After going through this unit you will learn :

How to find the whole length of the given curve between the given points.

• 8.1. FORMULAE FOR FINDING THE LENGTHS OF THE CURVES

(a) Let the equation of a curve be $y = f(x)$ and let A and B be two points on this curve between A and B, the length of curve is to be required. Let s be the length of an arc from a fixed point on the curve to any point on it.

Therefore, we have

$$\frac{ds}{dx} = \pm\sqrt{\left[1+\left(\frac{dy}{dx}\right)^2\right]},\quad ds = \pm\sqrt{\left[1+\left(\frac{dy}{dx}\right)^2\right]}\,dx$$

where positive and negative sign will have to take according as x increases and decreases as s increases. Thus the length of an arc between the points A and B where at A, $x = a$ and at B, $x = b$ is given by

$$s = \int_a^b \sqrt{\left[1+\left(\frac{dy}{dx}\right)^2\right]}\,dx, \qquad (a < b).$$

(b) If the equation of the curve is $x = f(y)$, then the length of an arc between C and D is given by

$$s = \int_c^d \sqrt{\left[1+\left(\frac{dx}{dy}\right)^2\right]}\,dy, \qquad (c < d).$$

(c) If the equation of the curve is in parametric form, *i.e.*, $x = f(t)$, $y = g(t)$, then we have

$$\frac{ds}{dt} = \sqrt{\left[\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2\right]} \quad \text{or} \quad ds = \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,.\,dt.$$

Thus the length of an arc between A and B where at A, $t = t_1$ and at B, $t = t_2$ is given by

$$s = \int_{t_1}^{t_2} \sqrt{\left[\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2\right]}\,dt.$$

REMARK

➤ If the curve is symmetrical about some lines, then in order to find the length of an arc, we first find the length of one of symmetrical part and multiply this length by the number of symmetrical parts.

SOLVED EXAMPLES

Example 1. *Find the length of the arc of the parabola $y^2 = 4ax$ cut off by its latus rectum.*

Solution. Latus rectum. A line which passes through the focus of the given parabola and perpendicular to the axis of that parabola.

Here the equation of the parabola is $y^2 = 4ax$ whose trace is shown adjoining figure and in the fig. LL' is the latus rectum, the co-ordinates of L and L' are respectively $(a, 2a)$ and $(a, -2a)$.

Since $y^2 = 4ax$ is symmetrical about the line OX. Therefore the required arc length $= 2 \times$ arc length OL.

Since $\quad y^2 = 4ax.$

$\therefore \quad y = 2\sqrt{a}\,\sqrt{x}.$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{\sqrt{a}}{\sqrt{x}}.$

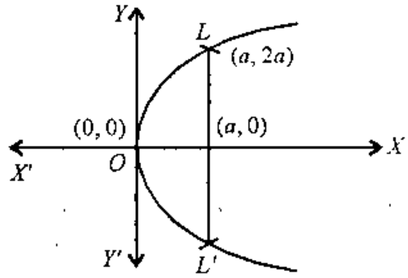

Fig. 1

Now arc length $OL = \displaystyle\int_0^a \sqrt{\left[1 + \left(\frac{dy}{dx}\right)^2\right]}\, dx$

($\because$ At point O, $x = 0$ and at point L, $x = a$)

$$= \int_0^a \sqrt{\left(1 + \frac{a}{x}\right)}\, dx = \int_0^a \frac{\sqrt{x+a}}{\sqrt{x}}\, dx$$

$$= \int_0^a \frac{x+a}{\sqrt{x^2+ax}}\, dx = \frac{1}{2}\int_0^a \frac{2x+2a}{\sqrt{(x^2+ax)}}\, dx$$

$$= \frac{1}{2}\int_0^a \frac{(2x+a)\,dx}{\sqrt{(x^2+ax)}} + \frac{a}{2}\int_0^a \frac{dx}{\sqrt{(x^2+ax)}}$$

$$= \frac{1}{2}\int_0^a \frac{(2x+a)\,dx}{\sqrt{(x^2+ax)}} + \frac{a}{2}\int_0^a \frac{dx}{\sqrt{\left[\left(x+\frac{a}{2}\right)^2 - \left(\frac{a}{2}\right)^2\right]}}$$

$$= \frac{1}{2}\left(2\sqrt{(x^2+ax)}\right)_0^a + \frac{a}{2}\left[\log\left\{\left(x+\frac{a}{2}\right) + \sqrt{x^2+ax}\right\}\right]_0^a$$

$$= a\sqrt{2} + \frac{a}{2}\log(3+2\sqrt{2}) = a\sqrt{2} + \frac{a}{2}\log(1+\sqrt{2})^2.$$

Arc length $\quad OL = a\sqrt{2} + a\log(1+\sqrt{2}).$

Hence the required arc length $= 2 \times$ arc length OL

$$= 2\sqrt{2}\,a + 2a\log(1+\sqrt{2}).$$

Example 2. *Find the whole length of the astroid*

$$x^{2/3} + y^{2/3} = a^{2/3}$$

or $\quad x = a\cos^3\theta,\ y = a\sin^3\theta.$

Solution. The equation of the curve is

$$x^{2/3} + y^{2/3} = a^{2/3}. \qquad \ldots(1)$$

Since the curve is symmetrical about both the axis, *i.e.*, curve lies in all the four quadrants as shown in fig. 2.

Therefore the whole length of the given curve

$= 4 \times$ arc length of the curve in first quadrant.

Since the co-ordinates of A and B are $(a, 0)$ and $(0, a)$ respectively. Thus in the first quadrant x varies from 0 to a.

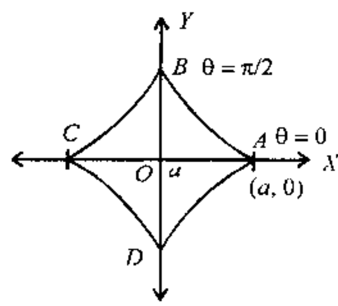

Fig. 2

Now differentiating (1) w.r.t. x, we get

$$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}\frac{dy}{dx} = 0$$

or

$$\frac{dy}{dx} = -\left(\frac{y}{x}\right)^{1/3}.$$

Therefore, the length of the curve in the first quadrant

$$= \int_0^a \sqrt{\left[1 + \left(\frac{dy}{dx}\right)^2\right]}\, dx = \int_0^a \sqrt{\left(1 + \frac{y^{2/3}}{x^{2/3}}\right)}\, dx$$

$$= \int_0^{a} \sqrt{\left(\frac{x^{2/3}+y^{2/3}}{x^{2/3}}\right)}\, dx = \int_0^{a} \sqrt{\left(\frac{a^{2/3}}{x^{2/3}}\right)}\, dx \quad \text{[using (1)]}$$

$$= a^{1/3} \int_0^{a} x^{-1/3}\, dx = a^{1/3} \left[\frac{3}{2} x^{2/3}\right]_0^a$$

$$= a^{1/3} \left[\frac{3}{2} a^{2/3}\right] = \frac{3}{2} a.$$

Hence, the whole length of the astroid $= 4 \times \frac{3a}{2} = 6a$.

• SUMMARY

- Length of an arc from $x = a$ to $x = b$ of th curve $y = f(x)$ is

$$s = \int_0^{b} \sqrt{1 - \left(\frac{dy}{dx}\right)^2}\, dx$$

- Length of an arc from $y = a$ to $y = b$ of the curve $x = f(y)$ is given by

$$s = \int_a^{b} \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

- Length of an arc from $t = t_1$ to $t = t_2$ of the curve $x = x(t)$, $y = y(t)$ is given by

$$s = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dy$$

• STUDENT ACTIVITY

1. Find the length of the curve $y = \log\left(\frac{e^x - 1}{e^x + 1}\right)$ from $x = 1$ to $x = 2$.

2. Find the whole length of the curve $x^2 + y^2 = a^2$.

• TEST YOURSELF

1. Find the whole length of the curve $x^2 + y^2 = b^2$.
2. Find the length of the arc of the curve $x^2 = 8y$ from the vertex to an extremity of the latus rectum.
3. Find the arc length of the curve $y = \frac{1}{2}x^2 - \frac{1}{4}\log x$ from $x = 1$ to $x = 2$.
4. Find the length of the arc from $\theta = 0$ to $\theta = 2\pi$ of the curve

$$x = a(\cos\theta + \theta\sin\theta),\ y = a(\sin\theta - \theta\cos\theta).$$

ANSWERS

1. $2\pi b$ 2. $2[\sqrt{2} + \log(1 + \sqrt{2})]$ 3. $\frac{3}{2} + \frac{1}{4}\log 2$ 4. $2a\pi^2$

OBJECTIVE EVALUATION

➤ FILL IN THE BLANKS :

1. The curve $y = f(x)$ is rectified between $x = a$ and $x = b$ if $f(x)$ is ....
2. The whole length of the arc of the curve $x^2 + y^2 = a^2$ is ....
3. If $x = f(y)$, then the formula for finding the length of the curve between $y = a$ and $y = b$ is ......

➤ TRUE OR FALSE :

Write 'T' for True and 'F' for False :

1. The length of the loop of the curve $x = t^2$, $y = t - \frac{1}{3}t^3$ is $4\sqrt{3}$. (T/F)
2. The whole length of the astroid $x^{2/3} + y^{2/3} = a^{2/3}$ is $7a$. (T/F)

➤ MULTIPLE CHOICE QUESTIONS :

Choose the most appropriate one :

1. The whole length of the curve $x^2 + y^2 - 2ax = 0$ is :
   (a) πa (b) $2\pi a$
   (c) πa^2 (d) $3\pi a$.
2. The whole length of the curve $x = a\cos^3 t$, $y = a\sin^3 t$ is :
   (a) $6a$ (b) $8a$
   (c) $3a$ (d) $2a$.
3. The length of an arc of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ is :
   (a) $6a$ (b) $5a$
   (c) $3a$ (d) $8a$.

ANSWERS

Fill in the Blanks :

1. Continuous 2. $2\pi a$ 3. $\int_a^b \sqrt{\left[1 + \left(\frac{dx}{dy}\right)^2\right]}\, dy$

True or False :

1. T 2. F

Multiple Choice Questions :

1. (b) 2. (a) 3. (d)

❑❑❑

UNIT 9

AREA OF BOUNDED CURVES

STRUCTURE

- Equation of Curve in Cartesian Form
  - Summary
  - Student Activity
  - Test Yourself

LEARNING OBJECTIVES

After going through this unit you will learn :

How determine the area of the bounded regions, bounded by the co-ordinate curves and the given curves.

• EQUATION OF CURVE IN CARTESIAN FORM

Let $y = f(x)$ be a continuous curve in cartesian form and let A be the area of the region bounded by the curve $y = f(x)$, the axis of x and the two ordinates $x = a$ and $x = b$. Then

$$A = \int_a^b y\,dx = \int_a^b f(x)\,dx.$$

Similarly, the area bounded by the curve $x = f(y)$, the axis of y and the ordinates $y = a$ and $y = b$ is given by

$$A = \int_a^b f(y)\,dy = \int_a^b x\,dy.$$

REMARK

- If the given curve is symmetrical either about x-axis or about y-axis or both axes, then find the area of one of the symmetrical part and multiply this area by the number of symmetrical parts, we get the whole area of the bounded region.
- Area bounded by two curves = | Area bounded by one curve − Area bounded by other curve |.

Answer - 1

SOLVED EXAMPLES

Example 1. *Find the area of the region bounded by the line $x = 2$ and the parabola $y^2 = 8x$.*

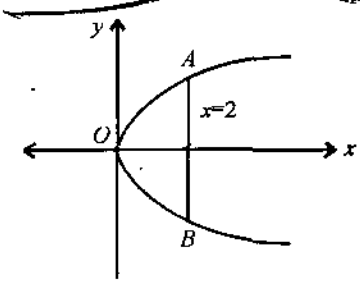

Fig. 1

Solution. Since the equation of the parabola is

$$y^2 = 8x$$

which is symmetrical about x-axis and the line $x = 2$ intersects the parabola $y^2 = 8x$ in two points (2, 4) and (2, − 4) as show in fig. 1.

$$\therefore \text{ The required area} = 2\int_0^2 y\,dx$$

$$= 2\int_0^2 \sqrt{8x}\,dx = 4\sqrt{2}\left[\frac{2}{3}x^{3/2}\right]_0^2$$

$$= \frac{32}{3} \text{ sq. units.}$$

End

Example 2. *Find the whole area of the ellipse*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Solution. Since the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is symmetrical about both axes and whose trace is shown in fig. 2.

$\therefore$ Required area $= 4\int_0^a y\,dx$

$$= 4\int_0^a b\sqrt{\left(1-\frac{x^2}{a^2}\right)}\,dx$$

$$= 4b\int_0^a \sqrt{\left(1-\frac{x^2}{a^2}\right)}\,dx.$$

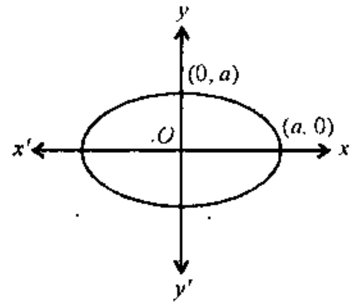

Fig. 2

Let us put $x = a\sin\theta$.

$\therefore$ $dx = a\cos\theta\,d\theta$ and θ varies from 0 to $\pi/2$.

$\therefore$ R. A. $= 4b\int_0^{\pi/2}\cos\theta\,.\,a\cos\theta\,d\theta = 4ab\int_0^{\pi/2}\cos^2\theta\,d\theta$

$$= 4ab\left[\frac{(2-1)}{2}\cdot\frac{\pi}{2}\right] \qquad \text{(By Walli's formula)}$$

$= \pi ab$ sq. units.

Example 3. *Find the common area between the curves $y^2 = 4ax$ and $x^2 = 4ay$.*

Solution. Since the curve $y^2 = 4ax$ is symmetrical about x-axis and the curve $x^2 = 4ay$ is symmetrical about y-axis. Both curves intersects at two points (0, 0) and $(4a, 4a)$ which are obtained by solving both equations of curves. The tracing of the curves is shown in fig. 3.

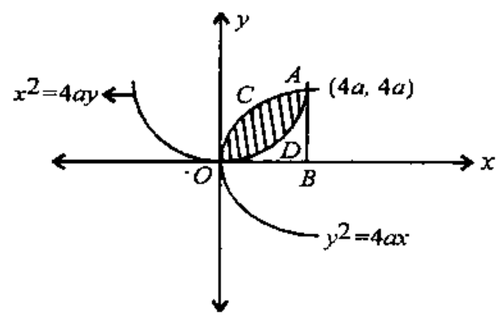

Fig. 3

The required area

= area of $OBACO$

− area of $OBADO$. ...(1)

Now area of $OBACO = \int_0^{4a} y\,dx$, where $y^2 = 4ax$

$$= \int_0^{4a}\sqrt{4ax}\,dx = 2\sqrt{a}\left[\frac{2}{3}x^{3/2}\right]_0^{4a} = \frac{32}{3}a^2$$

and area of $OBADO = \int_0^{4a} y\,dx$, where $y = \frac{x^2}{4a}$

$$= \int_0^{4a}\frac{x^2}{4a}\,dx = \frac{1}{4a}\left[\frac{x^3}{3}\right]_0^{4a} = \frac{16a^2}{3}.$$

From (1), we get

Required area $= \frac{32}{3}a^2 - \frac{16}{3}a^2 = \frac{16}{3}a^2$ sq. units.

• SUMMARY

- Area of bounded region, bounded by the curve $y = f(x)$ and x-axis between $x = a$ to $x = b$, is given by

$$A = \int_a^b y\,dx = \int_a^b f(x)\,dx$$

- Area of the region bounded by the curve $y = f(x)$ and y-axis between $y = a$ and $y = b$ is given by

$$A = \int_a^b f(x)\, dy$$

- Area bounded by two curves = | Area bounded by one curve – Area bounded by other curve |

• STUDENT ACTIVITY

1. Find the whole area of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

2. Find the area common to the curves $y^2 = 4ax$ and $x^2 = 4ay$.

• TEST YOURSLF

1. Find the area of the region bounded by the following curves, and the axis of x and the given ordinates :

(i) $y = \log x$;	$x = a$,	$x = b$
(ii) $y = c \cosh (x/c)$;	$x = 0$,	$x = a$
(iii) $y = \sin^2 x$;	$x = 0$,	$x = \pi/2$.

2. Find the area of the region bounded by the parabola $y^2 = 4x$ and the line $y = 2x$.
3. Find the area of a quadrant of the ellipse $x^2/a^2 + y^2/b^2 = 1$.
4. Find the area of a loop of the curve $xy^2 + (x + a)^2 (x + 2a) = 0$.
5. Find the area of the loop of the curve $3ay^2 = x (x - a)^2$.
6. Find the whole area of the curve $a^2x^2 = y^3 (2a - y)$.

ANSWERS

1. (i) $b \log (b/e) - a \log (a/e)$ (ii) $c^2 \sinh (a/c)$ (iii) $\pi/4$

2. $\frac{1}{3}$ 3. $\frac{1}{4}\pi ab$ 4. $2a^2 (1 - \pi/4)$ 5. $8a^2/(15\sqrt{3})$ 6. πa^2

OBJECTIVE EVALUATION

➤ FILL IN THE BLANKS :

1. The integral $\int_a^b f(x)\, dx$ represents ....

2. The area of the ellipse $x^2/a^2 + y^2/b^2 = 1$ is ....
3. The area of the curve $x^2 + y^2 - 2ax - 2ay = 0$ is ....
4. The area common to the curves $y^2 = 4ax$, $x^2 = 4ay$ is ....

➤ TRUE OR FALSE :

Write 'T' for True 'F' for False :

1. The integral $\int_a^b f(y)\, dy$ represents the area of a plane bounded by the curve $x = f(y)$ the y-axis and the lines $y = a$ and $y = b$. *(T/F)*
2. If the curve is symmetrical about x-axis, then the area of bounded portion is equal to twice the area of the upper portion of the curve above x-axis. *(T/F)*

➤ MULTIPLE CHOICE QUESTIONS :

Choose the most appropriate one :

1. The value of the integral $\int_1^2 f(x)\, dx$ if $f(x) = \frac{1}{x}$, is :

   (a) log 2 (b) log (2 − 1) (c) 0 (d) 1.

2. The integral $\int_a^b f(x)\, dy$ represents the area of bounded region between the lines :

   (a) $x = a, x = b$ (b) $x = 1, x = b$
   (c) $x = -a, y = b$ (d) $y = a, y = b$.

3. The area bounded by the curve $y = \sin^2 x$, x-axis and the lines $x = 0$, $x = \pi/2$ is :

   (a) $\pi/2$ (b) $\pi/4$ (c) π (d) $\pi/3$.

ANSWERS

Fill in the Blanks :

1. Area of bounded region **3.** πab **4.** $2\pi a^2$

True or False :

1. T **2.** T

Multiple Choice Questions :

1. (a) **2.** (d)

10

SURFACES AND VOLUMES OF SOLIDS OF REVOLUTION

STRUCTURE

LEARNING OBJECTIVES

After going through this unit you will learn :

How to calculate the surface areas and volumes of solids on revolution.

• 10.1. SURFACE OF REVOLUTION

Definition. When a plane curve is revolved about a certain fixed line which is lying in its own plane, a surface is generated. This surface is called a **surface of revolution**. Also the fixed line is called the **axis of revolution.**

• 10.2. REVOLUTION ABOUT x-AXIS

Let S be the surface area (curved surface) of a solid which is generated by the revolution of the curve $y=f(x)$ about x-axis between the ordinates $x=a$ and $x=b$ and let s be the arc length measured from the point $a, f(a)$ to any point $P(x, y)$. Then

$$S=\int_a^b 2\pi y\, ds=\int_a^b 2\pi y \frac{ds}{dx}\cdot dx.$$

Where $\dfrac{ds}{dx}=\sqrt{\left[1+\left(\dfrac{dy}{dx}\right)^2\right]}$ and S is the surface area of the solid of revolution of the curve $y=f(x)$ about x-axis between $x=a$ and $x=b$.

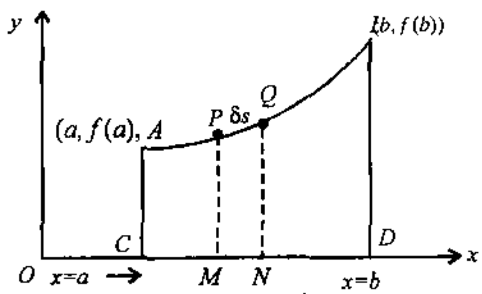

Fig. 1

• 10.3. REVOLUTION ABOUT y-AXIS

Let S be the surface area of a solid generated by the revolution of the curve $x=f(y)$ about y-axis between $y=a$ and $y=b$. Then

$$S=\int_a^b 2\pi x\, ds=\int_a^b 2\pi x \frac{ds}{dy}\cdot dy$$

where $\dfrac{ds}{dy}=\sqrt{\left[1+\left(\dfrac{dx}{dy}\right)^2\right]}$ and s the arc length being measured from the point $(f(a), a)$.

• 10.4. REVOLUTION ABOUT ANY LINE

Let S be the surface area of a solid generated by the curve about any line between certain points. Let s be the arc length of the curve measured from one of the two given points to any point P on the curve and let Q be any point very near to P such that $PQ = \delta s$. Now draw a perpendicular PM from the point P to the line of axis of the revolution. Then

$$S = \int 2\pi\,(PM)\,ds.$$

Here the limits of integration are taken the given points between them a solid is formed by the revolution.

• 10.5. SURFACE FORMULAE FOR DIFFERENT FORM OF EQUATIONS

Equation of a curve in parametric form. Suppose the equation of a curve is given in parametric form $x = f(t)$, $y = g(t)$, where t is the parameter, then the surface area of a solid generated by the revolution of the given curve about x-axis between the suitable limits is

$$S = \int 2\pi y \left(\frac{ds}{dt}\right) dt$$

where $$\frac{ds}{dt} = \sqrt{\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right]}.$$

Similarly for y-axis as the axis of revolution we may find the surface area.

REMARK

➤ If we have to find the surface area of a solid generated by the revolution of the curve about any line, then use the following steps :

1. Take any point P on the curve between the given points.
2. Draw the perpendicular from P to the line of axis which meets the axis at the point M (say).
3. Find the perpendicular PM.
4. And use the formula given is § Revolution about y-axis.

SOLVED EXAMPLES

Example 1. *Find the surface of a sphere of radius a.*

Solution. The sphere is generated, if a semi-circle is revolved about its diameter. Let the equation of a circle of radius a is

$$x^2 + y^2 = a^2. \qquad \text{...(1)}$$

$$\therefore \quad 2x + 2y\frac{dy}{dx} = 0 \quad \text{or} \quad \frac{dy}{dx} = -\frac{x}{y}.$$

$$\therefore \quad \frac{ds}{dx} = \sqrt{\left[1 + \left(\frac{dy}{dx}\right)^2\right]} = \sqrt{\left(1 + \frac{x^2}{y^2}\right)}$$

$$= \sqrt{\left(\frac{x^2+y^2}{y^2}\right)} = \frac{a}{y} \qquad \text{[using (1)]}$$

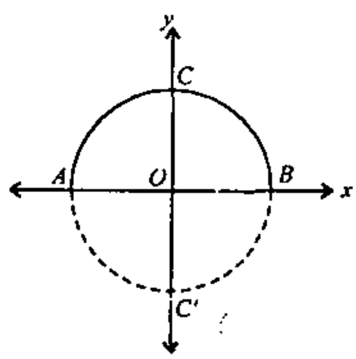

Fig. 2

Let $A\,(-a, 0)$ and $B\,(a, 0)$ be the bounding points of the semi-circle as shown in Fig. 2.

Here the diameter is taken as x-axis. Let S be the surface area of the sphere, then

$$S = \int_{-a}^{a} 2\pi y \cdot \frac{ds}{dx} \cdot dx$$

$$= \int_{-a}^{a} 2\pi y \cdot \frac{a}{y}\,dx \quad \left(\because \ \frac{ds}{dx} = \frac{a}{y}\right)$$

$$= 2\pi a \int_{-a}^{a} dx = 2\pi a \Big[x \Big]_{-a}^{a}.$$

$\therefore \qquad S = 4\pi a^2.$

Example 2. *Find the surface of the solid generated by the revolution of the curve $x = a \cos^3 t$ and $y = a \sin^3 t$ about the x-axis.*

Solution. We have

$$x = a \cos^3 t \text{ and } y = a \sin^3 t.$$

This curve is symmetrical about both the axes.

At $A(a, 0)$ $t = 0$ and at $B(0, a)$ $t = \pi/2$.

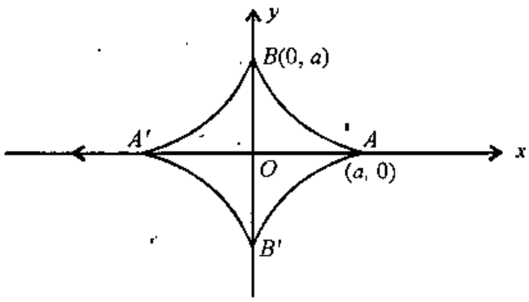

Fig. 3

Now $\quad \dfrac{dx}{dt} = -3a \cos^2 t \sin t$

$$\frac{dy}{dt} = 3a \sin^2 t \cos t$$

$$\therefore \quad \frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

$$= 3a \sqrt{\cos^4 t \sin^2 t + \sin^4 t \cos^2 t}$$

$$= 3a \sin t \cos t = \frac{3a}{2} \sin 2t.$$

Let S be the surface of a solid of revolution of the given curve about x-axis. Then

$$S = 2\int_0^{\pi/2} 2\pi y \frac{ds}{dt}\, dt = 4\pi \int_0^{\pi/2} a \sin^3 t \cdot \frac{3a}{2} \sin 2t\, dt$$

$$= 12\pi a^2 \int_0^{\pi/2} \sin^4 t \cos t\, dt = 12\pi a^2 \left[\frac{\sin^5 t}{5}\right]_0^{\pi/2} = \frac{12\pi a^2}{5}.$$

• TEST YOURSELF–1

1. Find the curved surface of a hemi-sphere of radius a.
2. Find the area of the surface formed by the revolution of the parabola $y^2 = 4ax$ about the x-axis by arc from the vertex to one of the latus rectum.
3. For a catenary $y = a \cosh (x/a)$, prove that
$$aS = \pi a\,(ax + sy)$$
where s is the length of the arc measured from the vertex, S is the area of curved surface of the solid generated by the revolution of the arc about x-axis.
4. Find the surface of the solid generated by the revolution of the astroid $x^{2/3} + y^{2/3} = a^{2/3}$ about x-axis.

ANSWERS

1. $2\pi a^2$ 2. $\dfrac{8}{3}\pi a^2 (2\sqrt{2} - 1)$ 4. $\dfrac{12\pi a^2}{5}$

• 10.6. VOLUME OF SOLIDS OF REVOLUTION

(a) Revolution about x-axis. Let V be the volume of a solid which is generated by the revolution of a plane area bounded by the curve $y = f(x)$, the ordinates $x = a$, $x = b$ and x-axis about the x-axis. Then

$$V = \int_a^b \pi y^2\, dx$$

where $y = f(x)$ is a continuous and single valued function defined on $[a, b]$.

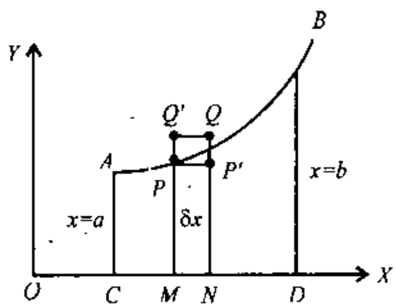

Fig. 4

(b) Revolution about y-axis. Similarly, the volume of a solid formed by the revolution of a plane area bounded by the curve $x = f(y)$ and the lines $y = a$ and $y = b$ and y-axis about y-axis is

$$V = \int_a^b \pi x^2 \, dy.$$

(c) Revolution about any line. The volume of a solid formed by the revolution of a plane area bounded by the arc AB and the lines AC and AD and the axis CD about any line CD (different from x-axis and y-axis) is

$$V = \int_{OC}^{OD} \pi (PM)^2 \, d(OM)$$

where PM is the length of perpendicular from any point P on the arc AB to the axis CD and O be any fixed point on the axis CD.

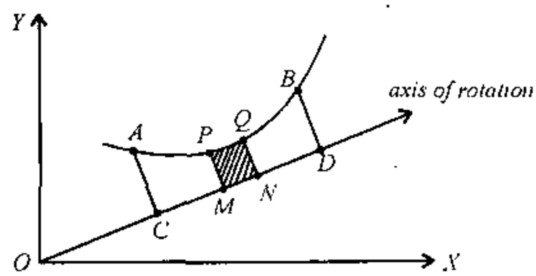

Fig. 5

• 10.7. VOLUME OF A SOLID OF REVOLUTION WHEN THE EQUATIONS OF THE CURVE ARE IN DIFFERENT FORMS

(a) Equation of curve in parametric form. Suppose the equation of a generating curve are in parametric form $x = f(t)$ and $y = g(t)$, then the volume of the solid generated by the revolution of the plane area bounded by the given curve, axis of x and the ordinates at the points where $t = a$ and $t = b$, about x-axis is

$$V = \int_a^b \pi y^2 \left(\frac{dx}{dt}\right) . \, dt = \int_a^b \pi \, [g(t)]^2 \, \frac{dx}{dt} \cdot dt$$

where $\qquad \dfrac{dx}{dt} = \dfrac{d}{dt}[f(t)].$

Similarly, the volume of a solid formed by the revolution of a plane area bounded by $x = f(t)$, $y = g(t)$ axis of y and the two absciassaes where $t = a$ and $t = b$ about y-axis is

$$V = \int_a^b \pi x^2 \left(\frac{dy}{dt}\right) . \, dt = \int_a^b \pi \, [f(t)]^2 \, \frac{dy}{dt} \cdot dt$$

where $\qquad \dfrac{dy}{dt} = \dfrac{d}{dt}[g(t)].$

SOLVED EXAMPLES

Example 1. *Find the volume of a spherical cap of height h cut off from a sphere of radius a.*

Solution. The spherical cap is generated by the revolution of a plane area bounded by the curve $x^2 + y^2 = a^2$, the axis of y and the line $y = a - h$ and $y = a$ in the first quadrant about y-axis as shown in fig. 6.

Let V be the volume of this spherical cap, then

$$V = \int_{a-h}^{a} \pi x^2 \, dy$$

$$= \pi \int_{a-h}^{a} (a^2 - y^2) \, dy \quad (\because \; x^2 + y^2 = a^2)$$

$$= \pi \left[a^2 y - \frac{y^3}{3} \right]_{a-h}^{a}$$

$$= \pi \left[\left(a^3 - \frac{a^3}{3} \right) - (a - h) \left\{ a^2 - \frac{(a-h)^2}{3} \right\} \right]$$

$$= \pi \left[\frac{2a^3}{3} - \frac{a-h}{3} \{ 3a^2 - (a-h)^2 \} \right]$$

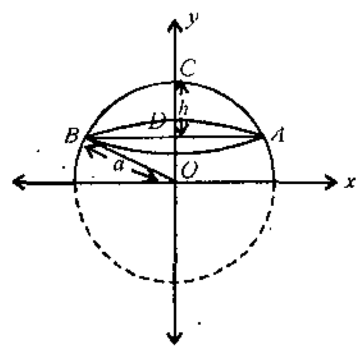

Fig. 6

$$\therefore \quad = \frac{\pi}{3}[2a^3 - (a-h)\{2a^2 - h^2 + 2ah\}]$$

$$= \frac{\pi}{3}[2a^3 - 2a^2(a-h) + (a-h)h^2 - 2ah(a-h)]$$

$$= \frac{\pi}{3}[3ah^2 - h^3]$$

$$V = \pi h^2\left(a - \frac{h}{3}\right).$$

Example 2. *Find the volume of the solid generated by the revolution of the tractrix*

$$x = a\cos t + \frac{a}{2}\log\tan^2(t/2),\ y = a\sin t.$$

about its asymptote.

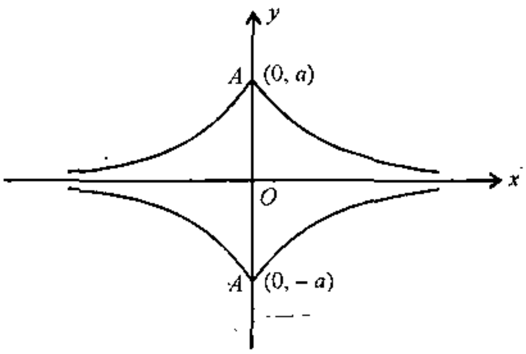

Fig. 7

Solution. We have

$$x = a\cos t + \frac{a}{2}\log\tan^2(t/2)$$

and $\quad y = a\sin t.$

Now $\quad \dfrac{dx}{dt} = -a\sin t + \dfrac{a}{\sin t} = \dfrac{a\cos^2 t}{\sin t}.$

Here x-axis is the aysmptote of the given curve so that for x-axis *i.e.*, $y = 0$, $t = \pi/2$ and at $A(0, a)$, $t = 0$.

Let V be the volume of the solid generated by the revolution of the tractrix. Then

$$V = 2\int_{\pi/2}^{0} \pi y^2 \frac{dx}{dt}\,dt \quad = 2\pi\int_{\pi/2}^{0} a^2\sin^2 t\,\frac{a\cos^2 t}{\sin t}\,dt$$

$$= 2\pi a^3\int_{\pi/2}^{0}\cos^2 t\sin t\,dt = 2\pi a^3\left[\frac{\cos^3 t}{3}\right]_{\pi/2}^{0}$$

$$= 2\pi a^3\left[\frac{1}{3} - 0\right] = \frac{2}{3}\pi a^3.$$

• SUMMARY

- Surface area of solid of revolution of the curve $y = f(x)$ about x-axis between $x = a$ to $x = b$ is given by

$$S = \int_a^b 2\pi y\left(\frac{ds}{dx}\right)dx = \int_a^b 2\pi y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx.$$

- Surface area of solid of revolution of the curve $y = f(x)$ about y-axis between $y = a$ and $y = b$ is given by

$$S = \int_a^b 2\pi x\frac{ds}{dy}\,dy = \int_a^b 2\pi x\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy$$

- Surface area of solid of revolution of the curve $x = x(t)$ and $y = y(t)$ about x-axis between $t = t_1$ and $t = t_2$ is given by

$$S = \int_{t_1}^{t_2} 2\pi y\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

- Volume of solid of revolution of the curve $y = f(x)$ about x-axis between $x = a$ to $x = b$ is given by

$$V = \int_a^b \pi y^2 dx$$

- Volume of solid of revolution of the curve $y = f(x)$ about y-axis between $y = a$ and $y = b$ is given by

$$V = \int_a^b \pi x^2 dy$$

• STUDENT ACTIVITY

1. Find the surface area of a sphere of radius a.

2. Find the volume of a sphere of radius a.

• TEST YOURSELF–2

1. Show that the volume of sphere of radius a is $\frac{4}{3}\pi a^3$.
2. Find the volume of a hemi-sphere.
3. The part of a parabola $y^2 = 4ax$ cut off by the latus rectum revolves about the tangnet at the vertex. Find the volume of a solid thus generated.
4. Find the volume of the solid generated by the revolution of an arc of the catenary $y = c\cosh(x/c)$ about x-axis.

ANSWERS

2. $\frac{2}{3}\pi a^3$ 3. $\frac{4}{5}\pi a^3$ 4. $\frac{\pi c^2}{2}\left[x + \frac{c}{2}\sinh\left(\frac{2x}{c}\right)\right]$

OBJECTIVE EVALUATION

➤ FILL IN THE BLANKS :

1. The integral $\int_a^b 2\pi y\, ds$ represents .... of a solid generated by the revolution about x-axis of the area bounded by the curve $y = f(x)$, the ordinates $x = a$, $x = b$ and x-axis.
2. The surface area of a hemi-sphere of radius a is ....

➤ TRUE OR FALSE :

Write 'T' for True and 'F' for False :

1. The volume of a hemi-sphere of radius a is $\frac{2}{3}\pi a^3$. *(T/F)*
2. The surface area of a sphere of radius a is $2\pi a^2$. *(T/F)*
3. The surface area of a solid generated by the revolution of the curve

$$x^{2/3} + y^{2/3} = a^{2/3} \text{ is } \frac{12}{5}\pi a^2.$$ *(T/F)*

➤ MULTIPLE CHOICE QUESTIONS :

Choose the most appropriate one :

1. The surface area of a sphere of radius r is :

   (a) $2\pi r^2$ (b) $4\pi r^2$

   (c) πr^2 (d) $3\pi r^2$.

2. The volume of a hemi-sphere of radius a is :

   (a) $\frac{2}{3}\pi a^2$ (b) $\frac{2}{3}\pi a^3$

   (c) $\frac{1}{3}\pi a^3$ (d) πa^3.

ANSWERS

Fill in the Blanks :

1. Surface 2. $2\pi a^2$

True or False :

1. T 2. F 3. T

Multiple Choice Questions :

1. (b) 2. (b)

❑❑❑

UNIT 11

DIFFERENTIAL EQUATION OF FIRST ORDER AND FIRST DEGREE

STRUCTURE

LEARNING OBJECTIVES

After going through this unit you will learn :

- How to determine the solution of the given differential equation of order one and degree

11.1. DEFINITIONS

1. Differential Equation. *An equation involving one dependent variable and its derivatives with respect to one or more independent variables is called a* ***differential equation***.

For Examples : (i) $e^x\,dx + e^y\,dy = 0$, (ii) $y = x\dfrac{dy}{dx}$, (iii) $\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{3/2} = c\,\dfrac{d^2y}{dx^2}$,

(iv) $\dfrac{\partial^2 y}{\partial x^2} = \dfrac{1}{c}\dfrac{\partial^2 y}{\partial t^2}$ *are all diffrential equations.*

2. Ordinary Differential Equation. *A differential equation which contains only one independent variable is called an ordinary* ***differential equation***

For Examples : (i) $x\,dx + y\,dy = 0$, (ii) $1+\left(\dfrac{dy}{dx}\right)^2 = \dfrac{d^3y}{dx^3}$ *are both ordinary differential equations as they have one independent variables.*

3. Partial Differential Equation. *A differential equation which contains more than one independent variables, is called* ***partial differential equation***.

For examples : (i) $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}+\dfrac{\partial^2 u}{\partial z^2}=0$, (ii) $\dfrac{\partial^2 u}{\partial x^2}=\dfrac{1}{c}\dfrac{\partial^2 u}{\partial y^2}$ *are both partial differential equation as they have more than one independent variable such as x, y, z.*

4. Order of Differential Equations. *The* **order** *of a differential equation is the order of the highest derivative appearing in it.*

5. Degree of Differential Equations. *The* **degree** *of a differential equation is the degree of the highest derivative occurring in it, after the equation has been expressed in a form free from radicals and fractions as far as the derivative are concerned.*

For examples : (i) $e^x\,dx+e^y\,dy=0$, is of the first order and first degree.

(ii) $\dfrac{d^2y}{dx^2}+x^2y=0$, is of the second order and first degree.

(iii) The differential equation $\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{3/2}=c\left(\dfrac{d^2y}{dx^2}\right)$ can be written as $\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^3=c^2\left(\dfrac{d^2y}{dx^2}\right)^2$, it is of second order and second degree.

SOLVED EXAMPLES

Example 1. *Determine the order and degree of the differential equation*

$$x\left(\frac{d^2y}{dx^2}\right)^3+y\left(\frac{dy}{dx}\right)^4+y^2=0.$$

Solution. The given differential equation contains second order derivative which is the highest derivatives, so the order of the differential equation is 2. The power of second order derivative is 3, so the degree of the differential equation is 3.

Hence, order = 2 and degree = 3.

Example 2. *Determine the order and degree of the differential equation*

$$\left[1+\left(\frac{dy}{dx}\right)^2\right]^{1/2}=\left(\frac{d^2y}{dx^2}\right)^{1/3}.$$

Solution. Making the differential equation free from redicals, we get

$$\left[1+\left(\frac{dy}{dx}\right)^2\right]^3=\left(\frac{d^2y}{dx^2}\right)^2.$$

Now in this equation, the order of the highest derivative is 2, so its order is 2.

The power of the highest derivative is 2, so its degree is 2.

Hence, the given differential equation is of order 2 and degree 2.

Example 3. *Determine the order and degree of the differential equation*

$$x\left(\frac{dy}{dx}\right)+\frac{2}{\left(\dfrac{dy}{dx}\right)}=y^2.$$

Solution. Making the given differential equation free from fraction, we get

$$x\left(\frac{dy}{dx}\right)^2+2=y^2\left(\frac{dy}{dx}\right).$$

Now, in this equation, the order of the highest derivative is one and its power is 2.

Hence, the differential equation of order 1 and degree 2.

• TEST YOURSELF–1

Determine the order and degree of each of the following differential equation :

1. $\left(\dfrac{dy}{dx}\right)^2+5y=\sin x.$
2. $\dfrac{d^2y}{dx^2}+3\left(\dfrac{dy}{dx}\right)^3+2y=0.$
3. $(x^3-y^3)\,dx+(xy^2-x^2y)\,dy=0.$
4. $x^2\left(\dfrac{dy}{dx}\right)+2xy-6x^3=0.$

ANSWERS

1. Order = 1, Degree = 2 **2.** Order = 2, Degree = 1 **3.** Order = 1, Degree = 1

4. Order = 1, Degree = 1

• 11.2. SOLUTION OF THE DIFFERENTIAL EQUATION

Any relation between the dependent and independent variables i.e., a function of the form $y = f(x) + C$, which satisfies the given differential equation is called its ***solution or primitive****.*

For example : Let $\frac{dy}{dx} = \cos x$ be a given differential equation. Then a function $y = \sin x + C$, is its solution.

1. General solution. *If the solution of nth order differential equation contains n arbitrary constants, then it is called* **general solution** *or* **complete primitive**.

2. Particular solution. *A solution obtained from the general solution by giving particular values to the arbitrary constants in the general solution is called* **particular solution** *or* **particular integral**.

Singular Solution. *The solution which can not be obtained from general solution by assigning particular values to the arbitrary constants, is called* **singular solution**.

Example 1. *Verify that $y = A \cos x - B \sin x$ is a solution of the differential equation*

$$\frac{d^2y}{dx^2} + y = 0.$$

Solution. We have, $\quad y = A \cos x - B \sin x.$ ...(1)

Differentiating (1) w.r.t. x, we get

$$\frac{dy}{dx} = -A \sin x - B \cos x.$$

Differentiating again, we get

$$\frac{d^2y}{dx^2} = -A \cos x + B \sin x$$

$$= -(A \cos x - B \sin x) = -y \qquad \text{[using (1)]}$$

$$\therefore \quad \frac{d^2y}{dx^2} + y = 0.$$

Hence, $y = A \cos x - B \sin x$ is a solution of $\frac{d^2y}{dx^2} + y = 0$.

Example 2. *Show that $y = Ae^x + Be^{-x}$ is a solution of the differential equation*

$$\frac{d^2y}{dx^2} - y = 0.$$

Solution. We have

$$y = Ae^x + Be^{-x}. \qquad ...(1)$$

Differentiating (1) w.r.t. x, we get

$$\frac{dy}{dx} = Ae^x - Be^{-x}.$$

Differentiating again, we get

$$\frac{d^2y}{dx^2} = Ae^x + Be^{-x} = y \qquad \text{[using (1)]}$$

$$\therefore \quad \frac{d^2y}{dx^2} - y = 0.$$

Hence, $y = Ae^x + Be^{-x}$ is a solution of $\frac{d^2y}{dx^2} - y = 0$.

Example 3. *Show that $y = ae^{2x} + be^{-x}$ is a solution of the differential equation*

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 0.$$

Solution. We have

$$y = ae^{2x} + be^{-x}. \qquad ...(1)$$

Differentiating (1) w.r.t. x, we get

$$\frac{dy}{dx} = 2ae^{2x} - be^{-x}. \qquad ...(2)$$

Differentiating again, we get

$$\frac{d^2y}{dx^2} = 4ae^{2x} + be^{-x}. \qquad ...(3)$$

Now, from (1), (2), (3), we get

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = (4ae^{2x} + be^{-x}) - (2ae^{2x} - be^{-x}) - 2(ae^{2x} + be^{-x}) = 0.$$

Hence, $y = ae^{2x} + be^{-x}$ is a solution of the differential equation $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 0.$

• TEST YOURSELF–2

1. Show that $y = ae^x$ is a solution of the differential equation $\frac{dy}{dx} - y = 0.$
2. Show that $y = ae^{-x}$ is a solution of the differential equation $\frac{dy}{dx} + y = 0.$
3. Show that $y = c(x - c)^2$ is a solution of the differential equation $8y^2 = 4xy\frac{dy}{dx} - \left(\frac{dy}{dx}\right)^3.$
4. Show that $y = 4 \sin 3x$ is a solution of the differential equation $\frac{d^2y}{dx^2} + 9y = 0.$

• 11.3. FORMATION OF A DIFFERENTIAL EQUATION

The general solution of a first order differential equation contains one arbitrary constant, which is called a **parameter**. If this parameter takes various values, then we get a **family of curve** of one parameter.

For example : The equation $x^2 + y^2 = c^2$ represent a one parameter family of circles if c takes all real values.

Suppose there is an equation, representing a family of curves, containing n arbitrary constants. Then in order to find its differential equation we proceed as follows :

(*i*) *Differentiate the given equation of family of curve n times to get n more equations containing n arbitrary constants and derivatives.*

(*ii*) *Eliminate all the n constants from all of these* $(n + 1)$ *equations to get an equation containing a nth order derivativie, which is the required differential equation of the given family of curves.*

Example 1. *Find the differential equation of the family of all straight lines passing through the origin.*

Solution. The general equation of the family of all straight lines passing through the origin is

$$y = mx \qquad ...(1)$$

where m is a parameter.

Differentiating (1) w.r.t. x, we get

$$\frac{dy}{dx} = m. \qquad ...(2)$$

Eliminating m between (1) and (2), we get

$$y = x\frac{dy}{dx}$$

which is the required differential equation.

Example 2. *Find the differential equation of the family of all straight lines making equal intercepts on the co-ordinate axes.*

Solution. The general equation of the family of all straight lines making equal intercepts on the axes is

$$\frac{x}{c} + \frac{y}{c} = 1 \quad \text{or} \quad x + y = c \qquad ...(1)$$

where c is a parameter.

Differentiating (1) w.r.t. x, we get

$$1 + \frac{dy}{dx} = 0$$

which is the required differential equation.

• TEST YOURSELF–3

Find the differential equations of the following families of the curves :

1. $y = 3x + c$, where c is a parameter.
2. $(2x + a)^2 + y^2 = a^2$, where a is a parameter.
3. $x^2 + y^2 = a^2$, where a is a parameter.
4. $x^2 + y^2 = 2ax$, where a is a parameter.
5. $y = a \sin (x + b)$, where a and b are parameters.
6. $y = a \cos (x + b)$, where a and b are parameters.

ANSWERS

1. $\frac{dy}{dx} = 3$ 2. $2xy \frac{dy}{dx} - y^2 + 4x^2 = 0$ 3. $x + y \frac{dy}{dx} = 0$

4. $2xy \frac{dy}{dx} + x^2 - y^2 = 0$ 5. $\frac{d^2y}{dx^2} + y = 0$ 6. $\frac{d^2y}{dx^2} + y = 0$

• 11.4. SOLUTION OF DIFFERENTIAL EQUATIONS

1. Method of Solving Differential Equation by Separation of Variables :

If any differential equation can be expressed as

$$f(x)\, dx = g(y)\, dy. \qquad ...(1)$$

Then we say that variables are separable.

In order to solve the given differential equation, using variable separable, we first reduce it to the form (1) *and then integrate both sides, we therefore obtain the required solution.*

Example 1. *Solve* $\frac{dy}{dx} = xy + x + y + 1$.

Solution. We have

$$\frac{dy}{dx} = xy + x + y + 1$$

$$\Rightarrow \qquad \frac{dy}{dx} = (1 + x)(1 + y).$$

Separating the variables, we get

$$\frac{dy}{1 + y} = (1 + x)\, dx.$$

Integrating, we get

$$\log |1 + y| = x + \frac{x^2}{2} + C$$

which is the required solution.

Example 2. *Solve* $\frac{dy}{dx} = \log (x + 1)$.

Solution. We have

$$\frac{dy}{dx} = \log (x + 1).$$

Separating the variables, we get

$$dy = \log (x + 1)\, dx.$$

Integrating both sides, we get

$$y = \int \log (x + 1)\, dx + C$$

$$= \log (x + 1) \int dx - \int \left\{ \frac{d}{dx} (\log (x + 1)) . \int dx \right\} dx + C$$

$$= x \log (x + 1) - \int \frac{x}{x + 1}\, dx + C$$

$$= x \log (x + 1) - \int \frac{x + 1 - 1}{x + 1}\, dx + C$$

$$= x \log(x+1) - \int dx + \int \frac{dx}{x+1}\, dx + C$$

$\therefore \qquad y = x \log(x+1) - x + \log|x+1| + C$

which is the required solution.

• TEST YOURSELF–4

Solve the following differential equations :

1. $x\dfrac{dy}{dx} = y.$ 2. $\dfrac{dy}{dx} + y = 1.$
3. $\dfrac{dy}{dx} = \sqrt{4-y^2}.$ 4. $\dfrac{dy}{dx} = e^{x+y}.$
5. $\dfrac{dy}{dx} = (e^x + 1)\, y.$ 6. $\dfrac{dy}{dx} = \dfrac{(1+y^2)}{(1+x^2)}.$

ANSWERS

1. $y = Cx$ 2. $\log|1-y| + x = C$ 3. $y = 2\sin(x+C)$
4. $e^x + e^{-y} = C.$ 5. $\log|y| = e^x + x + C$ 6. $\tan^{-1} y - \tan^{-1} x = C$

• 11.5. HOMOGENEOUS DIFFERENTIAL EQUATIONS

Homogeneous Function. *A function $f(x, y)$ in x and y is said to be a* **homogeneous function of degree n**, *if the degree of each term of $f(x, y)$ is n.*

For Example : $f(x, y) = ax^2 + 2hxy + by^2$ is a homogeneous function of degree 2.

REMARK

➤ In general, a homogeneous function of degree n can be expressed as

$$f(x, y) = x^n F\left(\frac{y}{x}\right).$$

Homogeneous differential equation. A differential equation of the form

$$\frac{dy}{dx} = \frac{f_1(x, y)}{f_2(x, y)}$$

is said to be **homogeneous**, if $f_1(x, y)$ and $f_2(x, y)$ are homogeneous functions of same degree.

For example : $\dfrac{dy}{dx} = \dfrac{x^2+y^2}{xy}$ is a homogeneous differential equation.

Method of Solving Homogeneous Differential Equation :

In order to solve a homogeneous differential equation we proceed through the following steps:

Step 1. Substitute $y = vx$ (or $x = vy$).

Step 2. Reduce the given differential equation in terms of v and x.

Step 3. Solve the reduced equation by the method of separation of variables.

Step 4. Replace v by $\dfrac{y}{x}\left(\text{or } v \text{ by } \dfrac{x}{y}\right)$ after integration.

Example 1. *Solve* $(x^2 + y^2)\dfrac{dy}{dx} = xy.$

Solution. The given differential equation can be expressed as

$$\frac{dy}{dx} = \frac{xy}{x^2+y^2}. \qquad \ldots(1)$$

Clearly (1) is a homogeneous differential equation.

Putting $y = vx$ and $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$ in (1), we get

$$v + x\frac{dv}{dx} = \frac{vx^2}{x^2 + v^2x^2} = \frac{v}{1+v^2}$$

$$\Rightarrow \qquad x\frac{dv}{dx} = \frac{v}{1+v^2} - v = \frac{-v^3}{1+v^2}.$$

Separating the variables, we get

$$-\frac{1+v^2}{v^3}\,dv = \frac{dx}{x}.$$

Integrating both sides, we get

$$\Rightarrow \quad -\int \frac{dv}{v^3} - \int \frac{dv}{v} = \int \frac{dx}{x} + \log|C|$$

$$\Rightarrow \quad \frac{1}{2v^2} - \log|v| = \log|x| + \log|C|$$

$$\Rightarrow \quad \frac{1}{2v^2} = \log|xvC|$$

$$\Rightarrow \quad xvC = e^{1/2v^2}$$

$$\Rightarrow \quad yC = e^{x^2/2y^2}$$

which is the required solution.

• TEST YOURSELF–5

Solve the following differential equations :

1. $\frac{dy}{dx} = \frac{y^2 - x^2}{2xy}$.
2. $\frac{dy}{dx} = \frac{x^2 + y^2}{2xy}$.
3. $x + y\frac{dy}{dx} = 2y$.
4. $x\frac{dy}{dx} = x + y$.
5. $\frac{dy}{dx} = \frac{x+y}{x-y}$.
6. $(x - y)\frac{dy}{dx} = x + 3y$.

ANSWERS

1. $(x^2 + y^2) = Cx$
2. $x = C(x^2 - y^2)$
3. $\log(y - x) = C + \frac{x}{y - x}$
4. $y = x\log|x| + Cx$
5. $\tan^{-1}\frac{y}{x} = \frac{1}{2}\log(x^2 + y^2) + C$
6. $\log|x + y| + \frac{2x}{x + y} = C$

• 11.6. LINEAR DIFFERENTIAL EQUATION

Definition. *A differential equation in which the dependent variable and all its derivatives are of first degree only and not multiplied together, is called a* **linear differential equation.**

For example : $x\frac{dy}{dx} + y = x$ is a linear differential equation.

General Form of Linear Differential Equation :

(i) The equation of the form

$$\frac{dy}{dx} + Py = Q$$

where P and Q are either constants or functions of x only, is most general form of linear differential equation.

(ii) The equation of the form

$$\frac{dx}{dy} + Px = Q$$

where P and Q are either constants or functions of y only, is most general form of linear differential equation.

Solution of $\frac{dy}{dx} + Py = Q$: Multiplying by $e^{\int P\,dx}$, which is known as **integrating factor** (*i.e.*, I.F.), we get

$$e^{\int P\,dx}\frac{dy}{dx} + e^{\int P\,dx}Py = Qe^{\int P\,dx}$$

$$\Rightarrow \quad \frac{d}{dx}\left[ye^{\int P\,dx}\right] = Qe^{\int P\,dx}.$$

Integrating both sides, we get

$$y \,.\, e^{\int P\,dx} = \int Qe^{\int P\,dx}\,dx + C$$

or $\quad$ $\boldsymbol{y \,.\, (\text{I.F.}) = \int Q \,.\, (\text{I.F.})\,dx + C}$, where I.F. $= e^{\int P\,dx}$

Working rule for solving $\frac{dy}{dx}+Py=Q$.

1. Find I.F. $= e^{\int P\,dx}$.
2. The solution is $y\,.\,(\text{I.F.}) = \int Q\,.\,(\text{I.F.})\,dx + C$.

SOLVED EXAMPLES

Example 1. *Solve* $(1+x^2)\frac{dy}{dx}+2xy=\cos x$.

Solution. The given differential equation can be written as

$$\frac{dy}{dx}+\frac{2x}{1+x^2}y=\frac{\cos x}{1+x^2}. \qquad ...(1)$$

Comparing (1) with $\frac{dy}{dx}+Py=Q$, we get

$$P=\frac{2x}{1+x^2},\quad Q=\frac{\cos x}{1+x^2}.$$

$$\therefore \qquad \text{I.F.} = e^{\int P\,dx} = e^{\int \frac{2x}{1+x^2}dx} = e^{\log(1+x^2)} = 1+x^2.$$

Thus, the solution is

$$y(1+x^2)=\int Q(1+x^2)\,dx+C$$

$$\Rightarrow \qquad y(1+x^2)=\int\frac{\cos x}{(1+x^2)}(1+x^2)\,dx+C$$

$$\Rightarrow \qquad y(1+x^2)=\int\cos x\,dx+C$$

$$\Rightarrow \qquad y(1+x^2)=\sin x+C.$$

Example 2. *Solve* $x\frac{dy}{dx}-y=x^2$.

Solution. The given differential equation can be written as

$$\frac{dy}{dx}-\frac{1}{x}y=x. \qquad ...(1)$$

Comparing (1) with $\frac{dy}{dx}+Py=Q$, we get

$$P=-\frac{1}{x} \text{ and } Q=x$$

so,
$$\text{I.F.} = e^{\int P\,dx} = e^{\int -\frac{1}{x}dx} = e^{-\log x} = \frac{1}{x}.$$

Thus, the solution of (1) is

$$y\,.\left(\frac{1}{x}\right)=\int Q\,.\left(\frac{1}{x}\right)dx+C$$

$$\Rightarrow \qquad y\,.\left(\frac{1}{x}\right)=\int x\,.\,\frac{1}{x}\,dx+C=\int dx+c=x+C.$$

$$\therefore \qquad y=x^2+Cx.$$

• TEST YOURSELF–6

Solve the following differential equations :

1. $\frac{dy}{dx}-y=e^{2x}$.
2. $\frac{dy}{dx}+y=e^{x}$.
3. $\frac{dy}{dx}-4y=e^{2x}$.
4. $\frac{dy}{dx}+y=e^{-2x}$.
5. $\frac{dy}{dx}+2y=e^{-x}$.
6. $\frac{dy}{dx}+2y=6e^{x}$.
7. $\frac{dy}{dx}-\frac{y}{x}=2x^2$.
8. $\frac{dy}{dx}+2y=xe^{4x}$.
9. $\frac{dy}{dx}+\frac{y}{x}=x^n$.

ANSWERS

1. $y=e^{2x}+Ce^{x}$
2. $y=\frac{1}{2}e^{x}+Ce^{-x}$
3. $y=-\frac{1}{2}e^{2x}+Ce^{4x}$

4. $y = -e^{-2x} + Ce^{-x}$ 5. $y = e^x + Ce^{-2x}$ 6. $y = 2e^x + Ce^{-2x}$

7. $y = x^3 + Cx$ 8. $y = \frac{1}{6}xe^{4x} - \frac{1}{36}e^{4x} + Ce^{-2x}$ 9. $(n+2)\,xy = x^{n+2} + (n+2)\,C$

• 11.7. EQUATION REDUCIBLE TO LINEAR FORM

A differential equation of the form

$$\frac{dy}{dx} + Py = Qy^n$$

is called an equation reducible to the linear form or **Bernoulli's equation.**

Solution of the equation. The given equation can be written as

$$y^{-n}\frac{dy}{dx} + Py^{-n+1} = Q. \qquad \ldots(1)$$

Put $\qquad y^{-n+1} = v$

$\Rightarrow \qquad (-n+1)\,y^{-n}\frac{dy}{dx} = \frac{dv}{dx}.$

Then (1), becomes

$$\frac{-1}{(n-1)}\frac{dv}{dx} + Pv = Q$$

or $\qquad \frac{dv}{dx} - (n-1)\,Pv = Q\,(1-n)$

which is a linear equation of first order, and can be solved in a usual manner.

Working Procedure :

(i) Write the given equation in the form $\frac{dy}{dx} + Py = Qy^n$.

(ii) Divide by y^n, *substitute* $y^{-n+1} = v$ *and get the equation of the form*

$$\frac{dv}{dx} - (n-1)\,Pv = Q\,(1-n).$$

(iii) Then, apply the method of solution of linear equation.

SOLVED EXAMPLES

Example 1. *Solve* $x\frac{dy}{dx} + y = y^2 \log x$.

Solution. The given equation can be written as

$$\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{x}\cdot\frac{1}{y} = \frac{1}{x}\log x$$

Put $\qquad \frac{1}{y} = v \Rightarrow -\frac{1}{y^2}\frac{dy}{dx} = \frac{dv}{dx}.$

Hence, the given equation becomes

$$-\frac{dv}{dx} - \frac{1}{x}v = -\frac{1}{x}\log x \quad \text{which in linear in } v.$$

$$\text{I.F.} = e^{\int P dx} = e^{-\int \frac{1}{x}dx} = e^{-\log x} = \frac{1}{x}$$

The solution is

$$\frac{1}{x}\cdot v = \int -\frac{1}{x^2}\log x\, dx + c$$

$$= \frac{1}{x}\log x - \int \frac{1}{x}\cdot\frac{1}{x}\,dx = \frac{1}{x}\log x + \frac{1}{x} + c$$

or $\qquad v = \log x + 1 + cx$ or $1 = cyx + y + y\log x$

or $\qquad 1 = y\,(1 + \log x) + cxy.$

Example 2. *Solve* $\frac{dy}{dx} + \frac{y}{x}\log y = \frac{y}{x^2}(\log y)^2$.

Solution. The given equation is of Bernoulli's type because of presence of non-linear part on right side, so dividing by $y\,(\log y)^2$, we get

$$\frac{1}{y\,(\log y)^2}\cdot\frac{dy}{dx} + \frac{1}{x}\cdot\frac{1}{\log y} = \frac{1}{x^2}. \qquad \ldots(1)$$

Now substituting $u = \frac{1}{\log y}$ or $\frac{du}{dx} = -\frac{1}{y(\log y)^2} \cdot \frac{dy}{dx}$ in (1), we get

$$-\frac{du}{dx} + \frac{1}{x} u = \frac{1}{x^2} \Rightarrow \frac{du}{dx} - \frac{1}{x} u = -\frac{1}{x^2}$$

which is a linear equation in u, therefore

$$\text{I.F.} = e^{-\int 1/x\, dx} = \frac{1}{x}. \qquad \text{...(2)}$$

Multiplying equation (2) by I.F. and after integrating, we get

$$\frac{u}{x} = \frac{1}{2x^2} + c$$

$\Rightarrow$ $\frac{1}{x \log y} = \frac{1}{2x^2} + c$, which is the required solution.

• TEST YOURSELF–7

Solve the following ordinary differential equation :

1. $\frac{dy}{dx} + \frac{1}{x} y = x^2 y^6$. 2. $x\left(\frac{dy}{dx}\right) + y = x^2 y^4$. 3. $\frac{dy}{dx} + \left(\frac{x}{1-x^2}\right) y = x\sqrt{y}$.

4. $\frac{dy}{dx} + \frac{y}{x} = y^3$. 5. $y(1+xy)\, dx - x\, dy = 0$.

6. $\frac{dy}{dx} + \frac{1}{x} = \frac{e^y}{x^2}$. 7. $(x^3 y^2 + xy)\, dx = dy$.

ANSWERS

1. $\frac{1}{x^5 y^5} = \frac{5}{2}(1/x^2) + c$ 2. $\frac{1}{y^3} = x^2(3 - cx)$

3. $\sqrt{y}(1-x^2)^{1/4} = -\frac{1}{3}(1-x^2)^{3/4} + c$ 4. $2xy^2 + cx^2y^2 = 1$

5. $-\frac{x}{y} = \frac{1}{2}x^2 + c$ 6. $2x = (2cx^2 + 1)\, e^y$

7. $1 = 2y\left(1 - \frac{x^2}{2}\right) - cye^{-x^2/2}$

• 11.8. EXACT DIFFERENTIAL EQUATION

The differential equation which can be derived from its primitive by direct differentiation without any further transformation (such as elimination or reduction), is called an exact differential equation.

For Example : The equation $ydx + xdy = 0$ is exact, as it is derived from it is primitive

$$yx = c.$$

Working Procedure :

(*i*) *Compare the given equation with* $Mdx + Ndy = 0$ *and find out M and N.*

(*ii*) *Show that* $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, *which conclude the exactness of the given equation.*

(*iii*) *Integrate the coefficient of dx (i.e., M) with respect to x, regarding y to be constant.*

(*iv*) *Omit the terms containing x is N and find the integral of the coefficient of dy with respect to y.*

(*v*) *Add the above two results and equate this sum to an arbitrary constant, i.e.,*

$$\int_{(y \text{ constant})} M\, dx + \int (\textit{terms in N not containing x})\, dy = c.$$

REMARK

➤ It is clear from the definition that the exact differential equation is formed from its general solution by direct differentiation and without any further operation of elimination or reduction.

- The necessary and sufficient condition for a differential equation (of first order and first degree)

$$Mdx + Ndy = 0,$$

where M and N are the functions of x and y

to be exact equation is that $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

- The equation $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ is true whenever both sides exist and are continuous. Here we take the hypothesis that all the functions, which we discuss are sufficiently continuous and differentiable to gurantee the validity of the operations we perform on them.

SOLVED EXAMPLES

Example 1. *Solve* $(ax + hy + g)\,dx + (hx + by + f)\,dy = 0$.

Solution. The given equations

$$(ax + hy + g)\,dx + (hx + by + f)\,dy = 0.$$

Compare with $Mdx + Ndy = 0$, we get

$$M = ax + hy + g \quad \text{and} \quad N = hx + by + f$$

$$\Rightarrow \quad \frac{\partial M}{\partial y} = h,\ \frac{\partial N}{\partial x} = h \Rightarrow \text{equation is exact.}$$

Hence, the solution is

$$\int_{(y\ \text{constant})} M\,dx + \int (\text{terms in } N \text{ not containing } x)\,dy = c$$

$$\Rightarrow \quad \int (ax + hy + g)\,dx + \int (by + f)\,dy = c$$

$$\Rightarrow \quad \frac{ax^2}{2} + hxy + gx + \frac{by^2}{2} + fy = c$$

$$\Rightarrow \quad ax^2 + 2hxy + by^2 + 2gx + 2fy = 2c.$$

Example 2. $(x^2 - ay)\,dx - (ax - y^2)\,dy = 0$.

Solution. Compare the given equation with $M\,dx + N\,dy = 0$, we get

$$M = x^2 - ay \quad \text{and} \quad N = y^2 - ax.$$

Now, we have

$$\frac{\partial M}{\partial y} = -a \quad \text{and} \quad \frac{\partial N}{\partial x} = -a \quad \Rightarrow \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$\Rightarrow$ The given equation is exact.

Therefore, the solution of the given equation is

$$\int_{y\ \text{is constant}} M\,dx + \int_{(\text{terms which do not contains } x)} N\,dy = c$$

$$\Rightarrow \quad \int (x^2 - ay)\,dx + \int y^2\,dy = c$$

$$\Rightarrow \quad \frac{1}{3}x^3 - ayx + \frac{y^3}{3} = c \quad \Rightarrow \quad x^3 + y^3 - 3axy = 3c.$$

• TEST YOURSELF–8

Solve the following ordinary differential equation :

1. $(4x + 3y + 1)\,dx + (3x + 2y + 1)\,dy = 0$.
2. $\frac{dy}{dx} = \frac{(2x - y)}{x + 2y - 5}$.
3. $x\,dx + y\,dy + \frac{x\,dy - y\,dx}{x^2 + y^2} = 0$.
4. $x\,dx + y\,dy = a^2 \frac{x\,dy - y\,dx}{x^2 + y^2}$.

ANSWERS

1. $2x^2 + 3xy + y^2 + x + y = c$.
2. $x^2 - y^2 - xy + 5y + c$.
3. $x^2 - 2\tan^{-1}(x/y) + y^2 = c$.
4. $x^2 + y^2 + 2a^2 \tan^{-1}\left(\frac{x}{y}\right) = c$.

• 11.9. INTEGRATING FACTOR

If the given equation $Mdx + Ndy = 0$ is not exact, then it can be made exact by multiplying some function of x and y. This **multiplier** is called an integrating factor (I.F.).

REMARKS

- The number of integrating factor for the equation

  $Mdx + Ndy = 0$, is infinite.
- If μ is an integrating factor. Then $\mu\,(Mdx + Ndy) = 0$ is an exact differential equation.
- Although an equation of the form

  $$Mdx + Ndy = 0$$

  always has integrating factor, there is no general method of finding them.

Here, we explain some rules for finding I.F. :

(A) By Inspection. The following list of exact differentials should be noted very carefully.

(1) $d\left(\dfrac{x}{y}\right) = \dfrac{y\,dx - x\,dy}{y^2}$ **(2)** $d\left(\dfrac{y}{x}\right) = \dfrac{x\,dy - y\,dx}{x^2}$

(3) $d\left(\tan^{-1}\dfrac{y}{x}\right) = \dfrac{x\,dy - y\,dx}{x^2 + y^2}$ **(4)** $d\left(\tan^{-1}\dfrac{x}{y}\right) = \dfrac{y\,dx - x\,dy}{x^2 + y^2}$

(5) $d\left(\log\dfrac{x}{y}\right) = \dfrac{ydx - xdy}{xy}$ **(6)** $d\left(\log\dfrac{y}{x}\right) = \dfrac{x\,dy - y\,dx}{xy}$

(7) $d(x, y) = x\,dy + y\,dx$ **(8)** $d\left(\dfrac{1}{xy}\right) = -\left[\dfrac{x\,dy + y\,dx}{x^2 y^2}\right]$

(9) $d\left(\dfrac{x^2}{y}\right) = \dfrac{2xy\,dx - x^2\,dy}{x^2 y^2}$ **(10)** $d\left(\dfrac{y^2}{x}\right) = \dfrac{2xy\,dy - y^2\,dx}{x^2}$

(11) $d\left(\dfrac{y^2}{x^2}\right) = \dfrac{2x^2 y\,dy - 2y^2 x\,dx}{x^4}$ **(12)** $d\left(\dfrac{x^2}{y^2}\right) = \dfrac{2xy^2\,dx - 2yx^2\,dy}{y^4}$

(13) $d\left(\dfrac{e^x}{y}\right) = \dfrac{ye^x\,dx - e^x\,dy}{y^2}$ **(14)** $d\,[\log(x^2 + y^2)] = \dfrac{2x\,dx + 2y\,dy}{x^2 + y^2}$.

There are several methods which are usually given for determining integrating factors in particular classes of equations.

If $$M\,dx + N\,dy = 0. \qquad \ldots(1)$$

is the given equation, which is not exact, then we proceed as follows.

Method (i) : If $Mx + Ny \neq 0$ and the equation is homogeneous, then $\dfrac{1}{Mx + Ny}$ is an integrating factor of (1).

Method (ii) : If equation (1) is of the form

$$f_1(x, y)\,y\,dx + f_2(x, y)\,xdy = 0.$$

Then $\dfrac{1}{Mx - Ny}$ is an integrating factor, provide $Mx - Ny \neq 0$.

Method (iii) : If $\dfrac{1}{N}\left[\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}\right]$ is a function of x alone say $f(x)$ or a constant c, then

$$\text{I.F.} = e^{\int f(x)\,dx} \text{ or } e^{\int C dx}.$$

Method (iv) : If $\dfrac{1}{M}\left[\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}\right]$ is a function of y alone say $f(y)$ or a constant, say c, then

$$\text{I.F.} = e^{\int f(y)\,dy} \text{ or } e^{\int C dy}.$$

Method (v) : If an equation is of the form

$$x^a y^b\,[my\,dx + nx\,dy] + x^r y^s\,(py\,dx + qx\,dy) = 0$$

where a, b, m, n, r, s, p and q are all constants, then the integrating factor will be $x^h y^k$, where h and k are such that after multiplying by the integrating factor that the condition of exactness is satisfied.

SOLVED EXAMPLES

Example 1. *Solve* $(y^2e^x + 2xy)\,dx - x^2\,dy = 0.$

Solution. The given equation is

$$(y^2e^x + 2xy)\,dx - x^2\,dy = 0.$$

Compare with $Mdx + Ndy = 0$

We get $M = y^2e^x + 2xy$ and $N = -x^2$

$$\frac{\partial M}{\partial y} = 2ye^x, \quad \frac{\partial N}{\partial x} = -2x$$

$$\Rightarrow \quad \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$\Rightarrow$ The given equation is not exact.

Now, if in the equation e^x is multiplied by some other function, then it must occur twice in the differential equation. But since it is occurring only once, therefore we should divide by y^2.

$$\therefore \quad e^x\,dx + \frac{2xy\,dx - x^2dy}{y^2} = 0 \quad \text{or} \quad \left(e^x + \frac{2x}{y}\right)dx + \left(\frac{-x^2}{y^2}\right)dy = 0$$

Now $\quad M = e^x + \dfrac{2x}{y}$ and $N = -\dfrac{x^2}{y^2}$

$$\Rightarrow \quad \frac{\partial M}{\partial y} = -\frac{2x}{y^2} = \frac{\partial N}{\partial x} \qquad \Rightarrow \text{Equation is exact.}$$

Hence, the solution is

$$\int_{y \text{ is constant}} M\,.\,dx + \int_{\text{only those terms does not contain } x} N\,.\,dy = c$$

$$\int\left(e^x + \frac{2x}{y}\right)dx = c \quad \Rightarrow \quad e^x + \frac{x^2}{y} = c.$$

Example 2. *Solve* $(x^2y - 2xy^2)\,dx - (x^3 - 3x^2y)\,dy = 0.$

Solution. The given equation is

$$(x^2y - 2xy^2)\,dx - (x^3 - 3x^2y)\,dy = 0.$$

Compare with $Mdx + Ndy = 0$

We get $\quad M = x^2y - 2xy^2$ and $N = 3x^2y - x^3$

$$\Rightarrow \quad \frac{\partial M}{\partial y} = x^2 - 4xy \quad \text{and} \quad \frac{\partial N}{\partial x} = 6xy - 3x^2$$

$$\Rightarrow \quad \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$\Rightarrow$ The given equation is not exact.

Here, we observe that M and N are the homogeneous functions of x and y and

$$\frac{1}{Mx + Ny} = \frac{1}{x^2y^2} \neq 0 \text{ therefore, I.F.} = \frac{1}{x^2y^2}.$$

Now, multiplying the given equation by I.F., we get

$$\left(\frac{1}{y} - \frac{2}{x}\right)dx + \left(\frac{3}{y} - \frac{x}{y^2}\right)dy = 0.$$

In this equation

$$M = \frac{1}{y} - \frac{2}{x} \quad \text{and} \quad N = \frac{3}{y} - \frac{x}{y^2}$$

$$\frac{\partial M}{\partial y} = -\frac{1}{y^2} = \frac{\partial N}{\partial x}$$

$\Rightarrow$ This equation is exact.

The solution is $\dfrac{x}{y} + \log\dfrac{y^3}{x^2} = c.$

• SUMMARY

- **Differential equation :** An equation involving one dependent variable and its derivatives with respect to one or more independent variables, is called a different equation.

  For example : $x\frac{dy}{dx}+y=0$

- **Order of *D.E* :** The order of the highest derivative appearing in *D.E* is called the order of *D.E*.
- **Degree of *D.E* :** The degree of the highest derivative appearing in *D.E*, after making *D.E* free from radicals and factors as for as the derivative are concerned, is called the degree of *D.E*.
- **General solution of *D.E* :** If the solution of n^{th} order *D.E* contains n arbitrary constants, then it is called general solution.
- **Particular solution of *D.E* :** The solution of *D.E* obtained from its gneral solution by assigning particular values to the arbitrary constants in it, is called particular solution of *D.E*.
- **Singular solution of *D.E* :** The solution of *D.E*, which can not be obtained from its general solution as obtained particular solution, is called singular solution of *D.E*.

• STUDENT ACTIVITY

1. Find the order and degree of the differential equation

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2}=\left(\frac{d^2y}{dx^2}\right)^{1/3}.$$

2. Solve $x\frac{dy}{dx}=y-x\tan\frac{y}{x}$.

• TEST YOURSELF–9

Solve the following ordinary differential equation :

1. $(1 + xy)\, y\, dx + (1 - xy)\, x\, dy = 0.$
2. $y\, dx - x\, dy + (1 + x^2)\, dx + x^2 \sin y\, dy = 0.$
3. $(x^3y^3 + x^2y^2 + xy + 1)\, y\, dx + (x^3y^3 - x^2y^2 - xy + 1)\, x\, dy = 0.$
4. $(y^2e^x + 2xy)\, dx - x^2\, dy = 0.$
5. $x^2y\, dx - (x^3 + y^3)\, dy = 0.$
6. $x\, dx + y\, dy + (x^2 + y^2)\, dy = 0.$
7. $(x^2 . y^2 + xy + 1)\, y\, dx + (x^2y^2 - xy + 1)\, x\, dy = 0.$
8. $(x^2 + y^2)\, dx - 2xy\, dy = 0.$
9. $(xy \sin xy + \cos xy)\, y\, dx + (xy \sin xy - \cos xy)\, x\, dy = 0.$
10. $(2y\, dx + 3x\, dy) + 2xy\, (3y\, dx + 4x\, dy) = 0.$

ANSWERS

1. $\log (x/y) = c + (1/xy)$ **2.** $\frac{y}{x} + \frac{1}{x} - x + \cos y = c$ **3.** $xy + (-1/xy) - 2 \log y = c$

4. $e^x + \frac{x^2}{y} = c$ **5.** $y = ce^{x^3/3y^3}$ **7.** $x^2 + y^2 = e^{c - 2y}$

8. $xy + \log x - \log y - \frac{1}{xy} = c$ **12.** $x^2 - y^2 = cx$ **9.** $x \sec (xy) = cy$

10. $x^2y^3 + 2x^3y^2 = c$

OBJECTIVE EVALUATION

➤ FILL IN THE BLANKS :

1. An equation, which contains, dependent, independent variables and different derivatives is called ....
2. A differential equation, which involve only one independent variable is called ....
3. A differential equation, which involve two or more independent variables and partial differential coefficients, is called ....

➤ TRUE OR FALSE :

Write 'T' for true and 'F' for false statement :

1. An exact differential equation can be derived from its primitive by direct differentiation. *(T/F)*
2. If μ is an integrating factor then $\mu\, (M\, dx + N\, dy) = 0$ is not necessarily exact. *(T/F)*
3. If the given equation is homogeneous and $Mx + Ny \neq 0$ then $1/Mx + Ny$ is the I.F. of the given equation. *(T/F)*

➤ MULTIPLE CHOICE QUESTIONS :

Choose the most appropriate one :

1. The general solution of a first degree equation contains :
   (a) one arbitrary constant (b) two arbitrary constant
   (c) three arbitrary constant (d) none of these.
2. The equation $(x^2 + y^2)\, \frac{dy}{dx} = xy$ is :
   (a) homogeneous (b) linear
   (c) reducible to linear (d) none of these.
3. The number of integrating factor for a differential equation of the type $M\, dx + N\, dy = 0$ is :
   (a) one (b) finite (c) infinite (d) none of these.

ANSWERS

Fill in the Blanks :

1. differential equation 2. *ODE* 3. *PDE*

True or False :

1. T 2. F 3. T

Multiple Choice Questions :

1. (a) 2. (a) 3. (c) 4. (b)

UNIT 12

DIFFERENTIAL EQUATION OF FIRST ORDER AND OF HIGHER DEGREE

STRUCTURE

LEARNING OBJECTIVES

After going through this unit you will learn :

How to determine the solution of differential equation of order one and degree more than one

In this chapter we shall deal with some special types of differential equations of the first order and of degree higher than the one.

Definition. A differential equation of the form

$$p^n + A_1 p^{n-1} + A_2 p^{n-2} + A_3 p^{n-3} + \ldots + A_n = 0$$

is called a differential equation of the first order and nth degree, where p is a symbol for $\frac{dy}{dx}$.

Some special types of equations are

(a) Solvable for p. **(b) Solvable for y.** **(c) Solvable for x.**

(A) EQUATIONS SOLVABLE FOR p

Consider, the differential equation

$$p^n + A_1 p^{n-1} + A_2 p^{n-2} + \ldots + A_{n-1} p + A_n = 0 \quad \ldots(1)$$

where $A_1, A_2 \ldots A_n$ are functions of x and y.

Working Procedure :

(*i*) *Put p for* $\frac{dy}{dx}$.

(*ii*) *Solve the equation for p.*

(*iii*) *Apply the method of solving the equation of first order and first degree.*

(*iv*) *Write down the result in closed brackets in the form of product and put this resulting equations equal to zero, by replacing $c_1, c_2, \ldots c_n$ by c.*

SOLVED EXAMPLES

Example 1. *Solve* $p^2 - 5p + 6 = 0$.

Solution. The given equation is

$$p^2 - 5p + 6 = 0 \quad \ldots(1)$$

The linear factors of (1) are given by

$$(p-2)(p-3)=0 \Rightarrow p=2,\ p=3$$

$$p=2 \Rightarrow \frac{dy}{dx}=2 \Rightarrow dy=2dx.$$

On integrating, we get $y=2x+c$.
Similarly $p=3$ gives $y=3x+c$.
Hence, the single combined solution of (1) is given by

$$(y-2x-c)(y-3x-c)=0.$$

Example 2. *Solve* $xyp^2-(x^2+y^2)p+xy=0$.

Solution. Here, the given differential equation is

$$xyp^2-(x^2+y^2)p+xy=0 \qquad \ldots(1)$$

which is the quadratic equation in p.

Solving (1) for p, we get

$$p=\frac{(x^2+y^2)\pm\sqrt{[(x^2+y^2)^2-4x^2y^2]}}{2xy}=\frac{(x^2+y^2)\pm(x^2-y^2)}{2xy}$$

so that
$$p=\frac{(x^2+y^2)+(x^2-y^2)}{2xy}=\frac{x}{y} \qquad \ldots(2)$$

and
$$p=\frac{(x^2+y^2)-(x^2-y^2)}{2xy}=\frac{y}{x} \qquad \ldots(3)$$

Now from (2), we get

$$p=\frac{dy}{dx}=\frac{x}{y} \Rightarrow xdx=ydy$$

On integrating, we get

$$x^2-y^2=c. \qquad \ldots(4)$$

Now from (3), we get

$$p=\frac{dy}{dx}=\frac{y}{x} \Rightarrow \frac{dy}{y}=\frac{dx}{x}$$

On integrating, we get $\log y-\log x=\log c$

$$\Rightarrow \qquad \frac{y}{x}=c. \qquad \ldots(5)$$

Hence, from (3) and (5), the single solution of the given differential equation is

$$(x^2-y^2-c)(y-cx)=0.$$

• TEST YOURSELF–1

Solve the following differential equations :

1. (a) $p^2-7p+12=0$ (b) $p^2-9p+18=0$.
2. $(x-y)^2p^2-3y(x-y)p+2y^2+xy-x^2=0$. 3. $(p-xy)(p-x^2)(p-y^2)=0$.

ANSWERS

1. (a) $(y-4x+c)(y-3x+c)=0$ (b) $(y-3x+c)(y-6x+c)=0$.
2. $\tan^{-1}\left(\frac{y}{x}\right)-\frac{1}{2}\log\left[1+\left(\frac{y}{x}\right)^2\right]=\log x+\log c$ 3. $(y-ce^{x^2/2})(3y-x^3-c)(xy+cy+1)=0$

• 12.2. (B) EQUATION SOLVABLE FOR y

Let the given differential equation is

$$f(x,y,p)=0 \qquad \ldots(1)$$

Solving for y, the equation (1) gives

$$y=F(x,p) \qquad \ldots(2)$$

Differentiating (2), with respect to x, we get an equation of the type

$$p=g\left(x,p,\frac{dp}{dx}\right). \qquad \ldots(3)$$

Equation (3), being an equation in two variables x and p, may possibly be solved. Let its solution is given by

$$\phi(x, p, c) = 0 \qquad \ldots(4)$$

The elimination of p between equations (2) and (4) gives a relation involving x, y and c, which is the required solution.

Working Procedure :

(*i*) *Solve the given equation for y and put it as F(x, p) only.*

(*ii*) *Differentiate it with respect to x.*

(*iii*) *Put p for* $\frac{dy}{dx}$ *and use the previous method.*

(*iv*) *Now eliminate p between* (1) *and* (3) *and get the required general solution.*

SOLVED EXAMPLES

Example 1. *Solve* $y + px = p^2x^4$.

Solution. Here, the given equation is

$$y + px = p^2x^4 \qquad \ldots(1)$$

Differentiating (1), w.r.t. x, we get

$$\frac{dy}{dx} = -p - x\frac{dp}{dx} + 2px^4\frac{dp}{dx} + 4p^2x^3.$$

Put $\frac{dy}{dx} = p$, we get

$$p = -p - x\frac{dp}{dx} + 2px^4\frac{dp}{dx} + 4p^2x^3$$

$$\Rightarrow \quad 2p - 4p^2x^3 + x\frac{dp}{dx}(1 - 2x^3p) = 0$$

$$\Rightarrow \quad 2p(1 - 2x^3p) + x\frac{dp}{dx}(1 - 2x^3p) = 0$$

$$\Rightarrow \quad (1 - 2x^3p)\left(2p + x\frac{dp}{dx}\right) = 0$$

$$\therefore \quad 2p + x\frac{dp}{dx} = 0 \text{ and } 1 - 2x^3p = 0.$$

Now, we solve these two equations, separately.

First consider

$$2p + x\frac{dp}{dx} = 0 \Rightarrow \frac{dp}{dx} = \frac{-2p}{x}.$$

On separating the variables, we get

$$\frac{dp}{p} = \frac{-2dx}{x}.$$

On integrating, we get

$$\log p = -2\log x + \log c$$

$$\Rightarrow \quad \log(px^2) = \log c \Rightarrow px^2 = c \quad \text{or} \quad p = \frac{c}{x^2}.$$

Put this value in eq. (1), we get

$$y = -x\frac{c}{x^2} + \frac{c^2}{x^4}x^4 \Rightarrow -y = -\left(\frac{c}{x}\right) + c^2$$

or $\quad xy = c^2x - c$, which is the required solution of (1).

Example 2. *Solve* $y - x = x\frac{dy}{dx} + \left(\frac{dy}{dx}\right)^2$.

Solution. Here, the given differential can be written as

$$y - x = xp + p^2, \quad \text{where } p = \frac{dy}{dx}$$

$$\Rightarrow \quad y = x(1 + p) + p^2. \qquad \ldots(1)$$

Differentiating (1), with respect to x, we get

$$\frac{dy}{dx} = p = 1 + p + x\frac{dp}{dx} + 2p\frac{dp}{dx}$$

or $$1 + x\frac{dp}{dx} + 2p\frac{dp}{dx} = 0 \quad \text{or} \quad \frac{dx}{dp} + x = -2p \qquad \ldots(2)$$

which is a linear differential equation of order one, with p as independent variable.

$$\text{I.F.} = e^{\int 1 \cdot dp} = e^p$$

Hence, the solution is

$$xe^p = \int -2pe^p \, dp + c \;\Rightarrow\; xe^p = -2(pe^p - e^p) + c$$

$$\Rightarrow \quad xe^p = -2e^p(p-1) + c \;\Rightarrow\; x = -2(p-1) + ce^{-p} \qquad \ldots(3)$$

Putting this value of x in equation (1), we get

$$y = (1+p)[-2(p-1) + ce^{-p}] + p^2$$

or $$y = c(1+p)e^{-p} + (2 - p^2) \qquad \ldots(4)$$

Hence, the equations (3) and (4) given the required solution of (1).

• TEST YOURSELF–2

Solve the following differential equations :

1. $y = 4p^3 + 3xp$. **2.** $yp = x^2 + p^2 x$. **3.** $y = x[p + \sqrt{(1+p^2)}]$.

4. $y = xp^2 + p^3$. **5.** $y = 3x + \log p$.

ANSWERS

1. $x = -\left(\frac{12}{7}\right)p^2 + cp^{-3/2}$ $\quad y = -\left(\frac{8}{7}\right)p^3 + 3cp^{-1/2}$

2. $x = -\frac{1}{3}p^3 + cp^{1/2}$ $\quad y = \dfrac{\left(-\frac{1}{3}p^2 - cp^{1/2}\right)^2}{p} + p^2\left(-\frac{1}{3}p^2 + cp^{1/2}\right)$

3. $x^2 + y^2 - 2cx = 0$

4. $x = \frac{1}{2}\frac{(2c + 3p^2 - 2p^3)}{(p-1)^2}$ $\quad y = \frac{2cp^2 + 2p^3 - p^4}{2(p-1)^2}$

5. $y = 3x + \log\frac{3}{1 - c\,e^{3x}}$

• 12.3. (C) EQUATION SOLVABLE FOR x

Suppose the given differential equation is

$$p^n + A_1 p^{n-1} + A_2 p^{n-2} + \ldots + A_{n-1} p + A_n = 0.$$

Let us suppose, it can be expressed in the form

$$x = f(y, p) \qquad \ldots(1)$$

Differentiating (1) with respect to y, we have

$$\frac{1}{p} = \frac{dx}{dy} = g\left(y, p, \frac{dp}{dy}\right) \qquad \ldots(2)$$

which is a new differential equation with variable y and p.

Let the general solution of (2) is of the form

$$\phi(x, p, c) = 0. \qquad \ldots(3)$$

Eliminating p between (1) and (3) we get, either

$$F(x, y, c) = 0$$

as the required solution or in parametric form.

SOLVED EXAMPLES

Example 1. *Solve* $y = 2px + p^2 y$.

Solution. Here, the given equation is

$$y = 2px + p^2 y.$$

Solving, it for x, we get

$$2x = -py + y/p. \qquad \ldots(1)$$

Differentiating (1) with respect to y, we get

$$\frac{2}{p} = -p - y\frac{dp}{dy} + \frac{1}{p} - \frac{y}{p^2}\frac{dp}{dy}$$

or $$p + \frac{1}{p} = -y\left(\frac{dp}{dy}\right)\left(1 + \frac{1}{p^2}\right)$$

or $$p\left(1 + \frac{1}{p^2}\right) + y\frac{dp}{dy}\left(1 + \frac{1}{p^2}\right) = 0$$

or $$\left(1 + \frac{1}{p^2}\right)\left[p + y\frac{dp}{dy}\right] = 0.$$

Now, since the first factor does not involve a derivative of p with respect to x or y, therefore it will be omitted (such factor always give the singular solution).

Now consider $\quad p + y\dfrac{dp}{dy} = 0.$

On separating the variables, we get

$$\frac{dp}{p} + \frac{dy}{y} = 0.$$

Integrating $\quad \log p + \log y = \log c \quad$ or $\quad py = c.$ ...(2)

To eliminate p between (1) and (2), solve (2) for p, which gives $p = \dfrac{c}{y}$. Putting this value of p in (1), we get

$$2x = -c + y^2/c \quad \Rightarrow \quad 2xc - y^2 + c^2 = 0.$$

Example 2. *Solve* $x = y + p^2$.

Solution. Differentiating the given differential equation with respect to y, we get

$$\frac{dx}{dy} = \frac{1}{p} = 1 + 2p\frac{dp}{dy}$$

or $$dy = \frac{2p^2}{1-p}\,dp = -2\left(p + 1 + \frac{1}{p-1}\right)dp.$$

Integrating, we have

$$y = -2\left[\frac{p^2}{2} + p + \log(p-1)\right] + c$$

or $$y = c - [\,p^2 + 2p + 2\log(p-1)].$$ ...(1)

Putting this value of y in the given equation, we get

$$x = c - [2p + 2\log(p-1)].$$ ...(2)

Relations (1) and (2) gives the required solution.

• TEST YOURSELF–3

Solve the following differential equations :

1. $p^3 - 4pxy + 8y^2 = 0.$ **2.** $y = 2px + p^2y.$ **3.** $x = py + ap^2.$

4. $4(xp^2 + py) = y^4.$ **5.** $x + p/\sqrt{1+p^2} = 0.$

ANSWERS

1. $y = c(x-c)^2$ **2.** $y^2 - 2cx - c^2 = 0$

3. $y = \dfrac{c - a\cosh^{-1}p}{\sqrt{(p^2-1)}} - ap, \quad x = \dfrac{p\,[c - a\cosh^{-1}p]}{\sqrt{(p^2-1)}}$

4. $y = 4c(xyc + 1)$ **5.** $(x-a)^2 + (y+b)^2 = 1$

• 12.4. LAGRANGE'S EQUATION

The differential equation of the form

$$y = x\,F(p) + f(p)$$ ...(1)

*is known as **Lagrange's equation.***

Method of Solution :

Differentiating the equation (1) w.r.t. x, we get

$$p = F(p) + xF'(p)\frac{dp}{dx} + f'(p)\frac{dp}{dx}$$

$$\Rightarrow \quad p - F(p) = [xF'(p) + f'(p)]\frac{dp}{dx} \quad \text{or} \quad \frac{dx}{dp} = \frac{xF'(p) + f'(p)}{p - F(p)}$$

which is linear in x and p and can be solved by the usual method.

• 12.5. CLAIRAUT'S EQUATION

The differential equation of the form

$$y = px + f(p)$$

is known as Clairaut's equation.

Method of Solution :

Here, the given differential equation is

$$y = px + f(p). \qquad \ldots(1)$$

Differentiating (1) with respect to x, we get

$$\frac{dy}{dx} = p = p + x\frac{dp}{dx} + f'(p)\frac{dp}{dx}$$

$$\Rightarrow \quad \frac{dp}{dx}[x + f'(p)] = 0$$

which gives either $\frac{dp}{dx} = 0$ or $x + f'(p) = 0$

$$\frac{dp}{dx} = 0 \Rightarrow p = c \qquad \ldots(2)$$

Now, the elimination of p between (1) and (2) gives

$y = cx + f(c)$, where c is any arbitrary constant.

SOLVED EXAMPLES

Q-8

Example 1. *Solve* $y = apx + bp^3$.

Solution. Here, the given differential equation is

$$y = apx + bp^3. \qquad \ldots(1)$$

Differentiating (1) with respect to x, we get

$$\frac{dy}{dx} = p = ap + ax\frac{dp}{dx} + 3bp^2\frac{dp}{dx}$$

or $$p(1 - a) = (ax + 3bp^2)\frac{dp}{dx}$$

or $$\frac{dx}{dp} = \frac{ax + 3bp^2}{(1-a)p} \Rightarrow \frac{dx}{dp} = \frac{a}{(1-a)p}\cdot x + \frac{3bp}{(1-a)}$$

which is linear in x and p.

$$\text{I.F.} = e^{\int\left(\frac{a\,dp}{(a-1)p}\right)} = e^{\frac{a}{(a-1)}\log p} = p^{\frac{a}{a-1}}.$$

Hence, the required solution is

$$xp^{\frac{a}{a-1}} = \frac{3b}{1-a}\int p^{\frac{a}{a-1}}p\,dp + c = \frac{3b}{1-a}\int p^{\frac{2a-1}{a-1}}\,dp + c$$

or $$xp^{\frac{a}{a-1}} = \frac{3b}{(1-a)}\cdot\frac{p^{\{(3a-2)/(a-1)\}}}{(3a-2)/(a-1)} + c$$

or $$x = (3bp^2)/(2-3a) + cp^{a/1-a}. \qquad \ldots(2)$$

End

The required solution is given by (1) and (2) in parametric form, being p as a parameter.

Example 2. *Solve* $y = px + \frac{a}{p}$.

Solution. The given differential equation is in Clairaut's form

Put $p = c$, we get

$$y = cx + \frac{a}{c},$$

which is the required general solution of the given equation.

Example 3. *Solve* $(y - px)(p - 1) = p$.

Solution. Here, the given equation is

$$(y - px)(p - 1) = p \qquad \ldots(1)$$

which can be rewritten as

$$y = px + \frac{p}{p-1}$$

which is of Clairaut's form. Hence, replacing p by c, the required general solution is

$$y = cx + \frac{c}{c-1}, \text{ where } c \text{ is any arbitrary constant.}$$

• TEST YOURSELF–4

Solve the following differential equations :

1. $y = px + ap\,(1-p)$. **2.** $y = px + p^2$.

3. $p^2 x = py - 1$. **4.** $p = \log\,(px - y)$.

5. $xp^3 - (y+3)\,p^2 + 4 = 0$.

6. $\sin px \cos y = \cos px \sin y + p$. **7.** $(y - px)^2/(1+p)^2 = a^2$.

ANSWERS

1. $y = cx + ac\,(1-c)$ **2.** $y = cx + c^2$ **3.** $y = cx + \dfrac{1}{c}$

4. $y = cx - e^c$ **5.** $y = cx + \dfrac{4}{c^2} - 3$ **6.** $y = cx - \sin^{-1} c$

7. $(y - cx)^2/(1+c)^2 = a^2$

• SUMMARY

- **Soluble for p :**
  (i) Put p for $\dfrac{dy}{dx}$ in *D.E.*
  (ii) Solve for p.
  (iii) Apply the method of solving the equation of first order and of first degree.
  (iv) Write down the result in closed brackets in the form of product and put this resulting equations equal to zero, by replacing the constants $c_1, c_2, \ldots c_4$ by c.
- **Soluble for y :**

  Given *D.E* is $\quad f(x, y, p) = 0 \quad \ldots(1)$

  Solving for y, we get

  $$y = F(x, p) \quad \ldots(2)$$

  After differentiating eq. (2) w.r.t. x, we get the equation of the form

  $$p = g\left(x, p, \frac{dp}{dx}\right) \quad \ldots(3)$$

  The solution of eq. (3) is given by

  $$\phi(x, p, c) = 0 \quad \ldots(4)$$

  Now eliminating p between eq. (2) and (4), we get the required solution.
- **Soluble for x :**

  Given *D.E* is $\quad f(x, y, p) = 0 \quad \ldots(1)$

  Solving for x, we get

  $$x = F(y, p) \quad \ldots(2)$$

  Differentiating eq. (2) w.r.t. y, we get the equation of the form

  $$\frac{1}{p} = g\left(y, p, \frac{dp}{dy}\right) \quad \ldots(3)$$

  The solution of eq. (3) is given by

  $$\phi(x, p, c) = 0 \quad \ldots(4)$$

  Eleminating p between eq. (2) and (4), we get the required solution.

• STUDENT ACTIVITY

1. Solve $p(p-y)=x(x+y)$.

2. Solve $y=2px+p^2y$.

• TEST YOURSELF–5

Find the singular solution of the following differential equations :

1. $y=px+a\sqrt{(1+p^2)}$.
2. $p=\log(px-y)$.
3. $y^2-2pxy+p^2(x^2-1)=m^2$.
4. $4p^2x(x-a)(x-b)=\{3x^2-2x(a+b)+ab\}^2$.
5. $x^2+y^2+2cx+2c^2-1=0$.
6. $p^2+y^2=1$.
7. $3p^2e^y-px+1=0$.
8. $y^2(1+4p^2)-2pxy-1=0$.

ANSWERS

1. $x^2+y^2=a^2$.
2. $x+y-x\log x=0$.
3. $y^2+m^2x^2=m^2$.
4. $x=0, x-a=0$ and $x-b=0$.
5. $x^2+2y^2=2$.
6. $y=\pm 1$.
7. $x^2-12e^{3y}=0$.
8. $x^2-4y^2+4=0$.

OBJECTIVE EVALUATION

➤ FILL IN THE BLANKS :

1. The differential equation $p^n+P_1p^{n-1}+\ldots+P_n=0$, where $P_1, P_2 \ldots P_n$ are functions of x and y, is said to be the differential equation of first order and .... degree.
2. The general solution of a nth order differential equation contains .... independent arbitrary constant.

➤ TRUE OR FALSE :

Write T for True and F for False statement :

1. The Lagrange's equation is a particular case of Clairaut's equation. *(T/F)*
2. The Clairaut's equation is a particular case of Lagrange's equation. *(T/F)*
3. The particular solution can be obtained by general solution. *(T/F)* *(T/F)*

➤ MULTIPLE CHOICE QUESTIONS :

Choose the most appropriate one :

1. A solution, which can not be obtained from general solution by assigning some particular value of arbitrary constant is known as :
   (a) Particular solution (b) Singular solution
   (c) Integrating factor (d) None of these.
2. A differential equation, in which both p and c-discriminant relations are same, it is said to be:
   (a) Legrange's equation (b) Equation solvable for p
   (c) Clairaut's equation (d) None of these.
3. A solution of a differential equation, which is given by the envelope of the family of curves represented by that differential equation is said to be :
   (a) Particular solution (b) Singular solution
   (c) General solution (d) None of these.
4. To find the singular solution of a differential equation, we use :
   (a) p-discriminant relation (b) c-discriminant relation
   (c) both (a) and (b) (d) None of these.

ANSWERS

Fill in the Blanks :

1. n 2. n

True or False :

1. F 2. T 3. T

Multiple Choice Questions :

1. (b) 2. (c)

❑❑❑

UNIT

13

LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

STRUCTURE

LEARNING OBJECTIVES

After going through this unit you will learn :

- How to determine the solution of differential equation with constant coefficients.
- About complementary function and finding particular integral of the given differential equation

• 13.1. LINEAR DIFFERENTIAL EQUATION

The equation $$\frac{d^n y}{dx^n} + A_1 \frac{d^{n-1} y}{dx^{n-1}} + A_2 \frac{d^{n-2} y}{dx^{n-2}} + \ldots + A_n y = B \qquad \ldots(1)$$

having $A_1, \ldots, A_n$ and B either constant or function of x, is called the linear differential equation of the n^{th} order.

If $A_0, A_1, \ldots, A_n$ are all constants and B may not be constant then equation (1) is said to be linear differential equation of n^{th} degree with constant coefficients.

If we take $B = 0$, then the corresponding equation is called **homogeneous equation.**

Using the symbols $D, D^2, \ldots D^n$ for $\frac{d}{dx}, \frac{d^2}{dx^2}, \ldots \frac{d^n}{dx^n}$ respectively in (1), then we get

$$D^n y + A_1 D^{n-1} y + A_2 D^{n-2} y + \ldots + A_n y = B$$

$$\Rightarrow \quad (D^n + A_1 D^{n-1} + A_2 D^{n-2} + \ldots + A_n) y = B \Rightarrow f(D) y = B \qquad \ldots(2)$$

where $\quad f(D) = D^n + A_1 D^{n-1} + A_2 D^{n-2} + \ldots + A_n.$

Now, consider the homogeneous differential equation

$$f(D) y = 0 \qquad \ldots(3)$$

(obtained by putting right hand side *i.e.*, B equal to zero).

Working Procedure :

(*i*) *Firstly, we find the general solution of* (2), *which is called the complimentry function* (*C.F.*), *contains as many arbitrary constants as is the order of the given differential equation.*

(ii) *Next, find the solution of* (1), *with no arbitrary constant which is called the particular integral (P.I.).*

(iii) *To find the general solution of* (1), *and C.F. & P.I., obtained in* (1) *and* (2).

i.e. $y = u + v = \text{C.F.} + \text{P.I.}$

Auxiliary Equation. Consider the differential equation (1) with $B = 0$ *i.e.*,

$$(D^n + A_1 D^{n-1} + A_2 D^{n-2} + \dots + A_n)\, y = 0 \quad \text{or} \quad f(D)\, y = 0. \qquad \dots(1)$$

Substitute $y = e^{mx}$ on a trial basis, then we get

$$e^{mx}(m^n + A_1 m^{n-1} + A_2 m^{n-2} + \dots + A_n) = 0$$

which holds if

$$m^n + A_1 m^{n-1} + A_2 m^{n-2} + \dots + A_n = 0 \quad \text{or} \quad f(m) \equiv 0. \qquad \dots(2)$$

Equation (2) is called the **auxiliary equation.**

From (1) and (2), we observe that the auxiliary equation $f(m) = 0$ will give the same value of m as the equation $f(D) = 0$ gives of D.

• 13.2. METHODS OF FINDING C.F.

To find the C.F., the roots of the auxiliary equation (2) are to be considered.

Three different cases arises :

(i) The roots of auxiliary equation (2) are real.

(ii) The roots of auxiliary equation (2) are complex *i.e.*, $\alpha + i\,\beta$ type.

(iii) The roots of auxiliary equation (2) are surds *i.e.*, $\alpha \pm \sqrt{\beta}$ type.

Case (i) : (a) Suppose that the auxiliary equation (2) has n distinct roots $m_1, m_2, \dots m_n$, then C.F. is given by

$$C_1 e^{m_1 x} + C_2 e^{m_2 x} + \dots + C_n e^{m_n x}$$

where $C_1, C_2, \dots, C_n$ are arbitrary constants.

(b) If the auxiliary equation having r roots are equal to m_1 (say) and remaining roots are distinct, then the C.F. is given by

$$[C_1 + C_2 x + C_3 x^3 + \dots + C_r x^{r-1}]\, e^{m_1 x} + C_{r+1} e^{m_{r+1} x} + \dots + C_n e^{m_n x}.$$

Case (ii) : If some of roots of the auxiliary equation are complex, then we shall follow the given procedure.

Let $\alpha \pm i\beta$ be the roots of the auxiliary equation, then the corresponding part becomes

$$= C_1 e^{(\alpha + i\beta)x} + C_2 e^{(\alpha - i\beta)x} = C_1 e^{\alpha x} . e^{i\beta x} + C_2 e^{\alpha x} . e^{-i\beta x}$$

$$= e^{\alpha x}[C_1 \cos \beta x + iC_1 \sin \beta x] + e^{\alpha x}[C_2 \cos \beta x - iC_2 \sin \beta x]$$

$$= e^{\alpha x}[(C_1 + C_2) \cos \beta x + (iC_1 - iC_2) \sin \beta x]$$

$$\textbf{C.F.} = e^{\alpha x}\,[B_1 \cos \beta x + B_2 \sin \beta x] \qquad \dots(1)$$

where B_1, B_2 are arbitrary constants.

The expression (1) can also be written as

(a) $B_1 e^{\alpha x} \cos(\beta x + B_2)$

(b) $B_1 e^{\alpha x} \sin(\beta x + B_2)$.

If, the equation has two equal pair of complex roots $\alpha + i\beta$ and $\alpha - i\beta$, say, occur twice, then the corresponding part of C.F. is written as

$$e^{\alpha x}\,[(B_1 + B_2 x) \cos \beta x + (B_3 + B_4 x) \sin \beta x].$$

In general, if $\alpha \pm i\,\beta$ occur K times, then the corresponding part of the C.F. can be written as

$$e^{\alpha x}\,\{(B_1 + B_2 x + \dots + B_K x^{K-1}) \cos \beta x + (B_{K+1} + B_{K+2} x + \dots + B_{2K} x^{K-1})\} \sin \beta x$$

where $B_1, B_2, \dots, B_K, B_{K+1}, \dots, B_{2K}$ are arbitrary constants.

Case (iii) : If a pair of the roots of the auxiliary equation involves surds, say $\alpha \pm \sqrt{\beta}$, where $\beta > 0$, then the corresponding part of C.F. in one of the following three forms

(a) $e^{\alpha x}\,[B_1 \cosh(x\sqrt{\beta}) + B_2 \sinh(x\sqrt{\beta})]$

(b) $B_1 e^{\alpha x} \cosh(x\sqrt{\beta} + B_2)$

(c) $B_1 e^{\alpha x} \sinh(x\sqrt{\beta} + B_2)$.

SOLVED EXAMPLES

Example 1. *Solve* $[D^3 + 6D^2 + 11D + 6]\, y = 0$.

Solution. Here, the given differential equation is

$$[D^3 + 6D^2 + 11D + 6]\, y = 0 \qquad \dots(1)$$

To find the auxiliary equation, replace D by m, then (1) becomes,

$$m^3 + 6m^2 + 11\,m + 6 = 0$$

$$\Rightarrow \quad (m+1)(m^2+5m+6) = 0$$

$$\Rightarrow \quad (m+1)(m+2)(m+3) = 0 \quad \Rightarrow \quad m = -1, -2, -3$$

i.e., Roots are real and unequal. Hence, the general solution is

$$y = C_1\,e^{-x} + C_2\,e^{-2x} + C_3\,e^{-3x}.$$

Example 2. *Solve* $[D^4 + 2D^3 - 3D^2 - 4D + 4]\,y = 0.$

Solution. Here, the auxiliary equation is

$$m^4 + 2m^3 - 3m^2 - 4m + 4 = 0$$

or $$(m-1)(m^3+3m^2-4) = 0$$

$$\Rightarrow \quad (m-1)(m-1)(m^2+4m+4) = 0$$

$$\Rightarrow \quad (m-1)(m-1)(m+2)^2 = 0$$

$$\Rightarrow \quad m = +1, +1, -2, -2$$

$\Rightarrow$ Repeated real roots exist.

Hence, general solution is

$$y_1 = (C_1 + C_2\,x)\,e^x + (C_3 + C_4\,x)\,e^{-2x}.$$

• TEST YOURSELF–1

Solve the following equations :

1. $\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} + 2y = 0.$
2. $(D^3 - 9D^2 + 23D - 15)\,y = 0.$
3. $(D^4 - D^3 - 9D^2 - 11D - 4)\,y = 0.$
4. $(D^2+1)^2\,(D-1)^2\,y = 0.$
5. $(D^3 - D^2 - 12D)\,y = 0.$
6. $(D^4 + 2n^2D^2 + n^4)\,y = 0.$

ANSWERS

1. $y = C_1\,e^{-x} + C_2\,e^{-2x}$
2. $y = C_1\,e^x + C_2\,e^{3x} + C_3\,e^{5x}$
3. $y = e^{-x}\,(C_1 + C_2\,x + C_3\,x^2) + C_4\,e^{4x}$
4. $y = (C_1 + C_2\,x)\sin x + (C_3 + C_4\,x)\cos x + (C_5 + C_6\,x)\,e^x$
5. $y = C_1 + C_2\,e^{4x} + C_3\,e^{-3x}$
6. $y = (C_1 + C_2\,x)\cos nx + (C_3 + C_4\,x)\sin nx$

Particular Integral :

Consider the differential equation

$$f(D)\,y = B \quad \Rightarrow \quad y = \frac{1}{f(D)}\,.\,B.$$

Let $\dfrac{1}{f(D)}\,.\,B$ denote some function of x, which when operated upon by $f(D)$ produces B.

Hence, $\quad \mathbf{P.I.} = \dfrac{1}{f(D)}.\,B.$

• 13.3. GENERAL METHOD OF FINDING P.I.

Result. *If B is a function of x, then*

$$\frac{1}{D-a}\,B = e^{ax}\int B\,e^{-ax}\,dx.$$

SOLVED EXAMPLES

Example 1. *Solve* $D^2 - 5D + 6 = e^{3x}.$

Solution. The given equation can be written as

$$(D-3)(D-2)\,y = e^{3x}$$

$$\text{C.F.} = C_1\,e^{3x} + C_2\,e^{2x}$$

and $$\text{P.I.} = \frac{1}{D-3}\cdot\frac{1}{D-2}\,e^{3x} = \frac{1}{D-3}\,e^{2x}\int e^{3x}\,e^{-2x}\,dx$$

$$= \frac{1}{D-3}\,e^{2x}\,.\,e^x = e^{3x}\int e^{3x}\,.\,e^{-3x}\,dx = xe^{3x}.$$

Now, General solution = C.F. + P.I.

$$\Rightarrow \qquad y = C_1 e^{3x} + C_2 e^{2x} + x e^{3x}.$$

Example 2. *Solve* $(D^2 + 1)\, y = sec^2 x.$

Solution. Here, the given equation is

$$(D^2 + 1)\, y = \sec^2 x \qquad \ldots(1)$$

To find the C.F. of (1).

The auxiliary equation of (1), we get

$$m^2 + 1 = 0 \quad \Rightarrow \quad m = \pm i$$

$$\Rightarrow \qquad \text{C.F.} = C_1 \cos x + C_2 \sin x$$

$$\text{P.I.} = \frac{1}{D^2+1} \sec^2 x = \frac{1}{(D+i)(D-i)} \sec^2 x = \frac{1}{2i}\left[\frac{1}{D-i} - \frac{1}{D+i}\right] \sec^2 x$$

$$= \frac{1}{2i}\left[e^{xi} \int e^{-ix} \sec^2 x dx - e^{-ix} \int e^{ix} \sec^2 x dx\right]$$

$$= \frac{1}{2i}\left\{e^{ix} \int \frac{\cos x - i \sin x}{\cos^2 x} dx - e^{-ix} \int \frac{\cos x + i \sin x}{\cos^2 x} dx\right\}$$

$$= \frac{1}{2i} \{e^{ix} \int (\sec x - i \sec x \tan x)\, dx - e^{-ix} \int (\sec x + i \sec x \tan x)\, dx\}$$

$$= \frac{1}{2i} \{(e^{ix} - e^{-ix}) \int \sec x\, dx - i\, (e^{ix} + e^{-ix}) \int \tan x \sec x\, dx\}$$

$$= \frac{1}{2i} (2i \sin x \log (\sec x + \tan x) - 2i \cos x \sec x)$$

$$= \sin x \log (\sec x + \tan x) - 1.$$

Hence, the general solution is,

$$y = \text{C.F.} + \text{P.I.}$$

$$\Rightarrow \qquad y = C_1 \cos x + C_2 \sin x + \sin x \log (\sec x + \tan x) - 1.$$

• TEST YOURSELF–2

Solve the following differential equation :

1. $(D^2 + a^2)\, y = \sec ax.$ **2.** $(D^2 + a^2)\, y = \tan ax.$ **3.** $(D^2 + 1)\, y = \text{cosec}\, x.$

ANSWERS

1. $y = C_1 \cos ax + C_2 \sin ax + \dfrac{x}{a} \sin ax + \dfrac{1}{a^2} \cos ax \log \cos ax$
2. $y = C_1 \cos ax + C_2 \sin ax - \dfrac{1}{a^2} \cos ax \log \tan \left(\dfrac{\pi}{4} + \dfrac{ax}{2}\right)$
3. $y = C_1 \cos x + C_2 \sin x + \sin x \log \sin x - x \cos x$

• 13.4. SHORT METHODS OF GETTING P.I.

The general method for getting P.I., discussed above requires lot of calculations. In certain cases the P.I. can be obtained by methods which are shorter than the general method.

(1) To evaluate P.I., when B is of the form e^{ax}, we use the following rule :

$$\frac{1}{f(D)} e^{ax} = \frac{e^{ax}}{f(a)}, \quad \textbf{provided } f(a) \neq 0.$$

SOLVED EXAMPLES

Example 1. *Solve* $(D^2 - 3D + 2)\, y = e^{5x}.$

Solution. Here, the given equation is

$$(D^2 - 3D + 2)\, y = e^{5x}.$$

Auxiliary equation is

$$m^2 - 3m + 2 = 0$$

$$\Rightarrow \qquad (m-1)(m-2) = 0 \quad \Rightarrow \quad m = 1, 2.$$

$$\therefore \qquad \text{C.F.} = C_1 e^x + C_2 e^{2x}$$

Now, $$\text{P.I.} = \frac{1}{D^2 - 3D + 2} \cdot e^{5x} = \frac{1}{25 - 3 \times 5 + 2} = \frac{1}{12} e^{5x}.$$

Hence, the general solution is

$$y = \text{C.F.} + \text{P.I.}$$

$$\Rightarrow \qquad y = C_1 e^x + C_2 e^{2x} + \frac{1}{12} \cdot e^{5x}.$$

Example 2. *Solve* $(D^3 + 1)\, y = (e^x + 1)^2$.

Solution. Here, the given equation is

$$(D^3 + 1)\, y = (e^x + 1)^2 \qquad \ldots(1)$$

The auxiliary equation is

$$m^3 + 1 = 0$$

$$\Rightarrow \qquad (m + 1)(m^2 - m + 1) = 0$$

$$\Rightarrow \qquad m = -1, \frac{1}{2} \pm \frac{i\sqrt{3}}{2}$$

therefore $$\text{C.F.} = C_1 e^{-x} + e^{x/2}\left[C_2 \cos\left(\frac{x\sqrt{3}}{2}\right) + C_3 \sin\left(\frac{x\sqrt{3}}{2}\right)\right].$$

Now, $$\text{P.I.} = \frac{1}{(D^3 + 1)}[e^x + 1]^2 = \frac{1}{(D^3 + 1)}[e^{2x} + 2e^x + 1]$$

$$= \frac{1}{D^3 + 1}(e^{2x} + 2e^x + e^{0x}) = \frac{1}{D^3 + 1} e^{2x} + 2\frac{1}{D^3 + 1} e^x + \frac{1}{D^3 + 1} e^{0x}$$

$$= \frac{1}{2^3 + 1} e^{2x} + 2\frac{1}{1^3 + 1} + \frac{1}{0 + 1} e^{0x} = \frac{1}{9} e^{2x} + e^x + 1.$$

Here, the general solution is

$$y = \text{C.F.} + \text{P.I.}$$

$$\Rightarrow \qquad y = C_1 e^{-x} + e^{x/2}\left[C_2 \cos\left(\frac{x\sqrt{3}}{2}\right) + C_3 \sin\left(\frac{x\sqrt{3}}{2}\right)\right] + \frac{1}{9} e^{2x} + e^x + 1.$$

• TEST YOURSELF–3

Solve the following differential equations :

1. $(D^2 - 4D + 1)\, y = e^{2x} - e^{-x}$.
2. $(D^2 + 5D + 6)\, y = e^{2x}$.
3. $(4D^2 + 4D - 3)\, y = e^{2x}$.
4. $-2D + 1)\, y = 2e^{5x/2}$.

ANSWERS

1. $y = e^{2x}(C_1 \cosh x\sqrt{3} + C_2 \sinh x\sqrt{3}) - \frac{1}{3} e^{2x} - \frac{1}{6} e^{-x}$
2. $y = C_1 e^{-2x} + C_2 e^{-3x} + \frac{1}{20} e^{2x}$
3. $y = C_1 e^{x/2} + C_2 e^{-3x/2} + \frac{1}{21} e^{2x}$
4. $y = (C_1 + C_2 x) e^x + \frac{8}{9} e^{5x/2}$

(2) To Evaluate P.I., when B is of the form $\sin ax$ or $\cos ax$:

Case (I) : If $f(D)$ contains even power of D :

$$\frac{1}{f(D^2)} \sin ax = \frac{1}{f(-a^2)} \sin ax$$

Case (II) : If $f(D)$ contains odd powers of D : If $f(D) = f_1(D^2) + Df_2(D^2)$

$$\frac{1}{f(D)} \sin ax = \frac{f_1(-a^2) \sin ax - f_2(-a^2)\, a \cos ax}{\{f_1(-a^2)\}^2 + a^2 \{f_2(-a^2)\}^2}.$$

SOLVED EXAMPLES

Example 1. *Solve* $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 2y = \cos 3x$.

Solution. Here, the given differential equation can be written as

$$(D^2 - 3D + 2)\, y = \cos 3x \qquad \ldots(1)$$

To find C.F., the auxiliary equation is

$$m^2 - 3m + 2 = 0$$

$\Rightarrow \qquad (m-1)(m-2) = 0$

which gives $m = 1$ and $m = 2$.

therefore, $\qquad$ C.F. $= C_1 e^x + C_2 e^{2x}$.

Now, $\qquad$ P.I. $= \dfrac{1}{D^2 - 3D + 2} \cos 3x = \dfrac{1}{-9 - 3D + 2} \cos 3x \qquad (\because \; D^2 = -a^2 = -9)$

$$= \frac{1}{-7 - 3D} \cos 3x = -\frac{(7 - 3D)}{(7^2 - 9D^2)} \cos 3x$$

$$= -\frac{(7 - 3D)}{7^2 - 9(-9)} \cos 3x = -\frac{1}{130} [7 \cos x - 3D \cos 3x]$$

$$= -\frac{7}{130} \cos 3x - \frac{9}{130} \sin 3x = -\frac{1}{130} (7 \cos 3x + 9 \sin 3x).$$

Hence, the general solution of (1) is given by

$$y = \text{C.F.} + \text{P.I.}$$

$\Rightarrow \qquad y = C_1 e^x + C_2 e^{2x} - \dfrac{1}{130} [7 \cos 3x + 9 \sin 3x].$

• TEST YOURSELF–4

Solve the following differential equations :

1. $(D^2 + 9)\, y = \cos 4x$.
2. $(D^2 - 2D + 5)\, y = \sin 3x$.
3. $(D^2 - 3D + 2)\, y = \sin 3x$.
4. $(D^4 + 2D^3 - 3D^2)\, y = 3e^{2x} + 4 \sin x$.

ANSWERS

1. $y = C_1 \cos 3x + C_2 \sin 3x - \dfrac{1}{7} \cos 4x$
2. $y = e^x [C_1 \cos 2x + C_2 \sin 2x] + \dfrac{1}{26} (3 \cos 3x - 2 \sin 3x)$
3. $y = C_1 e^x + C_2 e^{2x} + \dfrac{1}{130} (9 \cos 3x - 7 \sin 3x)$
4. $y = (C_1 + C_2 x) + C_3 e^x + C_4 e^{-3x} + \dfrac{3}{20} e^{2x} + \dfrac{4}{5} \sin x + \dfrac{2}{5} \cos x$

(3) To Evaluate P.I. when B is of the form of x^m, when m is positive integer :

i.e., **To evaluate $\dfrac{1}{f(D)} x^m$, $m \in Z^+$**

and $\qquad$ **$f(D) = A_0 D^n + A_1 D^{n-1} + \ldots + A_n$, we apply working procedure**

$$\frac{1}{D - a} x^m = -\frac{1}{a} \left[x^m + \frac{m x^{m-1}}{a} + \frac{m(m-1) x^{m-2}}{a^2} + \ldots + \frac{m(m-1)\, 2.1}{a^m} \right] \qquad \ldots(2)$$

Here, we observe that (1) and (2) are the same.

Working Procedure :

Take the lowest degree term from $f(D)$ and remaining factor will be of the form $[1 + f(D)]$ or $[1 - f(D)]$. Now, this factor can be taken in the numerator with a negative index, which can be expanded by Binomial theorem. Here, it should be noted that the expansion is to be carried upto the term D^m, since, we always have $D^{m+1} x^m = 0$, $D^{m+2} x^m = 0$ and all other higher differential coefficients of x^m are zero.

SOLVED EXAMPLES

Example 1. *Solve* $\quad (D^2 + D - 2)\, y = x + \sin x$.

Solution. Here, the given equation is

$$(D^2 + D - 2)\, y = x + \sin x \qquad \ldots(1)$$

To find C.F., the auxiliary equation is

$$m^2 + m - 2 = 0$$

$\Rightarrow \qquad (m-1)(m+2) = 0 \Rightarrow m = 1, -2$

$\therefore \qquad \text{C.F.} = C_1 e^x + C_2 e^{-2x}$

Now, $\qquad \text{P.I.} = \dfrac{1}{(D^2+D-2)}(x+\sin x) = \dfrac{1}{(D^2+D-2)}x + \dfrac{1}{(D^2+D-2)}\sin x$

$$= \frac{1}{-2\left(1-\frac{1}{2}D-\frac{1}{2}D^2\right)}x + \frac{1}{-1^2+D-2}\sin x$$

$$= -\frac{1}{2}\left[1-\left(\frac{1}{2}D+\frac{1}{2}D^2\right)\right]^{-1}x + \frac{(D+3)}{(D-3)(D+3)}\sin x$$

$$= -\frac{1}{2}\left(1+\frac{1}{2}D+\ldots\right)x + \frac{(D+3)}{D^2-9}\sin x = -\frac{1}{2}\left(x+\frac{1}{2}\right) + \frac{D+3}{-1-9}\sin x$$

$$= -\frac{1}{2}\left(x+\frac{1}{2}\right) - \left(\frac{1}{10}\right)[D(\sin x) + 3\sin x]$$

$$= -\frac{1}{2}x - \frac{1}{4} - \frac{1}{10}(\cos x + 3\sin x).$$

Hence, the complete solution is given by

$y = \text{C.F.} + \text{P.I.}$

$\therefore \qquad y = C_1 e^x + C_2 e^{-2x} - \dfrac{1}{2}x - \dfrac{1}{4} - \dfrac{1}{10}(\cos x + 3\sin x).$

• TEST YOURSELF–5

Solve the following differential equations :

1. $(D^3 - D^2 - 6D)\,y = x^2 + 1.$ 2. $(D^4 - a^4)\,y = x^4.$
3. $(D^3 + 2D^2 + D)\,y = e^{2x} + x^2 + x.$ 4. $(D^3 - 3D - 2)\,y = x^3.$

ANSWERS

1. $y = C_1 + C_2 e^{3x} + C_3 e^{-2x} - \dfrac{25}{108}x - \dfrac{1}{18}x^3 + \dfrac{1}{36}x^2$
2. $y = C_1 e^{ax} + C_2 e^{-ax} + C_3 \cos ax + C_4 \sin ax - \dfrac{x^4}{a^4} - \dfrac{24}{a^8}$
3. $y = C_1 + (C_2 + C_3 x)\,e^{-x} + \dfrac{1}{18}e^{2x} + \dfrac{1}{3}x^3 - \dfrac{3}{2}x^2 + 4x$
4. $y = (C_1 + C_2 x)\,e^{-x} + C_3 e^{2x} - \dfrac{1}{2}x^3 + \dfrac{9}{4}x^2 - \dfrac{27}{4}x + 15$

(4) To evaluate $\dfrac{1}{f(D)}e^{ax}.X$, where X is any function of x :

$$\frac{1}{f(D)}[e^{ax}.X] = e^{ax}\left[\frac{1}{f(D+a)}.X\right].$$

Working Procedure :

Replace D by $(D+a)$ and brought e^{ax} before the operator $\dfrac{1}{f(D)}$. After that, determine $\dfrac{1}{f(D+a)}.X$ as usual.

SOLVED EXAMPLES

Example 1. *Solve* $(D^2 + 4D - 12)\,y = (x-1)\,e^{2x}$.

Solution. Here, the given differential equation is

$$(D^2 + 4D - 12)\,y = (x-1)\,e^{2x} \qquad \ldots(1)$$

To find C.F. of (1), the auxiliary equation is

$$m^2 + 4m - 12 = 0 \;\Rightarrow\; (m-2)(m+6) = 0$$

which gives $m = 2$ and $m = -6$

$\therefore \qquad \text{C.F.} = C_1 e^{2x} + C_2 e^{-6x}$

Now, $\quad \text{P.I.} = \dfrac{1}{(D^2+4D-12)} e^{2x}(x-1) = e^{2x} \dfrac{1}{[(D+2)^2+4(D+2)-12]}(x-1)$

$$= e^{2x}\frac{1}{(D^2+8D)}(x-1) = e^{2x}\frac{1}{8D\left(1+\frac{D}{8}\right)}(x-1)$$

$$= \frac{1}{8}e^{2x}\frac{1}{D}\left(1+\frac{1}{8}D\right)^{-1}(x-1) = \frac{1}{8}e^{2x}\frac{1}{D}\left(1-\frac{1}{8}D+\ldots\right)(x-1)$$

$$= \frac{1}{8}e^{2x}\frac{1}{D}\left(x-1-\frac{1}{8}\right) = \frac{1}{8}e^{2x}\frac{1}{D}\left(x-\frac{9}{8}\right) = \frac{1}{8}e^{2x}\int\left(x-\frac{9}{8}\right)dx$$

$$= \frac{1}{8}e^{2x}\left(\frac{x^2}{2}-\frac{9}{8}x\right).$$

Hence, the general solution of (1), is given by

$$y = \text{C.F.} + \text{P.I.}$$

$\Rightarrow \quad y = C_1 e^{2x} + C_2 e^{-6x} + \dfrac{1}{8}e^{2x}\left[\dfrac{x^2}{2}-\dfrac{9}{8}x\right].$

• TEST YOURSELF–6

Solve the following differential equations :

1. $(D^2-2D+1)\,y = e^x . x^2.$
2. $(D^2-5D+6)\,y = x^3 . e^{2x}.$
3. $(D^2-1)\,y = e^x(1+x^2).$
4. $(D^2-4D+1)\,y = e^{2x}\sin x.$

ANSWERS

1. $y = (C_1 + C_2 x)\,e^x + \dfrac{1}{12}e^x . x^4$
2. $y = C_1 e^{2x} + C_2 e^{3x} - e^{2x}\left[\dfrac{x^4}{4}+x^3+3x^2+6x\right]$
3. $y = C_1 e^x + C_2 e^{-x} + \dfrac{1}{12}e^x[9x+2x^3-3x^2]$
4. $y = C_1 e^{(2+\sqrt{3})x} + C_2 e^{(2-\sqrt{3})x} - \dfrac{1}{4}e^{2x}\sin x$

(5) To evaluate $\dfrac{1}{f(D)}e^{ax}$, when $f(a)=0$:

Let us suppose $f(a) = 0$.

In this case $(D-a)$ is at least one factor of $f(D)$.

Let $\quad f(D) = (D-a)^r g(D)$, where $g(a) \neq 0$.

Then $\quad \dfrac{1}{f(D)}e^{ax} = \dfrac{1}{(D-a)^r}\cdot\dfrac{1}{g(a)}e^{ax} = \dfrac{1}{g(a)}\cdot\dfrac{1}{(D-a)^r}e^{ax}$

$$= \frac{1}{g(a)}\cdot\frac{1}{(D-a)^{r-1}}e^{ax}\int e^{ax}.e^{-ax}\,dx$$

$$= \frac{1}{g(a)}\cdot\frac{1}{(D-a)^{r-1}}xe^{ax} = \frac{1}{g(a)}\cdot\frac{1}{(D-a)^{r-2}}e^{ax}\int xe^{ax}.e^{-ax}\,dx$$

$$= \frac{1}{g(a)}\cdot\frac{1}{(D-a)^{r-2}}\cdot\frac{x^2}{2\,!}e^{ax}.$$

Proceeding in the same way, finally, we get

$$\boldsymbol{\frac{1}{f(D)}e^{ax} = \frac{1}{g(a)}\cdot\frac{x^r}{r\,!}e^{ax}.}$$

SOLVED EXAMPLES

Example 1. *Solve* $\quad (D^2+D-6)\,y = e^{2x}.$

Solution. Here, the given equation is

$$(D^2+D-6)\,y = e^{2x}. \qquad \ldots(1)$$

To find the C.F. of (1), the auxiliary equation is

$$m^2+m-6=0$$

$\Rightarrow \quad (m+3)(m-2) = 0 \Rightarrow m = 2, -3$

$\therefore \quad \text{C.F.} = C_1 e^{2x} + C_2 e^{-3x}$

Now, $\quad$ $$\text{P.I.} = \frac{1}{D^2+D-6} e^{2x} = \frac{1}{(D+3)(D-2)} e^{2x}$$
$$= \frac{1}{(2+3)(D-2)} e^{2x} = \frac{1}{5(D-2)} e^{2x} \cdot 1$$
$$= \frac{1}{5} e^{2x} \frac{1}{(D+2)-2} \cdot 1 = \frac{1}{5} e^{2x} \frac{1}{D} \cdot 1 = \frac{1}{5} x e^{2x}.$$

Hence, the complete solution of (1) is given by
$$y = \text{C.F.} + \text{P.I.}$$
$$\Rightarrow \quad y = C_1 e^{2x} + C_2 e^{-3x} + \frac{1}{5} x e^{2x}.$$

• TEST YOURSELF–7

Solve the following differential equations :

1. $(D^2 + 4D + 3)\, y = e^{-3x}$.
2. $(D^2 + 6D + 9)\, y = 2\, e^{-3x}$.
3. $(D^4 + D^3 + D^2 - D - 2)\, y = e^x$.
4. $(D^2 - 9D + 18)\, y = \cosh 3x$.
5. $(D-1)^2 (D^2+1)^2\, y = e^x$.

ANSWERS

1. $y = C_1 e^{-x} + C_2 e^{-3x} - \frac{x}{2} e^{-3x} x$
2. $y = (C_1 + C_2 x)\, e^{-3x} + x^2 e^{-3x}$
3. $y = C_1 e^x + C_2 e^{-x} + e^{-x/2}\left[C_3 \cos\left(\frac{\sqrt{7}}{2} x\right) + C_4 \sin\left(\frac{\sqrt{7}\, x}{2}\right)\right] + \frac{1}{8} x e^x$
4. $y = C_1 e^{3x} + C_2 e^{6x} - \frac{1}{6} x e^{3x} + \frac{1}{108} e^{-3x}$
5. $y = (C_1 + C_2 x)\, e^x + (C_3 + C_4 x) \cos x + (C_5 + C_6 x) \sin x + \frac{1}{8} x^2 e^x$

(6) To evaluate $\frac{1}{f(D^2)}$ sin ax, or cos ax, when $f(-a^2) = 0$, we use the following rules :

$$\Rightarrow \quad \frac{1}{D^2+a^2} \sin ax = -\frac{x}{2a} \cos ax = \frac{x}{2} \int \sin ax\, dx$$

and
$$\frac{1}{D^2+a^2} \cos ax = \frac{x}{2a} \sin ax = \frac{x}{2} \int \cos ax\, dx$$

SOLVED EXAMPLES

Example 1. *Solve* $\quad (D^2 + a^2)\, y = \sin ax$.

Solution. Here, the given equation is
$$(D^2 + a^2)\, y = \sin ax. \qquad \ldots(1)$$

To find the C.F. of (1), the auxiliary equation is
$$m^2 + a^2 = 0 \Rightarrow m = 0 \pm ai$$
$$\therefore \quad \text{C.F.} = e^{0x} [C_1 \cos ax + C_2 \sin ax] = [C_1 \cos ax + C_2 \sin ax]$$

Now, $\quad$ $$\text{P.I.} = \frac{1}{D^2+a^2} \sin ax$$
$$= \frac{x}{2} \int \sin ax\, dx = -\frac{x}{2a} \cos ax.$$

Hence, the complete solution of (1) is given by
$$y = \text{C.F.} + \text{P.I.}$$
$$\Rightarrow \quad y = C_1 \cos ax + C_2 \sin ax - \frac{x}{2a} \cos ax.$$

• SUMMARY

• Solution of $D.E$
$$\frac{d^2y}{dx^2} + P \frac{dy}{dx} + Qy = R$$
is of the form

$$y = C.F + P.I$$

where $\quad C.F = c_1 m^{m_1 x} + c_2 e^{m_2 x},$

and m_1 and m_2 are the roots of $m^2 + Pm + Q = 0$.

Also, $\quad P.I = \dfrac{R}{f(D)}, \quad f(D) = D^2 + PD + Q.$

If $m_1 = m_2 = m$, then

$$\text{C.F.} = (C_1 + C_2 x)\, e^{mx}$$

- If $R = e^{ax}$, $f(a) \neq 0$

$$\text{P.I.} = \frac{e^{ax}}{f(D)} = \frac{e^{ax}}{f(a)}$$

- If $R = a^{ax}, f(a) = 0$, then

$$\text{P.I.} = \frac{e^{ax}}{f(D)} = e^{ax}\left\{\frac{1}{f(D+a)}\right\}$$

- If $R = \sin ax$ or $\cos ax$, $f(-a^2) \neq 0$, then

$$\text{P.I.} = \frac{\sin ax}{f(D^2)} \text{ or } \frac{\cos ax}{f(D^2)} = \frac{\sin ax}{f(-a^2)} \text{ or } \frac{\cos ax}{f(-a^2)}$$

- If $R = \sin ax$ or $\cos ax, f(-a^2) = 0$, then

$$\frac{\sin ax}{D^2 + a^2} = -\frac{x}{2a}\cos ax, \quad \frac{\cos ax}{D^2 + a^2} = \frac{x}{2a}\sin ax$$

- If $R = x^m$, $m \in N$, then

$\text{P.I} = \dfrac{x^m}{f(D)} = [f(D)]^{-1}\, x^m$ [expanding $[f(D)]^{-1}\, x^m$ [expanding $[f(D)]^{-1}$ by Binomial theorem]

• STUDENT ACTIVITY

1. Solve $(D^4 + k^4)\, y = 0$

2. Solve $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 2y = \cos 3x.$

• TEST YOURSELF–8

Solve the following differential equations :

1. $(D^2 + a^2)\, y = \cos ax.$
2. $(D^2 + 4)\, y = \cos 2x.$
3. $(D^2 + 4)\, y = e^x + \sin 2x.$
4. $(D^2 + a^2)\, y = \sin ax.$

ANSWERS

1. $y = C_1 \cos ax + C_2 \sin ax + \dfrac{x}{2a} \sin ax.$
2. $y = C_1 \cos 2x + C_2 \sin 2x + \dfrac{x}{4} \sin 2x$
3. $y = C_1 \cos 2x + C_2 \sin 2x + \dfrac{1}{5} e^x - \dfrac{1}{4} x \cos 2x.$
4. $y = C_1 \cos ax + C_2 \sin ax - \dfrac{x}{2a} \cos ax.$

OBJECTIVE EVALUATION

➤ FILL IN THE BLANKS :

1. If the order of the given differential equation is n, than C.F. of this equation contains .... arbitrary constant.
2. If the order of the given differential equation is n, then .... of this equation does not contain any arbitrary constants.

➤ TRUE OR FALSE :

Write 'T' for True and F for False Statement :

1. The particular integral of an n^{th} order differential equation contains n independent arbitrary constants *(T/F)*
2. The complementary functions of a differential equation of order n contains n independent arbitrary constant. *(T/F)*

➤ MULTIPLE CHOICE QUESTIONS :

Choose the most appropriate one :

The solution of $(D^2 + 1)\, y = 0$ is $y =$:

(a) $A \cos x - B \sin x$ (b) $-A \cos x - B \sin x$
(c) $A \cos x + B \sin x$ (d) $-A \cos x + B \sin x.$

2. The general solution of the differential equation $(D^2 + 1)\, y = 0$ is :

(a) $y = \cos x$ (b) $y = C \cos x$
(c) $y = C_1 \cos (x + C_2)$ (d) $C_1 \cos (C_2 + C_3\, x).$

ANSWERS

Fill in the Blanks :

1. n 2. Particular integral

True or False :

1. T 2. F

Multiple Choice Questions :

1. (c) 2. (c)

❑❑❑

UNIT

14

HOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS

STRUCTURE

- Homogeneous Linear Differential Equations
- Solution of Homogeneous Linear Differential Equation
  - Test Yourself–1
  - Summary
  - Student Activity

LEARNING OBJECTIVES

After going through this unit you will learn :

How to get solution of homogenous linear differential equations.

• 14.1. HOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS

Definition. Any differential equation of the form

$$x^n \frac{d^ny}{dx^n} + A_1 x^{n-1} \frac{d^{n-1}y}{dx^{n-1}} + \ldots + A_n y = X$$

is called a homogeneous linear differential equation of n^{th} order, where $A_1, A_2, \ldots, A_n$ are constants and X is a function of x or a constant.

For example, consider the following differential equations :

(i) $x^2 \frac{d^2y}{dx^2} + 3x \frac{dy}{dx} + 4y = e^x$

(ii) $x^2 \frac{d^2y}{dx^2} + 2x \frac{dy}{dx} + 2y = e^x$

The above two differential equations are linear as the dependent variable y and its derivatives appear in their first degree and are not multiplied together.

• 14.2. SOLUTION OF HOMOGENEOUS LINEAR DIFFERENTIAL EQUATION

Consider the homogeneous linear differential equation

$$x^n \frac{d^ny}{dx^n} + A_1 x^{n-1} \frac{d^{n-1}y}{dx^{n-1}} + A_2 x^{n-2} \frac{d^{n-2}y}{dx^{n-2}} + \ldots + A_n y = X \qquad \ldots(1)$$

where $A_1, A_2, \ldots, A_n$ are constants and X is a function of x or constant.

Now, equation (1) can be transformed to an equivalent equation (linear differential equation) with constant coefficients by changing the independent variable by the relation.

$$x = e^z \;\; i.e., \;\; z = \log x$$

Working Procedure :

(*i*) *Put* $x = e^z, x\frac{d}{dx} = D = \frac{d}{dz}, x^2 \frac{d^2}{dx^2} = D(D-1)$ *and so on.*

(*ii*) *Obtained the equation in terms of D (linear equation).*

(*iii*) *To find the C.F. and P.I. used the usual method given in chapter* 16.

(*iv*) *Find general solution by adding C.F. and P.I.*

(*v*) *Finally, substitute* $z = \log x$.

SOLVED EXAMPLES

Example 1. *Solve* $x^2 \frac{d^2y}{dx^2} - 4x \frac{dy}{dx} + 6y = x.$ $\qquad \ldots(1)$

Solution. Putting $\quad x = e^z$

$\Rightarrow \quad z = \log x$ and $D = \frac{d}{dz}$.

Thus, the given equation becomes

$$[D(D-1) - 4D + 6]\, y = e^z \;\Rightarrow\; (D^2 - 5D + 6)\, y = e^z \qquad \ldots(2)$$

which is a linear equation in y.

To find the C.F. of (2), the auxiliary equation is

$$m^2 - 5m + 6 = 0$$

$\Rightarrow \quad (m-2)(m-3) = 0$ which gives $m = 2, 3$.

$\therefore \quad \text{C.F.} = C_1 e^{2z} + C_2 e^{3z}$

Now, $\quad \text{P.I.} = \dfrac{1}{D^2 - 5D + 6} e^z = \dfrac{1}{1 - 5 + 6} e^z = \dfrac{1}{2} e^z$.

Hence, the general solution is given by

$$y = \text{C.F.} + \text{P.I.}$$

$\Rightarrow \quad y = C_1 e^{2z} + C_2 e^{3z} + \dfrac{1}{2} e^z.$

Now put $e^z = x$

$\therefore \quad y = C_1 x^2 + C_2 x^3 + \dfrac{1}{2} x.$

Example 2. *Solve* $x^2 \dfrac{d^2y}{dx^2} - 3x \dfrac{dy}{dx} + 4y = 2x^2$.

Solution. Here, the given equation is

$$x^2 \frac{d^2y}{dx^2} - 3x \frac{dy}{dx} + 4y = 2x^2. \qquad \ldots(1)$$

Let $x = e^z$, then $z = \log x$, $x \dfrac{dy}{dx} = Dy$, $x^2 \dfrac{d^2y}{dx^2} = D(D-1)\, y$

Put in (1), then (1) becomes

$$(D(D-1) - 3D + 4)\, y = 2e^{2z}$$

$\Rightarrow \quad (D^2 - 4D + 4)\, y = 2e^{2z}. \qquad \ldots(2)$

To find the C.F. of (1), the auxiliary equation is

$$m^2 - 4m + 4 = 0 \;\Rightarrow (m-2)^2 = 0 \;\Rightarrow\; m = 2, 2$$

$\therefore \quad \text{C.F.} = (C_1 + C_2 z)\, e^{2z} = (C_1 + C_2 \log x)\, x^2$

Now, $\quad \text{P.I.} = \dfrac{1}{(D-2)^2} 2e^{2z} = 2 \dfrac{1}{(D-2)^2} e^{2z}$

$$= 2e^{2z} \frac{1}{[(D+2)-2]} \cdot 1 = 2e^{2z} \frac{1}{D} \cdot 1 = 2e^{2z} \cdot \frac{z^2}{2} = z^2 e^{2z} = (\log x)^2 \cdot x^2.$$

Hence, the complete solution of (1) is given by

$$y = \text{C.F.} + \text{P.I.}$$

$\Rightarrow \quad y = (C_1 + C_2 \log x)\, x^2 + x^2 (\log x)^2.$

• SUMMARY

- Homogeneous linear D.E

$$x^n \frac{d^ny}{dx^n} + A_n x^{n-1} \frac{d^{n-1}y}{dx^{n-1}} + \ldots\ldots + A_n y = x.$$

- Solution Homogeneous Linear D.E

(i) Put $x = e^z$ or $z = \log x$, and $x \dfrac{d}{dx} = D = \dfrac{d}{dz}$, $\dfrac{x^2 d^2}{dx^2} = D(D-1)$, $\dfrac{x^3 d^3}{dx^3} = D\, 9D - 1)(D-2)$ etc. in the given D.E.

(ii) The given D.E becomes

$$f(D)\, y = x, \; D = \frac{d}{dz}$$

(iii) Find C.F and P.I as usual obtained in previous chapter.

(iv) $y =$ C.F + P.I

(v) Put $z = \log x$ in $y =$ C.F + P.I

• STUDENT ACTIVITY

1. Solve $x^2 \frac{d^2y}{dx^2} - 4x \frac{dy}{dx} + 6y = x.$

2. Solve $d^2 \frac{d^2y}{dx^2} + 7x \frac{dy}{dx} + 13y = \log x.$

• TEST YOURSELF

Solve the following differential equations :

1. $x^2 \frac{d^2y}{dx^2} - \frac{dy}{dx} + y = x.$
2. $x^2 \frac{d^2y}{dx^2} - 4x \frac{dy}{dx} + 6y = x^4.$
3. $x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} - 4y = x^2.$
4. $x^4 \frac{d^3y}{dx^3} + 2x^3 \frac{d^2y}{dx^2} - x^2 \frac{dy}{dx} + xy = 1.$

ANSWERS

1. $y = x (C_1 + C_2 \log x) + \frac{1}{x} (\log x)^2.$
2. $y = C_1 x^2 + C_2 x^3 + \frac{1}{2} x^4.$
3. $y = C_1 x^2 + \frac{C_2}{x^2} + \frac{1}{4} x^2 \log x.$
4. $y = (C_1 + C_2 \log x) x + C_3 x^{-1} + \frac{1}{4x} \log x$

OBJECTIVE EVALUATION

➤ FILL IN THE BLANKS :

1. In any homogeneous equation $x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + y = X$, the X is either constant or a function of ...., only.
2. The homogeneous linear equation can be reduced to a linear differential equation with constant coefficiants by putting $x =$ ....

➤ TRUE OR FALSE :

Write T for True and F for False statement :

1. The homogeneous linear differential equation can be reduced to linear differential equation with constant coefficients. *(T/F)*
2. The differential equation, in which the power of x in the coefficients are greater than the order of its derivative, associated with them, is called the homogeneous linear equation. *(T/F)*

3. The differential equation in which the power of x in the coefficients are equal to the order of its derivative, associated with them is called the homogeneous linear equation. *(T/F)*

➤ MULTIPLE CHOICE QUESTIONS :

Choose the most appropriate one :

1. The particular integral of $(x^2 D^2 - 2xD)\, y = \log x$ is :

   (a) $\log x$ (b) $-\frac{1}{9}\log x - \frac{1}{6}(\log x)^2$

   (c) $A \log x$ (d) none of these.

2. The P.I. is $(x^2 D^2 + 5xD + 4)\, y = x \log x$ is given by :

   (a) $\frac{1}{9} x \log x$ (b) $\frac{2}{27} x$

   (c) $\frac{1}{9} x \log x - \frac{2}{27} x$ (d) none of these.

ANSWERS

Fill in the Blanks :

1. x **2.** e^x **3.** First **4.** equal

True or False :

1. T **2.** F **3.** T

Multiple Choice Questions :

1. (b) **2.** (c) **3.** (a)

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UNIT 15

WRONSKIAN

STRUCTURE

- Linear Dependence and Independence of Functions
  The Wronskian
  - Summary
  - Student Activity
  - Exercise

LEARNING OBJECTIVES

After going through this unit you will learn :

About linear independent and dependent functions.

How to find the nature of the given function through Wronskian.

• 15.1. LINEAR DEPENDENCE AND INDEPENDENCE OF FUNCTIONS

(a) *The functions $f_i(x), i = 1, 2 \ldots n$ are said to be linearly dependent on an interval $a \le x \le b$ if there exist a set $(a_1, a_2 \ldots a_n)$ of constants,* ***not all zero,*** *such that*

$$a_1 f_1(x) + a_2 f_2(x) + \ldots + a_n f_n(x) = 0 \text{ in } a \le x \le b \qquad \ldots(1)$$

(b) *The function $f_i(x), i = 1, 2, \ldots, n$ are said to be linearly independent if the only set of constants $a_1, a_2, \ldots a_n$ for which (1) holds, is the set $a_1 = a_2 = \ldots = a_n = 0$.*

Examples :

(i) $\sin x$, $3 \sin x$ and $-\sin x$ are linearly dependent on the interval $-1 \le x \le 2$.

(ii) x and x^2 are linearly independent on $0 \le x \le 1$ since

$$C_1 x + C_2 x^2 = 0 \ \ \forall x \text{ on } 0 \le x \le 1 \Rightarrow C_1 = C_2 = 0.$$

• 15.2. THE WRONSKIAN

Let $y_1, y_2, \ldots, y_n$ be n functions of x such that each possesses at least $(n-1)$ derivatives. Then the determinant

Calculus

$$\begin{vmatrix} y_1 & y_2 & \cdots\cdots & y_n \\ y_1' & y_2' & \cdots\cdots & y_n' \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots\cdots & y_n^{(n-1)} \end{vmatrix}$$

is called the ***Wronskian*** *of the functions $y_1, y_2 \ldots, y_n$* and denoted by $\mathbf{W(y_1, y_2, \ldots, y_n)}$ or simply W.

• SOLVED EXAMPLES

Example 1. *Show that the solutions e^x, e^{-x} and e^{2x} of*

$$\frac{d^3y}{dx^3} - 2\frac{d^2y}{dx^2} - \frac{dy}{dx} + 2y = 0$$

are linearly independent. Also, solve the given equation.

Solution. Since e^x, e^{-x} and e^{2x} satisfy the given differential equation. So they are solutions of the given equations.

Now, $$W[e^x, e^{-x}, e^{2x}] = \begin{vmatrix} e^x & e^{-x} & e^{2x} \\ \frac{d}{dx}(e^x) & \frac{d}{dx}(e^{-x}) & \frac{d}{dx}(e^{2x}) \\ \frac{d^2}{dx^2}(e^x) & \frac{d^2}{dx^2}(e^{-x}) & \frac{d^2}{dx^2}(e^{2x}) \end{vmatrix}$$

$$= \begin{vmatrix} e^x & e^{-x} & e^{2x} \\ e^x & -e^{-x} & 2e^{2x} \\ e^x & e^{-x} & 4e^{2x} \end{vmatrix} = e^x \cdot e^{-x} \cdot e^{2x} \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 1 & 1 & 4 \end{vmatrix}.$$

After simplify, we get

$$W = e^{2x} \begin{vmatrix} 1 & 1 & 1 \\ 2 & 0 & 3 \\ 0 & 0 & 3 \end{vmatrix} = e^{2x} \times (-6) \neq 0.$$

Hence, the solutions e^x, e^{-x}, e^{2x} are linearly independent solution of the given third order differential equation, so it can possesses at most three independent solutions. Therefore, the complete solution is given by $y = C_1 e^x + C_2 e^{-x} + C_3 2^{2x}$, with C_1, C_2 and C_3 as arbitrary constants.

Example 2. *Show that the Wronskian of the functions $x^2, x^2 \log x$ is non-zero. Can these functions be independent solutions of an ordinary differential equation, if so determine the equation.*

Solution. Let $\quad y_1 = x^2$

and $\quad y_2 = x^2 \log x.$

Then, the Wronskian is

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x_2 & x^2 \log x \\ 2x & 2x \log x + x \end{vmatrix} = x^3 \neq 0.$$

$\Rightarrow$ The functions y_1 and y_2 are linearly independent solution of an ordinary differential equation.

Now, we find the equation

Let $\quad y = C_1 y_1 + C_2 y_2 = C_1 x^2 + C_2 x^2 \log x$

$$\Rightarrow \quad \frac{dy}{dx} = 2C_1 x + 2C_2 x \log x + C_2 x$$

and

$$\frac{d^2y}{dx^2} = 2C_1 + 2C_2 \log x + 3C_2.$$

Now, eliminating C_1 and C_2 from the above three relations, we get

$$x^2 \frac{d^2y}{dx^2} - 3x \frac{dy}{dx} + 4y = 0.$$

• SUMMARY

- **Linear dependent functions :** Functions $f_1(x), f_2(x), f_3(x), \ldots\ldots, f_n(x)$, $x \in [a, b]$ are said to be linearly dependent on $[a, b]$ if these exist $a_1, a_2, \ldots, a_n$, not all zero such that

$$a_1 f_1(x) + a_2 f_2(x) + a_3 f_3(x) + \ldots + a_n f_n(x) = 0$$

- **Linear independent functions :** Function

$$a_1 f_1(x) + a_2 f_2(x) + f_3(x), \ldots, f_n(x) + \ldots + a_n f_n(x) = 0 \;\Rightarrow\; a_1 = 0 = a_2 = \ldots = a_n.$$

- **Wronskian :**

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$W(y_1, y_2, y_3) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} \text{ etc.}$$

• STUDENT ACTIVITY

1. Show that the functions x, x^2, x^3 are linearly independent.

2. Show that the functions $\sin x$, $\sin 3x$, $\sin^3 x$ are linearly dependent.

- **TEST YOURSELF**

1. Prove that the functions $1, x, x^2$ are linearly independent. Hence, form the differential equation whose roots are $1, x, x^2$.
2. Show that if $m_1 \neq m_2$, the functions $e^{m_1 x}$ and $e^{m_2 x}$ are linearly independent.
3. Show that the following functions are linearly independent $e^x \cos x, e^x \sin x$.
   Hence, form the differential equation of second order having these two functions as independent solutions.
4. Which of the following sets of functions are linearly independent for all x.
   (i) $\sin x, \cos x$ (ii) $\sin x, \sin 3x, \sin^3 x$
   (iii) $\sin x, \cos x, \sin 2x$ (iv) $x^2 - x + 1, x^2 - 1, 3x^2 - x - 1$.

ANSWERS

1. $\frac{d^3y}{dx^3} = 0$ 3. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$
4. (i) Linearly independent (ii) Linearly dependent
   (iii) Linearly independent (iv) Linearly independent.

UNIT 16

GENERAL EQUATION OF SECOND DEGREE

STRUCTURE

LEARNING OBJECTIVES

After going through this unit you will learn :

- How the given general equation of second degree reduces to pair of straight lines, circle, parabola, ellipse and hyperbola

• 16.1. INTRODUCTION

The general equation of second degree in x and y is given by

$$ax^2+2hxy+by^2+2gx+2fy+c=0 \quad ...(1)$$

where a, b, c, f, g and h are six constants. If the equation (1) is divided by any one of six constants, then the equation so obtained will have five constants, therefore in order to determine the values of these constants, there will required five equations between them. The solutions of these five equations will give the exact equation of degree two.

• 16.2. CONDITION FOR A PAIR OF STRAIGHT LINES

To find the condition that the general equation of second degree may represent a pair of straight lines.

Let the general equation of second degree in x and y be

$$ax^2+2hxy+by^2+2gx+2fy+c=0. \quad ...(1)$$

Suppose the equation (1) represents a pair of lines and let these lines be

$$l_1x+m_1y+n_1=0 \quad ...(2)$$

and

$$l_2x+m_2y+n_2=0. \quad ...(3)$$

Then we have

$$(l_1x+m_1y+n_1)(l_2x+m_2y+n_2) \equiv ax^2+2hxy+by^2+2gx+2fy+c=0.$$

Now comparing the coefficients of x^2, y^2, xy, x, y and constant term of both sides, we get

$$\left.\begin{aligned} l_1 l_2 = a,\ m_1 m_2 = b,\ n_1 n_2 = c \\ m_1 n_2 + n_1 m_2 = 2f,\ n_1 l_2 + n_2 l_1 = 2g \\ \text{and} \quad l_1 m_2 + l_2 m_1 = 2h \end{aligned}\right\} . \qquad ...(4)$$

Now eliminating l_1, m_1, n_1 and l_2, m_2, n_2 from (4), for this we have

Geometry and Vectors

$$\begin{vmatrix} l_1 & l_2 & 0 \\ m_1 & m_2 & 0 \\ n_1 & n_2 & 0 \end{vmatrix} \times \begin{vmatrix} l_2 & l_1 & 0 \\ m_2 & m_1 & 0 \\ n_2 & n_1 & 0 \end{vmatrix} = 0$$

($\because$ the value of each determinant is zero)

Multiplying both determinants row by row, we have

$$\begin{vmatrix} 2l_1l_2 & l_1m_2 + l_2m_1 & l_1n_2 + l_2n_1 \\ l_2m_1 + l_1m_2 & 2m_1m_2 & m_1n_2 + m_2n_1 \\ l_2n_1 + l_1n_2 & m_2n_1 + m_1n_2 & 2n_1n_2 \end{vmatrix} = 0. \qquad ...(5)$$

Putting the values given in (4) in (5), we get

$$\begin{vmatrix} 2a & 2h & 2g \\ 2h & 2b & 2f \\ 2g & 2f & 2c \end{vmatrix} = 0$$

$$\Rightarrow \quad \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

$$\Rightarrow \quad a(bc - f^2) - h(hc - gf) + g(hf - gb) = 0$$

$$\Rightarrow \quad \mathbf{abc + 2fgh - af^2 - bg^2 - ch^2 = 0.}$$

This is the required condition.

• 16.3. EQUATION OF A PAIR OF LINES THROUGH THE ORIGIN AND PARALLEL TO THE LINES $l_1x + m_1 y + n_1 = 0$ AND $l_2x + m_2 y + n_2 = 0$.

Let the equations through the origin and parallel to the lines $l_1x + m_1 y + n_1 = 0$ and $l_2x + m_2 y + n_2 = 0$ be

$$\left.\begin{aligned} l_1x + m_1 y = 0 \\ \text{and} \quad l_2x + m_2 y = 0 \end{aligned}\right\} \qquad ...(1)$$

where l_1, l_2, m_1, m_2 are governed by the equation (4) in above section **2**.

Now the combined equation is

$$(l_1x + m_1 y)(l_2x + m_2 y) = 0$$

or
$$l_1l_2x^2 + xy(l_1m_2 + l_2m_1) + m_1m_2 y^2 = 0. \qquad ...(2)$$

Putting $l_1l_2 = a$, $m_1m_2 = b$, $l_1m_2 + l_2m_1 = 2h$ in (2), we get

$$\mathbf{ax^2 + 2hxy + by^2 = 0.}$$

This is the required equation of a pair of lines.

• 16.4. ANGLE BETWEEN THE PAIR OF LINES $l_1x + m_1 y + n_1 = 0$ and $l_2x + m_2 y + n_2 = 0$.

The angle between the lines $l_1x + m_1 y + n_1 = 0$ and $l_2x + m_2 y + n_2 = 0$ is

$$\tan\theta = \frac{2\sqrt{h^2 - ab}}{a + b}.$$

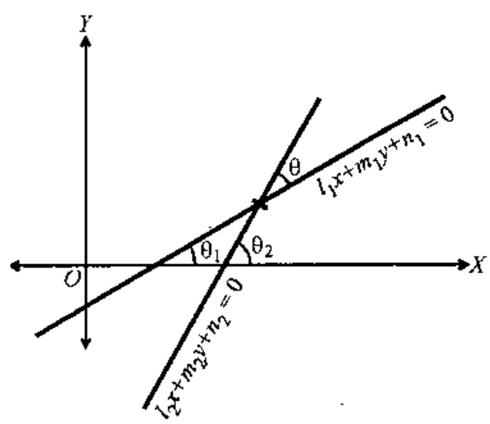


Fig. 1

• 16.5. CONDITION FOR PERPENDICULARITY OF A PAIR OF LINES $ax^2+2hxy+by^2+2gx+2fy+c=0$.

Let θ be the angle between the a pair of lines, then we have,

$$\tan\theta = \frac{2\sqrt{h^2-ab}}{a+b}.$$

From this equation, if $a+b=0$, then $\theta = 90°$ *i.e.* the lines are perpendicular. Hence, the required condition for the perpendicularity of the lines is given by

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

is

$$\boldsymbol{a+b=0}.$$

• 16.6. CONDITION FOR PARALLELISM OF THE LINES $ax^2+2hxy+by^2+2gx+2fy+c=0$.

Since the angle between the lines is governed by the relation

$$\tan\theta = \frac{2\sqrt{h^2-ab}}{a+b}.$$

If the lines are parallel, then $\tan\theta = 0$, then we get

$$h^2-ab=0 \quad \text{or} \quad \boldsymbol{h^2=ab}.$$

This is the required condition for parallelism.

• 16.7. CONDITION FOR COINCIDENCE OF THE LINES $ax^2+2hxy+by^2+2gx+2fy+c=0$.

The condition for coincidence of the pair of lines $ax^2+2hxy+by^2+2gx+2fy+c=0$ is given by

$$\boldsymbol{h^2-ab=0,\ g^2-ac=0,\ f^2-bc=0}.$$

• 16.8. POINT OF INTERSECTION OF THE LINES $ax^2+2hxy+by^2+2gx+2fy+c=0$.

The point of intersection of the lines represented by $ax^2+2hxy+by^2+2gx+2fy+c=0$ is given by

$$\left(\sqrt{\frac{f^2-bc}{h^2-ab}},\ \sqrt{\frac{g^2-ac}{h^2-ab}}\right).$$

• 16.9. CONDITION FOR THE GENERAL EQUATION OF SECOND DEGREE TO BE A CIRCLE

Let the general equation of second degree in x and y be

$$ax^2+2hxy+by^2+2gx+2fy+c=0. \qquad ...(1)$$

Definition. *The locus of a point in a plane, which moves in such a way that its distance from fixed point is always constant, is called* ***circle.***

Then by the definition of a circle, the equation of a circle is

$$(x-\alpha)^2+(y-\beta)^2=r^2 \qquad ...(2)$$

where (α, β) is the centre and r its radius.

Comparing (1) with (2) and then we observed that the general equation (1) represents a circle if

$$\boldsymbol{a=b\neq 0 \ \text{and}\ h=0}.$$

Therefore, this is the required condition for the general equation of second degree to be a circle.

• 16.10. CONDITION FOR THE GENERAL EQUATION OF SECOND DEGREE TO BE A PARABOLA

Let the general equation of second degree in x and y be

$$ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0. \qquad ...(1)$$

Definition. *The locus of a point, which moves so that its distance from a fixed point (called focus) is equal to the distance of this variable point from a fixed straight line, is called a* ***parabola.***

Let $lx + my + n = 0$ be a fixed straight line and (α, β) be a fixed point, then by the definition of the parabola, the equation of parabola is

$$\frac{lx + my + n}{\sqrt{l^2 + m^2}} = \sqrt{(x - \alpha)^2 + (y - \beta)^2}$$

or $\qquad (lx + my + n)^2 = (l^2 + m^2)\,[(x - \alpha)^2 + (y - \beta)^2]$

or $\qquad l^2x^2 + m^2y^2 + n^2 + 2lmxy + 2lnx + 2mny = (l^2 + m^2)\,(x^2 + y^2 - 2\alpha x - 2\beta y + \alpha^2 + \beta^2)$

or $\qquad m^2x^2 + l^2y^2 = 2lmxy + 2x\,(-\,ln - \alpha l^2 - \alpha m^2)$

$$+\, 2y\,(-\,mn - \beta l^2 - \beta m^2) + (\alpha^2 + \beta^2)\,(l^2 + m^2) - n^2 = 0$$

or $\qquad$ takes the form :

$$(mx - ly)^2 + 2Gx + 2Fy + C = 0. \qquad ...(2)$$

From (2) it has been observed that the second degree terms in the general equation of a parabola forms a perfect square. Therefore, using this conclusion on the general equation of second degree equation (1), we can say that

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

reduces to a parabola if

$$\boldsymbol{h^2 - ab = 0.}$$

This is the required condition.

• 16.11. CONDITION FOR THE GENERAL EQUATION OF SECOND DEGREE TO BE AN ELLIPSE

Let the general equation of second degree in x, y be

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0. \qquad ...(1)$$

Definition. *The locus of a point which* **means** *in such a way that its distance from a fixed point is always less than its distance from a fixed line, is called an* ***ellipse.***

On the **based** as discussed in above postulates, the required condition for the general equation given (1) to be an ellipse is

$$\boldsymbol{h^2 < ab.}$$

• 16.12. CONDITION FOR THE GENERAL EQUATION OF SECOND DEGREE TO BE A HYPERBOLA

The general equation of second degree in x, y is

$$ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0. \qquad ...(1)$$

Definition. *The locus of a point which moves so that its distance from a fixed point is always greater than its distance from a fixed line, is called* ***hyperbola.***

Therefore, (1) becomes the equation of a hyperbola if

$$\boldsymbol{h^2 > ab.}$$

This is the required condition.

SOLVED EXAMPLES

Example 1. *Show that the equation of second degree*

$$5x^2 - 2xy + 5y^2 + 2x - 10y - 7 = 0$$

represents an ellipse.

Solution. Compare the given equation with

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

we get $\qquad a = 5,\ b = 5,\ h = -\,1,\ f = -\,5,\ g = 1,\ c = -\,7.$

$$\begin{aligned}
\therefore \qquad \Delta &= abc + 2fgh - af^2 - bg^2 - ch^2 \\
&= (5)\,(5)\,(-\,7) + 2\,(-\,5)\,(1)\,(-\,1) - (5)\,(-\,5)^2 - 5\,(1)^2 - (-\,7)\,(-\,1)^2 \\
&= -\,175 + 10 - 125 - 5 + 7 \\
&= -\,288 \neq 0
\end{aligned}$$

and $\quad h^2 - ab = (-1)^2 - (5)(5) = 1 - 25 = -24 < 0$

$\therefore \quad h^2 < ab.$

Hence, the given equation is an ellipse.

Example 2. *Show that the equation*

$$8x^2 + 8xy + 2y^2 + 26x + 13y + 15 = 0$$

represents a pair of parallel lines.

Solution. Since the given equation is

$$8x^2 + 8xy + 2y^2 + 26x + 13y + 15 = 0. \qquad ...(1)$$

Compare (1) with the equation

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

we get $\quad a = 8, h = 4, b = 2, g = 13, f = 13/2, c = 15.$

$$\therefore \quad \Delta = abc + 2fgh - af^2 - bg^2 - ch^2$$

$$= (8)(2)(15) + 2\left(\frac{13}{2}\right)(13)(4) - 8\left(\frac{13}{2}\right)^2 - 2(13)^2 - 15(4)^2$$

$$= 240 + 676 - 338 - 338 - 240 = 0.$$

Also $\quad h^2 - ab = 16 - 16 = 0.$

Hence, the given equation represents a pair of parallel straight lines.

Example 3. *Show that the following equation represents a pair of lines. Find, also the angle between them :*

$$6x^2 + 13xy + 6y^2 + 8x + 7y + 2 = 0.$$

Solution. Since the given equation is

$$6x^2 + 13xy + 6y^2 + 8x + 7y + 2 = 0. \qquad ...(1)$$

Comparing (1) with the equation

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

we get, $\quad a = 6, h = 13/2, b = 6, g = 4, f = 7/2, c = 2.$

$$\therefore \quad \Delta = abc + 2fgh - af^2 - bg^2 - ch^2$$

$$= 6 \cdot 6 \cdot 2 + 2 \cdot \frac{7}{2} \cdot 4 \cdot \frac{13}{2} - 6 \cdot \left(\frac{7}{2}\right)^2 - 6 \cdot (4)^2 - 2 \cdot \left(\frac{13}{2}\right)^2$$

$$= 72 + 182 - \frac{147}{2} - 96 - \frac{169}{2}$$

$$= 158 - \frac{316}{2} = \frac{316 - 316}{2} = 0.$$

Thus the given equation represents a pair of straight lines.

The angle between these lines $= \tan^{-1}\left(\dfrac{2\sqrt{h^2 - ab}}{a + b}\right)$

$$= \tan^{-1}\left(\frac{2\sqrt{\left(\frac{13}{2}\right)^2 - 36}}{6 + 6}\right) = \tan^{-1}\left(\frac{5}{12}\right).$$

• SUMMARY

- General equation of second degree in x, y :

$$f(x, y, z) = ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$$

where a, b, c, f, g and h are constants.

- Equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of straight lines if $abc + 2fgh - af^2 - bg^2 + ch^2 = 0.$
- $ax^2 + 2hxy + by^2 = 0$ represents a pairs of straight lines passing through the origin.
- angle between the lines represented by $ax^2 + 2hxy + by^2 = 0$ is given by

$$\theta = \tan^{-1}\left[\frac{2\sqrt{h^2 - ab}}{a + b}\right]$$

- If $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of straight lines, then these lines are perpendicular if $a + b = 0$ and parallel to each other if $h^2 = ab$, $g^2 = ac$ and $f^2 = bc$.
- If the lines intersect each other then the point of intersection is

$$\left(\sqrt{\frac{f^2 - bc}{h^2 - ab}}, \sqrt{\frac{g^2 - ac}{h^2 - ab}} \right)$$

- The general equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ will represent :
  (i) a circle if $a = b \neq 0$ and $h = 0, \Delta \neq 0$
  (ii) a parabola if $h^2 = ab$, $\Delta \neq 0$
  (iii) a ellipse if $h^2 > ab$, $\Delta \neq 0$
  (iv) a hyperbola if $h^2 < ab$, $\Delta \neq 0$
  where $\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$.

• STUDENT ACTIVITY

1. Show that the equation $6x^2 + 13xy + 6y^2 + 8x + 7y + z = 0$ represents a pair of straight lines.

2. Show that $x^2 + 2xy + y^2 - 2x - 1 = 0$ represent a parabola.

• TEST YOURSELF

1. Show that the equation $25x^2 - 120xy + 144y^2 - 25x + 60y - 36 = 0$represents two parallel straight lines.
2. Show that the equation $3x^2 + 3y^2 - 2x + 3y - 5 = 0$ represents a circle.
3. Show that the following equation $11x^2 - 4xy + 14y^2 - 58x - 44y + 71 = 0$ represents an ellipse.
4. Show that the following equations :
   (i) $16x^2 - 24xy + 9y^2 - 104x - 172y + 44 = 0$ (i) $4x^2 - 4xy + y^2 - 8x - 6y + 5 = 0$.
   represents parabolas.

5. Show that the equation $3x^2 - 5xy - 2y^2 + 5x + 11y - 8 = 0$ represents a hyperbola.

OBJECTIVE EVALUATION

➤ FILL IN THE BLANKS :

1. The general equation of second degree in x and y has at least .... constants to find the nature of the equation.
2. The equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of straight lines if $abc + 2fgh - af^2 - bg^2 - ch^2 =$ ....

➤ TRUE OR FALSE :

Write T for true and F for false statement :

1. The equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a rectangular hyperbola if $a + b = 0$ and $ab - h^2 < 0$. (T/F)
2. The equation $x^2 + 2hxy + y^2 + 2gx + 2fy + c = 0$ represents a circle if $h = 1$. (T/F)
3. The equation $x^2 + 4xy + 4y^2 + 2gx + 2fy + c = 0$ represents a parabola. (T/F)

➤ MULTIPLE CHOICE QUESTIONS :

Choose the most appropriate are :

1. The equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a rectangular hyperbola if $h^2 > ab$ and :
   (a) $a + b = 0$ (b) $a - b = 0$ (c) $a = 0$ (d) $b = 0$.
2. The equation $9x^2 + 12xy + 16y^2 + 2gx + 2fy + c = 0$ represents :
   (a) Ellipse (b) Parabola (c) Hyperbola (d) Two straight lines.

ANSWERS

Fill in the Blanks :

1. Five 2. 0

True or False :

1. T 2. F

Multiple Choice Questions :

1. (a) 2. (b)

❑❑❑

UNIT 17

SYSTEM OF CONICS

STRUCTURE

- Standard Forms of Conics
- Reduction of General Equation of Second Degree into a Conic
- Determination of the Co-ordinates of the Centre of a Conic
  Equation of a Conic when the Origin is at the Centre
- Determination of the Lengths, Posiions and Direction of the Axes of the Central Conic
- Determination of the Eccentricity of a Central Conic
- Determination of the Co-ordinates of the Focus or Foci and the Equation of the directrix or Directrices
  - Summary
  - Student Activity
  - Exercise

LEARNING OBJECTIVES

After going through this unit you will learn :

What are conics in detail.

How to calculate the centre, lengths, directionof its areas and its eccentricity.

• 17.1. STANDARD FORMS OF CONICS

(I) Standard form of an ellipse. The standard form of an ellipse is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

Case 1. If $a > b$, then its major axis is x-axis and its minor axis is y-axis. Also the length of semi-major axis is a and semi-minor axis is b.

Centre is (0, 0), focus are $(\pm ae, 0)$ where $b^2 = a^2(1-e^2)$.

The equations of directrix are $x \pm a/e = 0$, also all the four vertices are $(\pm a, 0)$ $(0, \pm b)$.

Case 2. If $b > a$, then its major axis is y-axis and minor axis is x-axis of length $2b$ and $2a$ respectively.

Centre is (0, 0), focus are $(0, \pm be)$, where $a^2 = b^2(1-e^2)$.

Also, the equations of its directrix are $y \pm b/e = 0$, and all the four vertices are $(\pm a, 0)$, $(0, \pm b)$.

(II) Standard form of a parabola.

(i) $y^2 = 4ax$. Its vertex is (0, 0) and focus is $(a, 0)$.

Also the axis is x-axis and the equation of its directrix is $x + a = 0$.

(ii) $y^2 = -4ax$. Its vertex is (0, 0) and focus is $(-a, 0)$.

Also the axis is x-axis and the equation of its directrix is $x - a = 0$.

(iii) $x^2 = 4ay$. Its vertex is (0, 0) and its focus is $(0, a)$.

Also the axis is y-axis and its directrix is $y + a = 0$.

(iv) $x^2 = -4ay$. Its vertex is (0, 0) and its focus is $(0, -a)$.

Also the axis is y-axis and its directrix is $y - a = 0$.

(III) Standard form of a hyperbola. The standard form of a hyperbola is

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$$

Here, the centre is (0, 0) and focus are $(\pm ae, 0)$, where, $b^2 = a^2(e^2-1)$.

Also the vertex are $(\pm a, 0)$ and the equation of its directrix are $x \pm a/e = 0$.

• 17.2. REDUCTION OF GENERAL EQUATION OF SECOND DEGREE INTO A CONIC

Let the general equation of second degree in x and y be

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0. \qquad ...(1)$$

Change the equation (1) to the equation in which the term xy is removed. This can be done by putting $\theta = \frac{1}{2}\tan^{-1}\left(\frac{2h}{a-b}\right)$ such that the x change to $x\cos\theta - y\sin\theta$ and y to $x\sin\theta + y\sin\theta$. Thus transformed equation becomes

$$Ax^2 + By^2 + 2Gx + 2Fy + C = 0. \qquad ...(2)$$

Now there arises some cases :

Case I. If $A = B$, then the equation (2) becomes a circle.

Case II. If neither A nor B is zero, then (2) becomes

$$Ax^2 + 2Gx + By^2 + 2Fy + C = 0$$

or $$A\left(x^2 + 2\frac{G}{A}x\right) + B\left(y^2 + 2\frac{F}{B}y\right) + C = 0$$

or $$A\left(x^2 + 2\frac{G}{A}x + \frac{G^2}{A^2}\right) + B\left(y^2 + 2\frac{F}{B}y + \frac{F^2}{B^2}\right) - \frac{G^2}{A} - \frac{F^2}{B} + C = 0$$

or $$A\left(x + \frac{G}{A}\right)^2 + B\left(y + \frac{F}{B}\right)^2 = \frac{G^2}{A} + \frac{F^2}{B} - C.$$

Let $$\frac{G^2}{A} + \frac{F^2}{B} - C = D.$$

$\therefore$ $$A\left(x + \frac{G}{A}\right)^2 + B\left(y + \frac{F}{B}\right)^2 = D. \qquad ...(3)$$

Now by shifting the origin to the point $\left(-\frac{G}{A}, -\frac{F}{B}\right)$. The equation (5) is transformed to

$$Ax'^2 + By'^2 = D$$

or $$\frac{x'^2}{D/A} + \frac{y'^2}{D/B} = 1. \qquad ...(4)$$

Therefore the equation (6) represents an ellipse if both D/A and D/B are positive.

Also the equation (6) represents the hyperbola if D/A and D/B are of opposite sign

Case III. If either A or B is zero, let us suppose $A = 0$, then (4) becomes

$$By^2 + 2Fy + 2Gx + C = 0$$

or $$By^2 + 2Fy = -2Gx - C$$

or $$B\left[y^2 + \frac{2F}{B}y\right] = -2Gx - C$$

or $$B\left[y^2 + 2\frac{F}{B}y + \frac{F^2}{B^2}\right] = -2Gx + \frac{F^2}{B} - C$$

or $$B\left(y + \frac{F}{B}\right)^2 = -2G\left(x - \frac{F^2}{2BG} + \frac{C}{2G}\right)$$

or $$\left(y + \frac{F}{B}\right)^2 = -\frac{2G}{B}\left(x - \frac{F^2}{2BG} + \frac{C}{2G}\right).$$

Now shift the origin to $\left(\frac{F^2}{2BG} - \frac{C}{2G}, \frac{-F}{B}\right)$, then above equation is transformed to

$$y'^2 = -\frac{2G}{B}x'. \qquad ...(7)$$

This equation represents a parabola. Similarly if we take $B = 0$, then we shall again obtain a parabola.

• 17.3. DETERMINATION OF THE CO-ORDINATES OF THE CENTRE OF A CONIC

Working Procedure for Finding the Centre :

(I) *First treating the given conic as a function $\phi(x, y) = 0$.*

(II) *Now differentiate $\phi(x, y) = 0$ partially with respect to x and y respectively and obtain* $\frac{\partial\phi}{\partial x}$ *and* $\frac{\partial\phi}{\partial y}$.

(III) *And solving* $\frac{\partial\phi}{\partial x} = 0$ *and* $\frac{\partial\phi}{\partial y} = 0$ *for x and y. These value of x and y give the centre (x, y) of the conic.*

• 17.4. EQUATION OF A CONIC WHEN THE ORIGIN IS AT THE CENTRE

Let the equation of a conic be

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \quad ...(1)$$

The required equation of a conic referred to centre as origin is given by

or $$\mathbf{ax^2 + 2hxy + by^2 + \frac{\Delta}{ab - h^2} = 0} \quad ...(2)$$

where $\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$.

SOLVED EXAMPLE

Example 1. *Find the co-ordinates of the centre of the conic whose equation is* $32x^2 + 52xy - 7y^2 - 64x - 52y - 148 = 0$ *and hence the equation of the conic referred to centre as origin.*

Solution. Let us assume

$$\phi(x, y) \equiv 32x^2 + 52xy - 7y^2 - 64x - 52y - 148 = 0. (1)$$

Differentiating (1) partially w.r.t. x and y respectively, we get

$$\frac{\partial\phi}{\partial x} = 64x + 52y - 64$$

and $$\frac{\partial\phi}{\partial y} = 52x - 14y - 52.$$

Now solving $\frac{\partial\phi}{\partial x} = 0$ and $\frac{\partial\phi}{\partial y} = 0$, we get $x = 1$, $y = 0$.

Hence (1, 0) is the centre of the given conic.

Now the new constant $d = g\alpha + f\beta + c$

$$= (-32)(1) - 26(0) - 148 = -32 - 148 = -180.$$

Therefore, the equation of the conic referred to centre as origin is

$$32x^2 + 52xy - 7y^2 - 180 = 0$$

or $$\frac{32}{180}x^2 + \frac{52}{180}xy - \frac{7}{180}y^2 = 1.$$

• 17.5. DETERMINATION OF THE LENGTHS, POSITIONS AND DIRECTION OF THE AXES OF THE CENTRAL CONIC

Let the equation of a central conic be

$$Ax^2 + 2Hxy + By^2 = 1. \quad ...(1)$$

Let us choose a concentric circle of radius r of the equation

$$x^2 + y^2 = r^2$$

or $$\frac{x^2 + y^2}{r^2} = 1. \quad ...(2)$$

The equation of the lines joining the origin to the points of intersection of (1) and (2) is obtained by making (1) homogeneous with help of (2), we have

$$Ax^2 + 2Hxy + By^2 - \left(\frac{x^2 + y^2}{r^2}\right) = 0$$

or $$\left(A - \frac{1}{r^2}\right)x^2 + 2Hxy + \left(B - \frac{1}{r^2}\right)y^2 = 0. \quad ...(3)$$

These two lines given in (3) will coincide if and only if the circle and central conic touch each other at the extremities of either axis of the conic and therefore, we have

$$\left(A - \frac{1}{r^2}\right)\left(B - \frac{1}{r^2}\right) - H^2 = 0$$

($\because$ For a pair of coincident straight lines we know that $ab - h^2 = 0$)

or $$\frac{1}{r^4}-\frac{1}{r^2}(A+B)+(AB-H^2)=0. \qquad ...(4)$$

or $$(AB-H^2)\,r^4-(A+B)\,r^2+1=0 \qquad ...(5)$$

Since the equation (5) is a quadratic equation in r^2 so (5) gives two values of r^2. Let these values of r^2 be r_1^2 and r_2^2.

Now using (3) and (4), we get

$$\left[\left(A-\frac{1}{r^2}\right)x+Hy\right]^2=0$$

which is obtained by multiplying (3) by $\left(A-\frac{1}{r^2}\right)$ and using (4).

Thus the equation of either axis is

$$\left(A-\frac{1}{r^2}\right)x+Hy=0. \qquad ...(6)$$

Since r_1^2 and r_2^2 are the two values of r^2 so that there arises some cases.

Case I. If both r_1^2 and r_2^2 are positive and $r_1^2>r_2^2$ then the conic will be an ellipse and the greater value of r *i.e.* r_1 will be semi-major axis while the **smaller** value of r *i.e.* r_2 will be semi-minor axis. Hence the length of major axis is $2r_1$ and the length of minor axis is $2r_2$.

Case II. If $r_1^2>0$ and $r_2^2<0$, then the conic will be hyperbola and the real value of r *i.e.* r_1 will be semi-transverse axis while the imaginary value of r *i.e.* r_2 will be semi-conjugate axis. Hence, the lengths of transverse axis and conjugate axis are respectively $2r_1$ and $2r_2$.

Now the equation (6) corresponding to r_1^2 and r_2^2 will give the equations of semi-axes of the conic as follows :

$$\left(A-\frac{1}{r_1^2}\right)x+Hy=0 \qquad ...(7)$$

and $$\left(A-\frac{1}{r_2^2}\right)x+Hy=0. \qquad ...(8)$$

Direction of axes. The equation (1) can be rewritten as

$$ax^2+2hxy+by^2+d=0$$

where $$d=-\frac{\Delta}{ab-h^2}.$$

In order to remove the term xy from above equation we rotate the axes through an angle θ keeping the origin fixed. Then we get

$$\theta=\frac{1}{2}\tan^{-1}\left(\frac{2h}{a-b}\right)$$

or $$\tan 2\theta=\frac{2h}{a-b} \quad \text{or} \quad \frac{2\tan\theta}{1-\tan^2\theta}=\frac{2h}{a-b}$$

or $$h\tan^2\theta+(a-b)\tan\theta-h=0. \qquad ...(9)$$

This equation (9) is quadratic in $\tan\theta$ so it have two values of $\tan\theta$, say $\tan\theta_1$, $\tan\theta_2$. These two values give the slopes of the axes of the given conic.

• 17.6. DETERMINATION OF THE ECCENTRICITY OF A CENTRAL CONIC

1. Eccentricity of an ellipse. Since we know that if $r_1^2>r_2^2>0$, then the conic will be ellipse and its eccentricity e is given by

$$e=\sqrt{1-\frac{r_2^2}{r_1^2}}.$$

2. Eccentricity of hyperbola. If r_1^2 and r_2^2 are of opposite sign, then the conic will be hyperbola and its eccentricity e is given by

$$e=\sqrt{1+\left|\frac{r_2^2}{r_1^2}\right|}.$$

SOLVED EXAMPLES

Example 1. *Find the lengths and the equation of the axes of the conic whose equation is*

$$36x^2 + 24xy + 29y^2 - 72x + 126y + 81 = 0.$$

Solution. Let us assume

$$\phi(x, y) \equiv 36x^2 + 24xy + 29y^2 - 72x + 126y + 81 = 0 \quad ...(1)$$

Differentiating (1) partially w.r.t. x and y respectively, we get

$$\frac{\partial \phi}{\partial x} = 72x + 24y - 72$$

and
$$\frac{\partial \phi}{\partial y} = 24x + 58y + 126.$$

Now solving $\frac{\partial \phi}{\partial x} = 0$ and $\frac{\partial \phi}{\partial y} = 0$, we get $x = 2,\ y = -3$. Hence the centre of this conic is $(2, -3)$. Therefore the new constant

$$\boxed{d = \alpha g + f\beta + c}$$

Here $g = -36, f = 63, c = 81$ and $\alpha = 2, \beta = -3$

$\therefore \quad d = 2(-36) + 63(-3) + 81 = -72 - 189 + 81 = -180.$

Thus the equation of the conic referred to centre as origin is

$$36x^2 + 24xy + 29y^2 - 180 = 0$$

or
$$\frac{36}{180}x^2 + \frac{24}{180}xy + \frac{29}{180}y^2 = 1$$

which is the form of

$$Ax^2 + 2Hxy + By^2 = 1$$

where
$$A = \frac{36}{180}, H = \frac{12}{180}, B = \frac{29}{180}.$$

Now the lengths of the axes are given by

$$\frac{1}{r^4} - \frac{1}{r^2}(A + B) + (AB - H^2) = 0$$

or
$$\frac{1}{r^4} - \frac{1}{r^2}\left(\frac{36}{180} + \frac{29}{180}\right) + \left(\frac{36}{180} \cdot \frac{29}{180} - \left(\frac{12}{180}\right)^2\right) = 0$$

or
$$\frac{1}{r^4} - \frac{65}{180} \cdot \frac{1}{r^2} + \frac{900}{(180)^2} = 0$$

or
$$\left(\frac{1}{r^2} - \frac{45}{180}\right)\left(\frac{1}{r^2} - \frac{20}{180}\right) = 0$$

or
$$r^2 = 9 \text{ or } 4$$

i.e.
$$r_1^2 = 9, r_2^2 = 4 \Rightarrow r_1 = 3, r_2 = 2.$$

Obviously r_1^2 and r_2^2 both are positive, therefore the given conic is an ellipse.

Hence the lengths of major axis $= 2r_1 = 2(3) = 6$

and the length of minor axis $= 2r_2 = 2(2) = 4.$

Now the equation of major axis referred to $(2, -3)$ as origin is

$$\left(A - \frac{1}{r_1^2}\right)x + Hy = 0$$

or
$$\left(\frac{36}{180} - \frac{1}{9}\right)x + \frac{12}{180}y = 0 \quad \text{or} \quad 4x + 3y = 0.$$

Now shift the origin back from $(2, -3)$ by putting $x - 2$ for x and $y + 3$ for y, we get

$$4(x - 2) + 3(y + 3) = 0 \quad \text{or} \quad 4x + 3y + 1 = 0.$$

This is the required equation of major axis referred to $(0, 0)$ as origin.

And the equation of minor axis referred to $(2, -3)$ as origin is

$$\left(A - \frac{1}{r_2^2}\right)x + Hy = 0$$

or
$$\left(\frac{36}{180} - \frac{1}{4}\right)x + \frac{12}{180}y = 0 \quad \text{or} \quad 3x + 4y = 0.$$

Again shift the origin back from $(2, -3)$ by putting $x - 2$ for x and $y + 3$ for y, we get

$$3(x-2)-4(y+3)=0 \quad \text{or} \quad 3x-4y-18=0.$$

This is the required equation of minor axis referred to (0, 0) as origin.

• 17.7. DETERMINATION OF THE CO-ORDINATES OF THE FOCUS OR FOCI AND THE EQUATION OF THE DIRECTRIX OR DIRECTRICES

Let the equation of a conic be

$$ax^2+2hxy+by^2+2gx+2fy+c=0. \qquad ...(1)$$

Case I. If the conic (1) is an ellipse and let its semi-major axis be of the length a and its slope be $\tan\theta$. Let (α, β) be its centre, then the equation its axis is

$$\frac{x-\alpha}{\cos\theta}=\frac{y-\beta}{\sin\theta}=r \qquad ...(2)$$

where r is the distance any point $P(x, y)$ from (α, β) on this axis. In case of ellipse this distance $r=\pm ae$ from the centre (α, β). Then the co-ordinates of the foci of the ellipse are

$$(\alpha \pm ae\cos\theta, \beta \pm ae\sin\theta)$$

i.e. $\quad (\alpha + ae\cos\theta, \beta + ae\sin\theta)$ and $(\alpha - ae\cos\theta, \beta - ae\sin\theta)$.

Directrix. Let Q and Q' be the points on the semi-major axis, where the directrices of the ellipse meets its axis is at a distance $r=\pm a/e$ from its vertices. Thus the co-ordinates of Q and Q' are respectively,

$$Q\,(\alpha+(a/e)\cos\theta, \beta+(a/e)\sin\theta)$$

and $\quad Q'\,(\alpha-(a/e)\cos\theta, \beta-(a/e)\sin\theta).$

Equation (2) can be written as

$$x\sin\theta - y\cos\theta - \alpha\sin\theta + \beta\cos\theta = 0. \qquad ...(3)$$

Since the directrix is perpendicular to the axis (3) so it equation is given by

$$x\sin\theta + y\cos\theta + \lambda = 0. \qquad ...(4)$$

This equation (4) passes through the poin Q and Q'.

Hence the equation of the directrices of an ellipse are given by

$$(x-\alpha)\cos\theta + (y-\beta)\sin\theta \pm a/e = 0. \qquad ...(5)$$

Case II. If the given conic is a hyperbola, then the foci and the directrices are same as above, *only difference is that in this case a is the length of semi-transverse axis.*

Case III. If the conic is a parabola and let (α, β) be its vertex and $\tan\theta$ be the slope of its axis and $4a$ be the length of its latus rectum. Then the equation of the axis of the parabola is

$$\frac{x-\alpha}{\cos\theta}=\frac{y-\beta}{\sin\theta}=r$$

where r is the distance of the point $P(x, y)$ from (α, β).

Since the focus of the parabola lies on its axis and is at the distance 'a' from the vertex. Hence the co-ordinates of the focus of the parabola are

$$(\alpha + a\cos\theta, \beta + a\sin\theta).$$

Directrix. Now above equation of axis can also be written as

$$x\sin\theta - y\cos\theta - \alpha\sin\theta + \beta\cos\theta = 0.$$

Let Q be any point of its axis, through which the directrix passes and is at the distance $r=-a$ from the vertex. Further since we know that the directrix is perpendicular to the axis, so that the equation is

$$x\cos\theta + y\sin\theta + \lambda = 0.$$

This directrix passes through $Q\,(\alpha - a\cos\theta, \beta - a\sin\theta)$.

Hence the required equation of the directrix of the parabola is

$$(x-\alpha)\cos\theta + (y-\beta)\sin\theta + a = 0.$$

REMARKS

- For the co-ordinates of extremities of major axis put $r=\pm a$ in $(\alpha + r\cos\theta, \beta + r\sin\theta)$.
- For the co-ordinates of the extremities of minor axis put $r=\pm b$ in $(\alpha + r\cos\theta, \beta + r\sin\theta)$ where (α, β) is the centre and $\tan\theta$ is the slope of the minor axis.
- For the co-ordinates of the extremities of latus rectum, take (α, β) as focus and r = distance of focus from the extremities of latus rectum, and $\tan\theta$ is the slope of latus rectum.
- *The equation of latus rectum will be a line parallel to the minor or conjugate axis that* passes through the focus and hence its equation can be easily found.

➤ Latus rectum of ellipse or hyperbola $= 2\,\dfrac{|r_2^2|}{r_1}$.

SOLVED EXAMPLES

Example 1. *Find the centre, lengths, and equations of axes, eccentricity, foci, latus rectum and equations of the directrices of the ellipse*

$$40x^2 + 36xy + 25y^2 - 196x - 122y + 205 = 0.$$

Solution. Let us assume

$$\phi(x, y) \equiv 40x^2 + 36xy + 25y^2 - 196x - 122y + 205 = 0 \quad ...(1)$$

$\therefore$
$$\frac{\partial \phi}{\partial x} = 80x + 36y - 196$$

and
$$\frac{\partial \phi}{\partial y} = 36x + 50y - 122.$$

Solving $\dfrac{\partial \phi}{\partial x} = 0, \dfrac{\partial \phi}{\partial y} = 0$, we get

$$x = 2, y = 1.$$

Thus the centre is (2, 1).

The new constant $d = g\alpha + f\beta + c$.

Here $g = -98, f = -61, c = 205$ and $\alpha = 2, \beta = 1$, then

$$d = -98\,(2) - 61\,(1) + 205$$
$$= -196 - 61 + 205 = -52.$$

Hence the equation of the conic referred to (2, 1) as origin is

$$40x^2 + 36xy + 25y^2 - 52 = 0$$

or
$$\frac{40}{52}x^2 + \frac{36}{52}xy + \frac{25}{52}y^2 = 1$$

or
$$Ax^2 + 2Hxy + By^2 = 1$$

where
$$A = \frac{40}{52}, B = \frac{25}{52}, H = \frac{18}{52}.$$

Now the lengths of the axes are given by

$$\frac{1}{r^4} - \frac{1}{r^2}(A + B) + (AB - H^2) = 0$$

or
$$\frac{1}{r^4} - \frac{1}{r^2}\left(\frac{40}{52} + \frac{25}{52}\right) + \left(\frac{40}{52}\cdot\frac{25}{52} - \left(\frac{18}{52}\right)^2\right) = 0$$

or
$$\frac{1}{r^4} - \frac{65}{52}\cdot\frac{1}{r^2} + \frac{676}{(52)^2} = 0$$

$$\left(\frac{1}{r^2} - \frac{13}{52}\right)\left(\frac{1}{r^2} - \frac{52}{52}\right) = 0$$

$\Rightarrow$
$$r^2 = 4, 1 \ \ i.e.\ \ r_1^2 = 4, r_2^2 = 1.$$

Therefore, the lengths of major axis and minor axis are respectively given by 4, 2.

Now the equation of the major axis is

$$\left(A - \frac{1}{r_1^2}\right)x + Hy = 0$$

or
$$\left(\frac{40}{52} - \frac{1}{4}\right)x + \frac{18}{52}y = 0$$

or
$$3x + 2y = 0.$$

Shift the origin back from (2, 1) by putting $x - 2$ for x and $y - 1$ for y, we get

$$3\,(x - 2) + 2\,(y - 1) = 0$$

or
$$3x + 2y - 8 = 0.$$

And the equation of minor axis is

$$\left(A - \frac{1}{r_2^2}\right)x + Hy = 0$$

or
$$\left(\frac{40}{52} - 1\right)x + \frac{18}{52}y = 0$$

or $$-2x+3y=0.$$

Shift the origin back from (2, 1) by putting $x-2$ for x and $y-1$ for y, we get

$$-2(x-2)+3(y-1)=0$$

or $$-2x+3y+1=0$$

or $$2x-3y-1=0.$$

Hence the required major and minor axes are respectively

$$3x+2y-8=0 \text{ and } 2x-3y-1=0.$$

Eccentricity of the ellipse. The eccentricity e is given by

$$e=\sqrt{1-\frac{r_2^2}{r_1^2}}=\sqrt{1-\frac{1}{4}}=\sqrt{3/4}=\sqrt{3}/2.$$

Latus rectum. The latus rectum of an ellipse is given by

$$\text{L.R.}=\frac{2r_2^2}{r_1} \qquad (\because\ r_1>r_2)$$

$$=\frac{2(1)^2}{2}=1.$$

Foci. The foci lie on the major axis $3x+2y-8=0$.

Therefore, the slope of this axis is given by

$$\tan\theta=-\frac{3}{2}$$

$\therefore$ $$\sin\theta=3/\sqrt{13},\ \cos\theta=-2/\sqrt{13} \qquad (\because\ \theta \text{ is obtuse})$$

The distance of the foci from the centre (2, 1) is

$$r=\pm ae=\pm r_1e=\pm 2\cdot\frac{\sqrt{3}}{2}=\pm\sqrt{3}.$$

$\therefore$ The foci of the ellipse are

$$(\alpha+r\cos\theta,\beta+r\sin\theta)\equiv\left(2\pm\sqrt{3}\left(-\frac{2}{\sqrt{13}}\right),1\pm\sqrt{3}\left(\frac{3}{\sqrt{13}}\right)\right)$$

$$\equiv\left[2-\frac{2\sqrt{3}}{\sqrt{13}},1+\frac{3\sqrt{3}}{\sqrt{13}}\right],\left[2+\frac{2\sqrt{3}}{\sqrt{13}},1-\frac{3\sqrt{3}}{\sqrt{13}}\right].$$

Equation of the Directrices :

The equation of the directrices of an ellipse are given by

$$(x-\alpha)\cos\theta+(y-\beta)\sin\theta\pm r_1/e=0$$

$$(x-2)(-2/\sqrt{13})+(y-1)(3/\sqrt{13})\pm 2(2/\sqrt{3})=0$$

or $$-\frac{2}{\sqrt{13}}(x-2)+\frac{3}{\sqrt{13}}(y-1)\pm\frac{4}{\sqrt{3}}=0$$

or $$-2(x-2)+3(y-1)\pm 4\sqrt{\frac{13}{3}}=0$$

or $$-2x+4+3y-3+4\sqrt{\frac{13}{3}}=0$$

and $$-2x+4+3y-3-4\sqrt{\frac{13}{3}}=0$$

or $$2x-3y-\left(1+4\sqrt{\frac{13}{3}}\right)=0$$

and $$2x-3y-\left(1-4\sqrt{\frac{13}{3}}\right)=0$$

These are the required equation of directrices.

• SUMMARY

- Equation $ax^2+2hxy+by^2+2gx+2fy+c=0$ is transformed to the equation $Ax^2+By^2+2Gx+2Fy+c=0$ by putting $x\cos\theta-y\sin\theta$ for x and $x\sin\theta+y\cos\theta$ for x in given equation, where $\theta=\frac{1}{2}\tan^{-1}\left(\frac{2h}{a-b}\right)$.

- **Coordinates of centre of the given conic :**
  (i) Find $\frac{\partial \phi}{\partial x}$ and $\frac{\partial \phi}{\partial y}$
  (ii) Solve $\frac{\partial \phi}{\partial x} = 0$ and $\frac{\partial \phi}{\partial y} = 0$, the values of x and y so obtained give the centre (x, y).
- **Equation of conic with origin as centre**

$$ax^2 + 2hxy + by^2 = \frac{\Delta}{ab - h^2} = 0$$

  where $\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$.
- Equation of central conic is $Ax^2 + 2Hxy + By^2 = 1$
- Equation of semi-axes of a central conic $Ax^2 + 2Hxy + By^2 = 1$ are given by

$$\left(A - \frac{1}{r_1^2}\right)x + Hy = 0 \text{ and } \left(A - \frac{1}{r_2^2}\right)x + Hy = 0$$

  where r_1^2 and r_2^2 are the roots of the equation

$$(AB - H^2)\, r^4 - (A + B)\, r^2 + 1 = 0$$

- If $r_1^2 > r_2^2 > 0$ then the conic is an ellipse, inthis case length of major axis is $2r_1$ and the length of minor axis is $2r_2$ and its eccentricity is $\sqrt{1 - \frac{r_1^2}{r_2^2}}$. If $r_1^2 > 0$ and $r_2^2 < 0$, then the conic is a hyperbola, in this case length of transverse axis is $2r_1$ and the length of conjugate axis is $2r_2$ and its eccentricity is $\sqrt{1 + \left(\frac{r_2^2}{r_1^2}\right)}$

• STUDENT ACTIVITY

1. Solve the coordinates of the centre of the conic $32x^2 + 52xy - 7y^2 - 64x - 52y - 148 = 0$.

2. Show that the conic $36x^2 + 24xy + 2ay^2 - 72x + 126y + 81 = 0$ is an ellipse.

• TEST YOURSELF

1. Find the co-ordinate of the centre of the conic whose equation is
$$36x^2 + 24xy + 29y^2 - 72x + 126y + 81 = 0.$$
2. Find the centre of the conic section
$$13x^2 - 18xy + 37y^2 + 2x + 14y - 2 = 0.$$
Also, find the equation of the conic section referred its centre as origin.
3. Find the lengths and equations of the axes of the conic
$$8x^2 + 4xy + 5y^2 - 24x - 24y = 0.$$
4. Find the equation of the the directrices of the conic
$$55x^2 - 30xy + 39y^2 - 40x - 24y - 464 = 0.$$
5. Find the centre, foci and the equations of the axes of the ellipse
$$x^2 + xy + y^2 - x + 4y + 3 = 0.$$
Also trace it.
6. Find the lengths and the equations of the axes of the conic
$$5x^2 - 6xy + 5y^2 + 26x - 22y + 29 = 0.$$

ANSWERS

1. $(2, -3)$ **2.** $\left(-\frac{1}{4}, -\frac{1}{4}\right)$, $13x^2 - 18xy + 37y^2 - 4 = 0.$

3. Major axis : $2x + y = 4$, length $= 6$; Minor axis : $x - 2y + 3 = 0$, length $= 4$.

4. $3x + 5y = 36$, $3x + 5y = -28$.

5. Centre $(2, -3)$, foci $\left(2 \mp \frac{\sqrt{8}}{\sqrt{3}}, -3 \pm \frac{\sqrt{8}}{\sqrt{3}}\right)$

Major axis : $x + y + 1 = 0$, Minor axis : $x - y - 5 = 0$.

UNIT 18

CONFOCAL CONICS

STRUCTURE

- Confocal Conics
- Equation of Confocals to an Ellipse
- Some Properties of Confocal Conics
  - Summary
  - Student Activity
  - Test Yourself

LEARNING OBJECTIVES

After going through this unit you will learn :

How we define confocal conics and what are their properties.

• 18.1. CONFOCAL CONICS

Definition. *Those conics which have same two points as their* ***foci*** *or those conics having same focal points, are called* ***confocal conics*** *are also having same* ***axes.***

• 18.2. EQUATION OF CONFOCALS TO AN ELLIPSE

Let the equation of an ellipse be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1. \qquad \text{...(1)}$$

If $a > b$, then the co-ordinates of its foci are $(\pm ae, 0)$ where $e=\frac{\sqrt{a^2-b^2}}{a}$.

$\therefore \qquad (\pm ae, 0) \equiv (\pm \sqrt{a^2-b^2}, 0)$.

Let us consider the equation

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1. \qquad \text{...(2)}$$

Now the co-ordinates of the foci are $[\pm \sqrt{(a^2+\lambda)-(b^2+\lambda)}, 0]$ or $(\pm \sqrt{a^2-b^2}, 0)$, for all values of λ.

Therefore, the foci of (2) and (1) are same. Hence, the equation (2) are the confocal conics of (1).

• 18.3. SOME PROPERTIES OF CONFOCAL CONICS

Property I. *To prove that confocals cut at right angles. Let the two confocals be*

$$\frac{x^2}{a^2+\lambda_1}+\frac{y^2}{b^2+\lambda_1}=1 \qquad \text{...(1)}$$

and

$$\frac{x^2}{a^2+\lambda_2}+\frac{y^2}{b^2+\lambda_2}=1. \qquad \text{...(2)}$$

Suppose both confocals (1) and (2) intersects at (α, β) so that (α, β) will satisfy (1) and (2), we get

$$\frac{\alpha^2}{a^2+\lambda_1}+\frac{\beta^2}{b^2+\lambda_1}=1$$

and

$$\frac{\alpha^2}{a^2+\lambda_2}+\frac{\beta^2}{b^2+\lambda_2}=1.$$

Substracting these equations, we get

$$\alpha^2\left[\frac{1}{a^2+\lambda_1}-\frac{1}{a^2+\lambda_2}\right]+\beta^2\left[\frac{1}{b^2+\lambda_1}-\frac{1}{b^2+\lambda_2}\right]=0$$

or
$$\frac{(\lambda_2-\lambda_1)\,\alpha^2}{(a^2+\lambda_1)\,(a^2+\lambda_2)}+\frac{(\lambda_2-\lambda_1)\,\beta^2}{(b^2+\lambda_1)\,(b^2+\lambda_2)}=0$$

or
$$\frac{\alpha^2}{(a^2+\lambda_1)\,(a^2+\lambda_2)}+\frac{\beta^2}{(b^2+\lambda_1)\,(b^2+\lambda_2)}=0 \qquad ...(3)$$

$(\because\ \lambda_1 \neq \lambda_2)$

Now the tangents at (α, β) to the confocals (1) and (2), are given by

$$\frac{\alpha x}{a^2+\lambda_1}+\frac{\beta y}{b^2+\lambda_1}=1 \qquad ...(4)$$

and
$$\frac{\alpha x}{a^2+\lambda_2}+\frac{\beta y}{b^2+\lambda_2}=1. \qquad ...(5)$$

Slopes of (4) and (5) are respectively, m_1 and m_2; where

$$m_1=-\frac{\alpha/(a^2+\lambda_1)}{\beta/(b^2+\lambda_1)}=-\frac{\alpha\,(b^2+\lambda_1)}{\beta\,(a^2+\lambda_1)}$$

and
$$m_2=-\frac{\alpha/(a^2+\lambda_2)}{\beta/(b^2+\lambda_2)}=-\frac{\alpha(b^2+\lambda_2)}{\beta(a^2+\lambda_2)}$$

$\therefore$
$$m_1\,.\,m_2=\left(-\frac{\alpha\,(b^2+\lambda_1)}{\beta\,(a^2+\lambda_1)}\right)-\left(\frac{\alpha\,(b^2+\lambda_2)}{\beta\,(a^2+\lambda_2)}\right)$$
$$=\frac{\alpha^2\,(b^2+\lambda_1)\,(b^2+\lambda_2)}{\beta^2\,(a^2+\lambda_1)\,(a^2+\lambda_2)}=-1. \qquad \text{[using (3)]}$$

This implies tangents at (α, β) to the confocals (1) and (2) cut at right angles. Hence confocal cut at right angles.

Property II. *Through a given point in the plane of a conic, two confocals can be drawn, one of them is an ellipse and other is a hyperbola.*

Let the equation of a confocal to an ellipse be

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1. \qquad ...(1)$$

Suppose (α, β) is a given point in the plane of (1) so it will satisfy (1), we get

$$\frac{\alpha^2}{a^2+\lambda}+\frac{\beta^2}{b^2+\lambda}=1. \qquad ...(2)$$

The equation (2) is a quadratic in λ. Hence through (α, β) we can draw two confocals.

Further, let us take $b^2+\lambda=\mu$ *i.e.* $\lambda=\mu-b^2$.

Now putting the value of λ in (2), we get

$$\frac{\alpha^2}{a^2+\mu-b^2}+\frac{\beta^2}{\mu}=1$$

or
$$\frac{\alpha^2}{\mu+a^2-b^2}+\frac{\beta^2}{\mu}=1$$

or
$$\frac{\alpha^2}{\mu+a^2e^2}+\frac{\beta^2}{\mu}=1 \qquad (\because\ a^2e^2=a^2-b^2)$$

or
$$\mu^2+(a^2e^2-\alpha^2-\beta^2)\,\mu-\beta^2a^2e^2=0. \qquad ...(3)$$

Let μ_1 and μ_2 be the roots of (3), then we have

$$\mu_1\mu_2=-\beta^2a^2e^2$$

$\Rightarrow$ $\mu_1\mu_2$ is negative

$\Rightarrow$ one of $\mu=b^2+\lambda$ is positive and other is negative.

Again, let us take $a^2+\lambda=\mu$ *i.e.* $\lambda=\mu-a^2$.

Now put $\lambda=\mu-a^2$ in (2), we get

$$\frac{\alpha^2}{\mu}+\frac{\beta^2}{\mu+b^2-a^2}=1$$

or $$\frac{\alpha^2}{\mu}+\frac{\beta^2}{\mu-a^2e^2}=1 \qquad (\because\ a^2e^2=a^2-b^2)$$

or $$\mu^2-(a^2e^2+\alpha^2+\beta^2)+\alpha^2a^2e^2=0. \qquad ...(4)$$

Let μ_1 and μ_2 be the roots of (4), the we have

$$\left.\begin{aligned}\mu_1+\mu_2&=a^2e^2+\alpha^2+\beta^2\\ \text{and}\quad \mu_1\mu_2&=\alpha^2a^2e^2\end{aligned}\right\}. \qquad ...(5)$$

From (5) it is observed that both the roots of (4) are positive *i.e.* both value of $a^2+\lambda$ are positive.

Hence one of the confocals is an ellipse and other is hyperbola.

SOLVED EXAMPLES

Example 1. *If the confocals through* $P(\alpha, \beta)$ *to the ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ *are*

$$\frac{x^2}{a^2+\lambda_1}+\frac{y^2}{b^2+\lambda_1}=1 \quad \textit{and} \quad \frac{x^2}{a^2+\lambda_2}+\frac{y^2}{b^2+\lambda_2}=1$$

show that $$\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}-1=\frac{-\lambda_1\lambda_2}{a^2b^2}$$

and $$\alpha^2+\beta^2-a^2-b^2=\lambda_1+\lambda_2.$$

Solution. Let the equation of a confocal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ be

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1. \qquad ...(1)$$

This confocal (1) passes through (α, β), then we have

$$\frac{\alpha^2}{a^2+\lambda}+\frac{\beta^2}{b^2+\lambda}=1$$

or $$\alpha^2(b^2+\lambda)+\beta^2(a^2+\lambda)=(a^2+\lambda)(b^2+\lambda)$$

or $$\alpha^2b^2+\alpha^2\lambda+\beta^2a^2+\beta^2\lambda=a^2b^2+\lambda(a^2+b^2)+\lambda^2$$

or $$\lambda^2+\lambda(a^2+b^2-\alpha^2-\beta^2)+a^2b^2-\alpha^2b^2-\beta^2a^2=0. \qquad ...(2)$$

This is a quadratic equation in λ so it has two roots λ_1 and λ_2 (say).

Hence, we obtain two confocals

$$\frac{x^2}{a^2+\lambda_1}+\frac{y^2}{b^2+\lambda_1}=1 \quad \text{and} \quad \frac{x^2}{a^2+\lambda_2}+\frac{y^2}{b^2+\lambda_2}=1.$$

Also we have

$$\left.\begin{aligned}\lambda_1+\lambda_2&=-(a^2+b^2-\alpha^2-\beta^2)\\ \text{and}\quad \lambda_1\lambda_2&=a^2b^2-\alpha^2b^2-\beta^2a^2\end{aligned}\right\}. \qquad ...(3)$$

From (3) we obtain

$$\alpha^2+\beta^2-a^2-b^2=\lambda_1+\lambda_2$$

and $$\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}-1=-\frac{\lambda_1\lambda_2}{a^2b^2}.$$

• SUMMARY

- Confocal conic of $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ are given by

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}+1,\ \lambda\in R.$$

- Confocal conics cut each other at right angles.
- Through a given point in the plane of a conic, two confocals can be drawn one of them is an ellipse and the other is a hyperbola.

• STUDENT ACTIVITY

1. Prove that confocals cut at right angles.

2. If the confocals through $P(\alpha, \beta)$ to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ are $\frac{x^2}{a^2+\lambda_1}+\frac{y^2}{b^2+\lambda_1}=1$ and $\frac{x^2}{a^2+\lambda_2}+\frac{y^2}{b^2+\lambda_2}=1$ the prove that $\lambda_1+\lambda_2=\alpha^2+\beta^2-(a^2+b^2)$.

• TEST YOURSELF

1. Prove that the locus of the point of contact of the tangents from a given point to a system of confocals is a cubic curve which passes through the given point and through the foci.
2. Prove that the two conics $ax^2 + 2hxy + by^2 = 1$ and $a'x^2 + 2h'xy + b'y^2 = 1$ can be placed so as to be confocals if

$$\frac{(a-b)^2 + 4h^2}{(ab - h^2)^2} = \frac{(a'-b')^2 + 4h'^2}{(a'b' - h'^2)^2}.$$

OBJECTIVE EVALUATION

➤ FILL IN THE BLANKS :

1. Confocal have the same two points as their foci as well as same ....
2. Confocals to an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ are ....
3. Confocals cut one another at ....

➤ TRUE OR FALSE :

Write 'T' for true and 'F' for false statement :

1. Confocal have same foci but do not have same axis. *(T/F)*
2. Through a given point in the plane of a conic, two confocals can be drawn, one of them is an ellipse and the other a hyperbola. *(T/F)*
3. The foci of the confocal $\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1$ are $[\pm\sqrt{a^2 - b^2 - 2\lambda}, 0]$. *(T/F)*

➤ MULTIPLE CHOICE QUESTION :

Choose the most appropriate one :

1. The centre of the confocal $\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1$ is :
   (a) $(0, \lambda)$ (b) $(\lambda, 0)$
   (c) $(0, 0)$ (d) (λ, λ).
2. The foci of the confocal $\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1$ are :
   (a) $(\pm\sqrt{a^2 - b^2}, 0)$ (b) $(0, \pm\sqrt{a^2 - b^2})$
   (c) $(\pm\sqrt{a^2 - b^2 - 2\lambda}, 0)$ (d) None of these.
3. Confocal cut at an angles :
   (a) $\pi/4$ (b) $\pi/2$
   (c) $\pi/3$ (d) $\pi/6$.

ANSWERS

Fill in the Blanks :

1. Axis **2.** $\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1$ **3.** Right angles

True or False :

1. F **2.** T **3.** F

Multiple Choice Questions :

1. (c) **2.** (a) **3.** (b).

❑❑❑

UNIT 19

SYSTEM OF CO-ORDINATES

STRUCTURE

- Co-ordinate System in Space
- Co-ordinates of a Point in Space
- Octants
- Change of Origin (Shift of Origin)
- Distance between Two Points
- Division of the Join of Two Points
- Application of Division of Join of Two Points
  - Summary
  - Student Activity
  - Test Yourself

LEARNING OBJECTIVES

After going through this unit you will learn :

- About the 3-dimensional co-ordinate system
- How to locate a given point in space.
- How to find the distance between two given points in space.

• 19.1. CO-ORDINATE SYSTEM IN SPACE

It has been observed from review of two-dimensional geometry that an object having some volume can not be described by two dimensional geometry, because any object occupied some space will have some points which are not in a same plane. Therefore *to describe the location of the points of any volumetric object, we consider a space system in which there are three mutually perpendicular lines called the axes of coordinates and the geometry having three dimensions also known as solid geometry.*

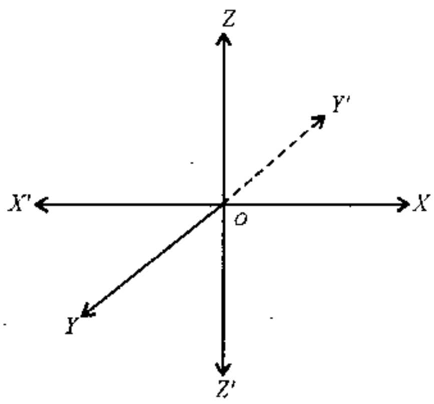


Fig. 1

Let XOX', YOY' and ZOZ' be three mutually perpendicular lines shown in fig. 1. The point O is known as *origin* and the lines are called *rectangular axis*, if these axes are taken in a pairs *i.e.*, YOZ, ZOX and XOY, these from co-ordinates planes known as YZ, ZX and XY-planes respectively.

• 19.2. CO-ORDINATES OF A POINT IN SPACE

Let P be any point in space. To find the location of this point P, let us draw the planes through P parallel to the co-ordinates planes as shown in fig. 2.

These planes cut the co-ordinate axis at points A, B and C on X-axis, Y-axis and Z-axis respectively. Let $OA = x$, $OB = y$ and $OC = z$. The co-ordinates of P be (x, y, z). Hence we can say that *the co-ordinates of any point are the perpendicular distances with proper signs from co-ordinates planes to that point.*

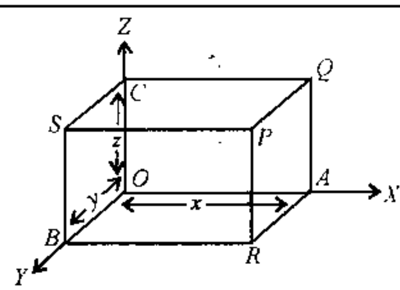


Fig. 2

• 19.3. OCTANTS

The co-ordinate planes i.e., YZ, ZX and XY planes divide the whole space into eight parts, called the octants. These octants contain negative axes also. Let XOX', YOY' and ZOZ' be three mutually perpendicular axes. Then the octants are :

1. $OXYZ$, 2. $OX'YZ$, 3. $OXY'Z$, 4. $OXYZ'$,
5. $OXY'Z'$, 6. $OX'YZ'$, 7. $OX'Y'Z$, 8. $OX'Y'Z'$.

Here dashes represent the negative side of the respective axis. In the first octant $OXYZ$ all the co-ordinates are positive. Let these co-ordinates for any point P be (x, y, z), then the co-ordinates of this point P with respect to other octants are $(-x, y, z)$, $(x, -y, z)$, $(x, y, -z)$, $(x, -y, -z)$, $(-x, y, -z)$, $(-x, -y, z)$ and $(-x, -y, -z)$ respectively.

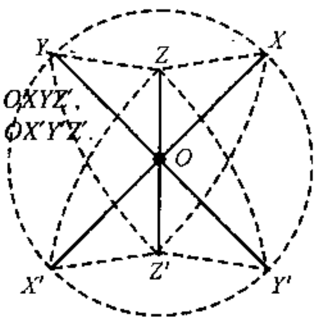


Fig. 3

To know about the octants, consider a sphere of any radius and let the centre be assumed the origin O of the co-ordinates axis and consider a great circle of this sphere. Now taking two perpendicular diameters of this circle as two axes XOX' and YOY' and a line through O perpendicular to the plane of great circle is taken as third axis ZOZ' as shown in fig. 3. In this way the whole sphere is divided into eight surfaces as written above in the beginning of this section. These surfaces are called the octants.

Examples 1. *Find the position of the following points :*

(*i*) $(4, 5, 7)$	(*ii*) $(1, -3, 2)$	(*iii*) $(-2, 1, 5)$
(*iv*) $(-3, -1, -6)$	(*v*) $(0, 0, 1)$	(*vi*) $(3, 2, -1)$
(*vii*) $(7, -1, -1)$	(*viii*) $(-5, 2, -2)$	(*ix*) $(-6, -6, 6)$
(*x*) $(1, 0, 0)$	(*xi*) $(0, 7, 0)$.	

Solution. (i) $(4, 5, 7)$ is in the octant $OXYZ$.
(ii) $(1, -3, 2)$ is in the octant $OXY'Z$.
(iii) $(-2, 1, 5)$ is in the octant $OX'YZ$.
(iv) $(-3, -1, -6)$ is in the octant $OX'Y'Z'$.
(v) $(0, 0, 1)$ is on the z-axis.
(vi) $(3, 2, -1)$ is in the octant $OXYZ'$.
(vii) $(7, -1, -1)$ is in the octant $OXY'Z'$.
(viii) $(-5, 2, -2)$ is in the octant $OX'YZ'$.
(ix) $(-6, -6, 6)$ is in the octant $OX'Y'Z$.
(x) $(1, 0, 0)$ is on the X-axis.
(xi) $(0, 7, 0)$ is on the Y-axis.

• 19.4. CHANGE OF ORIGIN (SHIFT OF ORIGIN)

Let OX, OY and OZ be three mutually perpendicular axes and let P be any point whose co-ordinates with respect to these axis be (a, b, c). Now we want to shift the origin O at P. For this purpose draw three lines through P parallel to OX, OY and OZ respectively and thus obtained new co-ordinates axes. Let these new axes be denoted by PX_1, PY_1 and PZ_1 as shown in fig. 4.

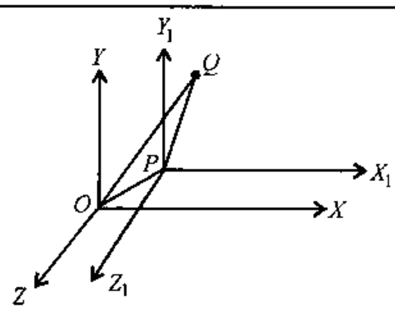


Fig. 4

Let Q be any other point whose co-ordinates with respect to OX, OY and OZ be (a_1, b_1, c_1). Now we shall find the co-ordinates of Q with respect to new axes PX_1, PY_1 and PZ_1. Let $\vec{r}$ be the position vector of a point P with respect to the origin O and $\vec{r_1}$ be the position vector of Q with respect to O. Then

$$\overrightarrow{OP} = \vec{r} = a\,\hat{i} + b\,\hat{j} + c\,\hat{k}$$

and $$\overrightarrow{OQ} = \vec{r_1} = a_1\,\hat{i} + b_1\,\hat{j} + c_1\,\hat{k}$$

where $\hat{i}$, $\hat{j}$ and $\hat{k}$ are the unit vectors along X-axis, Y-axis and Z-axis respectively.

In ΔOPQ, $\overrightarrow{OP} + \overrightarrow{PQ} = \overrightarrow{OQ}$ (By triangular property of vector)

$\therefore$ $$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \vec{r_1} - \vec{r} = (a_1\,\hat{i} + b_1\,\hat{j} + c_1\,\hat{k}) - (a\,\hat{i} + b\,\hat{j} + c\,\hat{k})$$

$$\overrightarrow{PQ} = (a_1 - a)\,\hat{i} + (b_1 - b)\,\hat{j} + (c_1 - c)\,\hat{k}.$$

Thus we obtained, the position vector of Q with respect to new origin P of new co-ordinate axes. Hence the co-ordinates of Q with respect to new co-ordinate axes are $(a_1 - a, b_1 - b, c_1 - c)$.

• 19.5. DISTANCE BETWEEN TWO POINTS

Let us consider two points P and Q in a space whose distance from each other is to be required. Let the co-ordinates of P and Q be (x_1, y_1, z_1) and (x_2, y_2, z_2).

Let $\mathbf{r}_1$ and $\mathbf{r}_2$ be the position vector of P and Q with respect to O.

Then we have

$$\mathbf{r}_1 = \overrightarrow{OP} = x_1\,\hat{i} + y_1\,\hat{j} + z_1\,\hat{k}$$

and
$$\mathbf{r}_2 = \overrightarrow{OQ} = x_2\,\hat{i} + y_2\,\hat{j} + z_2\,\hat{k}.$$

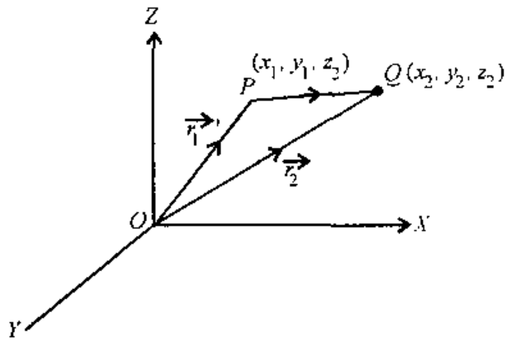


Fig. 5

In ΔOPQ, $\overrightarrow{OP} + \overrightarrow{PQ} = \overrightarrow{OQ}$

or $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \vec{r_2} - \vec{r_1}$

$$= (x_2\,\hat{i} + y_2\,\hat{j} + z_2\,\hat{k})$$

$$- (x_1\,\hat{i} + y_1\,\hat{j} + z_1\,\hat{k})$$

$$\overrightarrow{PQ} = (x_2 - x_1)\,\hat{i} + (y_2 - y_1)\,\hat{j} + (z_2 - z_1)\,\hat{k}.$$

Thus the distance between P and Q is the magnitude of the vector $\overrightarrow{PQ}$ *i.e.*, $PQ = |\overrightarrow{PQ}|$. Thus

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

REMARK

➤ Distance between two points, if one of them is origin and other is (x, y, z) is equal to $\sqrt{x^2 + y^2 + z^2}$.

• 19.6. DIVISION OF THE JOIN OF TWO POINTS

Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be two points. A line divide the PQ (*i.e.*, join of P and Q) in a ratio $m : n$ (where m and n are some integers) at the point $R(x, y, z)$ as shown in fig. 6.

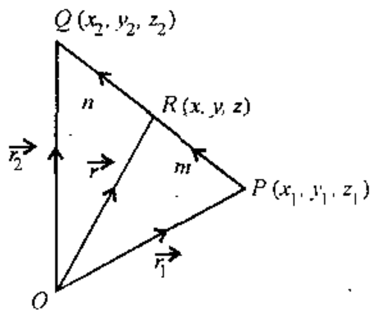


Fig. 6

Let $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}$ be the position vectors of P, Q and R respectively with respect to the origin O. Since we have

$$\frac{PR}{RQ} = \frac{m}{n} \quad \text{or} \quad m\,RQ = n\,PR \quad \text{or} \quad m\,\overrightarrow{RQ} = n\,\overrightarrow{PR}$$

or
$$m\,(\overrightarrow{OQ} - \overrightarrow{OR}) = n\,(\overrightarrow{OR} - \overrightarrow{OP})$$

or
$$m\,(\mathbf{r}_2 - \mathbf{r}) = n\,(\mathbf{r} - \mathbf{r}_1)$$

or
$$(m + n)\,\mathbf{r} = m\mathbf{r}_2 + n\mathbf{r}_1 \quad \text{or} \quad \mathbf{r} = \frac{m\mathbf{r}_2 + n\mathbf{r}_1}{m + n} \qquad \ldots(1)$$

Since $\mathbf{r} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$, $\mathbf{r}_1 = x_1\,\hat{i} + y_1\,\hat{j} + z_1\,\hat{k}$ and $\mathbf{r}_2 = x_2\,\hat{i} + y_2\,\hat{j} + z_2\,\hat{k}$.

Equation (1) becomes

$$x\,\hat{i} + y\,\hat{j} + z\,\hat{k} = \frac{m\,(x_2\,\hat{i} + y_2\,\hat{j} + z_2\,\hat{k}) + n\,(x_1\,\hat{i} + y_1\,\hat{j} + z_1\,\hat{k})}{m + n}$$

or
$$x\,\hat{i} + y\,\hat{j} + z\,\hat{k} = \frac{mx_2 + nx_1}{m + n}\,\hat{i} + \frac{my_2 + ny_1}{m + n}\,\hat{j} + \frac{mz_2 + nz_1}{m + n}\,\hat{k}.$$

Comparing the coefficients of $\hat{i}$, $\hat{j}$ and $\hat{k}$ of both sides, we get

$$x = \frac{mx_2 + nx_1}{m+n}$$
$$y = \frac{my_2 + ny_1}{m+n} \qquad \ldots(2)$$
$$z = \frac{mz_2 + nz_1}{m+n}$$

Hence, the co-ordinates of R are

$$\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}, \frac{mz_2 + nz_1}{m+n}\right).$$

Corollary. *The middle point of the join $P(x_1, y_1, z_1)$ and $Q(x_2, y_2\, z_2)$ is*

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right).$$

Corollary. *If a line divides the join of P and Q in a ratio λ : 1, then co-ordinates of intersection point are* $\left(\frac{\lambda x_2 + x_1}{\lambda+1}, \frac{\lambda y_2 + y_1}{\lambda+1}, \frac{\lambda z_2 + z_1}{\lambda+1}\right)$.

REMARKS

- If λ is positive real number, then the line divides the join of two points internally.
- If λ is negative real number, then the line divides the join of two points externally.
- If $\lambda = 0$, then the line will pass through one of the join of two points.
- For external intersection, the co-ordinates of intersection will be

$$\left(\frac{\lambda x_2 - x_1}{\lambda-1}, \frac{\lambda y_2 - y_1}{\lambda-1}, \frac{\lambda z_2 - z_1}{\lambda-1}\right).$$

• 19.7. APPLICATION OF DIVISION OF JOIN OF TWO POINTS

Some application of division of join of two points, are, to find the centroid of a triangle and a tetrahedron :

(i) Centroid of a triangle. Let the co-ordinates of A, B and C be (x_1, y_1, z_1), (x_2, y_2, z_3) and (x_3, y_3, z_3) forming a triangle ABC. Since we know that the centroid of ΔABC is the intersection of medians of ΔABC. Let G be the centroid of ΔABC, and let AD be a median of ΔABC. Then G divides the median AD in a ratio $AG : GD = 2 : 1$, where D is the middle point of BC as shown in fig. 7.

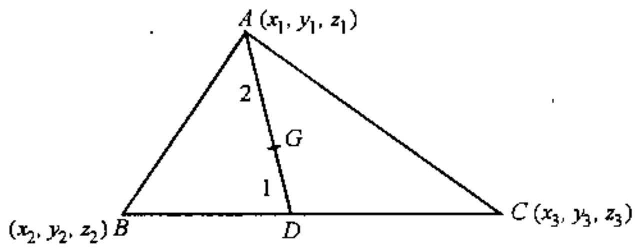


Fig. 7

$$\textbf{centroid of } \Delta ABC \textbf{ is } \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}, \frac{z_1 + z_2 + z_3}{3}\right).$$

(ii) Centroid of a tetrahedron. Let the co-ordinates of A, B, C and D be (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) and (x_4, y_4, z_4) respectively forming a tetrahedron $ABCD$. Let G_1 be the centroid of ΔBCD (face of tetrahedron). Since we know that the centroid of $ABCD$ will lie on AG_1 and will divide AG_1, in a ratio $AG : GG_1 = 3 : 1$, where G is the centroid of $ABCD$ as shown in fig. 8.

the centroid of tetrahedron $ABCD$ is

$$\left(\frac{x_1 + x_2 + x_3 + x_4}{4}, \frac{y_1 + y_2 + y_3 + y_4}{4}, \frac{z_1 + z_2 + z_3 + z_4}{4}\right)$$

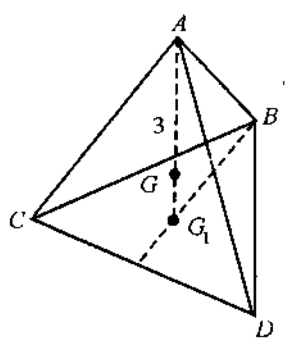


Fig. 8

SOLVED EXAMPLES

Example 1. *P is a variable point and the co-ordinates of two points A and B are* $(-2, 2, 3)$ *and* $(13, -3, 13)$ *respectively. Find the locus of P if* $3PA = 2PB$.

Solution. Let the co-ordinates P be (x, y, z). Then the distance between P and A; P and B respectively are

$$PA = \sqrt{(x+2)^2+(y-2)^2+(z-3)^2}$$

and $$PB = \sqrt{(x-13)^2+(y+3)^2+(z-13)^2}.$$

Since $3PA = 2PB$

$$\therefore \quad 3\sqrt{(x+2)^2+(y-2)^2+(z-3)^3} = 2\sqrt{(x-13)^2+(y+3)^2+(z-13)^3}.$$

Squaring of both sides, we get

$$9[(x+2)^2+(y-2)^2+(z-3)^2] = 4[(x-13)^2+(y+3)^2+(z-13)^2]$$

or $$9[x^2+4+4x+y^2+4-4y+z^2+9-6z]$$
$$= 4[x^2+169-26x+y^2+9+6y+z^2+169-26z]$$

or $$9(x^2+y^2+z^2+4x-4y-6z+17) = 4(x^2+y^2+z^2-26x+6y-26z+347)$$

or $$9x^2+9y^2+9z^2+36x-36y-54z+153 = 4x^2+4y^2+4z^2-104x+24y-104z+1388$$

or $$5x^2+5y^2+5z^2+140x-60y+50z-1235=0$$

or $$x^2+y^2+z^2+28x-12y+10z-247=0$$

which is the required locus.

Example 2. *Show that the points* $(1, 2, 3)$, $(2, 3, 1)$ *and* $(3, 1, 2)$ *form an equilateral triangle.*

Solution. Let ABC be triangle and the co-ordinates of A, B and C be $(1, 2, 3)$, $(2, 3, 1)$ and $(3, 1, 2)$.

$$\therefore \quad AB = \sqrt{(1-2)^2+(2-3)^2+(3-1)^2}$$
$$= \sqrt{(-1)^2+(-1)^2+(2)^2} = \sqrt{1+1+4} = \sqrt{6}$$

and $$BC = \sqrt{(3-2)^2+(1-3)^2+(2-1)^2}$$
$$= \sqrt{(1)^2+(-2)^2+(1)^2} = \sqrt{1+4+1} = \sqrt{6}$$
$$CA = \sqrt{(1-3)^2+(2-1)^2+(3-2)^2}$$
$$= \sqrt{(-2)^2+(1)^2+(1)^2} = \sqrt{1+4+1} = \sqrt{6}.$$

Thus $AB = BC = CA$. Hence ΔABC is an equilateral.

Example 3. *Find the co-ordinates of the point that divides the join of two points* $(2, -3, 1)$ *and* $(3, 4, -5)$ *in the ratio* $1 : 3$.

Solution. Let P and Q be two points whose co-ordinates are $(2, -3, 1)$ and $(3, 4, -5)$ and let R be that point which divides the join of P and Q in the ratio $1 : 3$.

Let $m : n = 1 : 3$ and co-ordinates of R be (x, y, z).

$$\therefore \quad x = \frac{mx_2+nx_1}{m+n} = \frac{1(3)+3(2)}{1+3} = \frac{3+6}{4} = \frac{9}{4}$$

$$y = \frac{my_2+ny_1}{m+n} = \frac{1(4)+3(-3)}{4} = \frac{4-9}{4} = -\frac{5}{4}$$

$$z = \frac{mz_2+nz_1}{m+n} = \frac{1(-5)+3(1)}{4} = \frac{-5+3}{4} = -\frac{2}{4} = -\frac{1}{2}.$$

Hence $\left(\frac{9}{4}, -\frac{5}{4}, -\frac{1}{2}\right)$ divides the join of $(2, -3, 1)$ and $(3, -4, 5)$ in the ratio $1 : 3$.

Example 4. *The mid-points of the sides of a triangle are* $(1, 5, -1)$, $(0, 4, -2)$ *and* $(2, 3, 4)$. *Find its vertices*

Solution. Let ABC be a triangle and let the co-ordinates of A, B and C be (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) respectively. Then we have

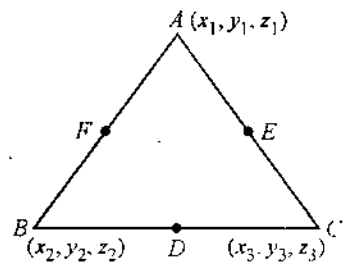


Fig. 9

$$\frac{x_1+x_2}{2} = 1, \frac{y_1+y_2}{2} = 5, \frac{z_1+z_2}{2} = -1$$

or $$x_1+x_2 = 2,\ y_1+y_2 = 10,\ z_1+z_2 = -2. \quad \text{...(1)}$$

Similarly,

$$x_2+x_3 = 0,\ y_2+y_3 = 8,\ z_2+z_3 = -4 \quad \text{...(2)}$$
$$x_3+x_1 = 4,\ y_3+y_1 = 6,\ z_3+z_1 = 8. \quad \text{...(3)}$$

Adding (1), (2), (3), we get

$$2(x_1 + x_2 + x_3) = 6,\ 2(y_1 + y_2 + y_3) = 24,$$
$$2(z_1 + z_2 + z_3) = 2$$

or $$x_1 + x_2 + x_3 = 3,\ y_1 + y_2 + y_3 = 12,\ z_1 + z_2 + z_3 = 1. \quad ...(4)$$

Subtracting successively (1), (2), (3) from (4), we get

$$x_1 = 3,\ x_2 = -1,\ x_3 = 1,\ y_1 = 4,\ y_2 = 6,\ y_3 = 2$$

and $$z_1 = 5,\ z_2 = -7,\ z_3 = 3.$$

Hence the vertices of the triangle are

$$A(3, 4, 5),\ B(-1, 6, -7),\ C(1, 2, 3).$$

• SUMMARY

- In 3-D geometry there are eight octants.
- Distance between (x_1, y_1, z_1) and (x_2, y_2, z_2) is given by $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$.
- If a point $P(x, y, z)$ divides a line segement AB with $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ in the $\lambda : 1$ then $x = \frac{Ax_2 + x_1}{\lambda + 1}$, $y = \frac{\lambda y_2 + y_1}{\lambda + 1}$, $z = \frac{\lambda z_2 + z_1}{\lambda + 1}$. If $\lambda > 0$, then the division is internally and if $\lambda < 0$ then the division is externally.
- Coordinates of centroid of the triangle ABC with $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ and $C(x_3, y_3, z_3)$ is $\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}, \frac{z_1 + z_3 + z_4}{3}\right)$.
- Co-ordinates of centroid of tetrahedron $ABCD$ with $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$, $C(x_3, y_3, z_3)$ and $D(x_4, y_4, z_4)$ is $\left(\frac{x_1 + x_2 + x_3 + x_4}{4}, \frac{y_1 + y_2 + y_3 + y_4}{4}, \frac{z_1 + z_3 + z_4}{4}\right)$.
- $x = r\cos\theta$, $y = r\sin\theta$, $z = z$ are the transformations for cylinderical polar coordinate systemj, where $\theta = \tan^{-1}\frac{y}{x}$.
- $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$ and $z = r\cos\theta$ are the transformations for spherical polar co-ordinate system, where $\tan\phi = \frac{y}{x}$, $\tan\theta = \frac{\sqrt{x^2 + y^2}}{z}$.

• STUDENT ACTIVITY

1. Show that point (1, 2, 3), (2, 3, 1) and (3, 1, 2) form an equilateral triangle.

2. Show that the points (3, – 2, 4), (1, 1, 1) and (– 1, 4, –2) are collinear.

• TEST YOURSELF

1. A variable point P moves in such a way that its distance from two fixed points A (3, 4, 5) and B (– 1, 3, – 7) are always same. Then find the locus of P.
2. Find the locus of a point P which moves in such a way that its distance from the point $A(a, b, c)$ is always equal to r.
3. A, B, C are three points on the axes of x, y and z respectively at distances a, b, c from the origin O; find the co-ordinates of the point which is equidistant from A, B, C and O.
4. Find the incentre of tetrahedron formed by planes $x = 0, y = 0, z = 0$ and $x + y + z = a$.
5. Show that (0, 7, 10), (– 1, 6, 6), (– 4, 9, 6) form an isosceles right angled triangle.
6. Find the co-ordinates of the point which divides the join of (2, 3, 4) and (3, – 4, 7) in the ratio 2 : – 4.

ANSWERS

1. $8x + 2y + 24z + 9 = 0$ **2.** $x^2 + y^2 + z^2 - 2ax - 2by - 2cz + a^2 + b^2 + c^2 - r^2 = 0$

3. $\left(\frac{a}{2}, \frac{b}{2}, \frac{c}{2}\right)$ **4.** $\left(\frac{a}{4}, \frac{a}{4}, \frac{a}{4}\right)$ **6.** (1, 10, 1)

OBJECTIVE EVALUATION

➤ FILL IN THE BLANKS :

1. $(\sqrt{2}, -\sqrt{2}, -\sqrt{2})$ lies in the octant ....
2. (– 1, – 2, – 3) lies in the octant ....
3. In the octant $OXY'Z$, the y-co-ordinate is ....

➤ TRUE OR FALSE :

Write 'T' for true 'F' for false statemet :

1. The number of octants are 6. *(T/F)*
2. (1, 0, 0) lies on the X-axis. *(T/F)*
3. (0, 3, 4) lies on the XY-axis. *(T/F)*

➤ MULTIPLE CHOICE QUESTIONS :

Choose the most appropriate one :

1. The point $(-x, -y, -z)$ lies :
   (a) $OXYZ$ (b) $OX'Y'Z'$ (c) $OX'Y'Z$ (d) $OX'YZ'$.
2. The number of octants are :
   (a) 6 (b) 7 (c) 8 (d) None of these.
3. The distance between (1, 2, 3) and (3, 2, 1) is :
   (a) $\sqrt{8}$ (b) $\sqrt{2}$ (c) $3\sqrt{2}$ (d) 1.
4. The middle point of the join of (1, 2, 3) and (– 1, – 2, – 3) is :
   (a) (1, – 2, 3) (b) (1, 1, 1) (c) (0, 0, 0) (d) (– 1, – 2, 1).

ANSWERS

Fill in the Blanks :

1. $OXY'Z'$ **2.** $OX'Y'Z'$ **3.** Negative

True or False :

1. F **2.** T **3.** F

Multiple Choice Questions :

1. (b) **2.** (c) **3.** (a) **4.** (c)

UNIT 20

DIRECTION COSINES AND DIRECTION RATIOS

STRUCTURE

- Direction Cosines of a Line
  Co-ordinates of a Point in Terms of D.C.'s
- Relation between Direction Cosines
- Direction Ratios
- Direction-Cosines of a Line through Two Points
- Projection of a Line through Two Points on Another Line whose D.C.'s are *l*, *m*, *n*
- Angle between Two Lines
  Angle between the Lines whose Direction Ratios are given
- Condition for Perpendicularity and Parallelism
  - Summary
  - Student Activity
  - Test Yourself

LEARNING OBJECTIVES

After going through this unit you will learn :

How to calculate the direction cosines and direction ratios of the given line.

How to determine the angle between the *l* given lines.

20.1. DIRECTION COSINES OF A LINE

Let AB be a line which makes the angles α, β and γ with the positive axes of X, Y and Z respectively. Draw a line OP through O and parallel to the line AB as shown in fig. 2.

The line OP makes the angle α, β, γ with positive axes of X, Y, Z respectively provided $\alpha + \beta + \gamma \neq 2\pi$. *Thus the non-dimensional quantities* $\cos\alpha, \cos\beta, \cos\gamma$ *are called the* ***direction cosines*** *of the line AB.*

The direction cosines of any line are usually denoted by the letters l, m and n respectively. *i.e.*,

$$\boldsymbol{l = \cos\alpha,\ m = \cos\beta,\ n = \cos\gamma.}$$

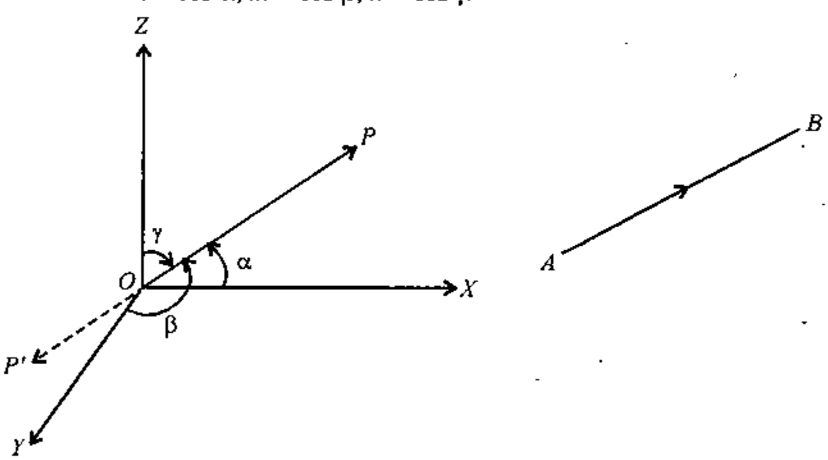


Fig. 2

Direction Cosines of Co-ordinate Axes.

(a) Direction cosines of X-axis. Since we know that X-axis makes the angle $\alpha = 0°$, $\beta = 90°$, $\gamma = 90°$ with positive axes *so its direction cosines are* $\cos 0°, \cos 90°, \cos 90°$ *i.e.*, **1, 0, 0.**

(b) Direction cosines of Y-axis. Y-axis makes the angle $\alpha = 90°$, $\beta = 0°$, $\gamma = 90°$ with positive axis respectively so that *its direction cosines are* $\cos 90°$, $\cos 0°$, $\cos 90°$ *i.e.*, **0, 1, 0**.

(c) Direction cosines of Z-axis. Likewise z-axis makes the angle $\alpha = 90°$, $\beta = 90°$, $\gamma = 0°$ with positive axis X, Y, Z respectively. Thus *direction cosines of z-axis are* $\cos 90°$, $\cos 90°$, $\cos 0°$ *i.e.*, **0, 0, 1**.

• 20.2. CO-ORDINATES OF A POINT IN TERMS OF D.C.'s

Let P be any point in a space and let OX, OY and OZ be rectangular axes. Suppose $OP = r$. Draw a perpendicular from P to X-axis which meets at M. Let the co-ordinates of P be (x, y, z). Then in ΔOPM, $\angle PMO = 90°$,

$$\frac{OM}{OP} = \cos\alpha$$

$$OM = OP\cos\alpha$$

$$x = r\cos\alpha \qquad (\because\ OM = x, OP = r)$$

$$\therefore\quad x = lr \qquad (\because\ l = \cos\alpha)$$

Similarly, $y = mr$, $z = nr$.

Hence the co-ordinates of P are (lr, mr, nr) where l, m and n are direction cosines of OP.

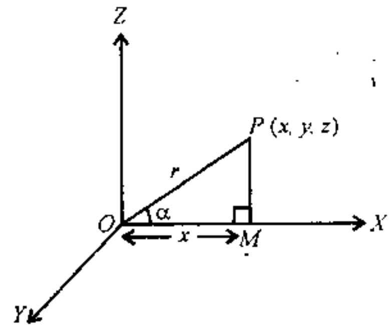


Fig. 3

• 20.3. RELATION BETWEEN DIRECTION COSINES

Let OX, OY and OZ be the system of co-ordinate axes mounted mutually perpendicular. Now draw a line OP through O and parallel to the given line AB. Let OP be r. Then the co-ordinates of P are (lr, mr, nr) where l, m, n are direction cosines of AB. Suppose that co-ordinates of P are assumed (x, y, z). Then

$$OP = r = \sqrt{x^2 + y^2 + z^2}$$

and we have $x = lr$, $y = mr$, $z = nr$

squaring and then adding, we get

$$x^2 + y^2 + z^2 = (lr)^2 + (mr)^2 + (nr)^2$$

or

$$x^2 + y^2 + z^2 = r^2(l^2 + m^2 + n^2)$$

or

$$r^2 = r^2(l^2 + m^2 + n^2) \qquad (\because\ r^2 = x^2 + y^2 + z^2)$$

$$\therefore\quad \boldsymbol{l^2 + m^2 + n^2 = 1.}$$

Hence proved the result.

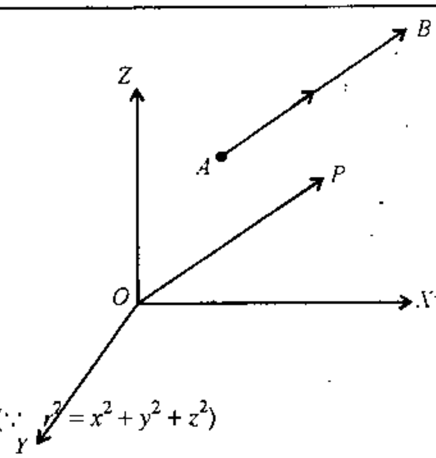


Fig. 4

• 20.4. DIRECTION RATIOS

Definition. *The three real numbers a, b, c which are proportional to the direction cosines l, m, n of a line, are called* ***direction ratios*** *of that line.*

Thus we have $\dfrac{l}{a} = \dfrac{m}{b} = \dfrac{n}{c}$.

$$\therefore\ l = \pm\frac{a}{\sqrt{a^2 + b^2 + c^2}},\quad m = \pm\frac{b}{\sqrt{a^2 + b^2 + c^2}},\quad n = \pm\frac{c}{\sqrt{a^2 + b^2 + c^2}}$$

• 20.5. DIRECTION-COSINES OF A LINE THROUGH TWO POINTS

Let P and Q be two points through which a line is passing. Let the co-ordinates of P and Q be (x_1, y_1, z_1) and (x_2, y_2, z_2) respectively, and let the direction cosines of PQ be l, m, n. Now draw two perpendicular PP' and QQ' from P and Q to X-axis respectively. Suppose the line, PQ makes the angles α, β and γ with the positive X-axis, Y-axis and Z-axis respectively; then

$$l = \cos\alpha,\ m = \cos\beta,\ n = \cos\gamma.$$

The direction cosines of a line through $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are $\frac{x_2 - x_1}{PQ}$, $\frac{y_2 - y_1}{PQ}$, $\frac{z_2 - z_1}{PQ}$, where PQ is distance between P and Q. Moreover it has been observed that $x_2 - x_1$, $y_2 - y_1$ and $z_2 - z_1$ are proportional to l, m and n, hence $x_2 - x_1$, $y_2 - y_1$, $z_2 - z_1$ are the direction ratios of a line passing through $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$.

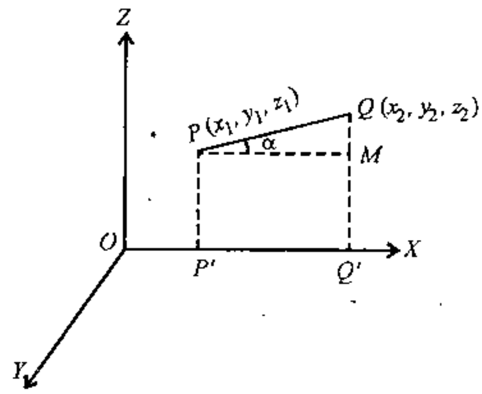


Fig. 5

• 20.6. PROJECTION OF A LINE THROUGH TWO POINTS ON ANOTHER LINE WHOSE D.C.'S ARE *l*, *m*, *n*

Let P and Q be two points with coordinates (x_1, y_1, z_1) and (x_2, y_2, z_2) respectively, through P and Q a line is passing whose projection on another line of direction cosines l, m, n is to be determined. Let OX, OY and OZ be the rectangular axes. Therefore,

$$\overrightarrow{OP} = x_1 \hat{i} + y_1 \hat{j} + z_1 \hat{k}$$

and $$\overrightarrow{OQ} = x_2 \hat{i} + y_2 \hat{j} + z_2 \hat{k}.$$

$$\therefore \quad \overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = (x_2 \hat{i} + y_2 \hat{j} + z_2 \hat{k}) - (x_1 \hat{i} + y_1 \hat{j} + z_1 \hat{k})$$

$$\overrightarrow{PQ} = (x_1 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}.$$

Now the unit vector along another line whose d.c.'s are l, m, n is

$$\hat{a} = l\hat{i} + m\hat{j} + n\hat{k}.$$

Thus the projection of PQ on another line is

$$= \frac{\overrightarrow{PQ} \cdot \hat{a}}{|\hat{a}|} \qquad \left(\because \text{ Projection of } \vec{B} \text{ on } \vec{A} \text{ is } \frac{\vec{B} \cdot \vec{A}}{|\vec{A}|}\right)$$

$$= \frac{[(x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}] \cdot (l\hat{i} + m\hat{j} + n\hat{k})}{|l\hat{i} + m\hat{j} + n\hat{k}|}$$

$$= \frac{(x_2 - x_1)\, l + (y_2 - y_1)\, m + (z_2 - z_1)\, n}{\sqrt{l^2 + m^2 + n^2}}$$

$$= (x_2 - x_1)\, l + (y_2 - y_1)\, m + (z_2 + z_1)\, n \qquad (\because\ l^2 + m^2 + n^2 = 1)$$

Corollary. *The projection of a line PQ on another line whose d.c.'s are l, m, n is, if* $P(0, 0, 0)$ *and* $Q(x_1, y_1, z_1)$

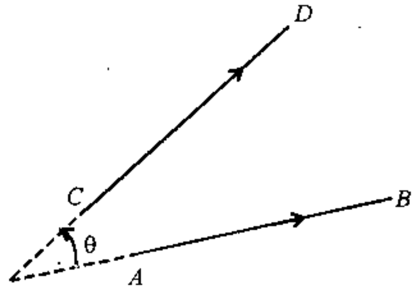
$$= x_1 l + y_1 m + z_1 n.$$

• 20.7. ANGLE BETWEEN TWO LINES

To prove that if l_1, m_1, n_1 and l_2, m_2, n_2 are the direction cosines of two lines and θ is the angle between them, then

$$\cos\theta = \boldsymbol{l_1 l_2 + m_1 m_2 + n_1 n_2}.$$

Let AB and CD be two lines whose direction cosines are l_1, m_1, n_1; l_2, m_2, n_2 respectively and let θ be an acute angle between AB and CD as shown in fig. 2.6 (a). Now draw two lines OP and OQ through O and parallel to AB and CD respectively. Let the co-ordinates of P and Q be (x_1, y_1, z_1) and (x_2, y_2, z_2) such that $OP = \mathbf{r}_1$ and $OQ = \mathbf{r}_2$. Thus the angle between OP and OQ will be equal to θ and also the direction cosines of OP and OQ are equal to l_1, m_1, n_1 and l_2, m_2, n_2 respectively. Since we know that



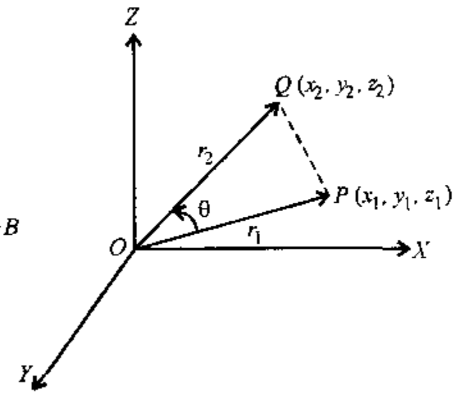


Fig. 6 (a) **Fig. 6 (b)**

$$x_1 = l_1 r_1,\ y_1 = m_1 r_1,\ z_1 = n_1 r_1;\ \ x_2 = l_2 r_2,\ y_2 = m_2 r_2,\ z_2 = n_2 r_2 \qquad ...(1)$$

$\therefore$ projection of OQ on $OP = (x_2 - 0)\, l_1 + (y_2 - 0)\, m_1 + (z_2 - 0)\, n_1$

$$= x_2 l_1 + y_2 m_1 + z_2 n_1 = r_2\,(l_1 l_2 + m_1 m_2 + n_1 n_2) \qquad ...(2)$$

$$(\because\ x_2 = l_2 r_2,\ y_2 = m_2 r_2,\ z_2 = n_2 r_2)$$

But we have

Projection of OQ on $OP = OQ \cos\theta = r_2 \cos\theta.$ ...(3)

From (2) and (3), we have

$$r_2 \cos\theta = r_2\,(l_1 l_2 + m_1 m_2 + n_1 n_2).$$

$$\therefore \qquad \boldsymbol{\cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2}.$$

Aliter. In ΔOPQ

$$\overrightarrow{OP} = x_1\,\hat{i} + y_1\,\hat{j} + z_1\,\hat{k}$$

and
$$\overrightarrow{OQ} = x_2\,\hat{i} + y_2\,\hat{j} + z_2\,\hat{k}.$$

Since $|\overrightarrow{OP}| = r_1$ and $|\overrightarrow{OQ}| = r_2$, then

$$\overrightarrow{OP}\,.\,\overrightarrow{OQ} = OP\ OQ \cos\theta \qquad (\because\ \vec{A}\,.\,\vec{B} = AB\cos\theta)$$

$$\therefore \qquad (x_1\,\hat{i} + y_1\,\hat{j} + z_1\,\hat{k})\,.\,(x_2\,\hat{i} + y_2\,\hat{j} + z_2\,\hat{k}) = r_1 r_2 \cos\theta$$

or
$$x_1 x_2 + y_1 y_2 + z_1 z_2 = r_1 r_2 \cos\theta$$

or
$$r_1 r_2\,(l_1 l_2 + m_1 m_2 + n_1 n_2) = r_1 r_2 \cos\theta \qquad \text{[From (}i\text{)]}$$

or
$$\boldsymbol{\cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2}.$$

• 20.8. ANGLE BETWEEN THE LINES WHOSE DIRECTION RATIOS ARE GIVEN

Let a_1, b_1, c_1 and a_2, b_2, c_2 be the direction ratios of two lines, then the direction cosines of these two lines will be

$$l_1 = \frac{a_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}},\ m_1 = \frac{b_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}},\ n_1 = \frac{c_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}}$$

and
$$l_2 = \frac{a_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}},\ m_2 = \frac{b_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}},\ n_2 = \frac{c_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}}.$$

Thus we get

$$\cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2.$$

$$\therefore \qquad \boldsymbol{\cos\theta = \frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\,.\,\sqrt{a_2^2 + b_2^2 + c_2^2}}}.$$

• 20.9. CONDITION FOR PERPENDICULARITY AND PARALLELISM

(a) Condition for perpendicularity. *If the two lines whose direction cosines are l_1, m_1, n_1 and l_2, m_2, n_2 are perpendicular, then we have*

$$\cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2.$$

Since $\theta = 90°$, *this implies* $\cos\theta = 0$, *hence the condition is*

$\therefore \quad l_1l_2 + m_1m_2 + n_1n_2 = 0.$

In terms of direction ratios this condition becomes

$$a_1a_2 + b_1b_2 + c_1c_2 = 0.$$

(b) Condition for parallelism. *If the two lines whose d.c.'s are respectively* l_1, m_1, n_1; l_2, m_2, n_2 *are parallel, then we have*

$$\sin\theta = \sqrt{[(l_1m_2 - m_1l_2)^2 + (m_1n_2 - n_1m_2)^2 + (n_1l_2 - n_2l_1)^2]}.$$

Since we know $\theta = 0$, *this implies* $\sin\theta = 0$

$\therefore \quad (l_1m_2 - m_1l_2)^2 + (m_1n_2 - n_1m_2)^2 + (n_1l_2 - n_2l_1)^2 = 0.$

This gives that

$$l_1m_2 - m_1l_2 = 0,\ m_1n_2 - n_1m_2 = 0,\ n_1l_2 - l_1n_2 = 0.$$

$$\therefore \quad \frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2} = \frac{\sqrt{l_1^2 + m_1^2 + n_1^2}}{\sqrt{l_2^2 + m_2^2 + n_2^2}} = \frac{1}{1} = 1.$$

$\therefore \quad l_1 = l_2,\ m_1 = m_2,\ n_1 = n_2.$

This is the required condition.

In terms of direction ratios this condition becomes

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}.$$

PERPENDICULAR DISTANCE OF A POINT FROM A LINE

To find the perpendicular distance of a point $P(x_1, y_1, z_1)$ *from a line passing through a point* $A(\alpha, \beta, \gamma)$ *whose direction cosines are* l, m, n.

Let AB be a line through $A(\alpha, \beta, \gamma)$ and whose d.c.'s are l, m, n. Let PN be the perpendicular from P to AB as shown in the adjoint figure 7.

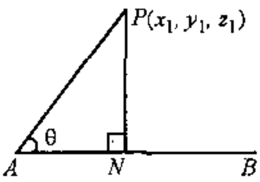


Fig. 7

Therefore AN = Projection of AP on AB

$$= (x_1 - \alpha)\, l + (y_1 - \beta)\, m + (z_1 - \gamma)\, n. \quad \text{...(1)}$$

Now $\quad AP = \sqrt{(x_1 - \alpha)^2 + (y_1 - \beta)^2 + (z_1 - \gamma)^2}.$

We have

$$PN^2 = AP^2 - AN^2$$

$$= [(x_1 - \alpha)^2 + (y_1 - \beta)^2 + (z_1 - \gamma)^2] - [(x_1 - \alpha)\, l + (y_1 - \beta)\, m + (z_1 - \gamma)\, n]^2$$

$$= (x_1 - \alpha)^2 (1 - l^2) + (y_1 - \beta)^2 (1 - m^2) + (z_1 - \gamma)^2 (1 - n^2)$$
$$- 2\, [lm\, (x_1 - \alpha)(y_1 - \beta) + mn\, (y_1 - \beta)(z_1 - \gamma) + ln\, (x_1 - \alpha)(z_1 - \gamma)]$$

$$= (x_1 - \alpha)^2 (m^2 + n^2) + (y_1 - \beta)^2 (l^2 + n^2) + (z_1 - \gamma)^2 (l^2 + m^2)$$
$$- 2\, [lm\, (x_1 - \alpha)(y_1 - \beta) + mn\, (y_1 - \beta)(z_1 - \gamma) + ln\, (x_1 - \alpha)(z_1 - \gamma)]$$

$$[\because \ l^2 + m^2 + n^2 = 1]$$

$$= \{n\, (y_1 - \beta) - m\, (z_1 - \gamma)\}^2 + \{l\, (z_1 - \gamma) - n\, (x_1 - \alpha)\}^2 + \{m\, (x_1 - \alpha) - l\, (y_1 - \beta)\}^2$$

$$PN = \sqrt{\sum \{n\, (y_1 - \beta) - m\, (z_1 - \gamma)\}^2}.$$

HOW TO SHOW THAT THREE GIVEN POINTS ARE COLLINEAR

In order to show that three points A, B and C are collinear, we proceed as follows :

(i) First we find the direction ratios of AB and AC.

(ii) If these direction ratios are proportional, then A, B and C are collinear.

SOLVED EXAMPLES

Example 1. *If* $\cos\alpha$, $\cos\beta$ *and* $\cos\gamma$ *are the direction cosines of a straight line, then prove that*

$$\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2.$$

Solution. Since we know that

$$l^2 + m^2 + n^2 = 1.$$

$\therefore \qquad \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1 \qquad (\because\ l = \cos\alpha,\ m = \cos\beta,\ n = \cos\gamma)$

or $\qquad (1 - \sin^2\alpha) + (1 - \sin^2\beta) + (1 - \sin^2\gamma) = 1$

or $\qquad 3 - \sin^2\alpha - \sin^2\beta - \sin^2\gamma = 1$

or $\qquad \sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2.$ *Hence proved.*

Example 2. *Find the direction cosines of a line which is equally inclined to the axes.*

Solution. Let l, m, n be the direction cosines of a line which is equally inclined to the axes, then

$$\alpha = \beta = \gamma.$$

$$\therefore \qquad \cos\alpha = \cos\beta = \cos\gamma \quad \text{or} \quad \frac{l}{1} = \frac{m}{1} = \frac{n}{1}$$

or
$$\frac{l}{1} = \frac{m}{1} = \frac{n}{1} = \pm\frac{\sqrt{l_2 + m^2 + n^2}}{\sqrt{1+1+1}} = \pm\frac{1}{\sqrt{3}}$$

$$\therefore \qquad l = \pm\frac{1}{\sqrt{3}},\ m = \pm\frac{1}{\sqrt{3}},\ n = \pm\frac{1}{\sqrt{3}}.$$

Example 3. *If $P(6, 3, 2)$, $Q(5, 1, 4)$, $R(3, -4, 7)$ and $S(0, 2, 5)$ are four points. Find the projection of PQ on RS.*

Solution. Firstly find d.c.'s of RS.

$\therefore$ d.r.'s of RS are $0 - 3,\ 2 + 4,\ 5 - 7$ *i.e.*, $-3, 6, -2$.

$\therefore$ d.c.'s of RS are $\dfrac{-3}{7}, \dfrac{6}{7}, \dfrac{-2}{7}$.

Thus the projection of PQ on RS

$$= |(x_2 - x_1)\,l + (y_2 - y_1)\,m + (z_2 - z_1)\,n|$$

$$= \left|(5-6)\left(-\frac{3}{7}\right) + (1-3)\left(\frac{6}{7}\right) + (4-2)\left(\frac{-2}{7}\right)\right|$$

$$= \left|\frac{3}{7} - \frac{12}{7} - \frac{4}{7}\right| = \left|\frac{-13}{7}\right| = \frac{13}{7}.$$

Example 4. *Find the direction cosines l, m, n of the two lines which are connected by the relation $l + m + n = 0$ and $mn - 2nl - 2lm = 0$. Also, show that angle between lines is $2\pi/3$.*

Solution. Since we have

$$l + m + n = 0 \qquad \text{...(1)}$$

$$mn - 2nl - 2lm = 0. \qquad \text{...(2)}$$

Eliminating n from (1) and (2), we have

$$-m(l+m) + 2l(l+m) - 2lm = 0$$

or
$$-ml - m^2 + 2l^2 + 2lm - 2lm = 0 \quad \text{or} \quad 2l^2 - lm - m^2 = 0$$

$$2l^2 - 2lm + lm - m^2 = 0$$

$$(2l + m)(l - m) = 0 \quad \text{or} \quad \frac{l}{1} = \frac{m}{1};\ \frac{l}{1} = \frac{m}{-2}. \qquad \text{...(3)}$$

Now eliminating l from (1) and (2), we have

$$mn + 2n(m+n) + 2m(m+n) = 0$$

or
$$mn + 2mn + 2n^2 + 2m^2 + 2mn = 0 \quad \text{or} \quad 2m^2 + 5mn + 2n^2 = 0$$

$$2m^2 + 4mn + mn + 2n^2 = 0$$

$$(2m + n)(m + 2n) = 0 \quad \text{or} \quad \frac{m}{1} = \frac{n}{-2};\ \frac{m}{-2} = \frac{n}{1}. \qquad \text{...(4)}$$

From (3) and (4) we obtained

$$\frac{l}{1} = \frac{m}{1} = \frac{n}{-2};\ \frac{l}{1} = \frac{m}{-2} = \frac{n}{1}.$$

Thus the direction cosines of two lines are respectively given by

$$\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}};\ \frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}.$$

Again, if θ is the angle between these lines, then

$$\cos\theta = \left(\frac{1}{\sqrt{6}}\right)\left(\frac{1}{\sqrt{6}}\right) + \left(\frac{1}{\sqrt{6}}\right)\left(-\frac{1}{\sqrt{6}}\right) + \left(-\frac{2}{\sqrt{6}}\right)\left(\frac{1}{\sqrt{6}}\right) = -\frac{3}{6} = -\frac{1}{2} \Rightarrow \theta = \frac{2\pi}{3}.$$

• SUMMARY

- $\cos\alpha, \cos\beta, \cos\gamma$ are the direction cosines of a straight line with $\alpha+\beta+\gamma \neq 2\pi$. We take $l=\cos\alpha$, $m=\cos\beta$, $x=\cos\gamma$. Then $l^2+m^2+n^2=1$.
- $(x, y, z) \to (lr, mr, nr)$.
- If a, b, c are the direction ratios of a line, then its direction cosines l, m, n are obtained by $l=\dfrac{\pm a}{\sqrt{a^2+b^2+c^2}}$, $m=\dfrac{\pm b}{\sqrt{a^2+b^2+c^2}}$, $n=\dfrac{\pm c}{\sqrt{a^2+b^2+c^2}}$
- Direction cosines of a line segment PQ with $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is given by $l=\dfrac{x_2-x_1}{PQ}$, $m=\dfrac{y_2-y_1}{PQ}$, $n=\dfrac{z_2-z_1}{PQ}$
- The projection of PQ, $P(x_1, y_1, z_1)$, $Q(x_2, y_2, z_2)$ on a line of d.c's l, m, n is given by $(x_2+x_1)\,l+(y_2-y_1)\,m+(z_2-z_1)\,n$.
- If l_1, m_1, n_1 and l_2, m_2, n_2 are the d.c's of this lines and θ is the angle between them, then $\cos\theta=l_1l_2+m_1m_2+n_1n_2$ if the direction ratios of the lines are given then
$$\cos\theta=\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\,\sqrt{a_2^2+b_2^2+c_2^2}}$$
where a_1, b_1, c_1 and a_2, b_2, c_2 are the direction ratios of two lines.
- Lines are perpendicular if
$$l_1l_2+m_1m_1+n_1n_2=0$$
or
$$a_1a_2+b_1b_2+c_1c_2=0.$$
- Lines are parallel to each other if
$$l_1=l_2, m_1=m_2, n_1=n_2$$
or
$$\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}.$$

• STUDENT ACTIVITY

1. Show that $\sin^2\alpha+\sin^2\beta+\sin^2\gamma=2$, where $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ are the direction cosines of a straight line.

2. Show that the lines whose d.c's are given by $l+m+n=0$ and $2mn+3ln-5lm=0$ are at right angles.

• TEST YOURSELF

1. Find the direction cosines of a line whose direction ratios are 2, 3, 6.
2. To show that the direction cosines of a line whose direction ratios are a, b, c are
$$\frac{a}{\sqrt{a^2+b^2+c^2}}, \frac{b}{\sqrt{a^2+b^2+c^2}}, \frac{c}{\sqrt{a^2+b^2+c^2}}.$$
3. Find the direction cosines of the line segment joining the points $P(2, 3, -6)$ and $Q(3, -4, 5)$.
4. If $P(2, 3, -6)$ and $Q(3, -4, 5)$ are two points, find the d.c.'s of OP, PO, OQ and PQ where O is the origin.
5. Find the length of a segment of a line whose projections on the axes are 2, 3, 6.
6. If the points P and Q are $(2, 3, -6)$ and $(3, -4, 5)$, then find the angle between OP and OQ where O is the origin.
7. If l, m, n are the direction cosines of a line, then prove that : $l^2+m^2+n^2=1$.
8. Find the direction cosines of two lines which are given by the relations
$$l-5m-3n=0 \text{ and } 7l^2+5m^2-3n^2=0.$$

ANSWERS

1. $\frac{2}{7}, \frac{3}{7}, \frac{6}{7}$. 3. $\frac{1}{\sqrt{171}}, \frac{-7}{\sqrt{171}}, \frac{11}{\sqrt{171}}$

4. d.c.'s of OP are $\frac{2}{7}, \frac{3}{7}, -\frac{6}{7}$ d.c.'s of PO are $-\frac{2}{7}, -\frac{3}{7}, \frac{6}{7}$

d.c.'s of OQ are $\frac{3}{5\sqrt{2}}, -\frac{4}{5\sqrt{2}}, \frac{5}{5\sqrt{2}}$ d.c.'s of PQ are $\frac{1}{\sqrt{171}}, \frac{-7}{\sqrt{171}}, \frac{11}{\sqrt{171}}$.

5. 7, 6. $\theta = \cos^{-1}\left(\frac{-18\sqrt{2}}{35}\right)$. 8. $-\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}; \frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}$

OBJECTIVE EVALUATION

➤ FILL IN THE BLANKS :

1. If a line makes the angle α, β, γ with the co-ordinate axes respectively, then $\alpha+\beta+\gamma \neq$ ....
2. The numbers which are proportional to the direction-cosines are called ....
3. The direction-cosines of z-axis are ....
4. If l, m, n are actual direction-cosines of a line, then $l^2+m^2+n^2=$.... .

➤ TRUE OR FALSE :

Write 'T' for true and 'F' for false statement :

1. The direction ratios of a line passing through the points (2, 3, 5) and (7, 6, 8) are 5, 3, 3. *(T/F)*
2. If $1, -2, -1$ are the direction-ratios of a line, then its direction cosines are $\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}$. *(T/F)*
3. The diretion-cosines of x-axis are 0, 0, 1. *(T/F)*

➤ MULTIPLE CHOICE QUESTIONS :

Choose the most appropriate one :

1. The direction-cosines of y-axis are :
(a) 0, 1, 0 (b) 0, 0, 1 (c) 1, 0, 0 (d) 1, 1, 1.
2. The direction-cosines of a line OP are l, m, n and $OP = r$, then the co-ordinates of P are :
(a) (0, 0, 0) (b) (l, m, n) (c) (r, r, r) (d) (lr, mr, nr).

ANSWERS

Fill in the Blanks :

1. 2π **2.** Direction-ratios **3.** 0, 0, 1 **4.** 1

True or False :

1. T **2.** F **3.** F

Multiple Choice Questions :

1. (a) **2.** (d) **3.** (a) **4.** (b)

UNIT 21

THE PLANE

STRUCTURE

LEARNING OBJECTIVES

After going through this unit you will learn :

- About the different forms of the plane
- How to find the plane passing through 3 points.
- How to calculate the angle between 2 planes and distance between planes.
- How to determine the planes bisecting the angle between the two given planes.

21.1. GENERAL FORMS OF A PLANE

(a) *General equation of first degree in x, y, z represents a plane.*

The general equation of a plane is given by

$$ax + by + cz + d = 0$$

(b) *General equation of a plane passing through a given point.*

Let $A(x_1, y_1, z_1)$ be a given point and let the equation of a plane be $ax + by + cz + d = 0$ that passes through $A(x_1, y_1, z_1)$, then we have

$$ax_1 + by_1 + cz_1 + d = 0$$

substracting these two equations, we get

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0.$$

This is the required general equation of a plane passing through (x_1, y_1, z_1).

(c) *Equation of a plane that passes through the origin is given by*

$$ax + by + cz = 0.$$

(d) *Equations of the plane parallel to the co-ordinate axes are given by*

(i) **If $a = 0$,** then $by + cz + d = 0$ is a plane parallel to x-axis.

(ii) **If $b = 0$,** then $ax + cz + d = 0$ is a plane parallel to y-axis.

(iii) **If $c = 0$,** then $ax + by + d = 0$ is a plane parallel to z-axis.

(e) *Equations of the co-ordinates planes* :

(i) The equation to yz-plane is $x = 0$

(ii) The equation to zx-plane is $y = 0$

(iii) The equation of xy-plane is $z = 0$.

(f) *Equation to the planes parallel to the co-ordinate planes*

(i) The equation of a plane parallel to yz-plane is $x = a$ (constant)

(ii) The equation of a plane parallel to zx-plane is $y = b$ (constant)

(iii) The equation of a plane parallel to xy-plane is $z = c$ (constant).

(g) *Equations of the planes perpendicular to the co-ordinate axes* :

(i) The equation of a plane perpendicular to x-axis is obviously parallel to yz-plane, so it equation is $\boldsymbol{x = a}$.

Similarly,

(ii)The equation of a plane perpendicular to y-axis is $\boldsymbol{y = b}$.

(iii)The equation of a plane perpendicular to z-axis is $\boldsymbol{z = c}$.

• 21.2. INTERCEPT FORM OF A PLANE

Definition. *The equation of a plane which cuts off intercepts on the coordinate axes, is called Intercept form of a plane.*

The equation of a plane in intercepted form is given by

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1,$$

where a = intercept on x-axis, b = intercept on y-axis, c = intercept on z-axis

• 21.3. NORMAL FORM OF A PLANE

Definition. *The equation of a plane in terms of a perpendicular distance from origin to the plane and direction-cosines of this perpendicular, is called Normal form.*

The equation of a plane in normal form is given by

or $$\boxed{lx + my + nz = p} \qquad (\because\ l^2+m^2+n^2=1)$$

• 21.4. EQUATION OF A PLANE PASSING THROUGH THREE POINTS

To obtain the equation of a plane passing though three points.

Let $P(x_1, y_1, z_1)$, $Q(x_2, y_2, z_2)$ and $R(x_3, y_3, z_3)$ be three points through these the equation of a plane is to be required.

The equation of required plane is given by

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0. \qquad \ldots(A)$$

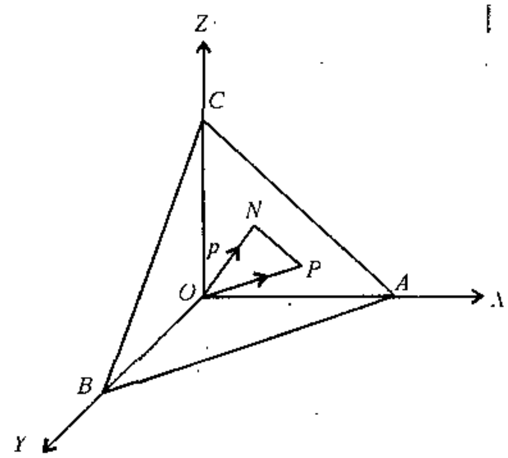


Fig. 1

• 21.5. ANGLE BETWEEN TWO PLANES

Definition. *An angle between two planes is equal to the angle between the normals to the planes.*

Let $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ be two planes and let θ be the angle between them. Then we have

$$\boxed{\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\cdot\sqrt{a_2^2+b_2^2+c_2^2}}}$$

where $a_1, b_1, c_1; a_2, b_2, c_2$ are the direction ratios of the normal to the planes.

Corollary 1. *The planes are perpendicular if* $a_1a_2 + b_1b_2 + c_1c_2 = 0$.

Corollary 2. *The planes are parallel if* $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$.

Corollary 3. *The equation parallel to a given plane* $ax + by + cz + d = 0$ *is* $ax + by + cz + \lambda = 0$, *where* λ *is to be determined by additional condition.*

SOLVED EXAMPLES

Example 1. *A plane meets the co-ordinate axes in A, B, C such that the centroid of the triangle ABC is the point* (p, q, r). *Show that the equation of the plane is*

$$\frac{x}{p}+\frac{y}{q}+\frac{z}{r}=3.$$

Solution. Let the equation of the plane be

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1. \qquad \ldots(1)$$

The plane (1) meets the co-ordinate axes in $A(a, 0, 0)$, $B(0, b, 0)$ and $C(0, 0, c)$. Therefore, the coordinates of the centroid of the ΔABC are

$$\left(\frac{a+0+0}{3}, \frac{0+b+0}{3}, \frac{0+0+c}{3}\right) \text{ or } \left(\frac{a}{3}, \frac{b}{3}, \frac{c}{3}\right).$$

Since centroid of ΔABC is given as (p, q, r).

$$\therefore \qquad p = \frac{a}{3} \text{ or } a = 3p.$$

Similarly, $\quad b = 3q$ and $c = 3r$.

Now substitute the values of a, b, c in (1), we get

$$\frac{x}{3p} + \frac{y}{3q} + \frac{z}{3r} = 1 \text{ or } \frac{x}{p} + \frac{y}{q} + \frac{z}{r} = 3.$$

Example 2. *Find the intercepts of the plane* $4x + 3y - 12z + 6 = 0$ *on the axes.*

Solution. First change the given plane into intercepted form,

$$4x + 3y - 12z + 6 = 0.$$

$$\therefore \qquad 4x + 3y - 12z = -6. \qquad ...(1)$$

Divide by -6 throughout (1), we get

$$\frac{x}{-3/2} + \frac{y}{-2} + \frac{z}{1/2} = 1.$$

Thus intercepts on x-axis, y-axis, z-axis are respectively $-\frac{3}{2}, -2, \frac{1}{2}$.

Example 3. *Find the equation of the plane passing through the points* $(1, -1, 2)$ *and* $(2, -2, 2)$ *and which is perpendicular to the plane* $6x - 2y + 2z = 9$.

Solution. The equation of any plane through the given point $(1, -1, 2)$ is

$$a(x-1) + b(y+1) + c(z-2) = 0 \qquad ...(1)$$

If the plane (1) passes through the second given point $(2, -2, 2)$, then

$$a(2-1) + b(-2+1) + c(2-2) = 0$$

or $$a - b + 0 \,.\, c = 0 \qquad ...(2)$$

If plane (1) is perpendicular to the given plane $6x - 2y + 2z = 9$ *i.e.,* their normals are perpendicular to each other, then

$$6a - 2b + 2c = 0 \text{ or } 3a - b + c = 0 \qquad ...(3)$$

Now solving (2) and (3) by cross-multiplication, we have

$$\frac{a}{-1-0} = \frac{b}{0-1} = \frac{c}{-1+3} \text{ or } \frac{a}{-1} = \frac{b}{-1} = \frac{c}{2}$$

or $$\frac{a}{1} = \frac{b}{1} = \frac{c}{-2} \qquad ...(4)$$

Eliminating a, b, c from (1) and (4), we get

$$1(x-1) + 1(y+1) - 2(3-2) = 0$$

or $$x + y - 2z + 4 = 0,$$

which is required equation of the plane.

• TEST YOURSELF–1

1. Find the intercepts made on the co-ordinate axes by the plane $x - 3y + 2z - 9 = 0$.
2. Reduce the equation of the plane $x + 2y - 2z - 9 = 0$ to the normal form and hence find the length of the perpendicular drawn from the origin to the given plane.
3. Find the dimension-cosines of the normal to the plane $2x + y + 2z = 3$.
   Find also the perpendicular distance from the origin to the plane.
4. Find the equation of a plane passing through the point $P(a, b, c)$ and perpendicular to OP, O being the origin.
5. The co-ordinates of a point A are $(2, 3, -5)$. Determine the equation to the plane through A at right angles to the line OA, where O is the origin.
6. Find the equation of the plane which cuts off intercepts $6, 3, -4$ from the axes of co-ordinates. Reduce it to normal form and find the perpendicular distance of the plane from the origin.
7. Find the equation of a plane passing through the point $(1, 1, 0)$, $(1, 2, 1)$ and $(-2, 2, -1)$.
8. Find the equation of a plane through the point $(1, 2, 3)$, perpendicular to the plane $x + 2y + 3z = 1$ and parallel to z-axis.

9. Find the equation of the plane through the points $(1, -2, 2)$, $(-3, 1, -2)$ and perpendicular to the plane $x + 2y - 3z = 5$.

ANSWERS

1. $9, -3, 9/2$. 2. $\frac{1}{3}x + \frac{2}{3}y - \frac{2}{3}z = 3, p = 3$.
3. d.r.'s are 2, 1, 2; perpendicular distance = 1.
4. $ax + by + cz = a^2 + b^2 + c^2$. 5. $2x + 3y - 5z = 38$.
6. $\frac{x}{6} + \frac{y}{3} - \frac{z}{4} = 1$; Normal form : $\frac{2}{\sqrt{29}}x + \frac{4}{\sqrt{29}}y - \frac{3}{\sqrt{29}}z = \frac{12}{\sqrt{29}}$; $p = \frac{12}{\sqrt{29}}$.
7. $2x + 3y - 3z - 5 = 0$.
8. $2x - y = 0$. 9. $x + 16y + 11z + 9 = 0$.

• 21.6. LENGTH OF PERPENDICULAR FROM A GIVEN POINT TO A GIVEN PLANE

Let $P(x_1, y_1, z_1)$ be any point and let the general equation of a plane be

$$ax + by + cz + d = 0. \qquad ...(1)$$

Then the perpendicular distance from (x_1, y_1, z_1) to (1) is equal to

$$\frac{ax_1 + by_1 + cz_1 + d}{\pm\sqrt{a^2 + b^2 + c^2}}.$$

• 21.7. DISTANCE BETWEEN TWO PARALLEL PLANES

To find the distance between two parallel planes.

The distance between two parallel planes is obtained by two methods.

Method I. Choose any point on one of the given planes and then find its perpendicular distance from other plane.

Method II. First of all find the perpendicular distance of each plane from origin and retain their signs and find the algebraic difference of these two perpendiculars which gives the required distance between the parallel planes. **But while applying this method we should be careful that the coefficient of x in both planes are of the same sign.**

SOLVED EXAMPLES

Example 1. *Find the distance between parallel planes*

$$2x - 2y + z + 3 = 0 \text{ and } 4x - 4y + 2z + 5 = 0.$$

Solution. Let p_1 and p_2 be the perpendicular distances from (0, 0, 0) to each planes.

$$\therefore \quad p_1 = \frac{3}{\sqrt{(2)^2 + (-2)^2 + (1)^2}} \qquad \left(\because\ p = \frac{d}{\sqrt{a^2 + b^2 + c^2}}\right)$$

$$= \frac{3}{3} = 1$$

and $$p_2 = \frac{5}{\sqrt{(4)^2 + (-4)^2 + (2)^2}} = \frac{5}{6}.$$

Thus distance between the planes $= p_1 - p_2 = 1 - \frac{5}{6} = \frac{1}{6}$.

Example 2. *A variable plane is at a constant distance p from the origin and meets the axes in A, B and C. Show that the locus of the centroid of the tetrahedron OABC is*

$$x^{-2} + y^{-2} + z^{-2} = 16p^{-2}.$$

Solution. Let the variable plane be

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \qquad ...(1)$$

$$\therefore \quad p = \frac{-1}{\sqrt{\left(\frac{1}{a}\right)^2 + \left(\frac{1}{b}\right)^2 + \left(\frac{1}{c}\right)^2}} \qquad \text{(perpendicular distance from (0, 0, 0) to (1))}$$

or $$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=\frac{1}{p^2}. \quad ...(2)$$

Since (1) meets the co-ordinate axes in A, B and C, so that the coordinates of A, B and C are $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$ respectively. Now the centroid of the tetrahedron $OABC$ is

$$\left(\frac{a+0+0+0}{4}, \frac{0+b+0+0}{4}, \frac{0+0+c+0}{4}\right) \text{ or } \left(\frac{a}{4}, \frac{b}{4}, \frac{c}{4}\right).$$

Let (α, β, γ) be the centroid of $OABC$, then

$$\frac{a}{4}=\alpha, \frac{b}{4}=\beta, \frac{c}{4}=\gamma.$$

$\therefore$ $$a=4\alpha, b=4\beta, c=4\gamma.$$

Substitute these values of a, b and c in (2), we get

$$\frac{1}{16\alpha^2}+\frac{1}{16\beta^2}+\frac{1}{16\gamma^2}=\frac{1}{p^2}$$

or $$\alpha^{-2}+\beta^{-2}+\gamma^{-2}=16p^{-2}.$$

Thus the locus of (α, β, γ) is

$$x^{-2}+y^{-2}+z^{-2}=16p^{-2}.$$

Hence proved.

• 21.8. BISECTORS OF THE ANGLE BETWEEN THE PLANES

Let $a_1x+b_1y+c_1z+d_1=0$ and $a_2x+b_2y+c_2z+d_2=0$ be two planes and let $P(x, y, z)$ be any point on the bisector plane as shown in fig. 2.

Draw PM and PN two perpendicular from P to the given planes respectivley. Therefore the plane PQ will be the angle bisector of $\angle MQN$ between two given planes if $PM = \pm PN$.

Thus the perpendicular from P to the plane $a_1x+b_1y+c_1z+d_1=0$ is given by

$$PM=\frac{a_1x+b_1y+c_1z+d_1}{\sqrt{a_1^2+b_1^2+c_1^2}} \quad ...(1)$$

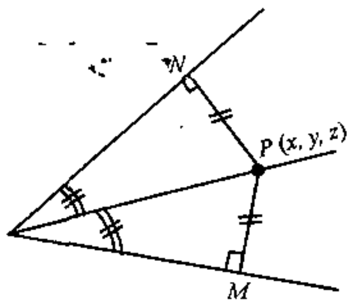


Fig. 2

and the perpendicular distance from P to the other plane $a_2x+b_2y+c_2z+d_2=0$ is given by

$$PN=\frac{a_2x+b_2y+c_2z+d_2}{\sqrt{a_2^2+b_2^2+c_2^2}}. \quad ...(2)$$

Therefore by (1) and (2) and using the result $PM = \pm PN$ the equation of angle bisectors or bisector planes are

$$PM = \pm PN$$

or $$\boxed{\frac{a_1x+b_1y+c_1z+d_1}{\sqrt{a_1^2+b_1^2+c_1^2}}=\pm\frac{a_2x+b_2y+c_2z+d_2}{\sqrt{a_2^2+b_2^2+c_2^2}}.}$$

Here positive and negative signs indicate that there are two angle bisectors, one of them is acute and other is obtuse bisector.

Distinction between Acute and Obtuse Bisector :

Let θ be an angle between one of the bisector planes and one of the given planes. If θ is obtained greater than $\pi/4$ (or 45°), then the chosen bisector plane is obtuse bisector otherwise acute bisector plane.

SOLVED EXAMPLES

Example 1. *Find the equation of the planes bisecting the angle between the planes $x+2y+2z=9$ and $4x-3y+12z+13=0$ and distinguish them.*

Solution. The equation of angle bisectors are given by

$$\frac{x+2y+2z-9}{\sqrt{1+4+4}}=\pm\frac{4x-3y+12z+13}{\sqrt{16+9+144}}$$

or $$\frac{1}{3}(x+2y+2z-9)=\pm\frac{1}{13}(4x-3y+12z+13)$$

Taking + ve sign, we get

$$\frac{1}{3}(x+2y+2z-9)=\frac{1}{13}(4x-3y+12z+13)$$

or $\quad 13x+26y+26z-117=12x-9y+36z+39$

or $\quad x+35y-10z-156=0. \qquad \ldots(1)$

Taking − ve sign, we get

$$\frac{1}{3}(x+2y+2z-9)=-\frac{1}{13}(4x-3y+12z+13)$$

or $\quad 13x+26y+26z-117=-(12x-9y+36z+39)$

or $\quad 13x+26y+26z-117=-12x+9y-36z-39$

or $\quad 25x+17y+62z-78=0.$

Distinction of Acute and Obtuse Bisector.

Let θ be an angle between $x+2y+2z-9=0$ and

$$x+35y-10z-156=0.$$

Then $\quad \cos\theta=\dfrac{1\times 1+35\times 2-10\times 2}{\sqrt{1+4+4}\,.\,\sqrt{1+1225+100}}=\dfrac{1+70-20}{3\sqrt{1326}}$

$$\cos\theta=\frac{51}{3\sqrt{1326}}=\frac{17}{\sqrt{1326}}.$$

$\therefore \quad \tan\theta=\dfrac{\sqrt{1037}}{17}>1.$

$\therefore \quad \theta>45°.$

Hence the plane $x+35y-10z-156=0$ is obtuse bisector and $25x+17y+62z-78=0$ is an acute bisector. Emo

Example 2. *Show that the origin lies in the acute angle between the planes* $x+2y+2z-9=0$ *and* $4x-3y+12z+13=0.$

Solution. First adjust the constant term in each plane to be of same sign. Thus we obtain,

$$-x-2y-2z+9=0 \quad \text{and} \quad 4x-3y+12z+13=0.$$

Now calculate $a_1a_2+b_1b_2+c_1c_2$.

$$\therefore \quad a_1a_2+b_1b_2+c_1c_2=(-1)\,4+(-2)(-3)+(-2)\,12$$

$$=-4+6-24=-22$$

which is obtained negative. Hence the origin lies in acute angle between the planes.

• SUMMARY

- General equation of a plane is $ax+by+cz+d=0$ where $a^2+b^2+c^2\neq 0$.
- Euation of yz plane is $x=0$.
- Equation of zx-plane is $y=0$.
- Equation of xy-plane is $z=0$.
- Intercepted form of plane is $\dfrac{x}{a}+\dfrac{y}{b}+\dfrac{z}{c}=1$.
- Normal form of a plane is $lx+my+xz=p$.
- Equation of a plane passing through three points (x_1,y_1,z_1), (x_2,y_2,z_2) and (x_3,y_3,z_3) is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_2-y_1 & z_3-z_1 \end{vmatrix}=0$$

- Length of perpendicular drawn from (x_1,y_1,z_1) to the plane $ax+by+cz+d=0$ is given by

$$\frac{ax_1+by_1+cz_2+d}{\sqrt{a^2+b^2+c^2}}$$

- Equations of bisector of angle between given planes $a_1x+b_1y+c_1z+d_1=0$ and $a_2x+b_2y+c_2z+d_2=0$ are given by

$$\frac{a_1x+b_1y+c_1z+d_2}{\sqrt{a_1^2+b_1^2+c_1^2}}=\pm\frac{a_2x+b_2y+c_2z+d_2}{\sqrt{a_2^2+b_2^2+c_2^2}}$$

- Equation of a plane through the planes

$P=a_1x+b_1y-c_1z+d_1=0$ and $Q\equiv a_2x+b_2y+c_2z+d_2=0$ is given by $P+\lambda Q=0$.

• STUDENT ACTIVITY

1. A plane meets the coordinate axes in A, B, C such that the centroid of the triangle ABC is the point (p, q, r). Show that the equation of the plane is $\frac{x}{p}+\frac{y}{q}+\frac{z}{r}=3$

2. A variable plane at a constant distance $3p$ from the origin meets the axes in A, B, C. Show that the locus of the centroid of th triangle ABC is $\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{1}{p^2}$.

• TEST YOURSELF–2

1. Find the distance between the parallel planes :
   (i) $2x-3y-6z-21=0$ and $2x-3y-6z+14=0$
   (ii) $2x-y+3z-4=0$ and $6x-3y+9z+13=0$.
2. Find the equation of the locus of a point P whose distance from the plane $6x-2y+3z+4=0$ is equal to its distance from the point $(-1, 1, 2)$.
3. A variable plane passes through a fixed point (α, β, γ) and meets the axes in A, B, and C. Show that the locus of the point of intersection of the planes through A, B and C parallel to the co-ordinate planes is

$$\frac{\alpha}{x}+\frac{\beta}{y}+\frac{\gamma}{z}=1.$$

4. Two systems of rectangular axes have the same origin. If a plane cuts them at a distances a, b, c and a', b', c' respectively from origin, show that

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=\frac{1}{a'^2}+\frac{1}{b'^2}+\frac{1}{c'^2}.$$

5. A variable plane is at a constant distance p from the origin and meets the axes in A, B and C. Show that the locus of the centroid of the triangle ABC is $x^{-2}+y^{-2}+z^{-2}=9p^{-2}$.

ANSWERS

1. (i) 5 (ii) $\frac{25}{3\sqrt{14}}$.

2. $13x^2 + 45y^2 + 40z^2 + 24xy - 36zx + 12yz + 50x - 82y - 220z + 278 = 0$.

OBJECTIVE EVALUATION

➤ FILL IN THE BLANKS :

1. The general form of a plane is ....
2. The plane $lx + my + nz = p$ is a ....
3. A plane cuts intercepts of length 2, – 3, 5 on co-ordinate axes, then the equation of plane is ....
4. The direction-ratios of the normal to the plane $ax + by + cz = 0$ are ....

➤ TRUE OR FALSE :

Write 'T' for true and 'F' for false statement :

1. The intercepts form of a plane is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$. *(T/F)*
2. The plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ meets the z-axis at $(0, 0, a)$. *(T/F)*
3. The normal form of the plane is $lx + my + nz = p$. *(T/F)*

➤ MULTIPLE CHOICE QUESTIONS :

Choose the most appropriate one :

1. The intercept on x-axis of the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ is :
   (a) b (b) a
   (c) 1 (d) c.
2. The perpendicular distance from the origin to the plane $lx + my + nz = p$ is :
   (a) p (b) l
   (c) n (d) 1.

ANSWERS

Fill in the Blanks :

1. $ax + by + cz + d = 0$ 2. normal form 3. $\frac{x}{2} - \frac{y}{3} + \frac{z}{5} = 1$

4. a, b, c

True or False :

1. T 2. F 3. T 4. T 5. F

Multiple Choice Questions :

1. (b) 2. (a)

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UNIT 22

THE STRAIGHT LINE

STRUCTURE

LEARNING OBJECTIVES

After going through this unit you will learn :

- About different forms of a straight line
- How to find the angle between the plane and the line
- How to find the plane through the given line
- How to find the projection of a given line on a given plane

22.1. EQUATIONS OF A LINE IN DIFFERENT FORMS

(i) General form (non-symmetrical form). Since every equation of first degree in x, y and z represents a plane, when two such planes intersect a line of intersection is formed, therefore the equations of two planes simultaneously represent a straight line.

If $a_1x + b_1 y + c_1z + d_1 = 0$ and $a_2x + b_2 y + c_2z + d_2 = 0$ be two planes in general form. Then the general form a straight line is given by

Geometry and Vectors

$$\left.\begin{aligned} a_1x + b_1 y + c_1z + d_1 = 0 \\ a_2x + b_2 y + c_2z + d_2 = 0 \end{aligned}\right\}.$$

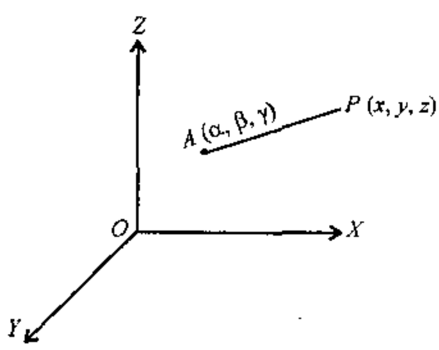

Fig. 1

(ii) Symmetrical form.

Symmetrical form of line is

$$\boxed{\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} = AP}$$

Corollary 1. *The equation of a line passing through a point* (α, β, γ) *having direction ratios* a, b, c. *Then its symmetrical form is*

$$\frac{x-\alpha}{a} = \frac{y-\beta}{b} = \frac{z-\gamma}{c}.$$

Corollary 2. *To obtain the equation of a straight line passing through two points.*

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}.$$

This is the required line in symmetrical form passing through two points (x_1, y_1, z_1) and (x_2, y_2, z_2).

SOLVED EXAMPLES

Example 1. *If the line* $\frac{x-1}{2}=\frac{y-3}{1}=\frac{z-2}{c}$ *is parallel to the plane* $3x-y+4z=7$, *then find* c.

Solution. Dr's of the given line are 2, 1, c.

Dr's of the normal to the plane $3x-y+4z=7$ are $3, -1, 4$.

The line is parallel to the plane if

$$2\times 3+1\times -1+c\times 4=0$$

$$\Rightarrow \qquad c=-5/4.$$

Example 2. *Find the point in which the line* $\frac{x-2}{3}=\frac{y+1}{4}=\frac{z-2}{12}$ *meets the plane*

$$x-2y+z=20.$$

Solution. Since the equations of the lines are

$$\frac{x-2}{3}=\frac{y+1}{4}=\frac{z-2}{12}=r \text{ (say)}.$$

Any point on this line is $P(3r+2, 4r-1, 12r+2)$.

The given line meets the plane $x-2y+z=20$ so P lies on this plane.

$$\therefore \qquad (3r+2)-2(4r-1)+(12r+2)=20$$

or $$7r=14 \quad \text{or} \quad r=2.$$

Putting this value of r in the co-ordinates of P, the required co-ordinates of the point are $(3(2)+2, 4(2)-1, 12(2)+2)$ *i.e.*, $(8, 7, 26)$.

Example 3. *Show that the line joining the points* $A(2,-3,-1)$ *and* $B(8,-1,2)$ *has equations* $\frac{x-2}{6}=\frac{y+3}{3}=\frac{z+1}{3}$.

Find two points on the line whose distance from A is 14.

Solution. The d.r.'s of the line AB are

$$8-2, -1-(-3), 2-(-1) \quad \text{or} \quad 6, 2, 3.$$

Thus the equation of a line AB through A whose d.r.'s are 6, 2, 3 are

$$\frac{x-2}{6}=\frac{y+3}{3}=\frac{z+1}{3}.$$

These are the required equations of the line joining the points A and B.

Let $$\frac{x-2}{6}=\frac{y+3}{3}=\frac{z+1}{3}=r \text{ (say)}$$

so any point on this line is $P(6r+2, 2r-3, 3r-1)$.

Since $$PA=14,$$

$$\therefore \qquad \sqrt{(6r+2-2)^2+(2r-3+3)^2+(3r-1+1)^2}=14$$

or $$\sqrt{36r^2+4r^2+9r^2}=14 \quad \text{or} \quad \sqrt{49r^2}=14$$

or $$\pm 7r=14 \quad \text{or} \quad r=\pm 2.$$

Putting $r=2$ and -2 successively in the co-ordinates of P, we get the required points as $(14, 1, 5)$ and $(-10, -7, -7)$.

• TEST YOURSELF–1

1. Find the co-ordinates of the point of intersection of the line $\frac{x+1}{1}=\frac{y+3}{3}=\frac{z-2}{2}$ with the plane $3x+4y+5z=20$.
2. Find the co-ordinates of the point where the line joining the points $(2,-3,1)$ and $(3,-4,-5)$ meets the plane $2x+y+z=7$.
3. Show that the distance of the point of intersection of the line $\frac{x-2}{3}=\frac{y+1}{4}=\frac{z-2}{12}$ and the plane $x-y+z=5$ from the point $(-1,-5,-10)$ is 13.
4. Find the equations of the line through the points (a, b, c) and (a', b', c') and prove that it passes through the origin if $aa'+bb'+cc'=rr'$ where r and r' are the distances of the given points from origin.

ANSWERS

1. $(0, 0, 4)$. 2. $(1, -2, 7)$. 4. $\frac{x-a}{a'-a}=\frac{y-b}{b'-b}=\frac{z-c}{c'-c}$

• 22.2. EQUATION OF A PLANE THROUGH A GIVEN LINE

(a) If the line is in symmetrical form. Let the line be

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}. \qquad ...(1)$$

Since the plane is to passes through the given line. Therefore the plane will pass through the fixed point (α, β, γ). Then the equation of a plane through (α, β, γ) is

$$a(x-\alpha)+b(y-\beta)+c(z-\gamma)=0. \qquad ...(2)$$

Since the plane (2) is passing throguh the line (1), then its normal is perpendicular to (1).

$$\therefore \quad al+bm+cn=0.$$

Hence the required plane is

$$a(x-\alpha)+b(y-\beta)+c(z-\gamma)=0,$$

where $\quad al+bm+cn=0.$

(b) If the line is in general form so let the equation of a line is

$$\left.\begin{array}{l} a_1x+b_1y+c_1z+d_1=0, \\ a_2x+b_2y+c_2z+d_2=0 \end{array}\right\}. \qquad ...(3)$$

Hence, the equation of a plane through (3) is given by

$$(a_1x+b_1y+c_1z+d_1)+\lambda(a_2x+b_2y+c_2z+d_2)=0.$$

REMARK

➤ If $u \equiv 0$ and $v \equiv 0$ are equations of two planes, then the equation of a plane through the intersection of these planes is $u+\lambda v=0$ where λ is a parameter.

• 22.3. ANGLE BETWEEN A LINE AND A PLANE

To find the angle between the line $\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$ *and the plane* $ax+by+cz+d=0$.

Let θ be the angle between the line AB and the plane, then the angle between the line AB and normal AN to the plane will be $90°-\theta$.

Direction ratios of AN are a, b, c and direction ratios of line AB are l, m, n.

$$\therefore \quad \cos(90°-\theta)=\frac{al+bm+cn}{\sqrt{(a^2+b^2+c^2)(l^2+m^2+n^2)}}$$

i.e.,

$$\sin\theta=\frac{al+bm+cn}{\sqrt{(a^2+b^2+c^2)(l^2+m^2+n^2)}}.$$

This gives the angle between the line and the plane.

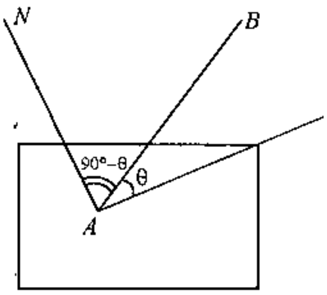

Fig. 5

SOLVED EXAMPLES

Example 1. *Find the equation of the plane through the line* $\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$ *and parallel to the line* $\frac{x-\alpha'}{l'}=\frac{y-\beta'}{m'}=\frac{z-\gamma'}{n'}$.

Solution. The equation of a plane through the line

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$$

is

$$a(x-\alpha)+b(y-\beta)+c(z-\gamma)=0 \qquad ...(1)$$

where a, b, c are the d.r.'s of the normal to the plane (1).

Since the plane (1) containing the first line whose d.r.'s are l, m, n and it is parallel to the second line whose d.r.'s are l', m', n', then we have

$$al+bm+cn=0 \qquad ...(2)$$

and

$$al'+bm'+cn'=0. \qquad ...(3)$$

Solving (2) and (3) by cross multiplication, we get

$$\frac{a}{mn'-m'n}=\frac{b}{l'n-ln'}=\frac{c}{lm'-l'm}.$$

Putting the proportionate values of a, b and c in (1), the equation of the required plane is

$$(mn'-m'n)(x-\alpha)+(l'n-ln')(y-\beta)+(lm'-l'm)(z-\gamma)=0.$$

Example 2. *Find the equation of a plane containing the line* $\frac{y}{b}+\frac{z}{c}=1, x=0$ *and parallel to the line* $\frac{x}{a}-\frac{z}{c}=1, y=0.$

Solution. The equation of a plane containing the line

$$\frac{y}{b}+\frac{z}{c}=1, x=0$$

is
$$\left(\frac{y}{b}+\frac{z}{c}-1\right)+\lambda x=0 \text{ or } \lambda x+\frac{y}{b}+\frac{z}{c}-1=0. \quad ...(1)$$

Now d.r.'s of normal to this plane are

$$\lambda, \frac{1}{b}, \frac{1}{c}.$$

Since this plane is parallel to the line

$$\frac{x}{a}-\frac{z}{c}=1, y=0 \quad i.e., \frac{x-a}{a}=\frac{y}{0}=\frac{z-0}{c}$$

whose d.r.'s are $a, 0, c$.

$$\therefore \quad \lambda a+\frac{1}{b}.0+\frac{1}{c}.c=0$$

or
$$\lambda a+1=0 \text{ or } \lambda=-1/a.$$

Putting this value of λ in (1), the equation of required plane is

$$-\frac{x}{a}+\frac{y}{b}+\frac{z}{c}-1=0$$

or
$$\frac{x}{a}-\frac{y}{b}-\frac{z}{c}+1=0.$$

• TEST YOURSLF–2

1. Find the equation of the plane through the line $P \equiv ax+by+cz+d=0$, $Q \equiv a'x+b'y+c'z+d'=0$ and parallel to the line $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$.
2. Find the equation of a plane through the line of intersection of the planes $x+2y+3z-4=0$ and $2x+y-z+5=0$ and perpendicular to the plane $5x+3y+6z+8=0$.
3. Find the direction cosines of the line whose equations are $x+y=3$ and $x+y+z=0$ and show that it makes an angle 30° with the plane $y-z+2=0$.
4. Prove that the lines
$3x+2y+z-5=0=x+y-2z-3$ and $2x-y-z=0=7x+10y-8z-15$ are perpendicular.

ANSWERS

1. $P(a'l+b'm+c'n)=Q(al+bm+cn)$ 2. $51x+15y-50z+173=0.$
3. $0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}$

• 22.4. COPLANAR LINES

(a) When both lines are in symmetrical form :

To obtain the condition that two lines may intersect and the equation of the plane in which they lie.

Let the equation of two lines be given by

$$\frac{x-\alpha_1}{l_1}=\frac{y-\beta_1}{m_1}=\frac{z-\gamma_1}{n_1} \quad ...(1)$$

and
$$\frac{x-\alpha_2}{l_2}=\frac{y-\beta_2}{m_2}=\frac{z-\gamma_2}{n_2}. \quad ...(2)$$

The equation of a plane through the line (1) is

$$a(x-\alpha_1)+b(y-\beta_1)+c(z-\gamma_1)=0 \quad ...(3)$$

where
$$al_1+bm_1+cn_1=0 \quad ...(4)$$

If the line (2) lies in the same plane, then the normal of this plane is perpendicular to (2), we have

$$al_2+bm_2+cn_2=0. \quad ...(5)$$

Further since the point $(\alpha_2, \beta_2, \gamma_2)$ also lies on the plane; then

$$a(\alpha_2-\alpha_1)+b(\beta_2-\beta_1)+c(\gamma_2-\gamma_1)=0. \qquad ...(6)$$

Now eliminating a, b, c from (4), (5) and (6) we get the required condition

$$\begin{vmatrix} \alpha_2-\alpha_1 & \beta_2-\beta_1 & \gamma_2-\gamma_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0.$$

Again eleminating a, b, c from (3), (4) and (5), the required plane is

$$\begin{vmatrix} x-\alpha_1 & y-\beta_1 & z-\gamma_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0.$$

(b) When one line in symmetrical form and other in general form :

To obtain the condition that the lines

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n} \qquad ... (1)$$

$$\left.\begin{aligned} a_1x+b_1y+c_1z+d_1=0, \\ a_2x+b_2y+c_2z+d_2=0 \end{aligned}\right\} \qquad ...(2)$$

are ***coplanar.***

If the given lines intersect, they are coplanar.

Any point on the first line is $(lr+\alpha, mr+\beta, nr+\gamma)$. This point will lie on the second line. Then we have

$$a_1(lr+a)+b_1(mr+\beta)+c_1(nr+\gamma)+d_1=0$$

or
$$r=-\frac{a_1\alpha+b_1\beta+c_1\gamma+d_1}{a_1l+b_1m+c_1n}.$$

Similarly
$$r=-\frac{a_2\alpha+b_2\beta+c_2\gamma+d_2}{a_2l+b_2m+c_2n}.$$

Equating the values of r, we get the required condition

$$\boldsymbol{\frac{a_1\alpha+b_1\beta+c_1\gamma+d_1}{a_1l+b_1m+c_1n}=\frac{a_2\alpha+b_2\beta+c_2\gamma+d_2}{a_2l+b_2m+c_2n}}.$$

• 22.5. PROJECTION OF A LINE ON A GIVEN PLANE

Let $\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$ be a line and $ax+by+cz+d=0$ be a plane on which the projection of the line is to be required. Therefore two ways to find the line of projection.

(i) The equation of any plane through the given line is

$$A(x-\alpha)+B(y-\beta)+C(z-\gamma)=0 \qquad ...(1)$$

where
$$Al+Bm+Cn=0. \qquad ...(2)$$

Since the given plane will be perpendicular to the plane (1) if

$$Aa+Bb+Cc=0. \qquad ...(3)$$

Solving (2) and (3), we get

$$\frac{A}{mc-nb}=\frac{B}{na-lc}=\frac{C}{lb-ma}.$$

Putting these values of A, B, C in (1), we get

$$(mc-nb)(x-\alpha)+(na-lc)(y-\beta)+(lb-ma)(z-\gamma)=0 \qquad ...(4)$$

Hence the equations $ax+by+cz+d=0$ and (4) together are the equations of the line of projection.

(ii) Any point on the given line is $P(lr+\alpha, mr+\beta, nr+\gamma)$.

If this point lies on the given plane, then

$$a(lr+\alpha)+b(mr+\beta)+c(nr+\gamma)+d=0$$

or
$$r=-\frac{a\alpha+b\beta+c\gamma+d}{al+mb+nc}.$$

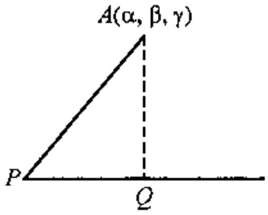

Fig. 6

Putting this value of r in the co-ordinates of P, so we get the point of intersection of the given line and the given plane.

Since the line passes through the point $A(\alpha, \beta, \gamma)$ so draw a perpendicular from A to the given plane, which meets at Q. Let Q be the foot of the perpendicular.

Now, the d.r.'s of the normal to the given plane are a, b, c and hence these are the d.r.'s of the line through (α, β, γ) and perpendicular to the given plane, therefore the equations of this perpendicular line are

$$\frac{x-\alpha}{a}=\frac{y-\beta}{b}=\frac{z-\gamma}{c}=r_1 \text{ (say).}$$

Any point on this line is $(ar_1+\alpha, br_1+\beta, c\gamma_1+\gamma)$ let this point be Q, then it will lie on the given plane $ax+by+cz+d=0$, so we have

$$a(ar_1+\alpha)+\beta(br+\beta)+c(cr_1+\gamma)=0$$

or
$$r_1=-\frac{a\alpha+b\beta+c\gamma}{a^2+b^2+c^2}.$$

Putting this value of r_1 in the co-ordinates of Q we get the point Q.

Thus the line passing through the point P and Q is the required line of projection.

SOLVED EXAMPLES

Example 1. *Find the distance of the point* $P(3, 8, 2)$ *from the line* $\frac{x-1}{2}=\frac{y-3}{4}=\frac{z-2}{3}$ *measured parallel to the plane* $3x+2y-3z+17=0$.

Solution. Any point on the given line is $N(2r+1, 4r+3, 3r+2)$.

Let it be the foot of the perpendicular drawn from P to the line.

The d.r.'s of PN are $2r-2, 4r-5, 3r$.

The d.r.'s of the normal to the given plane $3x+2y-2z+17=0$ are $3, 2, -2$.

Since PN is being measured parallel to the given plane.

$$\therefore \quad 3(2r-2)+2(4r+5)-2(3r)=0$$

or
$$6r-6+8r-10-6r=0$$

or
$$8r-16=0 \quad \text{or} \quad r=2.$$

Putting this value of r in the co-ordinates of N, the co-ordinates of N are $(5, 11, 8)$.

$$\therefore \quad PN=\sqrt{(5-3)^2+(11-8)^2+(8-2)^2}$$
$$=\sqrt{4+9+36}=\sqrt{49}=7.$$

Example 2. *Show that the lines*

$$\frac{x+3}{2}=\frac{y+5}{3}=\frac{z-7}{-3} \quad \textit{and} \quad \frac{x+1}{4}=\frac{y+1}{5}=\frac{z+1}{-1}$$

are coplanar. Find the equation of the plane containing them.

Solution. Let,

$$\frac{x+3}{2}=\frac{y+5}{3}=\frac{z-7}{-3}=r_1 \text{ (say)} \qquad \ldots(1)$$

and
$$\frac{x+1}{4}=\frac{y+1}{5}=\frac{z+1}{-1}=r_2 \text{ (say).} \qquad \ldots(2)$$

Any point on the line (1) is $P(2r_1-3, 3r_1-5, -3r_1+7)$ and any point on the line (2) is $Q(4r_2-1, 5r_2-1, -r_2-1)$.

If the lines (1) and (2) are coplanar, they will intersect

$$\therefore \quad 2r_1-3=4r_2-1 \quad \text{or} \quad 2r_1-4r_2=2 \qquad \ldots(3)$$
$$3r_1-5=5r_2-1 \quad \text{or} \quad 3r_1-5r_2=4 \qquad \ldots(4)$$

and
$$-3r_1+7=-r_2-1 \quad \text{or} \quad 3r_1-r_2=8. \qquad \ldots(5)$$

Solving (3) and (4), we get

$$r_1=3, r_2=1.$$

On putting these values in R.H.S. of (5), we get

$$3(3)-1=9-1=8=\text{R.H.S.}$$

Thus both points coincides. Hence the lines are coplanar. The point of intersection is $(3, 4, -2)$.

The equation of a plane containing (1) and (2) (*i.e.*, a plane through the line (1) and parallel to the line (2)) is,

$$\begin{vmatrix} x+3 & y+5 & z-7 \\ 2 & 3 & -3 \\ 4 & 5 & -1 \end{vmatrix}=0$$

or
$$(x+3)\{-3+15\}+(y+5)\{-12+2\}+(z-7)\{10-12\}=0$$

or
$$12(x+3)-10(y+5)-2(z-7)=0$$

or
$$12x-10y-2z=0 \quad \text{or} \quad 6x-5y-z=0.$$

• TEST YOURSELF–3

1. Find the projection of the line $3x - y + 2z - 1 = 0 = x + 2y - z = 2$ on the plane $3x + 2y + z = 0$.
2. Find the equations of the perpendicular from origin to the line
$$ax + by + cz + d = 0 = a'x + b'y + c'z + d'.$$
3. Prove that the lines :
$$\frac{x}{\alpha} = \frac{y}{\beta} = \frac{z}{\gamma};\ \frac{x}{a\alpha} = \frac{y}{b\beta} = \frac{z}{c\gamma};\ \frac{x}{l} = \frac{y}{m} = \frac{z}{n}$$
will lie in a plane if
$$\frac{l}{\alpha}(b - c) + \frac{m}{\beta}(c - a) + \frac{n}{\gamma}(a - b) = 0.$$
4. Prove that the lines :
$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4};\ \frac{x-2}{3} = \frac{y-3}{4} = \frac{z-4}{5}$$
are coplanar; find their point of intersection. Also find the equation of the plane in which they lie.

ANSWERS

1. $3x + 2y + z = 0 = 3x - 8y + 7z + 4$. 2. $(bc' - b'c)\,x + (ca' - c'a)\,y + (ab' - a'b)\,z = 0$.
4. $(-1, -1, -1)$; $x - 2y + z = 0$.

• 22.6. SHORTEST DISTANCE BETWEEN TWO LINES

Shortest distance between the lines can only have the meaning if, the lines do not intersect or the lines are not lying in the same plane, those lines which have this character, are called ***skew lines*** *and length between them which is perpendicular to both skew lines is called* ***shortest distance.***

Length and Equation of Shortest Distance :

To obtain the length and equations of shortest distance between two non-intersecting (or skew) lines.

(a) When both lines in symmetrical form.

(i) Projection Method. Let the two skew lines be given by
$$\frac{x-\alpha_1}{l_1} = \frac{y-\beta_1}{m_1} = \frac{z-\gamma_1}{n_1} \quad \text{and} \quad \frac{x-\alpha_2}{l_2} = \frac{y-\beta_2}{m_2} = \frac{z-\gamma_2}{n_2}.$$

Let l, m, n be the direction cosines of a shortest distance. Since shortest distance is perpendicular to both the given lines. Then we have
$$ll_1 + mm_1 + nn_1 = 0 \qquad ...(1)$$
and
$$ll_2 + mm_2 + nn_2 = 0. \qquad ...(2)$$

Solving (1) and (2) by cross multiplication method, we have
$$\frac{l}{m_1n_2 - m_2n_1} = \frac{m}{n_1l_2 - n_2l_1} = \frac{n}{l_1m_2 - l_2m_1}.$$

Thus, direction ratios of shortest distance are $m_1n_2 - m_2n_1, n_1l_2 - n_2l_1, l_1m_2 - l_2m_1$.

Therefore, the direction-cosines of shortest distance are
$$l = \frac{m_1n_2 - m_2n_1}{LM},\ m = \frac{n_1l_2 - n_2l_1}{LM},\ n = \frac{l_1m_2 - l_2m_1}{LM}$$
where
$$LM = \sqrt{[(m_1n_2 - m_2n_1)^2 + (n_1l_2 - n_2l_1)^2 + (l_1m_2 - l_2m_1)^2]}.$$

Further since, first line is passing through the point $P(\alpha_1, \beta_1, \gamma_1)$ and second line is passing through $Q(\alpha_2, \beta_2, \gamma_2)$. Suppose the shortest distance meet the given lines in R and S respectively as shown in fig. 7.

Therefore the proejction of PQ on the shortest distance gives the length of shortest distance which is given by

Shortest distance
$$= (\alpha_1 - \alpha_2)\,l + (\beta_1 - \beta_2)\,m + (\gamma_1 - \gamma_2)\,n.$$

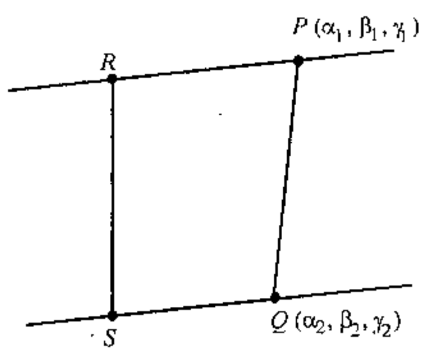

Fig. 7

Next, the equation of shortest distance is the plane containing first line and S.D. (shortest distance) and also containing second line and S.D. which are given by

$$\begin{vmatrix} x-\alpha_1 & y-\beta_1 & z-\gamma_1 \\ l_1 & m_1 & n_1 \\ l & m & n \end{vmatrix} = 0 \qquad ...(3)$$

and

$$\begin{vmatrix} x-\alpha_2 & y-\beta_2 & z-\gamma_2 \\ l_2 & m_2 & n_2 \\ l & m & n \end{vmatrix} = 0 \qquad ...(4)$$

Thus the equations (3) and (4) simultaneously gives the equation of shortest distance.

(ii) Co-ordinates Method (Point Method).

Let the equation of two lines be

$$\frac{x-\alpha_1}{l_1} = \frac{y-\beta_1}{m_1} = \frac{z-\gamma_1}{n_1} = r_1 \text{ (say)} \qquad ...(1)$$

and

$$\frac{x-\alpha_2}{l_2} = \frac{y-\beta_2}{m_2} = \frac{z-\gamma_2}{n_2} = r_2 \text{ (say)} \qquad ...(2)$$

Any point on (1) is $P(l_1r_1+\alpha_1, m_1r_1+\beta_1, n_1r_1+\gamma_1)$ and on (2) is $Q(l_2r_2+\alpha_2, m_2r_2+\beta_2, n_2r_2+\gamma_2)$. Let line PQ be the shortest distance. Then PQ will be perpendicular to both the lines (1) and (2) so that we have d.r.'s ratios of PQ are

$$l_2r_2 - l_1r_1 + \alpha_2 - \alpha_1,\ m_2r_2 - m_1r_1 + \beta_2 - \beta_1,\ n_2r_2 - n_1r_1 + \gamma_2 - \gamma_1.$$

Since PQ is perpendicular to (1) and (2), we have

$$l_1(l_2r_2 - l_1r_1 + \alpha_2 - \alpha_1) + m_1(m_2r_2 - m_1r_1 + \beta_2 - \beta_1) + n_1(n_2r_2 - n_1r_1 + \gamma_2 - \gamma_1) = 0 \qquad ...(3)$$

and

$$l_2(l_2r_2 - l_1r_1 + \alpha_2 - \alpha_1) + m_2(m_2r_2 - m_1r_1 + \beta_2 - \beta_1) + n_2(n_2r_2 - n_1r_1 + \gamma_2 - \gamma_1) = 0. \qquad ...(4)$$

Solving (3) and (4), we get r_1 and r_2. Substitute these value of r_1 and r_2 in the points P and Q we get the two points at which the shortest distance meets. Therefore, *the distance between the points P and Q gives the length of shortest distance and the equation of shortest distance is thus obtained by the line joining the point P and Q.*

(b) When one line in symmetrical form and other in general form.

Let the equation of two lines be given by

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} \qquad ...(1)$$

and

$$\left.\begin{aligned} u \equiv a_1x + b_1y + c_1z + d_1 = 0 \\ v \equiv a_2x + b_2y + c_2z + d_2 = 0 \end{aligned}\right\} \qquad ...(2)$$

The equation of a plane through the line (2) is given by

$$u + \lambda v = 0 \qquad ...(3)$$

where λ is a parameter.

Now find the λ in such a way that (1) is parallel to (3) and substitute this value of λ in (3) we get the equation of a plane parallel to the line (1). Therefore, *the perpendicular distance from any point on the line* (1) (α, β, γ) *(say) to the plane* (3) *gives the length of the shortest distance and the equation of shortest distance is obtained as the line of intersection of the two planes* :

(i) *the plane containing the line* (1) *and perpendicular to the plane* (3) **(ii)** *the plane containing the line* (2) *and perpendicular to the plane* (3).

(c) When both the lines are in general form.

Let $u_1 \equiv 0, v_1 \equiv 0$ and $u_2 \equiv 0, v_2 \equiv 0$ be the two lines in general form that is,

$$\left.\begin{aligned} u_1 \equiv a_1x + b_1y + c_1z + d_1 = 0 \\ v_1 \equiv a_2x + b_2y + c_2z + d_2 = 0 \end{aligned}\right\} \qquad ...(1)$$

and

$$\left.\begin{aligned} u_2 \equiv a_3x + b_3y + c_3z + d_3 = 0 \\ v_2 \equiv a_4x + b_4y + c_4z + d_4 = 0 \end{aligned}\right\}. \qquad ...(2)$$

The equation of a plane through the line (1) is given by

$$u_1 + \lambda_1 u_1 = 0 \qquad ...(3)$$

and the equation of a plane through the line (2) is given by

$$u_2 + \lambda_2 v_2 = 0. \qquad ...(4)$$

Now find the λ_1 and λ_2 is such a way that the plane (3) and the plane (4) are parallel. And substitute the values of λ_1 and λ_2 in (3) and (4) respectively. Thus (3) and (4) become parallel.

Therefore the distance between these parallel planes (3) and (4) gives the length of shortest distance between (1) and (2).

And the equation of shortest distance is obtained as the line of intersection of the two planes:

(i) The plane through (1) and perpendicular to the plane (3).

(ii) The plane through (2) and perpendicular to the plane (4).

SOLVED EXAMPLES

Example 1. *Find the shortest distance between the lines*

$$\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4};\ \frac{x-2}{3}=\frac{y-4}{4}=\frac{z-5}{5}.$$

Solution. Let l, m, n be the direction-cosines of shortest distance, since S.D. is perpendicular to both lines so we have

$$2l+3m+4n=0 \qquad ...(1)$$

and
$$3l+4m+5n=0. \qquad ...(2)$$

Solve (1) and (2), we get

$$\frac{l}{15-16}=\frac{m}{12-10}=\frac{n}{8-9}$$

or
$$\frac{l}{-1}=\frac{m}{2}=\frac{n}{-1} \quad \text{or} \quad \frac{l}{1}=\frac{m}{-2}=\frac{n}{1}.$$

$$\therefore \qquad l=\frac{1}{\sqrt{6}},\ m=-\frac{2}{\sqrt{6}},\ n=\frac{1}{\sqrt{6}}.$$

Since $P(1, 2, 3)$ and $Q(2, 4, 5)$ are the points on the given lines respectively. Thus the projection of PQ on the shortest distance is

$$\text{S. D.}=(2-1)\frac{1}{\sqrt{6}}+(4-2)\left(-\frac{2}{\sqrt{6}}\right)+(5-3)\frac{1}{\sqrt{6}}$$

$$=\frac{1}{\sqrt{6}}-\frac{4}{\sqrt{6}}+\frac{2}{\sqrt{6}}=-\frac{1}{\sqrt{6}}.$$

Hence, the length of S.D. $=\dfrac{1}{\sqrt{6}}$ numerically.

Example 2. *Show that the length of the shortest distance between the lines*

$$\frac{x-2}{2}=\frac{y+1}{3}=\frac{z}{4};$$

$$2x+3y-5z-6=0=3x-2y-z+3 \textit{ is } 97/(13\sqrt{6}).$$

Solution. Since, we have

$$\frac{x-2}{2}=\frac{y+1}{3}=\frac{z}{4} \qquad ...(1)$$

$$\left.\begin{aligned}2x+3y-5z-6=0\\ 3x-2y-z+3=0\end{aligned}\right\}. \qquad ...(2)$$

The equation of a plane through (2) is given by

$$(2x+3y-5z-6)+\lambda\,(3x-2y-z+3)=0$$

or
$$(2+3\lambda)\,x+(3-2\lambda)\,y+(-5-\lambda)\,z-6+3\lambda=0. \qquad ...(3)$$

Since (3) is parallel to (1) so d.r.'s of normal to (3) is perpendicular to (1). Then

$$2(2+3\lambda)+3(3-2\lambda)+4(-5-\lambda)=0$$

or
$$-4\lambda-7=0 \quad \text{or} \quad \lambda=-7/4.$$

Substitute this value of λ in (3), we get

$$\left(2-\frac{21}{4}\right)x+\left(3+\frac{14}{4}\right)y+\left(-5+\frac{7}{4}\right)z-6-\frac{21}{4}=0$$

or
$$-13x+26y-13z-45=0$$

or
$$13x-26y+13z+45=0. \qquad ...(4)$$

Now (1) and (4) is parallel and $P(2, -1, 0)$ is any point on (1) so length of shortest distance is given by.

S.D. = Perpendicular distance from $P(2, -1, 0)$ to the plane

$$13x-26y+13z+45=0$$

$$=\frac{13\times 2-26(-1)+13(0)+45}{\sqrt{(13)^2+(-26)^2+(13)^2}}=\frac{97}{13\sqrt{6}}.$$

• SUMMARY

- Equation of a straight line in general form is $a_1x + b_1y + c_1z + d_1 = 0 = a_2 + b_{2y} + c_1z + dz$.
- Equation of a straight line in symmetrical form is

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$$

where l, m, n are d.c's of the line.

- Let $ax + by + cz + d = 0$ be a plane and $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$ a line. Then

(i) Line is parallel to the plane if $al + bm + cn = 0$ and $a\alpha + b\beta + c\gamma + d \neq 0$

(ii) Line is perpendicular to the plane if $\frac{l}{a} = \frac{m}{b} = \frac{n}{c}$.

(iii) Line lies in the plane if $al + bm + cn = 0$ and $a\alpha + b\beta + c\gamma + d = 0$.

- Angle betwen the line $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$ and the plane $ax + by + cz + d = 0$ is given by

$$\sin^{-1}\left[\frac{al + bm + cn}{\sqrt{a^2 + bsup2 + c^2}\sqrt{l^2 + m^2 + n^2}}\right]$$

- Lines $\frac{x-\alpha_1}{l_1} = \frac{y-p_1}{m_1} = \frac{z-\gamma_1}{n_1}$ and $\frac{x-\alpha_2}{l_2} = \frac{y-\beta_2}{m_2} = \frac{z-\gamma_2}{n_2}$ are coplanar if

$$\begin{vmatrix} \alpha_2-\alpha_1 & \beta_2-\beta_1 & \gamma_2-\gamma_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

- Perpendicular distance of $\frac{x-\alpha}{l} = y - \beta ovrm = \frac{z-\gamma}{n}$ from $P(x_1, y_1, z_1)$ is given by

$$\therefore \qquad PN^2 = \begin{vmatrix} y_1-\beta & z_1-\gamma \\ m & n \end{vmatrix}^2 + \begin{vmatrix} z_1-\gamma & x_1-\alpha \\ n & l \end{vmatrix}^2 + \begin{vmatrix} x_1-\alpha & y_1-\beta \\ l & m \end{vmatrix}^2$$

- Shortest distance between the lines

$$\frac{x-\alpha_1}{l_1} = \frac{y-\beta_1}{m_1} = \frac{z-\gamma_1}{n_1} \text{ and } \frac{x-\alpha_2}{l_2} = \frac{y-\beta_2}{m_2} = \frac{z-\gamma_2}{n_2} \text{ is given by}$$

$$\text{S.D} = (\alpha_2-\alpha_1)\,l + (\beta_2-\beta_1)\,m + (\gamma_2-\gamma_1)\,n$$

where l, m, n satisfy the realtions

$$ll_1 + mm_1 + nn_1 = 0 \text{ and } ll_2 + mm_2 + nn_2 = 0$$

- Skew lines are $\frac{x}{1} = \frac{y}{\tan\alpha} = \frac{z-c}{0}$; $\frac{x}{1} = \frac{y}{\tan\alpha} = \frac{z+c}{0}$

• STUDENT ACTIVITY

1. Find the image of the point (1, 3, 4) in the plane $2x - y + z + 3 = 0$.

2. Find the shortest distance between the lines $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ and $\frac{x-2}{3}=\frac{y-4}{4}=\frac{z-5}{5}$.

• TEST YOURSELF–5

1. Find the shortest distance between the lines
$$\frac{x-3}{3}=\frac{y-8}{-1}=\frac{z-3}{1};\quad \frac{x+3}{-3}=\frac{y+7}{2}=\frac{z-6}{4}.$$
Find also its equations and the points in which it meets the given lines.
2. Find the length and equation of the shortest distance between the following lines
   (i) $\frac{x-3}{1}=\frac{y-5}{-2}=\frac{z-7}{1};\ \frac{x+1}{7}=\frac{y+1}{-6}=\frac{z+1}{-1}.$
   (ii) $\frac{x-3}{1}=\frac{y-5}{-2}=\frac{z-2}{1};\ \frac{x-1}{7}=\frac{y+1}{-6}=\frac{z+1}{1}.$
   (iii) $\frac{x-3}{1}=\frac{y-5}{-2}=\frac{z-7}{1};\ \frac{x+1}{7}=\frac{y+1}{-6}=\frac{z+1}{1}.$

ANSWERS

1. $3\sqrt{30};\ \frac{x-3}{2}=\frac{y-8}{5}=\frac{z-3}{1}$; (3, 8, 3) and (– 3, – 7, 6)
2. (i) $2\sqrt{29},\ \frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$.
   (ii) $\frac{30}{\sqrt{29}},\ -11x-2y+7z+9=0=27x+26y-33z+22$
   (iii) $2\sqrt{29},\ \frac{x-3}{2}=\frac{y-5}{3}=\frac{z-7}{4}$

OBJECTIVE EVALUATION

➤ FILL IN THE BLANKS :

1. The equation of a straight line through the point (α, β, γ) having the d.c.'s l, m, n is ....
2. The equation of x-axis in symmetrical form is ....
3. The equation of the straight line passing through the two given points (x_1, y_1, z_1) and (x_2, y_2, z_3) is ....
4. The direction ratios of the line $\frac{x-0}{2}=\frac{y-1}{3}=\frac{z-2}{3}$ are ....

➤ TRUE OR FALSE :

Write 'T' for true 'F' for false :

1. $\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$ is the equation of a straight line in symmetrical form. *(T/F)*
2. The equation of y-axis is $\frac{x}{0}=\frac{y}{0}=\frac{z}{1}$. *(T/F)*
3. The line of intersection of the planes $a_1x+b_1y+c_1z+d_1=0$ and $a_2x+b_2y+c_2z+d_2=0$ gives the line in non-symmetrical form. *(T/F)*
4. The equation $u+\lambda v=0$ represents the line of intersection through the planes $u\equiv 0$ and $v\equiv 0$. *(T/F)*

➤ MULTIPLE CHOICE QUESTIONS :

Choose the most appropriate one :

1. The line $\frac{x-1}{2}=\frac{y-3}{3}=\frac{z+1}{2}$ passing through the point :

(a) (1, 3, 1) (b) (− 1, 3, − 1)
(c) (1, 3, − 1) (d) (− 1, − 3, − 1).

2. The equation $\frac{x}{1}=\frac{y}{0}=\frac{z}{0}$ represents the equation of :

(a) x-axis (b) y-axis
(c) z-axis (d) None of these.

3. The shortest distance between the lines $\frac{x+1}{1}=\frac{y+2}{2}=\frac{z+3}{3}$ and $\frac{y+2}{4}=\frac{x+1}{5}=\frac{z+3}{6}$ is :

(a) 1 (b) 6 (c) 0 (d) 2.

4. The shortest distance between two coplanar lines is :

(a) 0 (b) 1 (c) 2 (d) None of these

ANSWERS

Fill in the Blanks :

1. $\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$ **2.** $\frac{x}{1}=\frac{y}{0}=\frac{z}{0}$ **3.** $\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$

4. 2, 3, 3

True or False :

1. T **2.** F **3.** T **4.** T

Multiple Choice Questions :

1. (c) **2.** (a) **3.** (c) **4.** (a)

❑❑❑

UNIT 23

THE SPHERE

STRUCTURE

LEARNING OBJECTIVES

After going through this unit you will learn :

How to calculate the radius of the circle defined as the section of the sphere of the plane

How to find the tangent plane

How to find the condition that two spheres cut orthogonally

• 23.1. EQUATION OF A SPHERE

(*a*) *To obtain the equation of a sphere whose centre and radius are given.*

Let C be the centre of a sphere whose co-ordinates are assumed (α, β, γ) and let r be the radius of this sphere. Let $P(x, y, z)$ be any variable on the surface of the sphere whose equation to be determined as shown in fig. 1.

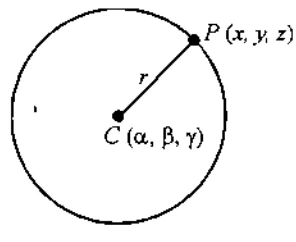

Fig. 1

Since we have

$$CP = r \quad \text{or} \quad CP^2 = r^2. \qquad \text{...(1)}$$

Further since CP is the distance between two points $C(\alpha, \beta, \gamma)$ and $P(x, y, z)$. Then

$$CP = \sqrt{(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2}.$$

Now substitute this value of CP in (1), we get

$$(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2 = r^2. \qquad \text{...(2)}$$

This is the required equation of a sphere.

Corollary. *If the centre of the sphere is origin i.e.,* $(0, 0, 0)$ *and radius is r then its equation is*

$$x^2 + y^2 + z^2 = r^2.$$

Proof. In the formula

$$(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2 = 2^2.$$

Putting $\alpha = 0, \beta = 0, \gamma = 0$, we get

$$x^2 + y^2 + z^2 = r^2. \quad \textit{Hence obtained.}$$

(*b*) *To obtain the equation of a sphere whose end points of a diameter are given.*

Let $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ be the end points of a diameter of a sphere as shown in fig. 2.

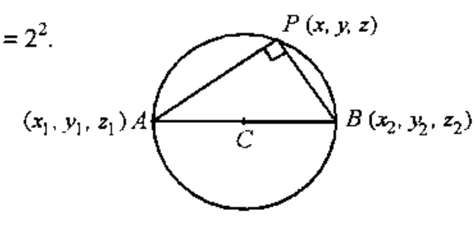

Fig. 2

Let $P(x, y, z)$ be any point on the surface of the sphere. Join P to A and P to B such that ΔAPB is a right angled triangle right angled at P. Therefore the lines PA and PB are perpendicular to each other. Then

d.r.'s of PA are $x - x_1, y - y_1, z - z_1$

and d.r.'s of PB are $x - x_2, y - y_2, z - z_2$

since PA is perpendicular to PB, then we have

$$\boxed{(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + (z - z_1)(z - z_2) = 0.}$$

This is the required equation of the sphere.

• 23.2. GENERAL FORM OF A SPHERE

To obtain the equation of a sphere in general form.

The equation of a sphere of radius r having centre (α, β, γ) is given by

$$(x - \alpha)^2 + (y - \beta)^2 + (z - \gamma)^2 = r^2$$

or $$x^2 + y^2 + z^2 - 2\alpha x - 2\beta y - 2\gamma z + \alpha^2 + \beta^2 + \gamma^2 - r^2 = 0 \quad ...(1)$$

From (1) we observed that the equation (1) is a second degree in x, y, z having no terms of xy, yz and zx and coefficients of x^2, y^2 and z^2 equal to 1. Therefore the equation of the type

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \quad ...(2)$$

having the same character as (1) has, is called general equation of a sphere.

Now comparing (2) with (1), we get

$$\alpha = -u, \beta = -v, \gamma = -w \text{ and } d = \alpha^2 + \beta^2 + \gamma^2 - r^2.$$

Thus the centre and radius of (2) are given by $(-u, -v, -w)$ and $\sqrt{u^2 + v^2 + w^2 - d}$ respectively.

SOLVED EXAMPLES

Example 1. *Find the equation of the sphere whose centre is* $(2, -3, 4)$ *and radius* $\sqrt{51}$.

Solution. Since we know the equation of a sphere whose centre is (α, β, γ) and radius r as follows :

$$(x - \alpha)^2 + (y - \beta)^2 + (z - \gamma)^2 = r^2.$$

Here $\alpha = 2, \beta = -3, \gamma = 4$ and $r = \sqrt{51}$, then

$$(x - 2)^2 + (y + 3)^2 + (z - 4)^2 = 51$$

or $$x^2 + y^2 + z^2 - 4x + 6y - 8z - 22 = 0.$$

Example 2. *Find the centre and radius of the sphere given by*

$$x^2 + y^2 + z^2 - 4x + 6y + 2z + 5 = 0.$$

Solution. The given sphere is

$$x^2 + y^2 + z^2 - 4x + 6y + 2z + 5 = 0.$$

Compare this equation with the equation

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0,$$

we get $2u = -4$ or $u = -2$

$2v = 6$ or $v = 3$

$2w = 2$ or $w = 1$ and $d = 5$.

$\therefore$ Centre is $(-u, -v, -w) = (2, -3, -1)$

and $$\text{radius} = \sqrt{u^2 + v^2 + w^2 - d}$$
$$= \sqrt{(-2)^2 + 3^2 + 1^2 - 5}$$
$$= \sqrt{4 + 9 + 1 - 5} = \sqrt{9} = 3.$$

Example 3. *Find the equation of the sphere on the point* $(2, -3, 1)$ *and* $(3, -1, 2)$ *as diameter.*

Solution. Since we have the equation of a sphere whose end points of a diameter are (x_1, y_1, z_1) and (x_2, y_2, z_2). The equation is

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + (z - z_1)(z - z_2) = 0$$

Here $(x_1, y_1, z_1) \equiv (2, -3, 1)$ and $(x_2, y_2, z_2) \equiv (3, -1, 2)$.

Then we get

$$(x - 2)(x - 3) + (y + 3)(y + 1) + (z - 1)(z - 2) = 0$$

or $$x^2 + y^2 + z^2 - 5x + 4y - 3z + 11 = 0.$$

Example 4. *A plane passes through a fixed point* (p, q, r) *and cuts off the axes in* A, B, C. *Show that the locus of the centre of the sphere OABC is*

$$\frac{p}{x} + \frac{q}{y} + \frac{r}{z} = 2.$$

Solution. Let the equation of a plane be

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1 \quad ...(1)$$

which cuts the axes in $A(a, 0, 0)$, $B(0, b, 0)$ and $C(0, 0, c)$ and also passes through the fixed point (p, q, r), then we have

$$\frac{p}{a}+\frac{q}{b}+\frac{r}{c}=1. \quad ...(2)$$

The equation of a sphere $OABC$ is

$$x^2+y^2+z^2-ax-by-cz=0$$

its centre is $\left(\frac{a}{2}, \frac{b}{2}, \frac{c}{2}\right)$.

Let (α, β, γ) be the centre of a sphere $OABC$, then

$$\alpha=\frac{a}{2},\ \beta=\frac{b}{2},\ \gamma=\frac{c}{2}$$

or $$a=2\alpha, b=2\beta, c=2\gamma. \quad ...(3)$$

Eliminating a, b, c from (2) and (3), we get

$$\frac{p}{2\alpha}+\frac{q}{2\beta}+\frac{r}{2\gamma}=1$$

or $$\frac{p}{\alpha}+\frac{q}{\beta}+\frac{r}{\gamma}=2.$$

Thus the locus of (α, β, γ) is

$$\frac{p}{x}+\frac{q}{y}+\frac{r}{z}=2.$$ **Hence proved.**

• TEST YOURSELF–1

1. Find the centre and radius of the following sphere :
$$2x^2+2y^2+2z^2-2x+4y-6z-15=0.$$
2. Find the equation of the sphere whose centre is $(-3, 4, 5)$ and radius 7.
3. Find the equation of a sphere whose centre is $(2, -3, 4)$ and which passes through the point $(1, 2, -1)$.
4. Find the equation of the sphere whose centre is $(1, 3, 5)$ and which passes through the point $(3, 5, 6)$.
5. Find the equation of the sphere which passes through the points $(1, -3, 4)$, $(1, -5, 2)$, $(1, -3, 0)$ and whose centre lies on the plane $x+y+z=0$.
6. Find the equation to the sphere through the points $(0, 0, 0)$, $(0, 1, -1)$, $(-1, 2, 0)$ and $(1, 2, 3)$.
7. (a) Find the equation of the sphere on the join $(3, -1, 5)$ and $(4, 5, 1)$ as diameter.
   (b) Find the equation of a sphere whose extremities of a diameter are $(1, 1, 1)$ and $(2, 3, 5)$.

ANSWERS

1. $\left(\frac{1}{2}, -1, \frac{3}{2}\right)$; $\sqrt{11}$. 2. $x^2+y^2+z^2+6x-8y-10z+1=0$.
3. $x^2+y^2+z^2-4x+6y-8z-22=0$. 4. $x^2+y^2+z^2-2x-6y-10z+26=0$.
5. $x^2+y^2+z^2-2x+6y-4z+10=0$.
6. $8(x^2+y^2+z^2)-15x-25y-11z=0$
7. (a) $x^2+y^2+z^2-7x-4y-6z+12=0$. (b) $x^2+y^2+z^2-3x-4y-6z+10=0$.

• 23.3. THE PLANE SECTION OF A SPHERE

When a plane cuts the sphere, a cross-section of a sphere is obtained as circle.

To prove that the cross section of a sphere cuts by a plane is a circle and find also its centre and radius.

Let the equation of the sphere and the plane be

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0 \quad ...(1)$$

and $$ax+by+cz+d_1=0. \quad ...(2)$$

In the fig. 4 the dotted line represents the cross-section of a sphere cuts off by a plane (2). Let C be the centre of the sphere (1). Draw a perpendicular from C to the plane (2) which meets the plane at C', join C' to P, where P is on the surface of the sphere and on the dotted line. Then $C'P$ lies in the plane and CC' is perpendicular to the plane. Thus CC' is perpendicular to every line which is in the plane. Therefore $\angle CC'P = 90°$.

$\therefore$ In $\Delta CC'P$, $\angle CC'P = 90°$, then

$$C'P^2 = CP^2 - CC'^2 \text{ or } C'P = \sqrt{CP^2 - CC'^2}.$$

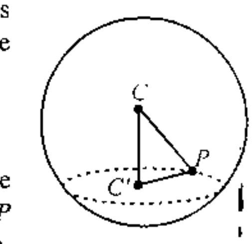

Fig. 4

Since CP is the radius of the sphere which is constant and CC' is the length of the perpendicular drawn from C to the given plane. Thus $C'P$ comes out to be a constant for all positions of P on the cross-section. Therefore, the point P moves in plane (2) in such a way that the distance of P from fixed point C' is always contant. Hence P traces out a circle whose centre is C' and the radius is $C'P$. Consequently the equation (1) and (2) simultaneously give the equation of a circle.

Great Circle. *The cross section of a sphere by a plane through the centre of the sphere is called the* ***a great circle****. The centre and radius of a great circle are same as centre and radius of the sphere.*

• 23.4. INTERSECTION OF TWO SPHERES

To prove that the intersection of two spheres comes out be a circle.

Let the equation of two spheres be given

$$S_1 \equiv x^2 + y^2 + z^2 + 2u_1x + 2v_1y + 2w_1z + d_1 = 0 \quad ...(1)$$

and
$$S_2 \equiv x^2 + y^2 + z^2 + 2u_2x + 2v_2y + 2w_2z + d_2 = 0. \quad ...(2)$$

Subtracting (1) and (2), we get

$$S_1 - S_2 = 0.$$

$$\therefore \quad \boxed{\mathbf{2\,(u_1 - u_2)\,x + 2\,(v_1 - v_2)\,y + 2\,(w_1 - w_2)\,z + d_1 - d_2 = 0.}} \quad ...(3)$$

This equation (3) represents a ***plane***. Thus the points of intersection of two spheres are as same as obtained by the intersection of either (1) or (2) and the plane (3). Hence these points of intersection lie on a circle.

• 23.5. SPHERES THROUGH A GIVEN CIRCLE

From above it has been observed that the intersection of a sphere and a plane give a circle or the intersection of two spheres give a circle.

Let the given circle be represented by the equations

$$S \equiv x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$$

and
$$P \equiv ax + by + cz + d_1 = 0$$

then the equation $S + \lambda P = 0$ is satisfied by those points which are common to both $S = 0$ and $P = 0$ for all values of λ. In fact $S + \lambda P = 0$ represents the equation of a sphere through the given circle $S = 0$ and $P = 0$ together.

Likewise the equation

$$\boxed{\mathbf{S_1 + S_2\lambda = 0}}$$

represents a sphere through the circle $S_1 = 0$ and $S_2 = 0$ together.

SOLVED EXAMPLES

Example 1. *Find the co-ordinates of the centre and the radius of the circle*

$$x + 2y + 2z = 15, x^2 + y^2 + z^2 - 2y - 4z - 11 = 0.$$

Solution. The centre of the sphere

$$x^2 + y^2 + z^2 - 2y - 4z - 11 = 0$$

is (0, 1, 2) and its radius is

$$r = \sqrt{0 + 1 + 4 + 11} = 4.$$

Let p be the perpendicular distance from (0, 1, 2) to the plane $x + 2y + 2z = 15$.

$$\therefore \quad p = \frac{0 + 2(1) + 2(2) - 15}{\sqrt{1 + 4 + 4}} = -\frac{9}{3} = -3.$$

Thus the radius of the circle is

$= \sqrt{(4)^2 - (-3)^2} = \sqrt{16-9} = \sqrt{7}.$

Centre of the circle. The equation of a straight line through the point (0, 1, 2) and perpendicular to the plane $x + 2y + 2z = 15$ is given by

$$\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{2} = r \text{ (say)}.$$

Any point on this line is $(r, 2r+1, 2r+2)$, let this point be the centre of the circle so $(r, 2r+1, 2r+2)$ lies on the plane $x + 2y + 2z = 15$. Then

$$r + (2r+1)2 + 2(2r+2) = 15$$

$$9r = 9.$$

$$\therefore \quad r = 1.$$

Hence the centre is (1, 3, 4).

Example 2. *Find the equation of the sphere through the circle*

$$x^2 + y^2 + z^2 = 9, 2x + 3y + 4z = 5$$

and the point (1, 2, 3).

Solution. Since the equation of the circle is given by

$$x^2 + y^2 + z^2 = 9, \; 2x + 3y + 4z = 5$$

i.e., $\quad S \equiv x^2 + y^2 + z^2 - 9 = 0$

and $\quad P \equiv 2x + 3y + 4z - 5 = 0.$

The equation of the sphere through the circle is

$$S + \lambda P = 0$$

or $\quad x^2 + y^2 + z^2 - 9 + \lambda(2x + 3y + 4z - 5) = 0. \quad \text{...(1)}$

Since (1) is passing through the point (1, 2, 3) also. Then

$$1 + 2^2 + 3^2 - 9 + \lambda(2 + 6 + 12 - 5) = 0 \quad \text{or} \quad \lambda = -\frac{1}{3}$$

Substitute the value of λ in (1), we get

$$3(x^2 + y^2 + z^2) - 2x - 3y - 4z - 22 = 0.$$

This is the required equation of the sphere.

• TEST YOURSELF–2

1. Find the radius and centre of the circle of intersection of the sphere $x^2 + y^2 + z^2 - 2y - 4z = 11$ and the plane $x + 2y + 2z = 15$.
2. Find the radius of the circle given by the equations $3x^2 + 3y^2 + 3z^2 + x - 5y - 2 = 0$, $x + y = 2$.
3. A variable plane is parallel to the given plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 0$ and meets the axes in A, B, C respectively. Prove that the circle ABC lies on the surface
$$yz\left(\frac{b}{c} + \frac{c}{b}\right) + zx\left(\frac{c}{a} + \frac{a}{c}\right) + xy\left(\frac{a}{b} + \frac{b}{a}\right) = 0.$$
4. Find the equation of the sphere passing through the circles $y^2 + z^2 = 9, x = 4$ and $y^2 + z^2 = 36$, $x = 1$.

ANSWERS

1. $\sqrt{7}$; (1, 3, 4). **2.** $1/\sqrt{2}$. **4.** $x^2 + y^2 + z^2 + 4x - 41 = 0$.

• 23.6. EQUATION OF THE TANGENT PLANE

The equation of the tangent plane to a sphere $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ *at the point* (x_1, y_1, z_1) *on the sphere is*

$$\boxed{xx_1 + yy_1 + zz_1 + u(x + x_1) + v(y + y_1) + w(z + z_1) + d = 0.}$$

Corollary. *The equation of a tangent plane at* (x_1, y_1, z_1) *to the sphere* $x^2 + y^2 + z^2 = r^2$ *is given by*

$$xx_1 + yy_1 + zz_1 = r^2.$$

• 23.7. CONDITION FOR TANGENCY OF A PLANE TO THE SPHERE

Let the equation of a sphere be given

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0 \quad ...(1)$$

and let the plane be $$ax+by+cz+d_1=0. \quad ...(2)$$

If the plane (2) is the tangent plane of (1), then the perpendicular distance from the centre $(-u,-v,-w)$ of the sphere (1) to the plane (2) will be equal to the radius of the sphere

Since the radius of the sphere (1) is $\sqrt{u^2+v^2+w^2-d}$.

$\therefore$ $\sqrt{u^2+v^2+w^2-d}$ = perpendicular distance from $(-u,-v,-w)$ to the plane $ax+by+cz+d_1=0$

or $$\sqrt{u^2+v^2+w^2-d}=\frac{-au-bv-cw+d_1}{\sqrt{a^2+b^2+c^2}}.$$

Squaring of both the sides, we get the required condition

$$\boxed{(u^2+v^2+w^2-d)(a^2+b^2+c^2)=(au+bv+cw-d_1)^2.}$$

SOLVED EXAMPLES

Example 1. *Find the equation of a tangent plane to the sphere*

$$x^2+y^2+z^2-2x-4y-6z+9=0 \text{ at } (1,1,1).$$

Solution. Equation of tangent at (x_1, y_1, z_1) to the sphere

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0 \text{ is}$$

$$xx_1+yy_1+zz_1+u(x+x_1)+v(y+y_1)+w(z+z_1)+d=0.$$

Here $(x_1, y_1, z_1)=(1,1,1)$ and the sphere is

$$x^2+y^2+z^2-2x-4y-6z+9=0.$$

So, the required tangent plane is

$$xx_1+yy_1+zz_1-(x+x_1)-2(y+y_1)-3(z+z_1)+9=0$$

i.e., $$x+y+z-(x+1)-2(y+1)-3(z+1)+9=0$$

or $$-y-2z+3=0$$

or $$y+2z-3=0.$$

Example 2. *Show that the plane* $2x-2y+z+12=0$ *touches the sphere*

$$x^2+y^2+z^2-2x-4y+2z-3=0.$$

Solution. The given sphere is

$$x^2+y^2+z^2-2x-4y+2z-3=0.$$

Centre is $(1,2,-1)$ *and its radius is* $\sqrt{1+4+1+3}=3$.

Now, the perpendicular distance from $(1,2,-1)$ to the plane $2x-2y+z+12=0$ is

$$=\frac{2(1)-2(2)-1+12}{\sqrt{4+4+1}}$$

$$=3$$

which is equal to the radius of the sphere. Hence the plane $2x-2y+z+12=0$ touches the given sphere.

Example 3. *Find the equation of the two tangent planes to the sphere* $x^2+y^2+z^2-2y-6z+5=0$ *which are parallel to the plane* $2x+2y-z=0$.

Solution. The equation of any plane parallel to the given plane $2x+2y-z=0$ is

$$2x+2y-z=\lambda. \quad ...(1)$$

Since (1) touches the given spheres. Then the perpendicular distance from the centre (0, 1, 3) to the plane (1) is equal to the radius $\sqrt{5}$ of the sphere.

$$\therefore \quad \frac{2\times 0+2\times 1-3-\lambda}{\sqrt{4+4+1}}=\pm\sqrt{5}$$

or $$-3-\lambda=\pm 3\sqrt{5} \quad \text{or} \quad \lambda=-3-(\pm 3\sqrt{5}).$$

Substitute the value of λ in (1), we get

$$2x+2y-z=-3-(\pm 3\sqrt{5})$$

or $$2x+2y-z+3\pm 3\sqrt{5}=0.$$

This is the required equations of tangent planes.

• TEST YOURSELF–3

1. Find the equation of tangent planes of the sphere $x^2+y^2+z^2+2x-4y+6z-7=0$ which intersect in the line

$$6x - 3y - 23 = 0 = 3z + 2.$$

2. Find the equation of the sphere described on the line joining the points (3, 4, 1) and (− 1, 0, 5) as diameter and find also the equation of the tangent plane at (− 1, 0, 5).
3. Prove that the equation of the sphere which touches the sphere $4(x^2 + y^2 + z^2) + 10x - 25y - 2z = 0$ at (1, 2, − 2) and passes through the point (− 1, 0, 0) is
$$x^2 + y^2 + z^2 + 2x - 6y + 1 = 0.$$
4. Find the equation of the sphere touching three co-ordinate axes. How many such spheres can be drawn.
5. Find the equations of the spheres which pass through the circle $x^2 + y^2 + z^2 = 5$, $x + 2y + 3z = 5$ and touch the plane $4x + 3y = 15$.
6. Show that the sum of the squares of the intercepts made by a given sphere on any three mutually perpendicular straight lines through the fixed point is constant.
7. A sphere touches the three co-ordinate planes and passes through the point (2, 1, 5). Find its equation.

ANSWERS

2. $2x - y + 4z - 5 = 0$ and $4x - 2y - z - 16 = 0$.
3. $x^2 + y^2 + z^2 - 2x - 4y - 6z + 2 = 0$; $x + y - z + 6 = 0$.
5. $x^2 + y^2 + z^2 \pm 2rx \pm 2ry \pm 2rz + 2r^2 = 0$; Eight spheres.
5. $x^2 + y^2 + z^2 + 2x + 4y + 6zx - 11 = 0$; $5(x^2 + y^2 + z^2) - 4x - 8y - 12z - 3 = 0$.
7. $x^2 + y^2 + z^2 - 10x - 10y - 10z + 50 = 0$.

• 23.8. INTERSECTION OF TWO SPHERES

(a) Angle of intersection of two spheres.

When two spheres intersect each other, then the angle of intersection of two spheres at the common point of intersection is equal to the angle between their tangent planes at the common point.

(b) Condition for orthogonal intersection of two spheres.

When two spheres intersect each other, the intersection is said to be orthogonal if the angle of intersection is a right angle.

Let C_1 and C_2 be the centres of two spheres and P be the common point of intersection as shown in fig. 5.

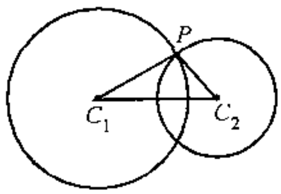

Fig. 5

Since $\angle C_1PC_2 = 90°$, then in ΔC_1PC_2
$$C_1C_2^2 = C_1P^2 + C_2P^2.$$
Let r_1 and r_2 be the radius of two spheres respectively then
$$r_1^2 + r_2^2 = C_1C_2^2. \quad ...(1)$$
Let the equation of two spheres be
$$x^2 + y^2 + z^2 + 2u_1x + 2v_1y + 2w_1z + d_1 = 0 \quad ...(2)$$
and
$$x^2 + y^2 + z^2 + 2u_2x + 2v_2y + 2w_2z + d_2 = 0. \quad ...(3)$$
Then centres C_1 and C_2 are respectively $(-u_1, -v_1, -w_1)$ and $(-u_2, -v_2, -w_2)$ and
$$r_1^2 = u_1^2 + v_1^2 + w_1^2 - d_1 \text{ and } r_2^2 = u_2^2 + v_2^2 + w_2^2 - d_2.$$
The distance between the centres C_1 and C_2 is
$$C_1C_2^2 = (u_2 - u_1)^2 + (v_2 - v_1)^2 + (w_2 - w_1)^2.$$
∴ (1) becomes
$$u_1^2 + v_1^2 + w_1^2 - d_1 + u_2^2 + v_2^2 + w_2^2 - d_2 = (u_2 - u_1)^2 + (v_2 - v_1)^2 + (w_2 - w_1)^2$$
or
$$\mathbf{2u_1u_2 + 2v_1v_2 + 2w_1w_2 = d_1 + d_2.}$$
This is the required condition for orthogonality of two spheres.

REMARKS

- ➤ If $|r_1 + r_2| = C_1C_2$, then the spheres touch externally.
- ➤ If $|r_1 - r_2| = C_1C_2$, then the spheres tocuh internally.
- ➤ If θ be the angle between the tangents at the point of intersection of the spheres, then
$$\cos\theta = \frac{r_1^2 + r_2^2 - c_1c_2^2}{2r_1r_2}.$$

SOLVED EXAMPLES

Example 1. *Find the equation of the sphere that passes through the circle* $x^2+y^2+z^2-2x+3y-4z+6=0,\quad 3x-4y+5z-15=0$ *and cuts the sphere* $x^2+y^2+z^2+2x+4y-6z+11=0$ *orthogonally.*

Solution. The equation of a sphere through the circle

$$x^2+y^2+z^2-2x+3y-4z+6=0,\ 3x-4y+5z-15=0 \text{ is}$$

$$(x^2+y^2+z^2-2x+3y-4z+6)+\lambda(3x-4y+5z-15)=0$$

or $$x^2+y^2+z^2+(-2+3\lambda)\,x+(3-4\lambda)\,y+(-4+5\lambda)\,z+6-15\lambda=0 \quad \ldots(1)$$

Since (1) cuts the given sphere

$$x^2+y^2+z^2+2x+4y-6z+11=0 \text{ orthogonally.}$$

Then, $2u_1u_2+2v_1v_2+2w_1w_2=d_1+d_2.$

$\therefore$ $$2\left[\frac{-2+3\lambda}{2}\right]\left[\frac{2}{2}\right]+2\left[\frac{3-4\lambda}{2}\right]\times 2+2\left[\frac{-4+5\lambda}{2}\right]\times(-3)=6-15\lambda+11$$

or $$2(-2+3\lambda)+2(3-4\lambda)-3(-4+5\lambda)=17-15\lambda$$

or $$-5\lambda=1 \quad \text{or} \quad \lambda=-1/5.$$

Putting the value of λ in (1), we get

$$x^2+y^2+z^2-\frac{13}{5}x+\frac{19}{5}y-5z+9=0$$

or $$5(x^2+y^2+z^2)-13x+19y-25z+45=0.$$

This is the required equation of a sphere.

Example 2. *Two spheres of radii* r_1 *and* r_2 *cut orthogonally. Prove that the radius of the common circle is* $\frac{r_1r_2}{\sqrt{r_1^2+r_2^2}}$.

Solution. Let the equation of the common circle be

$$x^2+y^2=a^2,\ z=0. \quad \ldots(1)$$

Therefore, the equation of the given spheres through the circle (1) are

$$x^2+y^2+z^2+2\lambda z-a^2=0 \quad \ldots(2)$$

and $$x^2+y^2+z^2+2\mu z-a^2=0. \quad \ldots(3)$$

$\therefore$ $$r_1^2=\lambda^2+a^2,\ r_2^2=\mu^2+a^2.$$

Since (2) and (3) cut orthogonally, then

$$2u_1u_2+2v_1v_2+2w_1w_2=d_1+d_2$$

$\therefore$ $$2\lambda\times\mu=-a^2-a^2 \quad \text{or} \quad \lambda^2\mu^2=a^4 \text{ (squaring)}$$

or $$(r_1^2-a^2)(r_2^2-a^2)=a^4 \qquad (\because\ r_1^2=\lambda^2+a^2,\ r_2^2=\mu^2+a^2)$$

or $$r_1^2r_2^2=a^2(r_1^2+r_2^2) \quad \text{or} \quad a=\frac{r_1r_2}{\sqrt{r_1^2+r_2^2}}.$$

$\therefore$ radius of the circle is $\frac{r_1r_2}{\sqrt{r_1^2+r_2^2}}$.

• SUMMARY

- Equation of a sphere of centre (α,β,γ) and radius r is $(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2=r^2$.
- Equation of a sphere having (x_1, y_1, z) and (x_2, y_2, z_2) as end points of its diameter is
$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)+(z-z_1)(z-z_2)=0.$$
- General quation of a sphere is $x^2+y^2+z^2+2ux+2vy+2wz+d=0$ with its centre $(-u,-v,-w)$ and radius $=\sqrt{u^2+v^2+w^2-d}$.
- Equation of sphere through
$$S\equiv x^2+y^2+z^2+2ux+2vy+2wz+d=0$$
and $$P\equiv ax+by+cz+d=0$$
is givn by $S+\lambda P=0$.
- Tangent plane at (x_1, y_1, z_1) to th sphere $x^2+y^2+z^2+2ux+2vy+2wz+d=0$ is given by $xx_2+yy_1+zz_1+u(x+x_1)+v(y+y_1)+w(z+z_1)+d=0$.

- Two spheres $x^2+y^2+z^2+2u_1x+2v_1y+2w_1z+d_1=0$ and $x^2+y^2+z^2+2u_2x+2v_2y+2w_2z+d_2=0$ cut orthogonally if $2u_1u_2+2v_1v_2+2w_1w_2-d_1+d_2$.

• STUDENT ACTIVITY

1. A plane passes through a fixed point (p, q, r) and cuts off th axes in A, B, C. Show that the locus of the centre of the sphere $OABC$ is $\frac{p}{x}+\frac{q}{y}+\frac{r}{z}=2$.

2. Find the equation of the tangent plane to the sphre $x^2+y^2+z^2-2x-4y-6z+9=0$ at $(1, 1, 1)$.

TEST YOURSELF–4

1. Find the equation of a sphere which touches the plane $3x+2y-z+2=0$ at the point $(1, -2, 1)$ and also cuts orthogonally the sphere $x^2+y^2+z^2-4x+6y+4=0$.
2. Prove that the sphere which cuts two spheres $S_1=0$ and $S_2=0$ at right angles will cut the sphere $\lambda_1S_1+\lambda_2S_2=0$ at right angles.
3. Show that the spheres

$$x^2+y^2+z^2-4x-2y+2z-3=0$$

and

$$x^2+y^2+z^2-8x-8y-10z+41=0$$

touch externally.

4. Show that the two spheres

$$x^2+y^2+z^2+6y+2z+8=0$$

and

$$x^2+y^2+z^2+6x+8y+4z+20=0$$

are orthogonal.

ANSWERS

1. $x^2+y^2+z^2+7x+10y-5z+12=0$.

OBJECTIVE EVALUATION

➤ FILL IN THE BLANKS :

1. In the equation of a sphere the coefficients of x^2, y^2, z^2 must be ....
2. The equation $ax^2 + by^2 + cz^2 + 2ux + 2vy + 2wz + d = 0$ represents a sphere if ....
3. The equation of a sphere whose centre is (0, 0, 0) and radius is a is ....

➤ TRUE OR FALSE :

Write 'T' for true and 'F' for false statement :

1. The centre of the sphere $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ is $(-u, -v, -w)$. (T/F)
2. The centre of the sphere $(x-2)(x-4) + (y-1)(y-3) + (z-3)(z-5) = 0$ is (3, 2, 4). (T/F)
3. The centre of the great circle is the centre of the sphere. (T/F)

➤ MULTIPLE CHOICE QUESTIONS :

Choose the most appropriate one :

1. The centre of the sphere $x^2 + y^2 + z^2 - ax - by - cz = 0$ is :
   (a) (0, 0, 0) (b) $\left(\frac{a}{2}, \frac{b}{2}, \frac{c}{2}\right)$
   (c) $\left(-\frac{a}{2}, -\frac{b}{2}, -\frac{c}{2}\right)$ (d) (a, b, c).
2. The radius of the sphere $2x^2 + 2y^2 + 2z^2 = 50$ is :
   (a) 5 (b) 25
   (c) 50 (d) 2.
3. The number of spheres that are touching the co-ordinate axis are :
   (a) Infinite (b) 0
   (c) 1 (d) None of these.

ANSWERS

Fill in the Blanks :

1. unity 2. $a = b = c$ 3. $x^2 + y^2 + z^2 = a^2$

True or False :

1. T 2. T 3. T

Multiple Choice Questions :

1. (b) 2. (a) 3. (a)

UNIT 24

THE CONE

STRUCTURE

LEARNING OBJECTIVES

After going through this unit you will learn :

How to find equation of a cone with given vertex and given base curve.

How to find the cone reciprocal of the given cone.

• 24.1. THE CONE

Definition. *A surface generated by a moving straight line which passes through a fixed point and touches a given surface or intersect a given curve, is called* ***the cone.***

Vertex. *A fixed point through which a moving line passes, is called the* ***vertex*** *of the cone.*

Guiding curve. *The moving line which interest the given curve, this curve is called the* ***guiding curve.***

Generator. *The moving line in any position is called the* ***generator.***

• 24.2. EQUATION OF A CONE WITH ITS VERTEX AT THE ORIGIN

To obtain the equation of a cone having its vertex at the origin is a homogeneous equation of degree two in x, y, z.

Let us assume that the general equation of degree two in x, y and z represents a cone with its vertex at origin O. Let its equation be

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2ux+2vy+2wz+d=0. \quad ...(1)$$

Let P be any point on the cone whose co-ordinates is (x_1, y_1, z_1). Then the equation of its generator OP is given by

$$\frac{x}{x_1}=\frac{y}{y_1}=\frac{z}{z_1}. \quad ...(2)$$

Any point Q on this line is (rx_1, ry_1, rz_1) where

$$\frac{x}{x_1}=\frac{y}{y_1}=\frac{z}{z_1}=r \text{ (say)}.$$

Since OP is the generator of (1) and Q is any point on this line OP, then Q will satisfy the equation (1) for all values of r.

$$\therefore \quad a(rx_1)^2+b(ry_1)^2+c(rz_1)^2+2fr^2y_1z_1+2gr^2z_1x_1+2hr^2x_1y_1$$
$$+2urx_1+2vry_1+2wrz_1+d=0$$

or $\quad r^2(ax_1^2+by_1^2+cz_1^2+2fy_1z_1+2gz_1x_1+2hx_1y_1)+r(2ux_1+2vy_1+2wz_1)+d=0.$ ...(3)

For all values of r (3) must be an identity. Therefore we must have

$$ax_1^2+by_1^2+cz_1^2+2fy_1z_1+2gz_1x_1+2hx_1y_1=0 \quad ...(4)$$

$$2ux_1+2vy_1+2wz_1=0 \quad ...(5)$$

and $\quad d=0.$ ...(6)

From equation (5) it has been observed that the point $P(x_1, y_1, z_1)$ satisfies an equation of degree one which gives that the surface is a plane if u, v and w are not all zero. But the surface is assumed to be a cone, hence u, v, w all should be zero. *i.e.*, $u=v=w=0$. Now from equation (6) we obtained that $d=0$. Therefore substituting $u=v=w=0$ and $d=0$ in (1) we get (1) reduces to a homogeneous equation of degree 2 in x, y, z.

i.e., $\quad ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0.$

Hence the equation of a cone with its vertex is a homogeneous equation of degree two in x, y, z.

Conversely, *Every equation which is homogeneous of degree two in x, y, z represents a cone with its vertex at the origin.*

Let the homogeneous equation of degree two in x, y, z be given by

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0. \quad ...(1)$$

Let $P(x_1, y_1, z_1)$ be any point. Which obviously satisfies (1). Therefore for all values of r, the point (rx_1, ry_1, rz_1) also satisfies the equation (1). Thus any point (rx_1, ry_1, rz_1) on the line OP, where O is the origin satisfies the equation (1). Hence the line OP is the generator of the surface (1) through the origin. Consequently the surface (1) represents a cone with vertex at the origin O.

• 24.3. AN IMPORTANT RESULT

The direction-cosines of the generator of the cone satisfy the equation of the cone with its vertex at the origin.

Let the equation of a line representing the generator of a cone with its vertex at the origin be

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n} \quad ...(1)$$

and let the equation of a cone be

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0. \quad ...(2)$$

Any point on the line (1) is (lr, mr, nr), where r is the distance of this point from the origin O, because l, m, n are the actual direction-cosines. Since any point on the generator will satisfies the equation of a cone (2), then

$$r^2[al^2+bm^2+cn^2+2fmn+2gnl+2hlm]=0$$

or $\quad \mathbf{al^2+bm^2+cn^2+2fmn+2gnl+2hlm=0} \quad (\because r\neq 0)$

Conversely. If $al^2+bm^2+cn^2+2fmn+2gnl+2hlm=0$, then the line with direction-cosines l, m, n is a generator of the cone (2) with its vertex at origin.

• 24.4. EQUATION OF A CONE PASSING THROUGH THE AXES

Since we know that the equation of a cone with its vertex at the origin is satisfied by the direction-cosines of its generator. Here the x-axis, y-axis, and z-axis are the generators of a required cone.

Let the equation of a cone with its vertex at the origin of degree two be given as

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0. \quad ...(1)$$

Since x, y, z axes are the generators of (1) and d.c.'s of x, y and z are respectively, 1, 0, 0; 0, 1, 0; 0, 0, 1. then we have $a=0$, $b=0$ and $c=0$.

Thus (1) becomes

$$2fyz+2gzx+2hxy=0$$

or $\quad \mathbf{fyz+gzx+hxy=0}.$

This is the required equation of a cone of degree two with its vertex at the origin passing through the co-ordinates axes.

SOLVED EXAMPLES

Example 1. *The plane* $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ *meets the co-ordinate axes in A, B and C. Prove that the equation to the cone generated by lines drawn from O to meet the circle ABC is*

$$yz\left(\frac{b}{c}+\frac{c}{b}\right)+zx\left(\frac{c}{a}+\frac{a}{c}\right)+xy\left(\frac{a}{b}+\frac{b}{a}\right)=0.$$

Solution. Since the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ intersects the co-ordinate axes in A, B and C, then the co-ordinates of A, B, C are respectively $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$.

The equation of a sphere passing through $A(a, 0, 0)$, $B(0, b, 0)$, $C(0, 0, c)$ and $O(0, 0, 0)$ is

$$x^2+y^2+z^2-ax-by-cz=0. \quad \text{...(1)}$$

Therefore, the circle ABC is the intersection of the sphere (1) and the plane

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1.$$

Now making (1) homogeneous with the help of $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ as follows :

$$x^2+y^2+z^2-(ax+by+cz)\times 1=0$$

or $$x^2+y^2+z^2-(ax+by+cz)\left(\frac{x}{a}+\frac{y}{b}+\frac{z}{c}\right)=0$$

or $$yz\left(\frac{b}{c}+\frac{c}{b}\right)+zx\left(\frac{c}{a}+\frac{a}{c}\right)+xy\left(\frac{a}{b}+\frac{b}{a}\right)=0.$$ **Hence proved.**

Example 2. *Prove that the equation of the cone, whose vertex is the origin and base the curve $z=k, f(x, y)=0$ is*

$$f\left(\frac{xk}{z}, \frac{yk}{z}\right)=0.$$

Solution. Let the equation of the generator through the origin O be

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n} \quad \text{...(1)}$$

Since this line meets the plane $z=k$, then

$$\frac{x}{l}=\frac{y}{m}=\frac{k}{n}.$$

$\therefore$ $$x=\frac{lk}{n}, y=\frac{mk}{n}.$$

Now putting the values of x and y in $f(x, y)=0$, we have

$$f\left(\frac{l}{n}k, \frac{m}{n}k\right)=0. \quad \text{...(2)}$$

Eliminating l, m, n from (1) and (2), we get

$$f\left(\frac{xk}{z}, \frac{yk}{z}\right)=0.$$ **Hence proved.**

Example 3. *Find the equation to the cone whose vertex is origin and which passes through the curve given by $ax^2+by^2=2z$, $lx+my+nz=p$.*

Solution. The equation of given curve are

$$ax^2+by^2=2z \quad \text{...(1)}$$

and $$lx+my+nz=p. \quad \text{...(2)}$$

Equation (2) can be written as

$$\frac{lx+my+nz}{p}=1. \quad \text{...(3)}$$

Now making (1) homogeneous with the help of (3), the required equation of the cone with vertex at the origin is

$$ax^2+by^2=2z\left(\frac{lx+my+nz}{p}\right)$$

or $$p(ax^2+by^2)=2z\,(lx+my+nz).$$

• TEST YOURSELF–1

1. Find the equation of a cone whose vertex is (0, 0, 0) and which passes through the curve on intersection of the plane $lx+my+nz=p$ and the surface $ax^2+by^2+cz^2=1$.
2. Find the equation of the cone with vertex at the origin and which passes through the following curve :

$$ax^2+by^2+cz^2=1, \alpha x^2+\beta y^2=2z.$$

3. Find the equation of the cone with vertex at (0, 0, 0) and passing through the circle given by $x^2 + y^2 + z^2 + x - 2y + 3z - 4 = 0, x - y + z = 2$.

ANSWERS

1. $p^2 (ax^2 + by^2 + cz^2) = (lx + my + nz)^2$. 2. $4z^2 (ax^2 + by^2 + cz^2) = (\alpha x^2 + \beta y^2)^2$.
3. $x^2 + 2y^2 + 3z^2 + xy - yz = 0$.

• 24.5. EQUATION OF A CONE HAVING A GIVEN VERTEX AND GIVEN BASE (GUIDING CURVE)

To obtain the equation of a cone whose vertex is (α, β, γ) and the guiding curve (base) the conic

$$ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0, z = 0.$$

Let the equation of any line in symmetrical form through the point (α, β, γ) be

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}. \quad ...(1)$$

Then the required tone is given by

or

$$\boxed{\begin{aligned} &a(\alpha z - \gamma x)^2 + b(\beta z - \gamma y)^2 + 2h(\alpha z - \gamma x)(\beta z - \gamma y) \\ &\quad + 2g(\alpha z - \gamma x)(z - \gamma) + 2f(\beta z - \gamma y)(z - \gamma) + c(z - \gamma)^2 = 0 \end{aligned}}$$

• 24.6. CONDITION FOR THE GENERAL EQUATION OF DEGREE TWO TO BE A CONE

Working Procedure

(1) *Make the equation homogeneous by multiplying a new variable t with proper power.*
(2) *Differentiating the equation so obtained partially with respect to x, y, z and t.*
(3) *Now equating the differential coefficeints to zero and then put $t = 1$.*
If $F \equiv F(x, y, z, t)$, then at $t = 1$
put $\frac{\partial F}{\partial x} = 0, \frac{\partial F}{\partial y} = 0, \frac{\partial F}{\partial z} = 0$ *and* $\frac{\partial F}{\partial t} = 0$

first three equations $\frac{\partial F}{\partial x} = 0, \frac{\partial F}{\partial y} = 0, \frac{\partial F}{\partial z} = 0$ *give the vertex and if this vertex satisfies* $\frac{\partial F}{\partial t} = 0$, *then the eqaution $F(x, y, z) = 0$ is the equation of a cone.*

SOLVED EXAMPLES

Example 1. *Find the equation of a cone whose vertex is the point $P(\alpha, \beta, \gamma)$ and whose generating lines pass through the conic*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, z = 0.$$

If the section of this cone by the plane $x = 0$ is a rectangular hyperbola show that the locus of P is

$$\frac{x^2}{a^2} + \frac{y^2 + z^2}{b^2} = 1.$$

Solution. The equation of a line through the point (α, β, γ)

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}. \quad ...(1)$$

Since the line (1) meets the plane $z = 0$, then

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{-\gamma}{n} \quad ...(2)$$

and the co-ordinates of the point at which (1) meets $z = 0$ is given by (2), are

$$\left(\alpha - \frac{l\gamma}{n}, \beta - \frac{m\gamma}{n}, 0\right).$$

This point will lie on the given conic, if

$$\frac{1}{a^2}\left(\alpha - \frac{l\gamma}{n}\right)^2 + \frac{1}{b^2}\left(\beta - \frac{m\gamma}{n}\right)^2 = 1. \quad ...(3)$$

Eliminating l, m, n from (1) and (3), we get

$$\frac{1}{a^2}\left[\alpha - \gamma\left(\frac{x-\alpha}{z-\gamma}\right)\right]^2 + \frac{1}{b^2}\left[\beta - \gamma\left(\frac{y-\beta}{z-\gamma}\right)\right]^2 = 1$$

or
$$\frac{(\alpha z - \gamma x)^2}{a^2} + \frac{(\beta z - \gamma y)^2}{b^2} = (z-\gamma)^2. \quad ...(4)$$

This is the required equation of a cone.

Further, the section of this cone (4) by $x = 0$ is

$$\frac{\alpha^2 z^2}{a^2} + \frac{(\beta z - \gamma y)^2}{b^2} = (z-\gamma)^2 \qquad \text{(By putting } x = 0 \text{ in (4))}$$

or
$$\frac{\gamma^2}{b^2}y^2 + \left(\frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} - 1\right)z^2 - \frac{2\beta\gamma yz}{b^2} + 2z\gamma - \gamma^2 = 0$$

If this equation represents a rectangular hyperbola in yz-plane then, we have

$$\left(\frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} - 1\right) + \frac{\gamma^2}{b^2} = 0 \qquad (\because \text{ coefficient of } y^2 + \text{ coefficient of } z^2 = 0)$$

$\therefore$ The locus of $P(\alpha, \beta, \gamma)$ is obtained by generating α, β, γ to x, y, z. That is

$$\frac{x^2}{a^2} + \frac{y^2 + z^2}{b^2} = 1.$$

Hence proved.

Example 2. *Prove that the equation*

$$ax^2 + by^2 + cz^2 + 2ux + 2vy + 2wz + d = 0$$

represents a cone if $\frac{u^2}{a} + \frac{v^2}{b} + \frac{w^2}{c} = d.$

Solution. Making the equation homogeneous of degree 2 by multiplying the proper power of t, we get

$$f(x, y, z, t) \equiv ax^2 + by^2 + cz^2 + 2uxt + 2vyt + 2wzt + dt^2 = 0. \quad ...(1)$$

Differentiating (1) partially w.r.t. x, y, z and t, we get

$$\frac{\partial f}{\partial x} = 2ax + 2ut, \qquad \frac{\partial f}{\partial y} = 2by + 2vt,$$

$$\frac{\partial f}{\partial z} = 2cz + 2wt, \qquad \frac{\partial f}{\partial t} = 2ux + 2vy + 2wz + 2dt$$

Now putting $t = 1$ and taking $\frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 0, \frac{\partial f}{\partial z} = 0, \frac{\partial f}{\partial t} = 0$, we get

$$2ax + 2u = 0 \qquad \left(\because \frac{\partial f}{\partial x} = 0 \text{ and } t = 1\right)$$

or
$$ax + u = 0 \quad \text{or} \quad x = -\frac{u}{a}.$$

Similarly, $y = -\frac{v}{b},\ z = -\frac{w}{c}$

and
$$ux + vy + wz + d = 0. \quad ...(2)$$

Substitute the values of x, y and z in (2), we get

$$-\frac{u^2}{a} - \frac{v^2}{b} - \frac{w^2}{c} + d = 0 \quad \text{or} \quad \frac{u^2}{a} + \frac{v^2}{b} + \frac{w^2}{c} = d.$$

• TEST YOURSELF–2

1. Find the equation to the cone whose vertex is the point (a, b, c) and whose generating lines intersects the conic $px^2 + qy^2 = 1, z = 0$.
2. Find the equation of the cone, whose vertex is (α, β, γ) and base $ax^2 + by^2 = 1, z = 0$.
3. Prove that the equation of a cone whose vertex is the point (α, β, γ) and base curve as $z^2 = 4by, x = 0$ is $(\gamma x - \alpha z)^2 = 4b(\alpha - x)(\alpha y - \beta x)$.
4. Find the equation to the cone whose vertex is (α, β, γ) and base $y^2 = 4ax, z = 0$.

5. Find the equation to the cone whose vertex is (α, β, γ) and whose generating lines pass through the conic $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ z=0$.

6. The vertex of a cone is (a, b, c) and the yz-plane cuts it in the curve $F(y, z)=0,\ x=0$. Show that the zx-plane cuts it in the curve

$$y=0,\ F\left(\frac{bx}{x-a}, \frac{cx-az}{x-a}\right)=0.$$

ANSWERS

1. $c^2(px^2+qy^2)+(pa^2+qb^2-1)z^2-2c(apzx+bqyz-z)=c^2$.
2. $a(\alpha z-\gamma x)^2+b(\beta z-\gamma y)^2=(z-\gamma)^2$. 4. $(\beta z-\gamma y)^2=4a(\alpha z-\gamma x)(z-y)$.
5. $b^2(\alpha z-\gamma x)^2+a^2(\beta z-\gamma y)^2=a^2b^2(z-\gamma)^2$.

• 24.7. THE TANGENT PLANE

The equation of a plane to the cone $ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$ at the point (α, β, γ) is

$$\boxed{x(a\alpha+h\beta+g\gamma)+y(h\alpha+b\beta+f\gamma)+z(g\alpha+f\beta+c\gamma)=0}$$

• 24.8. CONDITION OF TANGENCY

To obtain the condition for a plane $lx+my+nz=0$ to be a tangent plane to the cone

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0.$$

Since the equation of a cone with vertex at the origin is given by

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0 \qquad ...(1)$$

and the equation of a plane is

$$lx+my+nz=0. \qquad ...(2)$$

The equation of a tangent plane at (α, β, γ) to the cone (1) is

$$x(a\alpha+h\beta+g\gamma)+y(h\alpha+b\beta+f\gamma)+z(g\alpha+f\beta+c\gamma)=0. \qquad ...(3)$$

Let us suppose (2) and (3) represent the same plane, then we have

$$\frac{a\alpha+h\beta+g\gamma}{l}=\frac{h\alpha+b\beta+f\gamma}{m}=\frac{g\alpha+f\beta+c\gamma}{n}=k \text{ (say)}$$

$$\therefore \qquad a\alpha+b\beta+g\gamma=lk$$

or
$$a\alpha+h\beta+g\gamma-lk=0. \qquad ...(4)$$

Similarly,
$$h\alpha+b\beta+f\gamma-mk=0 \qquad ...(5)$$
$$g\alpha+f\beta+c\gamma-nk=0. \qquad ...(6)$$

Since the point (α, β, γ) also lies on the plane (2), then

$$l\alpha+m\beta+n\gamma=0$$

or
$$l\alpha+m\beta+n\gamma-0.k=0. \qquad ..(7)$$

Now eliminating α, β, γ and $-k$ from (4), (5), (6) and (7), we get

$$\begin{vmatrix} a & h & g & l \\ h & b & f & m \\ g & f & c & n \\ l & m & n & o \end{vmatrix}=0. \qquad ...(8)$$

This is the required condition.

• 24.9. THE RECIPROCAL CONE

Definition. *The locus of the lines passing through the vertex of a given cone and perpendicular to the tangent planes is obtained a surface, which is called the* ***reciprocal cone.***

Or

The locus of the normals drawn through the vertex of a given cone to the tangent planes is called the ***reciprocal cone.***

Equation of the Reciprocal Cone :

To obtain the reciprocal cone to the given cone

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0.$$

Let the equation of a tangent plane to the given equation of a cone be

$$lx + my + nz = 0. \quad ...(1)$$

The equation of a cone is

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0. \quad ...(2)$$

Since (1) is a tangent plane of (2) provided the condition

$$Al^2 + Bm^2 + Cn^2 + 2Fmn + 2Gnl + 2Hlm = 0 \quad ...(3)$$

where $A = bc - f^2, B = ca - g^2, C = ab - h^2, F = gh - af,$
$G = hf - bg, H = fg - ch.$

The direction ratios of a normal to the plane (1) are l, m, n.

$\therefore$ The equation of the normal to the plane (1) passing through the vertex (0, 0, 0) of the given cone (2) is

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n}. \quad ...(4)$$

Eliminating l, m, n from (3) and (4), we get

$$\boxed{Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy = 0.}$$

This is the required equation of the reciprocal cone of the given cone.

Reciprocal Cone of $Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy = 0$.

Let the reciprocal cone of

$$Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy = 0 \quad ...(1)$$

be the cone $A'x^2 + B'y^2 + C'z^2 + 2F'yz + 2G'zx + 2H'xy = 0 \quad ...(2)$

where $A' = BC - F^2, \quad B' = CA - G^2, \quad C' = AB - H^2,$
$F' = GH - AF, \quad G' = HF - BG, \quad H' = FG - CH.$

$$\therefore \quad A' = BC - F^2 = (ca - g^2)(ab - h^2) - (gh - af)^2$$
$$= a^2bc - ach^2 - abg^2 + g^2h^2 - g^2h^2 - a^2f^2 + 2afgh$$
$$= a(abc + 2fgh - af^2 - bg^2 - ch^2)$$
$$= a\Delta, \text{ where } \Delta = abc + 2fgh - af^2 - bg^2 - ch^2.$$

Similarly, $B' = b\Delta, C' = c\Delta, F' = f\Delta, G' = g\Delta, H' = h\Delta.$

Now substitute the values of A', B', C', F', G', H' in (2), we get

$$\Delta\,(ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy) = 0.$$

Since $\Delta \neq 0$

$$\therefore \quad \boxed{ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0.}$$

This is the required equation of the reciprocal cone of

$$Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy = 0.$$

REMARK

➤ $$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$$

and $$Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy = 0$$

are the reciprocal cone to each other. That is why the word "Reciprocal" means.

SOLVED EXAMPLES

Example 1. *Find the equation of a cone reciprocal to the cone*

$$ax^2 + by^2 + cz^2 = 0.$$

Solution. Let the reciprocal cone be

$$Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy = 0. \quad ...(1)$$

Since the equation of the given cone is

$$ax^2 + by^2 + cz^2 = 0. \quad ...(2)$$

Now compare this equation (2) with the equation

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$$

we get $a = a, b = b, c = c, f = 0, g = 0, h = 0.$

Further since, $A = bc - f^2, B = ca - g^2, C = ab - h^2,$
$F = gh - af, G = hf - bg, H = fg - ch.$

$$\therefore \quad A = bc - f^2 = bc - 0 = bc$$
$$B = ca - g^2 = ca - 0 = ca$$
$$C = ab - h^2 = ab - 0 = ab$$

and $$F = gh - af = 0 - 0 = 0.$$

Similarly $\quad G = 0, H = 0.$

Substitute these values in (1), we get

$$bcx^2 + cay^2 + abz^2 = 0.$$

Divide by abc, we get

$$\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = 0.$$

This is the required equation of a reciprocal cone.

Example 2. *Prove that the equation*

$$f^2x^2 + g^2y^2 + h^2z^2 - 2ghyz - 2hfzx - 2fgxy = 0$$

or $$\sqrt{(fx)} \pm \sqrt{(gy)} \pm \sqrt{(hz)} = 0$$

represents a cone which touches the co-ordinate planes.

Solution. The given equation is

$$\sqrt{(fx)} \pm \sqrt{(gy)} \pm \sqrt{(hz)} = 0 \text{ or } \sqrt{(fx)} \pm \sqrt{(gy)} = \mp \sqrt{(hz)}.$$

Squaring of both sides, we get

$$fx + gy \pm 2\sqrt{(fx)}\sqrt{(gy)} = hz \text{ or } fx + gy - hz = \mp 2\sqrt{fgxy}.$$

Again squaring of both sides, we have

$$(fx + gy - hz)^2 = 4fgxy$$

or $$f^2x^2 + g^2y^2 + h^2z^2 + 2fgxy - 2ghyz - 2fhzx = 4fgxy$$

or $$f^2x^2 + g^2y^2 + h^2z^2 - 2fgxy - 2ghyz - 2fhzx = 0. \qquad ...(1)$$

This is a homogeneous equation of degree 2, hence this represents a cone.

If the plane $x = 0$ intersect (1), then

$$g^2y^2 + h^2z^2 - 2ghyz = 0 \quad \text{or} \quad (gy - hz)^2 = 0.$$

It is therefore obtained a perfect sqaure hence $x = 0$ touches (1). Similarly $y = 0, z = 0$ also touch the cone (1)

• SUMMARY

- Equation of a cone with its vertex at the origin is given by
$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0.$$
- If $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$ is the generator of a cone $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$, then
$$al^2 + bm^2 + cn^2 + 2fmn + 2glx + 2hlm = 0.$$
- Equation of a cone passing through coordinate axes is
$$fyz + gzx + hxy = 0.$$
- Tangent plane at (α, β, γ) to the cone $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$ is given by
$$x(a\alpha + h\beta + g\gamma) + y(h\alpha + b\beta + f\gamma) = z(g\alpha + f\beta + c\gamma) = 0.$$
- The equation of a cone reciprocal to the given cone $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$ is given by
$$Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy = 0.$$
where $A = bc - f^2, B = ca - g^2, C = ab - h^2, F = gh - af, G = hf - by, H = fg - ch.$
- Angle between the plane $ux + uy + wz = 0$ and the cone
$f(x, y, z) = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$ is given by
$$\theta = \tan^{-1}\left[2p\frac{\sqrt{u^2 + v^2 + w^2}}{(a + b + c)(u^2 + v^2 + w^2) - f(u, v, w)}\right]$$
where $$p^2 = \begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & o \end{vmatrix}$$
- The cone $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hzx + 2hxy = 0$ *has three mutually perpendicular* generators if $a + b + c = 0$.
- The cone $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$ has three mutually perpendicular tangent planes if $f^2 + g^2 + h^2 = ab + bc + ca$.

• STUDENT ACTIVITY

1. Find the equation of the cone whose vertex is the origin and which passes through the curve given by $ax^2 + by^2 = 2z$, $lx + my + nz = p$.

2. Prove that the equation $ax^2 + by^2 + cz^2 + 2ux + 2vy + 2wz + d = 0$ represents a cone if $\frac{u^2}{a} + \frac{v^2}{b} + \frac{w^2}{c} = d$.

• TEST YOURSELF–3

1. Show that the reciprocal cone of the cone

$$\sqrt{fx} \pm \sqrt{gy} \pm \sqrt{hz} = 0$$

is $$fyz + gzx + hxy = 0.$$

2. Prove that perpendicular drawn from the origin to the tangent planes to the cone $ax^2 + by^2 + cz^2 = 0$ lie on the cone

$$\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = 0.$$

3. Find the equation of the cone reciprocal to the cone $fyz + gzx + hxy = 0$.

ANSWERS

1. $f^2x^2 + g^2y^2 + h^2z^2 - 2ghyz - 2hfzx - fgxy = 0.$

OBJECTIVE EVALUATION

➤ FILL IN THE BLANKS :

1. The equation of a cone with its vertex (0, 0, 0) is homogeneous of degree ....
2. Every homogeneous equation of second degree represents a cone whose vertex is ....
3. The equation of a cone of second degree passing through the axis is ....
4. The equation of the cone whose vertex is (0, 0, 0) and base the curve $z = k, f(x, y) = 0$ is ....

➤ TRUE OR FALSE :

Write 'T' for true and 'F' for false statement :

1. If the line $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$ is a generator of the cone $f(x, y, z) = 0$ (a homogeneous equation of second degree), then $f(l, m, n) = 0$. *(T/F)*
2. The equation of a cone with its vertex at (0, 0, 0) is not homogeneous. *(T/F)*
3. Every homogeneous equation of second degree in x, y, z represents a cone. *(T/F)*

➤ MULTIPLE CHOICE QUESTIONS :

Choose the most appropriate one :

The vertex of the cone $fyz + gzx + hxy = 0$ is :

(a) (0, 0, 0) (b) (1, 1, 1) (c) (0, 1, 0) (d) (0, 0, 1).

2. The degree of every homogeneous equation of a cone is :

(a) One (b) Two (c) Three (d) None of these.

ANSWERS

Fill in the Blanks :

1. Two **2.** (0, 0, 0) **3.** $fyz + gzx + hxy = 0$ **4.** $f\left(\frac{xk}{z}, \frac{yk}{z}\right) = 0$

True or False :

1. T **2.** F **3.** T

Multiple Choice Questions :

1. (a) **2.** (b)

UNIT 25

THE CYLINDER

STRUCTURE

- Cylinder
- Right Circular Cylinder
- Equation of a Cylinder Passing through a Given Conic
- Equation of a Right Circular Cylinder
- Equation of a Tangent Plane to th Cylinder
- Summary
- Student Activity
- Test Yourself

LEARNING OBJECTIVES

After going through this unit you will learn :

- About the cylinder and how to get the equation of the cylinder passing through the given conic
- How to find the equation of right circular cylinder
- How to find the tangent plane to the given cylinder

25.1. CYLINDER

Definition. *A surface generated by a variable straight line which moves parallel to a fixed line and intersecting a given curve or touching a given surface, is called a* **cylinder.**

Axis of the cylinder. *The fixed line parallel to which a variable straight line moves, is called the* **axis** *of the cylinder.*

Guiding curve. A straight line which intersect a given curve. This given curve is called the **guiding curve.**

25.2. RIGHT CIRCULAR CYLINDER

Definition. *A surface generated by a moving straight line (generator) which moves in such a way that it is always at a constant distance from the fixed line (axis), is called a* **right circular cylinder** *and the constant distance is called the* **radius** *of this right circular cylinder.*

25.3. EQUATION OF A CYLINDER PASSING THROUGH A GIVEN CONIC

To obtain the equation of a cylinder whose generator are parallel to the fixed line (axis) and intersecting the given conic.

Let the equation of fixed line through the origin O and having the direction-ratios l, m, n be

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n} \qquad ...(1)$$

and that of the given conic be

$$ax^2+2hxy+by^2+2gx+2fy+c=0, z=0. \qquad ...(2)$$

Let $P(\alpha, \beta, \gamma)$ be any point on the surface of the cylinder. Then the equation of its generator is

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}. \qquad ...(3)$$

This generator (moving straight line) meets the plane $z=0$ at the point whose co-ordinates are $\left(\alpha-\frac{l\gamma}{n}, \beta-\frac{m\gamma}{n}, 0\right)$. This point lies on the conic (2), we get

$$a\left(\alpha-\frac{l\gamma}{n}\right)^2+2h\left(\alpha-\frac{l\gamma}{n}\right)\left(\beta-\frac{m\gamma}{n}\right)+b\left(\beta-\frac{m\gamma}{n}\right)^2+2g\left(\alpha-\frac{l\gamma}{n}\right)+2f\left(\beta-\frac{m\gamma}{n}\right)+c=0$$

or $$a\,(\alpha n-l\gamma)^2+2h(\alpha n-l\gamma)\,(\beta n-m\gamma)+b(\beta n-m\gamma)^2$$
$$+\,2gn(\alpha n-l\gamma)+2fn\,(\beta n-m\gamma)+n^2c=0 \quad ...(4)$$

Thus the locus of the point $P(\alpha, \beta, \gamma)$ is obtained by generating $\alpha = x$, $\beta = y$, $\gamma = z$ in (4), we get

$$\boxed{\begin{aligned}&a(nx-lz)^2+2h(nx-lz)\,(ny-mz)+b(ny-mz)^2\\&\qquad+2gn(nx-lz)+2fn(ny-mz)+n^2c=0.\end{aligned}} \quad ...(5)$$

This is the required equation of a cylinder.

Corollary. *The equation of a cylinder whose axis is the axis of z passing the given conic*

$$ax^2+2hxy+by^2+2gx+2fy+c=0,\ z=0$$

is $f(x, y) \equiv ax^2+2hxy+by^2+2gx+2fy+c=0.$

Proof. Since the equation of a cylinder whose axis is $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$ and passing the conic $ax^2+by^2+2hxy+2gx+2fy+c=0,\ z=0$ is

$$a(nx-lz)^2+2h(nx-lz)\,(ny-mz)+b(ny-mz)^2+2gn(nx-lz)+2fn(ny-mz)+n^2c=0$$

[From (5)]

Here the axis is axis of z, then putting $l = 0$, $m = 0$, $n = 1$ in above equation, we get

$$f(x, y) \equiv ax^2+2hxy+by^2+2gx+2fy+c=0.$$

Hence proved.

REMARKS

- The equation $f(x, y) \equiv ax^2+2hxy+by^2+2gx+2fy+c=0$ represents a conic in plane geometry whereas in solid geometry it represents the equation of a cylinder whose aixs is the axis of z.
- The equation of a cylinder which intersects the curves $f_1(x, y, z) = 0$, $f_2(x, y, z) = 0$, whose axis is the axis of z is obtained by eliminating z between f_1 and f_2.

• 25.4. EQUATION OF A RIGHT CIRCULAR CYLINDER

To obtain the equation of a right circular cylinder.

Let the equation of the axis through a point $A(\alpha, \beta, \gamma)$ be

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n} \quad ...(1)$$

and let r be the radius of the cylinder let $P(x, y, z)$ be any point on the surface of the cylinder and Q be on the axis of the cylinder such that $PQ = r$. Since AQ is the projection of AP on the axis whose direction-cosines or ratios are l, m, n. Then

$$AQ=[(x-\alpha)\,l+(y-\beta)\,m+(z-\gamma)\,n]/\sqrt{l^2+m^2+n^2}$$

and $$PA=\sqrt{[(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2]}.$$

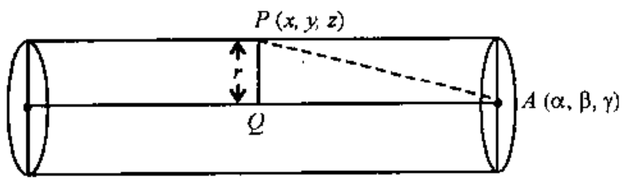

Fig. 1

In ΔAQP, $\angle PQA = 90°$.

$\therefore$ $$PQ^2 = PA^2 - AQ^2$$

$$r^2=[(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2]-\frac{[(x-\alpha)\,l+(y-\beta)\,m+(z-\gamma)\,n]^2}{l^2+m^2+n^2}$$

or $$[(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2]\,(l^2+m^2+n^2)$$
$$-\,[(x-\alpha)\,l+(y-\beta)\,m+(z-\gamma)\,n]^2=r^2\,(l^2+m^2+n^2). \quad ...(2)$$

This is the required equation of a right circular cylinder.

REMARK

- If the axis of the right circular cylinder is z-axis, then the equation of right circular cylinder is obtained by putting $\alpha = 0, \beta = 0, \gamma = 0$ and $l = 0, m = 0, n = 1$ in (2) above, is
$$x^2 + y^2 = r^2.$$

• 25.5. EQUATION OF A TANGENT PLANE TO THE CYLINDER

To obtain the equation of a tangent plane to the cylinder whose equation is
$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$
at the point $P(\alpha, \beta, \gamma)$.
$$a\alpha x + h(x\beta + y\alpha) + by\beta + g(x + \alpha) + f(y + \beta) + c = 0. \qquad \text{... (1)}$$

Corollary. *The tangent plane* (1) *touches the cylinder*
$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$
along a generator.

Proof. Since the equation of a cylinder whose axis is the axis of z is given by
$$f(x, y) \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0. \quad \text{...(2)}$$
The equation of a generator through the point (α, β, γ) parallel to the axis of the cylinder is
$$\frac{x-\alpha}{0} = \frac{y-\beta}{0} = \frac{z-\gamma}{1} = r_1 \text{ (say)} \qquad \text{... (3)}$$
any point on this line is $(\alpha, \beta, r_1 + \gamma)$.

Now the equation of a tangent plane at $(\alpha, \beta, r_1 + \gamma)$ to (1) using the formula (1), we get
$$ax\alpha + 2h(x\beta + y\alpha) + by\beta + g(x + \alpha) + f(y + \beta) + c = 0 \qquad \text{... (4)}$$
Thus (1) and (4) are identical. Hence the tangent plane (1) touches the cylinder along the generator.

SOLVED EXAMPLES

Example 1. *Find the equation to the cylinder whose generators are parallel to the line*
$$\frac{x}{1} = \frac{y}{-2} = \frac{z}{3}$$
and the guiding curve is the ellipse $x^2 + 2y^2 = 1, z = 3$.

Solution. Let $P(\alpha, \beta, \gamma)$ be any point on the cylinder. Then the equation of the generator through $P(\alpha, \beta, \gamma)$ and parallel to the line
$$\frac{x}{1} = \frac{y}{-2} = \frac{z}{3}$$
is given by
$$\frac{x-\alpha}{1} = \frac{y-\beta}{-2} = \frac{z-\gamma}{3} = r \text{ (say)}. \qquad \text{...(1)}$$
Any point on this line is $(r + \alpha, -2r + \beta, 3r + \gamma)$. This point lies on the ellipse $x^2 + 2y^2 = 1, z = 3$, then we get
$$(r + \alpha)^2 + 2(-2r + \beta)^2 = 1. \qquad \text{...(2)}$$
and
$$3r + \gamma = 3. \qquad \text{...(3)}$$
Eliminating r between (2) and (3), we get
$$\left(\alpha + \frac{3-\gamma}{3}\right)^2 + 2\left(-\frac{2(3-\gamma)}{3} + \beta\right)^2 = 1$$
or
$$(3\alpha + 3 - \gamma)^2 + 2\,(-6 + 2\gamma + 3\beta)^2 = 9.$$
$\therefore$ Thus the locus of $P(\alpha, \beta, \gamma)$ is
$$(3x - z + 3)^2 + 2(3y + 2z - 6)^2 = 9$$
or
$$9x^2 + z^2 + 9 - 6zx - 6z + 18x + 18y^2 + 8z^2 + 72 + 24yz - 48z - 72y = 9$$
or
$$9x^2 + 18y^2 + 9z^2 + 24yz - 6zx + 18x - 54z - 72y + 72 = 0.$$

Example 2. *Find the equation of the right circular cylinder of radius 2 and whose axis is the line*
$$\frac{x-1}{2} = \frac{y}{3} = \frac{z-3}{1}.$$

Solution. Let $P(x, y, z)$ be any point on the cylinder. The eqution of its axis is
$$\frac{x-1}{2} = \frac{y}{3} = \frac{z-3}{1}$$

d.r.'s of this line 2, 3, 1.

$\therefore$ d.c.'s of this line are $\frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}, \frac{1}{\sqrt{14}}$.

Let $A(1, 0, 3)$ be any point on the axis and let Q be a point on the axis a such that $PQ = 2$.

$\therefore$ The projection of AP on the axis $= QA$

or $QA = (x-1)\frac{2}{\sqrt{14}} + (y-0)\cdot\frac{3}{\sqrt{14}} + (z-3)\cdot\frac{1}{\sqrt{14}} = \frac{1}{\sqrt{14}}(2x+3y+z-5)$

and $\qquad PA^2 = (x-1)^2 + y^2 + (z-3)^2.$

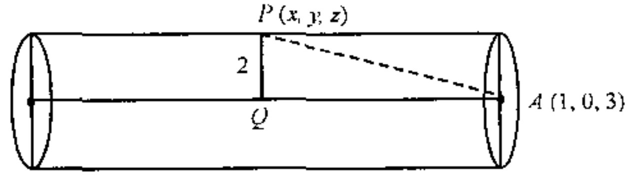

Fig. 2

In ΔPQA, $\angle PQA = 90°$, then

$$PQ^2 = PA^2 - QA^2$$

or $$4 = [(x-1)^2 + y^2 + (z-3)^2] - \frac{1}{14}(2x+3y+z-5)^2$$

or $$56 = 14\,[(x-1)^2 + y^2 + (z-3)^2] - (2x+3y+z-5)^2$$

or $$14\,(x^2+y^2+z^2-2x-6z+10) - (4x^2+9y^2+z^2+25+12xy+6yz + 4zx - 20x - 30y - 10z) = 56$$

or $$10x^2 + 5y^2 + 13z^2 - 6yz - 4zx - 12xy - 8x + 30y - 74z + 59 = 0.$$

Example 3. *Find the equation of the cylinder which intersects the curve $ax^2 + by^2 + cz^2 = 1$, $lx + my + nz = p$ whose generators are parallel to the axis of x.*

Solution. The equations of guiding curve are

$$ax^2 + by^2 + cz^2 = 1 \qquad ...(1)$$

and $$lx + my + nz = p \qquad ...(2)$$

Since the generators of the cylinder are parallel to x-axis, therefore the required equation of the cylinder will not contain the terms of x. So eliminating x between (1) and (2), we get

$$a\left[\frac{p - my - nz}{l}\right]^2 + by^2 + cz^2 = 1$$

or $$a\,(p^2 + m^2y^2 + n^2z^2 - 2pmy - 2pxz + 2mnyz) + bl^2y^2 + cl^2z^2 = l^2$$

or $$(am^2 + bl^2)\,y^2 + (an^2 + cl^2)\,z^2 + 2amnyz - 2apmy - 2apnz + (ap^2 - l^2) = 0,$$

which is the required equation of the cylinder.

• SUMMARY

- Equation of a cylinder whose generator is $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$ and intersecting the given conic $ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$ is given by
  $a\,(nx - lz)^2 + 2h\,(nx - lz)\,(ny - mz) + b\,(ny - mz)^2 + 2gn\,(nx - lz) + 2fn\,(ny - mz) + n^2c = a.$
- Equation of a right circular cylinder of radius r and axis $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$ is given by
  $[(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2]\,)l^2 + m^2 + n^2) - [l\,(x-\alpha) + m\,(y-\beta) + n\,(z-\gamma)]^2 = r^2\,(l^2 + m^2 + n^2)$
- Equation of the tangent plane at (α, β, γ) to the cylinder $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ is given by
  $$a\alpha x + h\,(x\beta + y\alpha) + by\beta + g\,(x+\alpha) + f\,(y+\beta) + c = 0.$$

• STUDENT ACTIVITY

1. Find the equation of the cylinder whose generators are parallel to the line $\frac{x}{1} = \frac{y}{-2} = \frac{z}{3}$ and the guiding curve is the ellipse $x^2 + 2y^2 = 1, z = 3$.

2. Find the equation of the cylinder which intersects the curve $ax^2 + by^2 + cz^2 = 1$, $lx + my + nz = p$ whose generators are parallel to the axis x.

• TEST YOURSELF

1. Prove that the equation to the cylinder whose generators are parallel to the line $\frac{x}{1} = \frac{y}{-2} = \frac{z}{3}$ and passing through the curve $x^2 + 2y^2 = 1, z = 0$ is
$$x^2 + 2y^2 + z^2 - \frac{2}{3}xz + \frac{8}{3}yz - 1 = 0.$$
2. Find the equation to the surface generated by a straight line which is parallel to the line $y = mx, z = nx$ and intersects the ellipse
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, z = 0.$$
3. Find the equation of the cylinder with generators parallel to z-axis and passing through the curve $ax^2 + by^2 = 2cz, lx + my + nz = p$.
4. Find the equation of a cylinder whose generators are parallel to the line $x = y/2 = -z$ and passing through the curve $3x^2 + 2y^2 = 1, z = 0$.

ANSWERS

2. $b^2(nx - z)^2 + a^2(ny - mz)^2 = a^2b^2n^2$.
3. $anx^2 + bny^2 + 2c(lx + my) - 2pc = 0$.
4. $3x^2 + 2y^2 + 11z^2 + 8yz + 6zx - 1 = 0$.

OBJECTIVE EVALUATION

➤ FILL IN THE BLANKS :

1. The equation $f(x, y) \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a ....

2. The axis of the cylinder $f(x, y) \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ is ....

3. The equation of a right circular cylinder of radius r having its axis as z-axis is ....
4. The equation of the right circular cylinder of radius 5 whose axis is the y-axis is ....

➤ TRUE OR FALSE :

Write 'T' for true and 'F' for false statement :

1. The axis of the cylinder $f(y, z) \equiv ay^2 + 2fyz + cz^2 + 2vy + 2wz + c = 0$ is x-axis. *(T/F)*
2. The intersection of the curve $f(x, y, z) \equiv 0$ and $\phi(x, y, z) \equiv 0$ gives the equation of a cylinder with its axis z-axis. *(T/F)*
3. The equation $y^2 + z^2 = a^2, x = 0$ represents a cone. *(T/F)*

➤ MULTIPLE CHOICE QUESTIONS :

Choose the most appropriate one :

1. The axis of the cylinder $f(x, y) \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ is :
   (a) x-axis (b) y-axis
   (c) z-axis (d) None of these.
2. The generators of the cylinder $f(y, z) \equiv 0$ are parallel to the axis :
   (a) x-axis (b) y-axis
   (c) z-axis (d) None of these.

ANSWERS

Fill in the Blanks :

1. Cylinder **2.** z-axis **3.** $x^2 + y^2 = r^2$
4. $x^2 + z^2 = 25$

True or False :

1. T **2.** F **3.** F

Multiple Choice Questions :

1. (c) **2.** (a).