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B. Sc. (Part I) Mathematics

ALGEBRA, TRIGONOMETRY & VECTORS

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UNIT

1

THEORY OF EQUATIONS

STRUCTURE

LEARNING OBJECTIVES

After going through this unit you will learn :

- The theory of equations which comprises of roots, relations between roots and coefficient of an algebriac equation
- Horner's synthetic division, transformation of equationand Descarte's rule of sign along with some results drawn

1.1. NUMBER OF ROOTS OF ANY EQUATION

Theorem 1. *Every equation of degree n has n roots and no more.*

Proof. Let the equation of degree n be

$$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n = 0 \qquad \ldots(1)$$

provided $a_0 \neq 0$.

The equation $f(x) = 0$ has the roots, real as well as imaginary. Therefore, Let α_1 be any root of the equation (1), then $f(x)$ can be written as

$$f(x) = (x - \alpha_1)(a_0x^{n-1} + \ldots)$$

or $$f(x) = (x - \alpha_1)\,\phi_1(x) \qquad \ldots(2)$$

where $\phi_1(x)$ is a function of x of degree $n - 1$, such that $\phi_1(\alpha_1) \neq 0$. Further let α_2 be a root of $\phi_1(x) = 0$, then $\phi_1(x)$ can be written

$$\phi_1(x) = (x - \alpha_2)\,\phi_2(x)$$

$$\therefore \quad f(x) = (x - \alpha_1)(x - \alpha_2)\,\phi_2(x). \qquad \ldots(3)$$

Continuing this process upto n times, we obtain

$$f(x) = a_0(x - \alpha_1)(x - \alpha_2)\ldots(x - \alpha_n). \qquad \ldots(4)$$

From equation (4) it is clear that when x take the values from α_1 to α_n, $f(x)$ comes out be zero.

Hence the equation $f(x)$ has n roots. Moreover if x takes any value different from $\alpha_1, \alpha_2 \ldots \alpha_n$, $f(x)$ can not be zero so that $f(x) = 0$ has exactly n roots. Hence the theorem.

• 1.2. RELATION BETWEEN THE ROOTS AND COEFFICIENTS

Let the general equation of degree n be given by

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n = 0 \qquad \ldots(1)$$

where $a_0, a_1, a_2 \ldots a_n$ are the coefficients and $a_0 \neq 0$ and let $\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_n$ be the roots of the equation (1). Then the equation (1) can be identically written as

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n = a_0(x - \alpha_1)(x - \alpha_2) \ldots (x - \alpha_n)$$

or $$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n$$

$$= a_0[x^n - (\Sigma\alpha_1)x^{n-1} + (\Sigma\alpha_1\alpha_2)x^{n-2} + \ldots + (-1)^n\alpha_1\alpha_2\ldots\alpha_n]$$

where $\Sigma\alpha_1 = \alpha_1 + \alpha_2 + \ldots + \alpha_n$

$\Sigma\alpha_1\alpha_2 = \alpha_1\alpha_2 + \alpha_1\alpha_3 + \ldots$ etc.

Now equating the coefficients of like power of x of both sides we get

$$\Sigma\alpha_1 = -\frac{a_1}{a_0},\ \Sigma\alpha_1\alpha_2 = \frac{a_2}{a_0},\ \Sigma\alpha_1\alpha_2\alpha_3 = -\frac{a_3}{a_0},\ \alpha_1\alpha_2\ldots\alpha_n = (-1)^n\frac{a_n}{a_0}. \qquad \ldots(2)$$

Hence the equation (2) gives the required relation between the roots and the coefficients of equation.

REMARK

If the equation is not complete *i.e.*, some of the terms are missing, then we should first make this equation complete by adding the missing terms with zero coefficients.

• 1.3. HORNER'S SYNTHETIC DIVISION

In order to find the quotient and the remainder when a polynomial

$$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n,\ (a_0 \neq 0) \qquad \ldots(1)$$

of degree n is devided by a linear factor $(x - \alpha)$, we use a method given by **Horner**, called *synthetic division*. This method is being dicussed as follows :

$$\begin{array}{c|cccccc}
\alpha & a_0 & a_1 & a_2 \ldots\ldots\ldots & a_{n-1} & a_n & \\
 & & \alpha a_0 & \alpha b_1 \ldots\ldots\ldots & \alpha b_{n-2} & \alpha b_{n-1} & \\
\hline
 & a_0 & b_1 & b_2 & b_{n-1} & \multicolumn{1}{|c}{R} &
\end{array}$$

(1) If the equation (1) is not complete, then first make it complete by adding missing terms with zero coefficient.

(2) In the first horizontal line (row) we should write the coefficients $a_0, a_1, a_2, \ldots a_{n-1}, a_n$ of the polynomial $f(x)$.

(3) Since we have to divide the polynomial $f(x)$ by $x - \alpha$, so we should write α to the left of the vertical line as shown above.

(4) In the third horizontal line (row) we should write a_0 and the first term of the second horizontal line (row) is obtained by multiplying a_0 to α and then add this term with a_1 we obtain b_1 which is the second term of the third row. Next, we multiply b_1 and α and obtained the second term of the second row now adding this αb_1 with a_2 we obtain third terms of the third row. Continue the process in the same way we obtain the last term in the third row which is in fact the remainder R while the second last term in the same is b_{n-1}.

REMARK

If the remainder R comes out be zero, then α will be a root of the equation $f(x) = 0$.

• SOLVED EXAMPLES

Example 1. *Find the condition that two of the roots* α, β *of the equation* $x^3 - px^2 + qx - r = 0$ *are connected by the relation* $\alpha + \beta = 0$.

Solution. Let α, β and γ be the roots of the equation $x^3 - px^2 + qx - r = 0$. Then

$$\alpha + \beta + \gamma = p \qquad \ldots(1)$$

From (1) $\quad \alpha + \beta + \gamma = p$

$\Rightarrow \quad 0 + \gamma = p \qquad [\because \alpha + \beta = 0]$

$\Rightarrow \quad \gamma = p$

γ is a root of the equation $x^3 - px^2 + qx - r = 0$ if

$$\gamma^3 - p\gamma^2 + q\gamma - r = 0$$

$$\Rightarrow \quad p^3 - p(p^2) + qp - r = 0 \qquad [\because \gamma = p]$$

$$\Rightarrow \quad qp - r = 0$$

$$\therefore \quad pq = r$$

which is the required condition.

Example 2. *If the two roots* α, β *of the equation* $x^3 + px^2 + qx + r = 0$ *are connected as* $\alpha\beta + 1 = 0$, *then show that* $1 + q + pr + r^2 = 0$.

Solution. Let α, β and γ be the roots of the equation

$$x^3 + px^2 + qx + r = 0 \qquad ...(1)$$

$$\alpha\beta\gamma = -r$$

$$\Rightarrow \quad (-1)\gamma = -r \qquad [\because \alpha\beta + 1 = 0]$$

$$\Rightarrow \quad \gamma = r$$

γ is a root of equation (1) if

$$\gamma^3 + p\gamma^2 + q\gamma + r = 0. \qquad ...(2)$$

Putting the value of γ in (2), we get

$$r^3 + pr^2 + qr + r = 0$$

$$\therefore \quad r^2 + pr + q + 1 = 0.$$

which is the required condition.

Example 3. *Find the condition that the sum of two roots of the equation* $x^4 + px^3 + qx^2 + rx + s = 0$ *be equal to the sum of the other two roots.*

Solution. Let α, β, γ and δ be the roots of the equation

$$x^4 + px^3 + qx^2 + rx + s = 0 \qquad ...(1)$$

Then

$$\alpha + \beta + \gamma + \delta = -p \qquad ...(2)$$

$$\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = q$$

or

$$(\alpha + \beta)(\gamma + \delta) + \alpha\beta + \gamma\delta = q \qquad ...(3)$$

$$\alpha\beta\gamma + \alpha\gamma\delta + \beta\gamma\delta + \alpha\beta\delta = -r$$

or

$$\alpha\beta(\gamma + \delta) + \gamma\delta(\alpha + \beta) = -r \qquad ...(4)$$

and

$$\alpha\beta\gamma\delta = s \qquad ...(5)$$

Also

$$\alpha + \beta = \gamma + \delta \text{ (given).} \qquad ...(6)$$

From (2) and (6), we get

$$\alpha + \beta = \gamma + \delta = -\frac{p}{2}. \qquad ...(7)$$

From (4) and (7), we get

$$\alpha\beta\left(-\frac{p}{2}\right) + \gamma\delta\left(-\frac{p}{2}\right) = -r$$

or

$$\alpha\beta + \gamma\delta = \frac{2r}{p}. \qquad ...(8)$$

From (3), (7) and (8), we get

$$\left(-\frac{p}{2}\right)\left(-\frac{p}{2}\right) + \frac{2r}{p} = q$$

or

$$p^3 - 4pq + 8r = 0$$

which is the required condition.

Example 4. *If* α, β, γ *are the roots of the cubic* $x^3 + px^2 + qx + r = 0$, *find the value of* $(\beta + \gamma)(\gamma + \alpha)(\alpha + \beta)$.

Solution. We have

$$\alpha + \beta + \gamma = -p$$

$$\alpha\beta + \beta\gamma + \alpha\gamma = q$$

$$\alpha\beta\gamma = -r$$

$$\begin{aligned}(\beta + \gamma)(\gamma + \alpha)(\alpha + \beta) &= (\alpha + \beta + \gamma - \alpha)(\alpha + \beta + \gamma - \beta)(\alpha + \beta + \gamma - \gamma) \\ &= (-p - \alpha)(-p - \beta)(-p - \gamma) \\ &= -[p^3 + p^2(\alpha + \beta + \gamma) + p(\alpha\beta + \beta\gamma + \alpha\gamma) + \alpha\beta\gamma] \\ &= -[p^3 + p^2(-p) + p(q) - r] \\ &= r - pq.\end{aligned}$$

• TEST YOURSELF–1

1. If one root of the equation $x^3 - px^2 + qx - r = 0$ be n times of the other, show that it may be found from a quadratic.
2. Find the condition that two roots of the cubic $x^3 - px^2 + qx - r = 0$ be equal.
3. Solve the equation $4x^3 + 20x^2 - 23x + 6 = 0$, two of its roots being equal.
4. If the equation $ax^3 + 3bx^2 + 3cx + d = 0$ has two equal roots, show that each of them equals $\dfrac{bc - ad}{2(ac - b^2)}$.

ANSWERS

2. $(pq - 9r)^2 = 4(p^2 - 3q)(q^2 - 3pr)$ 3. $\frac{1}{2}, \frac{1}{2}, -6$

• 1.4. TRANSFORMATION OF EQUATION

Sometimes there arises some difficulties to find the roots of a given equation. In that case a process of transformation of a given equation into another equation plays an important role for finding the roots of given equation.

In this section we shall discuss some important transformation.

(i) To transform an equation into another equation whose roots are the roots of the given equation with different sign.

Let the given equation be

$$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n = 0 \qquad \ldots(1)$$

and let $\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_n$ be the roots of the equation (1).

Now put $x = -y$ in (1), we get

$$f(-y) = a_0(-y)^n + a_1(-y)^{n-1} + a_2(-y)^{n-2} + \ldots + a_{n-1}(-y) + a_n = 0$$

or $$f(-y) = (-1)^n \{a_0y^n - a_1y^{n-1} + a_2y^{n-2} - \ldots + (-1)^{n-1}a_{n-1}y + (-1)^n a_n = 0. \qquad \ldots(2)$$

This is the transformed equation.

(ii) To transform an equation into another equation whose roots are equal to the roots of the given equation multiplied by a given constant number m.

Let the given equation be

$$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n = 0 \qquad \ldots(1)$$

and let $\alpha_1, \alpha_2, \ldots \alpha_n$ be its roots, then (1) can be written as

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n = a_0(x - \alpha_1)(x - \alpha_2) \ldots (x - \alpha_n) \qquad \ldots(2)$$

putting $y = mx$ or $x = \dfrac{y}{m}$ in (1), we get

$$f\left(\frac{y}{m}\right) = a_0\left(\frac{y}{m}\right)^n + a_1\left(\frac{y}{m}\right)^{n-1} + a_2\left(\frac{y}{m}\right)^{n-2} + \ldots + a_{n-1}\left(\frac{y}{m}\right) + a_n = 0$$

or $$f\left(\frac{y}{m}\right) = \frac{1}{m^n}[a_0y^n + ma_1y^{n-1} + m^2a_2y^{n-2} + \ldots + m^{n-1}ya_{n-1} + m^na_n] = 0$$

or $$a_0y^n + ma_1y^{n-1} + m^2a_2y^{n-2} + \ldots + m^{n-1}a_{n-1}y + m^na_n = 0. \qquad \ldots(3)$$

This is the transformed equation.

(iii) To transform an equation into another equation whose roots are the reciprocals of the roots of the given equation.

Let the given equation be

$$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n = 0 \qquad \ldots(1)$$

and let $\alpha_1, \alpha_2, \ldots \alpha_n$ be its roots, then we have

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n = a_0(x - \alpha_1)(x - \alpha_2) \ldots (x - \alpha_n) \qquad \ldots(2)$$

putting $x = \dfrac{1}{y}$ in (1), we get

$$f\left(\frac{1}{y}\right) = a_0\left(\frac{1}{y}\right)^n + a_1\left(\frac{1}{y}\right)^{n-1} + a_2\left(\frac{1}{y}\right)^{n-2} + \ldots + a_{n-1}\left(\frac{1}{y}\right) + a_n = 0$$

or $$f\left(\frac{1}{y}\right)=\frac{1}{y^n}[a_0+a_1y+a_2y^2+\ldots+a_{n-1}y^{n-1}+a_ny^n]=0$$

or $$a_ny^n+a_{n-1}y^{n-1}+\ldots+a_1y+a_0=0. \qquad \ldots(3)$$

This is the transformed equation.

(iv) **Reciprocal equation.** *An equation which remains unchanged when x is replaced by $\frac{1}{x}$, is called a **reciprocal equation.***

Let the given equation be

$$f(x)\equiv a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\ldots+a_{n-1}x+a_n=0. \qquad \ldots(1)$$

Replace x by $\frac{1}{x}$, we obtain,

$$f\left(\frac{1}{x}\right)\equiv a_0+a_1x+a_2x^2+\ldots+a_{n-1}x^{n-1}+a_nx^n=0. \qquad \ldots(2)$$

• 1.5. REMOVAL OF TERMS OF AN EQUATION

Let the given equation be

$$f(x)\equiv a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\ldots+a_{n-1}x+a_n=0 \qquad \ldots(1)$$

if we put $x=y+h$, we get

$$a_0(y+h)^n+a_1(y+h)^{n-1}+a_2(y+h)^{n-2}+\ldots+a_{n-1}(y+h)+a_n=0.$$

This equation can be written in the decending powers of y as follows :

$$a_0y^n+(na_0h+a_1)y^{n-1}+\left\{\frac{n(n-1)}{2!}a_0h^2+(n-1)a_1h\,y^{n-2}+a_2\right\}+\ldots=0.$$

Now we want to remove second term, then we shall equate to zero the coefficient of y^{n-1}, we get

$$na_0h+a_1=0 \quad \text{or} \quad h=-\frac{a_1}{na_0}.$$

Hence we decreased all the roots of the given equation by a constant $-\frac{a_1}{na_0}$, the second term of the given equation can be removed.

Similarly if we want to remove third term, we put

$$\frac{n(n-1)}{2!}a_0h^2+(n-1)a_1h+a_2=0.$$

Solve this equation we get two values of h and similarly we can remove any term of the given equation.

• SOLVED EXAMPLES

Example 1. *Change the signs of the roots of the equation*

$$x^7+5x^5-x^3+x^2+7x+3=0.$$

Solution. First making the equation complete by adding missing terms with zero coefficients, we get

$$f(x)\equiv x^7+0\cdot x^6+5x^5+0\cdot x^4-x^3+x^2+7x+3=0 \qquad \ldots(1)$$

Put $x=-y$, in (1), we get

$$(-y)^7+0\cdot(-y)^6+5(-y)^5+0\cdot(-y)^4-(-y)^3+(-y)^2+7(-y)+3=0$$

or $$-y^7+0\cdot y^6-5y^5+0\cdot y^4+y^3+y^2-7y+3=0$$

or $$y^7+5y^5-y^3-y^2+7y-3=0.$$

This is the required equation whose roots are same to the roots of the given equation with contrary signs.

Example 2. *Transform the equation* $72x^3-54x^2+45x-7=0$ *into another equation with integral coeff· ·nts ar ·aving the leading coefficient unity.*

Sol· ·he g·ven equation can be written as

$$x^3-\frac{54}{72}x^2+\frac{45}{72}x-\frac{7}{72}=0$$

or $$x^3-\frac{3}{4}x^2+\frac{5}{8}x-\frac{7}{72}=0. \qquad \ldots(1)$$

Put $y = xm$ or $x = \dfrac{y}{m}$ in (1), we get

$$\left(\frac{y}{m}\right)^3 - \frac{3}{4}\left(\frac{y}{m}\right)^2 + \frac{5}{8}\left(\frac{y}{m}\right) - \frac{7}{72} = 0$$

or $$y^3 - \frac{3}{4}my^2 + \frac{5}{8}m^2y - \frac{7}{72}m^3 = 0. \quad \ldots(2)$$

Now to remove fractional coefficients let us put $m = 12$ in (2), we get

$$y^3 - \frac{3}{4}(12)\,y^2 + \frac{5}{8}(12)^2y - \frac{7}{72}(12)^3 = 0$$

or $$y^3 - 9y^2 + 90y - 168 = 0.$$

This is the required equation.

Example 3. *Form the equation whose roots are the reciprocals of the roots of the equation*

$$x^4 - 3x^3 + 7x^2 + 5x - 2 = 0.$$

Solution. The given equation is

$$x^4 - 3x^3 + 7x^2 + 5x - 2 = 0 \quad \ldots(1)$$

Putting $x = \dfrac{1}{y}$ in (1), we get

$$\left(\frac{1}{y}\right)^4 - 3\left(\frac{1}{y}\right)^3 + 7\left(\frac{1}{y}\right)^2 + 5\left(\frac{1}{y}\right) - 2 = 0$$

or $$1 - 3y + 7y^2 + 5y^3 - 2y^4 = 0$$

or $$2y^4 - 5y^3 - 7y^2 + 3y - 1 = 0.$$

This is the required equation whose roots are the reciprocal of the roots of (1).

Example 4. *Remove the fractional coefficients from the equation*

$$2x^3 - \frac{3}{2}x^2 - \frac{1}{8}x + \frac{3}{16} = 0.$$

Solution. The given equation is

$$2x^3 - \frac{3}{2}x^2 - \frac{1}{8}x + \frac{3}{16} = 0. \quad \ldots(1)$$

Putting $x = \dfrac{y}{m}$ in (1), we get

$$2\left(\frac{y}{m}\right)^3 - \frac{3}{2}\left(\frac{y}{m}\right)^2 - \frac{1}{8}\left(\frac{y}{m}\right) + \frac{3}{16} = 0$$

or $$2y^3 - \frac{3}{2}my^2 - \frac{1}{8}m^2y + \frac{3}{16}m^3 = 0$$

Let us put $m = 4$, we get

$$2y^3 - \frac{3}{2}(4)\,y^2 - \frac{1}{8}(4)^2\,y + \frac{3}{16}(4)^3 = 0$$

or $$2y^3 - 6y^2 - 2y + 12 = 0 \quad \text{or} \quad y^3 - 3y^2 - y + 6 = 0.$$

This is the required equation.

Example 5. *Solve the following reciprocal equation*

$$x^4 - 10x^3 + 26x^2 - 10x + 1 = 0.$$

Solution. The given equation can be written as

$$x^4 + 1 - 10\,(x^3 + x) + 26x^2 = 0.$$

Divide by x^2, we get

$$\left(x^2 + \frac{1}{x^2}\right) - 10\left(x + \frac{1}{x}\right) + 26 = 0. \quad \ldots(1)$$

Let us put $x + \dfrac{1}{x} = y$ and $x^2 + \dfrac{1}{x^2} = y^2 - 2$ in (1), we get

$$y^2 - 2 - 10y + 26 = 0 \quad \text{or} \quad y^2 - 10y + 24 = 0$$

or $$y^2 - 6y - 4y + 24 = 0 \quad \text{or} \quad (y - 6)\,(y - 4) = 0$$

or $$y = 4, 6.$$

Since $$x + \frac{1}{x} = y \ \text{ if } \ y = 4, \ \text{ then } x + \frac{1}{x} = 4$$

or $\qquad x^2 - 4x + 1 = 0$ or $x = \dfrac{4 \pm \sqrt{16-4}}{2} = \dfrac{4 \pm 2\sqrt{3}}{2} = 2 \pm \sqrt{3}$

if $\qquad y = 6$, then $x + \dfrac{1}{x} = 6$

or. $\qquad x^2 - 6x + 1 = 0$ or $x = \dfrac{6 \pm \sqrt{36-4}}{2} = \dfrac{6 \pm 4\sqrt{2}}{2} = 3 \pm 2\sqrt{2}$.

Hence, the roots of the given equation are $2 \pm \sqrt{3}, 3 \pm 2\sqrt{2}$.

Example 6. *If α, β, γ are the roots of the cubic $x^3 - px^2 + qx - r = 0$, form the equation whose roots are* $\beta\gamma + \dfrac{1}{\alpha}, \gamma\alpha + \dfrac{1}{\beta}, \alpha\beta + \dfrac{1}{\gamma}$.

Solution. Since the given equation is

$$x^3 - px^2 + qx - r = 0 \qquad \ldots(1)$$

and α, β, γ are its roots, then

$$\alpha + \beta + \gamma = p, \ \alpha\beta + \beta\gamma + \alpha\gamma = q, \ \alpha\beta\gamma = r.$$

Let y be a root of the required equation. Then

$$y = \beta\gamma + \frac{1}{\alpha} = \frac{\alpha\beta\gamma + 1}{\alpha}$$

$$y = \frac{r+1}{\alpha} \quad y = \frac{r+1}{x} \qquad (\because x = \alpha)$$

$$\therefore \quad x = \frac{r+1}{y}.$$

Substitute this value of x in (1), we get

$$\left(\frac{r+1}{y}\right)^3 - p\left(\frac{r+1}{y}\right)^2 + q\left(\frac{r+1}{y}\right) - r = 0$$

or $$\frac{(r+1)^3}{y^3} - \frac{p(r+1)^2}{y^2} + \frac{q(r+1)}{y} - r = 0$$

or $$(r+1)^3 - p(r+1)^2 y + q(r+1)y^2 - ry^3 = 0$$

or $$ry^3 - q(r+1)y^2 + p(r+1)^2 y - (r+1)^3 = 0.$$

This is the required equation.

Example 7. *Remove the second term of the equation $x^4 + 4x^3 + 2x^2 - 4x - 2 = 0$.*

Solution. Suppose the roots of the given equation are diminished by h so put $y = x - h$ or $x = y + h$ in the given equation, we get

$$(y+h)^4 + 4(y+h)^3 + 2(y+h)^2 - 4(y+h) - 2 = 0$$

$$(y^4 + 4hy^3 + 6h^2y^2 + 4h^3y + h^4) + 4(y^3 + 3hy^2 + 3h^2y + h^3) + 2(y^2 + 2yh + h^2) - 4y - 4h - 2 = 0$$

or $$y^4 + (4h+4)y^3 + (6h^2 + 12h + 2)y^2 + (4h^3 + 12h^2 + 4h - 4)y + (h^4 + 4h^3 + 2h^2 - 4h - 2) = 0. \qquad \ldots(1)$$

In order to remove the second term let us put

$$4h + 4 = 0 \quad \text{or} \quad h = -1$$

substitute this value h in (1), we get

$$y^4 - 4y^2 + 1 = 0.$$

• 1.6. DESCARTE'S RULE OF SIGNS

Before discussion of descarte's Rule of sign, we must remember the following theorems (without proof) :

Theorem 1: *Let $f(x)$ be any polynomial. If a and b two real numbers such that $f(a)$ and $f(b)$ are found of opposite signs, then atleast one or an odd number of real roots of the equation $f(x) = 0$ lie between a and b.*

If $f(a)$ and $f(b)$ are of the same sign, then either no real root or an even number of roots of the equation $f(x) = 0$ lie between a and b.

Theorem 2 : *Let $f(x)$ be any polynomial. If $f(x)$ keeps its sign constant for all real values of x, then the equation $f(x) = 0$ has no real root.*

Descarte's Rule of Sign determines the nature of the roots of the equation $f(x) = 0$ without actually finding its roots. before going into the detail of the rule, we first try to understand the changes of signs in a given polynomial whose terms are arranged in descending or ascending order.

Changes of Signs : Consider the polynomials

(i) $f(x) = x^5 + 3x^4 - 5x^3 + 6x^2 + 2x - 7$

(ii) $g(x) = x^4 - 4x^3 + 5x^2 - 7x - 3$

(i) In $f(x)$ we write each terms with sign as follows :

$f(x) \longrightarrow$	x^5	$3x^4$	$5x^3$	$6x^2$	$2x$	7
Signs $\longrightarrow$	+	+	−	+	+	−

Here we observed that the change of signs occurs with terms are as follows :

x^5	$3x^4$	$5x^3$	$6x^2$	$2x$	7
+	+	−	+	+	−

No change | Change | Change | No change | Change

Clearly, there are three changes of signs in $f(x)$.

(ii) Similarly

$g(x) \longrightarrow$	x^4	$4x^3$	$5x^2$	$7x$	3
Sign $\longrightarrow$	+	−	+	−	−

Change | Change | Change | No change

Clearly, there are three changes of signs in $g(x)$.

Next, we try to understand, what is Descarte's rule of sign ?

For posiive roots : An equation $f(x) = 0$ cannot have more positive roots than the number of changes of signs in $f(x)$.

For negative roots : An equation $f(x) = 0$ cannot have more negative roots than the number of changes of signs in $f(-x)$.

For complex roots : If $f(x) = 0$ is an incomplete equations of degree N and $f(x)$ has p changes of signs and $f(-x)$ has n changes of signs, then $f(x) = 0$ has atmost $p + n$ real roots and has atleast $N - (p + n)$ complex roots.

Remark : *If $f(x) = 0$ is a complete equation of degree n, then we cannot draw any definite conclusion regarding the existence of complex roots.*

IMPORTANT RESULTS DRAWN FROM DESCARTE'S RULE OF SIGN

Result 1 : *Every equation of an odd degree has at least one real root whose sign is opposite to that of its last term, the coefficient of the first terms being positive.*

Proof : Let $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-1} + \ldots\ldots + a_{n-1}x + a_n$ with $a_0 \neq 0$ and n is odd, be a polynomial of odd degree.

Then we have

$$f(-\infty) < 0 \qquad (\because n \text{ is odd})$$

$$f(0) = a_n$$

and

$$f(\infty) > 0$$

Now we have two cases here.

Case I : If a_n is positive, then $f(-\infty)$ and $f(0)$ have opposite signs, thus $f(x) = 0$ has at least one real root between $-\infty$ and 0 which is negative *i.e.*, opposite to the sign of a_n.

Case II : If a_n is negative, then $f(0)$ and $f(\infty)$ have opposite signs, thus $f(x) = 0$ has at least one real root between 0 and ∞ which is positive *i.e.*, opposite to the sign of a_n..

Result 2 : *Every equation of even degree, whose last term is negative and the coefficient of the first term is positive, has at least two real roots, one positive and one negative.*

Proof : Let $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots\ldots + a_{n-1}x + a_n$ $(a_0 \neq 0)$ be a polynomial of degree n (n is even), then

$$f(-\infty) > 0, \quad f(0) = a_n \quad \text{and} \quad f(\infty) > 0$$

As $a_n < 0$ so that $f(-\infty)$ and $f(0)$ have opposite signs, thus $f(x) = 0$ has atleast one real root lying between $-\infty$ and 0 and it is negative.

Also $f(0)$ and $f(\infty)$ are of opposite signs, thus the equation $f(x) = 0$ has atleast one real root lying between 0 and ∞ and it is positive.

Hence, $f(x) = 0$ has atleast two real roots one positive and one negative.

Result 3 : *If an equation has only one change of sign, it must have only one positive root and no more.*

Proof : Let $f(x) = 0$ be an equation. Without any loss of generalty we may assume that the leading coefficient (coefficient of highest degree term) is positive.

Since $f(x) = 0$ has only one change of sign, then $f(x) = 0$ must have a set of positive terms followed by a set of negative terms, which gives a conclusion that the constant term of $f(x) = 0$ is negative.

Therefore, we have

$$f(\infty) > 0 \text{ and } f(0) < 0 \qquad \text{(As } f(0) \text{ is the constant term)}$$

Since $f(\infty)$ and $f(0)$ are of opposite signs, thus $f(x) = 0$ has at least one real root lying between 0 and ∞. But $f(x)$ has only one change of sign so hat the number of positive root of $f(x) = 0$ can not more than one. Hence $f(x) = 0$ has only one positive root.

Result 4 : *If all the terms of an equation are positive and the equation involves no odd powers of x, then all its roots are complex.*

Proof : Let $f(x) = 0$ be an equation having all terms positive and it involves no odd powers of x, then $f(x)$ and $f(-x)$ will have no changes of signs. Thus, by Descarte's rule of signs, $f(x) = 0$ will not have any positive or negative root. Hence all the roots of $f(x) = 0$ must be complex.

Result 5 : *If all the terms of an equation are positive and the equation involves only odd powers of x, then 0 is the only real root of the given equation.*

Proof : Let $f(x) = 0$ be an equation having all terms positive and it involves only odd powers of x.

As $f(x) = 0$ has only odd powers of x, so that $f(x) = 0$ has no constant term, which implies that $x = 0$ must be a root of $f(x) = 0$,

Thus, $f(x)$ can be written as

$$f(x) = xg(x)$$

Now $g(x) = 0$ has all terms positive and it involves no odd terms, then by Result 4, all the roots of $g(x) = 0$ must be complex. Hence, $x = 0$ is the only real root of $f(x) = 0$.

• SOLVED EXAMPLES

Example 1. *Show that the equation*

$$x^6 + 3x^2 - 5x + 1 = 0$$

has atleast four imaginary roots.

Sol. Let $\quad f(x) \equiv x^6 + 3x^2 - 5x + 1 = 0$

Clearly, $f(x)$ has two changes of signs so $f(x) = 0$ has not more than two positive roots.

Also $f(-x) = x^6 + 3x^2 + 5x + 1$ has no changes of signs so $f(x) = 0$ has no negative root. As degree of $f(x) = 0$ is 6, hence $f(x) = 0$ has at least $6 - (2 + 0) = 4$ imaginary roots.

Example 2. *Apply Descarte's rule of signs to discuss the nature of the roots of the equation $x^4 + 15x^2 + 7x - 11 = 0$.*

Sol. Let $\quad f(x) = x^4 + 15x^2 + 7x - 11 = 0$

Clearly, $f(x)$ has only one change of signs, so by Descarte's rule of signs, $f(x) = 0$ has at most one positive root.

Also $\quad f(\infty) > 0 \quad$ and $\quad f(0) = -11 < 0$.

Since $f(0)$ and $f(\infty)$ are of opposite signs, then $f(x) = 0$ has one or odd number of real roots lying between 0 and ∞. But $f(x) = 0$ has atmost one positive root.

Hence $f(x) = 0$ has only one positive root.

Next $\quad f(-x) = x^4 + 15x^2 - 7x - 11$.

Clearly, $f(-x)$ has only one change of signs, so that $f(x) = 0$ has atmost one negative root.

Also $f(-\infty) > 0$ and $f(0) < 0$. which implies that $f(x) = 0$ has one or odd number of real roots lying between $-\infty$ and 0. But $f(x) = 0$ has atmost one negative root. Hence $f(x) = 0$ has only one negative root. Further, degree of $f(x) = 0$ is 4 so that $f(x) = 0$ has other two roots imaginary.

Example 3. *Locate the positions of the roots of the equation $x^3 + x^2 - 2x - 1 = 0$.*

Sol. Let $\quad f(x) \equiv x^3 + x^2 - 2x - 1 = 0$.

Clearly, $f(x)$ has one change of signs so that $f(x) = 0$ has atmost one positive root. But $f(0) = -1, f(1) = -1$ and $f(2) = 7$, which implies that $f(x) = 0$ has one or odd number of roots lying between 1 and 2.

Therefore, $f(x) = 0$ has none positive root lying between 1 and 2.

Now, $f(-x) \equiv -x^3 + x^2 + 2x - 1 = 0$

Clearly, $f(-x)$ has two changes of signs so that $f(x) = 0$ has atmost two negative roots. But $f(0) = -1, f(-1) = 1, f(-2) = -1$.

Since $f(-1)$ and f $f(0)$ are of opposite signs so that one negative root less beween -1 and 0. Also $f(-2)$ and $f(-1)$ are of opposite signs, so that one negative root lies between -2 and -1.

Hence, all the three roots of $f(x) = 0$ are real and lying in open intervals $(-2, -1)$, $(-1, 0)$ and $(1, 2)$.

• STUDENT ACTIVITY

1. Find the value of r so that the root α, β of the equation $x^3 - 3x^2 + 2x - r = 0$ are connected by the relation $\alpha + \beta = 0$.

2. Reduce the equaion $4x^4 - 85x^3 + 357x^2 - 340x + 64 = 0$ into reciprocal quation.

• SUMMARY

- Every equation of degree n has n roots no more.
- If $\alpha_1, \alpha_2, \alpha_3, \ldots\ldots \alpha_n$ are the roots of the equation

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots\ldots + a_{n-1}x + a_n = 0 \qquad (a_0 \neq 0)$$

then

$$\Sigma\alpha_i = -\frac{a_1}{a_0}, \quad \Sigma\alpha_i\alpha_j = \frac{a_2}{a_0}, \quad \Sigma\,\alpha_i\alpha_j\alpha_k = -\frac{a_3}{a_0} \ldots\ldots \alpha_1\alpha_2\alpha_3 \ldots\ldots \alpha_n = (-1)^n \frac{a_n}{a_0}.$$

- ***Reciprocal equation :*** An equation $f(x) = 0$ is said to be reciprocal if $f(x) = 0$ remains the same when x is replaced by $\frac{1}{x}$.
- **Descarte's Rule of signs :** (i) An equation $f(x) = 0$ can not have more positive roots than the number of changes of signs in $f(x)$.
(ii) An equation $f(x) = 0$ can not have more negative roots than the number fo changes of signs in $f(-x)$.
(iii) If an equation $f(x) = 0$ has atmost p positive roots and has atmost n negative roots, then $f(x) = 0$ will have atleast $N - (p + n)$ complex roots if N is the degree of $f(x) = 0$.

• TEST YOURSELF–2

1. Change the signs of the roots of the equation $x^5 - 4x^3 + 3x^2 + 8x - 9 = 0$.
2. Transform the equation $x^3 - 4x^2 + \frac{1}{4}x - \frac{1}{9} = 0$ into another equation with integral coefficients and having leading coefficient unity.
3. Transform the equation $3x^4 - 5x^3 + x^2 - x + 1 = 0$ into another equation with integral coefficients having leading coefficient unity.
4. *Find the equation whose roots are twice the reciprocals of the roots of*
$$x^4 + 3x^3 - 6x^2 + 2x - 4 = 0.$$
5. Remove the fractional coefficients from the equation
$$x^3 - \frac{5}{2}x^2 - \frac{7}{18}x + \frac{1}{108} = 0.$$
6. Remove the fractional coefficients from the equation
$$x^4 - \frac{5}{6}x^3 - \frac{13}{12}x^2 + \frac{1}{300} = 0.$$
7. Solve the following reciprocal equations :
   (i) $6x^6 - 25x^5 + 31x^4 - 31x^2 + 25x - 6 = 0$
   (ii) $x^5 - 5x^4 + 9x^3 - 9x^2 + 5x - 1 = 0$.
8. Find the equation whose roots are the squares of the roots of the equation $x^4 + x^3 + 2x^2 + x + 1 = 0$.
9. Remove the second term form the following equations :
   (i) $x^3 - 6x^2 + 10x - 3 = 0$ (ii) $x^4 + 8x^3 + x - 5 = 0$
   (iii) $x^5 + 5x^4 + 3x^3 + x^2 + x - 1 = 0$
10. If α, β, γ are the roots of the equation $x^3 + qx + r = 0$, form the equation whose roots are
    (i) $\alpha(\beta+\gamma), \beta(\gamma+\alpha), \gamma(\alpha+\beta)$ (iii) $\left(\alpha - \frac{1}{2}\right), \left(\beta - \frac{1}{2}\right), \left(\gamma - \frac{1}{2}\right)$
11. Show that the equation $2x^7 - x^4 + 4x^3 - 5 = 0$ has atleast four imaginary roots.
12. Apply Descarte's Rule of signs to discuss the nature of the roots of the equation $x^3 + 4x^2 + 9x + 10 = 0$.
13. Locate the positives of the roots of the equation $4x^3 - 13x^2 - 31x - 275 = 0$.
14. Prove that the equation $x^5 - x + 16 = 0$ has two pairs of complex roots.

ANSWERS

1. $y^5 - 4y^3 - 3y^2 + 8y + 9 = 0$.
2. $y^3 - 24y^2 + 9y - 24 = 0$.
3. $y^4 - 5y^3 + 3y^2 - 9y + 27 = 0$.
4. $y^4 - y^3 + 6y' - 6y - 4 = 0$.
5. $y^3 - 15y^2 - 14y + 2 = 0$.
6. $y^4 - 25y^3 - 975y^2 + 2700 = 0$.
7. (i) $\pm 1, 2, \frac{1}{2}, \frac{5 \pm i\sqrt{11}}{6}$ (ii) $1, \frac{1}{2}(1 \pm i\sqrt{3}), \frac{1}{2}(3 \pm \sqrt{5})$.
8. $y^4 + 3y^3 + 4y^2 + 3y + 1 = 0$.
9. (i) $y^3 - 2y + 1 = 0$ (ii) $y^4 - 24y^2 + 65y - 55 = 0$
   (iii) $y^5 - 7y^3 + 12y^2 - 7y = 0$
10. (i) $y^3 - 2qy^2 + q^2y + r^2 = 0$ (ii) $8y^3 + 12y^2 + (6 + 8q)y + (8r + 4q + 1) = 0$.

OBJECTIVE EVALUATION
FILL IN THE BLANKS :

1. If $\alpha_1, \alpha_2, \ldots \alpha_n$ are the roots of the equation $f(x) = 0$ where $f(x) \equiv a_0x^n + a_1x^{n-1} + \ldots + a_n$, then the product of the roots is ....
2. If $1, \alpha_1, \alpha_2, \ldots \alpha_{n-1}$ are the roots of $x^n - 1 = 0$, then the value of $(1-\alpha_1) . (1-\alpha_2) \ldots (1-\alpha_{n-1})$ is ....

3. If α, β are the roots of $ax^2 + bx + c = 0$, then the equation whose roots are $\frac{1}{\alpha}, \frac{1}{\beta}$ is ....

4. If α, β, γ are the roots of the equation $x^3 - 5x - 3 = 0$, then the equation whose roots are $-\alpha, -\beta, -\gamma$ is ....

TRUE OR FALSE :

Write 'T' for True and 'F' for False :

1. Every equation of odd degree has at least two real roots. *(T/F)*
2. Every equation of even degree with last term negative has at least two real roots. *(T/F)*
3. To remove the second term of the equation $a_0x^n + a_1x^{n-1} + \ldots + a_n = 0$, we diminish its all roots by $h = \frac{-a_1}{na_0}$. *(T/F)*
4. If α and β are the roots of $x^2 + bx + c = 0$, then the equation $x^2 + (b-2)x + c - b + 1 = 0$ has the roots $\alpha + 1, \beta + 1$. *(T/F)*

MULTIPLE CHOICE QUESTIONS :

***Choose the most appropriate one* :**

1. If $1, \alpha_1, \alpha_2, \ldots \alpha_{n-1}$ are the roots of $x^n - 1 = 0$, then the value of $(1-\alpha_1)(1-\alpha_2)\ldots(1-\alpha_{n-1})$ is :

   (a) $n-1$ (b) n (c) $n+1$ (d) n^2.

2. If α, β, γ are the roots of the equation $x^3 + qx + r = 0$, then the equation whose roots are $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$:

   (a) $rx^3 + qx^2 + 1 = 0$ (b) $rx^3 - qx^2 + 1 = 0$

   (c) $rx^3 + qx^2 - 1 = 0$ (d) $x^3 - qx + r = 0$.

3. If α, β are the roots of $x^2 - x + 1 = 0$, then the equation whose roots are α^2, β^2 is :

   (a) $x^4 + x^2 + 1 = 0$ (b) $x^2 + x + 1 = 0$

   (c) $x^4 - x^2 + 1 = 0$ (d) $x^2 + x - 1 = 0$.

4. To remove second term of the equation $x^4 + 8x^3 + x - 5 = 0$ we diminish its all roots by :

   (a) 2 (b) 3 (c) -2 (d) -3.

5. If α, β, γ are the roots of the equation $x^3 + px + r = 0$ then $\alpha + \beta + \gamma$ is :

   (a) p (b) $-p$ (c) 0 (d) 1.

ANSWERS

Fill in the Blanks :

1. $(-1)^n \frac{a_n}{a_0}$ **2.** n **3.** $cx^2 + bx + a = 0$ **4.** $x^3 - 5x + 3 = 0$

True or False : **1.** F **2.** T **3.** T **4.** T

Multiple Choice Questions : **1.** (b) **2.** (a) **3.** (b) **4.** (c) **5.** (c)

UNIT 2

SOLUTION OF CUBIC EQUATIONS

STRUCTURE

- Cardan's Method to Find the Roots of A Cubic Equation
- Solved Examples
  - *Student Activity*
  - Summary
  - *Test Yourself*

LEARNING OBJECTIVES

After going through this unit you will be learn :

- How to calculate the roots of a cubic equation using Cardon's Method.

2.1. CARDAN'S METHOD TO FIND THE ROOTS OF A CUBIC EQUATION

Let the general cubic equation be

$$a_0x^3 + 3a_1x^2 + 3a_2x + a_3 = 0. \quad \ldots(1)$$

First reduce this equation (1) into an equation having no second degree term *i.e.*, $3a_1x^2$. The *equation (1) is reduced to the following equation.*

$$z^3 + 3Hz + G = 0 \quad \ldots(2)$$

where $\mathbf{H} = a_0a_2 - a_1^2$, $\mathbf{G} = a_0^2a_3 - 3a_0a_1a_2 + 2a_1^3$ and $\mathbf{z} = a_0x + a_1$.

Let us assume $z = u + v$. ...(3)

Cubing both the sides of (3), we get

$$z^3 = (u+v)^3 = u^3 + v^3 + 3uv\,(u+v) = u^3 + v^3 + 3uv\,(z)$$

$$z^3 = u^3 + v^3 + 3uvz \quad \text{or} \quad z^3 - 3uvz - (u^3 + v^3) = 0. \quad \ldots(4)$$

Comparing (2) and (4), we get

$$uv = -H,\ u^3 + v^3 = -G \quad \text{or} \quad u^3v^3 = (-H)^3,\ u^3 + v^3 = -G$$

hence u^3, v^3 *are the roots of the quadratic equation given by*

$$t^2 + Gt - H^3 = 0. \quad \ldots(5)$$

Solving (5), we get

$$t = \frac{-G \pm \sqrt{G^2 + 4H^3}}{2}$$

$$\therefore \quad u^3 = \frac{-G + \sqrt{G^2 + 4H^3}}{2} \quad \ldots(6)$$

and

$$v^3 = \frac{-G - \sqrt{G^2 + 4H^3}}{2} \quad \ldots(7)$$

From (3), we get

$$z = \left\{-\frac{G}{2} + \frac{1}{2}\sqrt{G^2 + 4H^3}\right\}^{1/3} + \left\{-\frac{G}{2} - \frac{1}{2}\sqrt{G^2 + 4H^3}\right\}^{1/3}. \quad \ldots(8)$$

From (6) and (7) it is obvious that each u and v will have three cube roots and hence from (8), z will have nine values. But the degree of the equation (2) in z is three so it must have three roots *i.e., three values of* z. Since we have that $uv = -H$, therefore the cube roots are taken in pairs so that $uv = -H$. Hence we shall take the pair of cube roots as

$$u, v\ ;\ u\omega, v\omega^2;\ u\omega^2, v\omega$$

where w and w^2 are the imaginary cube roots of unity. Therefore the roots of the equation (2) are

$$u + v,\ u\omega + v\omega^2,\ u\omega^2 + v\omega$$

and hence we can find the roots of the equation (1) by the relation $z = a_0x + a_1$ corresponding to $u + v$, $u\omega + v\omega^2$ and $u\omega^2 + v\omega$.

• SOLVED EXAMPLES

Example 1. *Solve the equation $x^3 - 15x - 126 = 0$ by Cardan's method.*

Solution. Since the given equation is

$$x^3 - 15x - 126 = 0 \quad \ldots(1)$$

and let the solution of (1) be $\quad x = u + v \quad \ldots(2)$

Cubing (2), we get

$$x^3 = (u + v)^3 = u^3 + v^3 + 3uv(u + v)$$

or $\quad x^3 = u^3 + v^3 + 3uv(x) \quad [\because\ x = u + v]$

or $\quad x^3 - 3uvx - (u^3 + v^3) = 0. \quad \ldots(3)$

The equations (1) and (3) are same so comparing the coefficients of like terms, we get

$$3uv = 15 \quad \text{or} \quad uv = \frac{15}{3} \quad \text{or} \quad u^3v^3 = 125$$

and $\quad u^3 + v^3 = 126$

hence u^3, v^3 are the roots of the quadratic

$$t^2 - 126t + 125 = 0$$

$\therefore \quad (t - 125)(t - 1) = 0,\ t = 125,\ t = 1$

$\therefore \quad u^3 = 125,\ v^3 = 1 \ \text{ or } \ u = 5,\ v = 1.$

Thus the roots of (1) are given by

$$u + v,\ u\omega + v\omega^2,\ u\omega^2 + v\omega$$

where $\quad \omega = -\dfrac{1}{2} + \dfrac{i\sqrt{3}}{2}$

$\therefore \quad u + v = 5 + 1 = 6$

$$u\omega + v\omega^2 = 5\omega + \omega^2 = 4\omega + \omega + \omega^2$$

$$= 4\omega - 1 \quad (\because\ 1 + \omega + \omega^2 = 0)$$

$$= 4\left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right) - 1 = -3 + i\,2\sqrt{3}$$

and $\quad u\omega^2 + v\omega = 5\omega^2 + \omega = 4\omega^2 + \omega^2 + \omega = 4\omega^2 - 1$

$$= 4\left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^2 - 1 = -3 - i\,2\sqrt{3}.$$

Hence, roots are $6,\ -3 + 2i\sqrt{3},\ -3 - 2i\sqrt{3}$.

Example 2. *Solve the equation $x^3 - 15x^2 - 33x + 847 = 0$ by Cardan's method.*

Solution. Since the given equation is

$$x^3 - 15x^2 - 33x + 847 = 0. \quad \ldots(1)$$

First we remove the second term *i.e.*, $-15x^2$ by diminishing each of its roots by the constant

$$h = -\frac{a_1}{na_0} = -\frac{-15}{3 \times 1} = 5.$$

Now using synthetic division method

5	1	– 15	– 33	847
		5	– 50	– 415
	1	– 10	– 83	4
			32	
		+	– 25	
	5			
	1	– 5	**–108**	
		5		
	1	**0**		
	1			

Thus the transformed equation is (without second degree term)

$$z^3 - 108z + 432 = 0 \qquad ...(2)$$

where $\quad z = x - 5$

Let the solution of (2) be

$$z = u + v \qquad ...(3)$$

Cubing (3) of both sides, we get

$$z^3 - 3uvz - (u^3 + v^3) = 0. \qquad ...(4)$$

The equation (2) and (4) are same so we have

$$uv = 36,\ u^3 + v^3 = -432 \quad \text{or} \quad u^3v^3 = (36)^3.$$

$\therefore\quad u^3, v^3$ are the roots of the equation $t^2 + 432t + (36)^3 = 0$

$\therefore$
$$t = \frac{-432 \pm \sqrt{(432)^2 - 4(36)^3}}{2} = -\frac{432}{2} = -216$$

$\therefore$
$$u^3 = -216,\ v^3 = -216$$

$\therefore$
$$u = (-216)^{1/3} = -6,\ v = (-216)^{1/3} = -6.$$

$\therefore\quad$ The roots of (2) are

$$u + v,\ u\omega + v\omega^2,\ u\omega^2 + v\omega \quad i.e., \quad z_1 = u + v = -6 - 6 = -12$$

$$z_2 = -6\omega - 6\omega^2 = -6(\omega + \omega^2) = -6(-1) = 6$$

$$z_3 = -6\omega^2 - 6\omega = -6(\omega^2 + \omega) = -6(-1) = 6.$$

Therefore the roots of given equation (1) are

$$x_1 = z_1 + 5 = -12 + 5 = -7$$

$$x_2 = z_2 + 5 = 6 + 5 = 11$$

$$x_3 = z_3 + 5 = 6 + 5 = 11$$

Hence the roots of the given cubic equation, are $-7, 11, 11$.

Example 3. *Show that the roots of the equation $x^3 - 3x + 1 = 0$ are*

$$2\cos\frac{2\pi}{9},\ 2\cos\frac{8\pi}{9},\ \cos\frac{14\pi}{9}.$$

Solution. Since the given equation is

$$x^3 - 3x + 1 = 0 \qquad ...(1)$$

Let $\quad x = u + v. \qquad ...(2)$

Cubing (2) of both sides, we get

$$x^3 - 3uvx - (u^3 + v^3) = 0. \qquad ...(3)$$

Since (1) and (3) are same so we have

$$uv = 1,\ u^3 + v^3 = -1$$

or
$$u^3v^3 = 1,\ u^3 + v^3 = -1.$$

$\therefore\quad u^3, v^3$ are the roots of the following equation $t^2 + t + 1 = 0$.

$\therefore$
$$t = \frac{-1 \pm \sqrt{1-4}}{2}$$

$$t = \frac{-1 \pm i\sqrt{3}}{2}.$$

$\therefore$
$$u^3 = -\frac{1}{2} + \frac{i}{2}\sqrt{3},\ v^3 = -\frac{1}{2} - \frac{i}{2}\sqrt{3}.$$

From (2), we get

$$x = \left(-\frac{1}{2} + \frac{i}{2}\sqrt{3}\right)^{1/3} + \left(-\frac{1}{2} - \frac{i}{2}\sqrt{3}\right)^{1/3} \qquad ...(4)$$

Change the complex number on R.H.S. of (4) into polar form by putting

$$-\frac{1}{2} = r\cos\theta,\ \frac{\sqrt{3}}{2} = r\sin\theta$$

$\therefore$
$$r^2 = 1 \Rightarrow r = 1$$

and
$$\tan\theta = -\sqrt{3} \Rightarrow \theta = \frac{2\pi}{3}$$

$\therefore$
$$x = (r\cos\theta + ir\sin\theta)^{1/3} + (r\cos\theta - ir\sin\theta)^{1/3}$$

$$= r^{1/3}\left[\cos\frac{2n\pi + \theta}{3} + i\sin\frac{2n\pi + \theta}{3} + \cos\frac{2n\pi + \theta}{3} - i\sin\frac{2n\pi + \theta}{3}\right]$$

$$= 2r^{1/3}\cos\frac{2n\pi+\theta}{3},\ n = 0, 1, 2$$

$$x_1 = 2(1)^{1/3}\cos\frac{\theta}{3} = 2\cos\frac{2\pi}{9}$$

$$x_2 = 2\cos\frac{2\pi+\theta}{3} = 2\cos\left(\frac{2\pi}{3}+\frac{2\pi}{9}\right) = 2\cos\frac{8\pi}{9}$$

$$x_3 = 2\cos\frac{4\pi+\theta}{3} = 2\cos\left(\frac{4\pi}{3}+\frac{2\pi}{9}\right) = 2\cos\frac{14\pi}{9}.$$

• STUDENT ACTIVITY

1. Explain Cardon's method.

2. Find the roots of the equation $x^3 - 6x^2 + 11x - 6 = 0$ using Cardon's method.

• SUMMARY

- **Cardon's Method**

Step I : First reduce the equation $a_0x^3 + 3a_1x^2 + 3a_2x + a_0 = 0$ to the form $z^3 + 3Hz + G = 0$ where $H = a_0a_2 - a_1^2$, $C_2 = a_0^2a_3 - 3a_0a_1a_2 + 2a_1^3$ and $z = a_0x + a_1$.

Step II : Put $z = u + v$ in $z^3 + 3Hz + G = 0$ and find a quadratic whose roots are u^3 and v^3.

Step III : Find u^3 and v^3 and then find u and v.

Step IV : The roots of $z^3 + 3Hz + G = 0$ are given by $u + v$, $u\omega + v\omega^2$, $u\omega^2 + v\omega$.

Step V : The roots of the given equation are given by the equation $z = a_0x + a_1$.

• TEST YOURSELF

Solve the following cubic equations by Cardan's method :

1. $x^3 + 6x^2 + 9x + 4 = 0$.
2. $x^3 + 6x^2 - 12x + 32 = 0$.

3. $x^3 - 21x - 344 = 0$.
4. $x^3 - 12x^2 - 6x - 10 = 0$.
5. $27x^3 + 54x^2 + 198x - 73 = 0$.
6. $x^3 - 18x - 35 = 0$.
7. $x^3 - 6x - 9 = 0$.
8. $x^3 - 15x^2 - 357x + 5491 = 0$.
9. $x^3 + 3x^2 - 27x + 104 = 0$.
10. $x^3 - 6x - 9 = 0$.
11. $2x^3 + 3x^2 + 3x + 1 = 0$.
12. $8a^3x^3 - 6ax + 2\sin 3A = 0$.
13. $64x^3 - 144x^2 + 108x - 27 = 0$.

ANSWERS

1. $-4, -1, -1$. **2.** $-8, (1 \pm i\sqrt{3})$. **3.** $8, (-4 \pm i\,3\sqrt{3})$.

4. $4 + 3(2)^{1/3} + 3(4)^{1/3}, 4 + 3\omega(2)^{1/3} + 3\omega^2(4)^{1/3}, 4 + 3\omega^2(2)^{1/3} + 3\omega(4)^{1/3}$ where $\omega = \left(-\frac{1}{2} \pm \frac{i}{2}\sqrt{3}\right)$

5. $\frac{1}{3}, \left(-\frac{7}{6} \pm \frac{3\sqrt{3}}{2} i\right)$. **6.** $5, \left(-\frac{5}{2} \pm \frac{i\sqrt{3}}{2}\right)$. **7.** $\frac{1}{3}, -\frac{1}{6}(3 \pm \sqrt{3}i)$. **8.** $-19, 17, 17$

9. $-8, \frac{1}{2}(5 \pm i\,3\sqrt{3})$. **10.** $3, \left(-\frac{3}{2} \pm \frac{i\sqrt{3}}{2}\right)$. **11.** $-\frac{1}{2}, \left(-\frac{1}{2} \pm \frac{i\sqrt{3}}{2}\right)$

12. $\frac{1}{a}\sin A, \frac{1}{a}\sin\left(\frac{\pi}{3} - A\right), -\frac{1}{a}\sin\left(\frac{\pi}{3} + A\right)$. **13.** $\frac{3}{4}, \frac{3}{4}, \frac{3}{4}$.

OBJECTIVE EVALUATION

Fill in the blanks :

1. To solve the cubic equation $a_0x^3 + 3a_1x^2 + 3a_2x + a_3 = 0$ by Cardan's method, we first remove the second term by diminishing its roots by $h = \ldots\ldots$
2. The cubic equation $a_0x^3 + 3a_1x^2 + 3a_2x + a_3 = 0$ reduces to $z^3 + 3Hz + G = 0$ by $Z = a_0x + a_1$, then G equals ...
3. If $z = u + v$ is a solution of the cubic equation $z^3 + 3Hz + G$, then $u^3 + v^3 = \ldots\ldots$ and $u^3v^3 = \ldots\ldots$.

TRUE OR FALSE :

Write 'T' for True and 'F' for False :

1. The equation $Z^3 + 3Hz + G = 0$ has two equal roots if $G^2 + 4H^3 = 0$. ***(T/F)***
2. A cubic equation with real coefficient has at least one real root. ***(T/F)***
3. The equation $x^3 + 3Hx + G = 0$ all its roots real if $G^2 + 4H^3 > 0$. ***(T/F)***

MULTIPLE CHOICE QUESTIONS :

Choose of the most appropriate one :

1. If $z = u + v$ is a solution of $z^3 + 3Hz + G = 0$, then $u^3 + v^3$ equals :
   (a) G (b) $-G$ (c) H (d) $-H$.
2. If $z = u + v$, $z = u\omega + v\omega^2$ are two roots of $z^3 + 3Hz + G = 0$ then its third roots is
   (a) $u\omega^2 + v\omega$ (b) $u\omega^2 - v\omega$ (c) $u\omega - v\omega^2$ (d) $u - v$.
3. If $z = u + v$ is a solution of $z^3 - 12z - 65 = 0$, then u and v are the roots of the quadratic :
   (a) $t^2 + 65t - 64 = 0$ (b) $t^2 - 65t + 64 = 0$
   (c) $t^2 - 64t + 65 = 0$ (d) $t^2 + 64t + 65 = 0$.

ANSWERS

Fill in the Blanks : **1.** $\frac{-a_1}{na_0}$ **2.** $a_0^2a_3 - 3a_0a_1a_2 + 2a_1^3$ **3.** $-G, -H^3$

True or False : **1.** T **2.** T **3.** F

Multiple Choice Questions : **1.** (b) **2.** (a) **3.** (b)

UNIT

3

SOLUTION OF BIQUADRATIC EQUATIONS

STRUCTURE

- Descarte's Method for Finding the Roots of a Biquadratic Equation
- Ferrari's Method for Finding the Roots of a Biquadratic Equation
- Solved Examples
  - Student Activitiy
  - Summary
  - Test Yourself

LEARNING OBJECTIVES

After going through this unit you will learn :

- How to calculate the roots of the biquadratic equations using Discarte's and Ferrari's Method

• 3.1. DESCARTE'S METHOD FOR FINDING THE ROOTS OF A BIQUADRATIC EQUATION

Let the equation of a biquadratic be

$$a_0x^4 + 4a_1x^3 + 6a_2x^2 + 4a_3x + a_4 = 0. \quad \ldots(1)$$

First we remove the second term *i.e.*, $4a_1x^3$ from (1) be diminishing each of root of (1) by a constant $h = -\frac{a_1}{na_0}$, we get

$$z^4 + 6Hz^2 + 4GZ + a_0^2 I - 3H^2 = 0 \quad \ldots(2)$$

where $\mathrm{H} = a_0a_2 - a_1^2$, $\mathrm{G} = a_0^2 a_3 - 3a_0a_1a_2 + 2a_1^3$, $\mathrm{I} = a_0a_4 - 4a_1a_3 + 3a_2^2$ and $z = a_0x + a_1$.

Let us assume

$$z^4 + 6Hz^2 + 4Gz + a_0^2I - 3H^2 \equiv (z^2 + kz + l)(z^2 - kz + m).$$

Now equating the coefficients of like powers of z, we get

$$l + m - k^2 = 6H,\ k(m - l) = 4G,\ lm = a_0^2 I - 3H^2.$$

Solving first two of these equations for l and m, we get

$$\left.\begin{aligned} 2l &= k^2 + 6H - \frac{4G}{k} \\ \text{and}\quad 2m &= k^2 + 6H + \frac{4G}{k} \end{aligned}\right\} \quad \ldots(\mathrm{A})$$

Substitute these values of l, m in the following equation $lm = a_0^2 I - 3H^2$, we get

$$\left(k^2 + 6H - \frac{4G}{k}\right)\left(k^2 + 6H + \frac{4G}{k}\right) = 4\,(a_0^2 I - 3H^2)$$

or

$$(k^3 + 6Hk - 4G)\,(k^3 + 6Hk + 4G) = 4\,(a_0^2 I - 3H^2)\,k^2$$

$$k^6 + 12Hk^4 + 4k^2(12H^2 - a_0^2 I) - 16G^2 = 0. \quad \ldots(3)$$

This is a cubic equation in k^2 so it will always have one positive real value of k^2. when k^2 is known, then the values of l and m are obtained from the equation (A). Thus the biquadratic (2) is obtained as the product of quadratics $(z^2 + kz + l)$ and $(z^2 - kz + m)$.

Now solving these two quadratics

$$z^2 + lz + l = 0 \quad \text{and} \quad z^2 - kz + m = 0$$

and finally from the transformation $z = a_0 x + a_1$ we obtain the solution of the given biquadratic (1) corresponding to the roots of the equations

$$z^2 + kz + l = 0 \text{ and } z^2 - kz + m = 0.$$

• 3.2. FERRARI'S METHOD FOR FINDING THE ROOTS OF A BIQUADRATIC EQUATION

Let the equation of a biquadratic be

$$x^4 + 2a_1x^3 + a_2x^2 + 2a_3x + a_4 = 0. \qquad \ldots(1)$$

Now adding $(ax+b)^2$ to each side of (1), we get

$$x^4 + 2a_1x^3 + a_2x^2 + 2a_3x + a_4 + (ax+b)^2 = (ax+b)^2$$

or
$$x^4 + 2a_1x^3 + (a_2 + a^2)x^2 + 2(a_3 + ab)x + (a_4 + b^2) = (ax+b)^2. \qquad \ldots(2)$$

In order to determine a and b make the left side of above equation *a perfect square.* Suppose the perfect square of left side of (2) is $(x^2 + a_1x + k)^2$, then

$$x^4 + 2a_1x^3 + (a_2 + a^2)x^2 + 2(a_3 + ab)x + (a_4 + b^2) \equiv (x^2 + a_1x + k)^2. \qquad \ldots(3)$$

Comparing the coefficients of like powers of x of (3), we get

$$a_1^2 + 2k = a_2 + a^2,\ a_1k = a_3 + ab,\ k^2 = a_4 + b^2.$$

Eliminating a and b between these equations, we get

$$(2k + a_1^2 - a_2)(k^2 - a_4) = (a_1k - a_3)^2$$

or
$$2k^3 - a_2k^2 + 2(a_1a_3 - a_4)k - a_1^2a_4 + a_2a_4 - a_3^2 = 0. \qquad \ldots(4)$$

This is a cubic equation in k so it must have one real values of k. This real value is obtained by trial method. Once we obtained the value of k we thus obtain a and b and then put these values in (3) and using (2), we get

$$(x^2 + a_1x + k)^2 = (ax+b)^2$$

or
$$x^2 + a_1x + k = \pm(ax+b).$$

Thus the given biquadratic is obtained as the product of two quadratics

$$\left.\begin{aligned} &x^2 + (a_1 - a)\,x + (k - b) = 0 \\ \text{and} \quad &x^2 + (a_1 + a)\,x + (k + b) = 0 \end{aligned}\right\}. \qquad \ldots(5)$$

On solving these quadratics we finally obtained the solution of the given quadratic.

• SOLVED EXAMPLES

Example 1. *Solve the equation* $x^4 - 3x^2 - 42x - 40 = 0$ *by Descarte's method.*

Solution. Since the given equation is

$$x^4 - 3x^2 - 42x - 40 = 0. \qquad \ldots(1)$$

Let us assume $\quad x^4 - 3x^2 - 42x - 40 \equiv (x^2 + kx + l)(x^2 - kx + m) = 0. \qquad \ldots(2)$

Equating the coefficients of like powers of x, we get

$$l + m - k^2 = -3 \quad \text{or} \quad l + m = -3 + k^2 \qquad \ldots(3)$$

and
$$k(m - l) = -42 \quad \text{or} \quad m - l = -\frac{42}{k} \qquad \ldots(4)$$

and
$$lm = -40. \qquad \ldots(5)$$

Solving (3) and (4), we get

$$2m = -3 + k^2 - \frac{42}{k}$$

and
$$2l = -3 + k^2 + \frac{42}{k}.$$

Substitute the values of l and m in (5) we get

$$\left(-3 + k^2 - \frac{42}{k}\right)\left(-3 + k^2 + \frac{42}{k}\right) = 4(-40)$$

or
$$(k^3 - 3k - 42)(k^3 - 3 + 42) = -160k^2$$

or
$$(k^3 - 3k)^2 - (42)^2 = -160k^2 \quad \text{or} \quad k^6 - 6k^4 + 169k^2 - 1764 = 0.$$

Let $k^2 = t$, then we get

$$t^3 - 6t^2 + 169t - 1764 = 0.$$

By trial method it is obvious that $t = 9$ satisfies above equation.

Hence $\qquad k^2 = 9$ or $k = \pm 3$.

Taking $k = 3$, then (3) and (4), we get

$$l + m = 6 \quad \text{and} \quad m - l = -14.$$

Solving these equation for l and m, we get $l = 10$, $m = -4$ therefore from (2) we obtain the given biquadratic as the product of two quadratics

$$(x^2 + 3x + 10)(x^2 - 3x - 4) = 0.$$

Solving these quadratics respectively we get the required solutions

$$x = 4, -1, \frac{-2 \pm i\sqrt{31}}{2}.$$

Example 2. *Solve the equation* $x^4 + 8x^3 + 9x^2 - 8x - 10 = 0$ *by Descarte's method.*

Solution. Since the equation is

$$x^4 + 8x^3 + 9x^2 - 8x - 10 = 0. \qquad \ldots(1)$$

First we remove the second term *i.e.*, $8x^3$ by diminishing each of its roots by a constant

$$h = -\frac{a_1}{na_0} = -\frac{8}{4 \cdot 1} = -2.$$

Using synthetic division method :

$$\begin{array}{r|rrrrr}
-2 & 1 & 8 & 9 & -8 & -10 \\
 & & -2 & -12 & 6 & 4 \\
\hline
 & 1 & 6 & -3 & -2 & \mathbf{-6} \\
 & & -2 & -8 & 22 & \\
\hline
 & 1 & 4 & -11 & \mathbf{20} & \\
 & & -2 & -4 & & \\
\hline
 & 1 & 2 & \mathbf{-15} & & \\
 & & -2 & & & \\
\hline
 & 1 & \mathbf{0} & & & \\
 & \mathbf{1} & & & &
\end{array}$$

Thus the transformed equation is

$$z^4 - 15z^2 + 20z - 6 = 0 \qquad \ldots(2)$$

where $z = x + 2$

Let us assume $\quad z^4 - 15z^2 + 20z - 6 \equiv (z^2 + kz + l)(z^2 - kz + m) = 0. \qquad \ldots(3)$

Comparing the coefficients of like powers of z, we get

$$l + m - k^2 = -15 \quad \text{or} \quad l + m = -15 + k^2 \qquad \ldots(4)$$

$$k(m - l) = 20 \quad \text{or} \quad m - l = \frac{20}{k} \qquad \ldots(5)$$

and $\qquad lm = -6.$

Solving (4) and (5), we get

$$2l = k^2 - 15 - \frac{20}{k}$$

$$2m = k^2 - 15 + \frac{20}{k}.$$

Substitute these values of l and m in (6), we get

$$\left(k^2 - 15 - \frac{20}{k}\right)\left(k^2 - 15 + \frac{20}{k}\right) = -24$$

or $\qquad (k^3 - 15k - 20)(k^3 - 15k + 20) = -24k^2$

or $\qquad (k^3 - 15k)^2 - 400 = -24k^2$ or $k^6 - 30k^4 + 249k^2 - 400 = 0$

let $k^2 = t$, then $\qquad t^3 - 30t^2 + 249t - 400 = 0. \qquad \ldots(7)$

From (7) it is obvious that $t = 16$ satisfies the equation (7)

$\therefore \qquad k^2 = 16$ or $k = \pm 4$.

Taking $k = 4$, in from (4) and (5), we get

$$l + m = 1$$
$$m - l = 5.$$

On solving these equations, we get

$$l = -2, m = 3.$$

Substitute the values of l, m and k in (3), we get

$$z^4 - 15z^2 + 20z - 6 \equiv (z^2 + 4z - 2)(z^2 - 4z + 3) = 0$$

$\therefore$ $(z^2 + 4z - 2)(z^2 - 4z + 3) = 0$ and $z = 1, 3, -2 \pm \sqrt{6}$

But $z = x + 2$

$\therefore$ $x = z - 2.$

Hence the solution of the given biquadratic are

$$x = -1, 1, -4 \pm \sqrt{6}.$$

Example 3. *Solve the equation $x^4 - 2x^3 - 5x^2 + 10x - 3 = 0$ by Ferrari's method.*

Solution. Since the equation is

$$x^4 - 2x^3 - 5x^2 + 10x - 3 = 0 \qquad \ldots(1)$$

Adding $(ax + b)^2$ of both sides we get

$$x^4 - 2x^3 - 5x^2 + 10x - 3 + (ax + b)^2 = (ax + b)^2$$

or

$$x^4 - 2x^3 + (a^2 - 5)x^2 + 2(ab + 5)x + b^2 - 3 = (ax + b)^2. \qquad \ldots(2)$$

Let us assume that L.H.S. of (2) must be a perfect square therefore suppose $(x^2 - a_1 x + k)^2$ is a perfect square of L.H.S. of (2)

$$\therefore \quad x^4 - 2x^3 + (a^2 - 5)x^2 + 2(ab + 5)x + b^2 - 3 \equiv (x^2 - x + k)^2 \qquad \ldots(3)$$

$(\because \ a_1 = -1)$

Equating the coefficients of like powers of x, we get

$$a^2 = 2k + 6, ab = -k - 5, b^2 = k^2 + 3.$$

Now eliminating a and b between these three equations, we get

$$(2k + 6)(k^2 + 3) = (k + 5)^2 \quad \text{or} \quad 2k^3 + 5k^2 - 4k - 7 = 0.$$

It is a cubic in k so it must have one real root, then by trial method, we get

$$k = -1$$

and hence $a^2 = 4, b^2 = 4, ab = -4$ or $a = 2, b = -2.$

Substitute the values of k, a, and b in (3) and (4), we get

$$(x^2 - x - 1)^2 = (2x - 2)^2 \quad \text{or} \quad x^2 - x - 1 = \pm (2x - 2)$$

or

$$x^2 - 3x + 1 = 0 \text{ and } x^2 + x - 3 = 0.$$

Solving these quadratics, we get $x = \dfrac{3 \pm \sqrt{5}}{2}, \dfrac{-1 \pm \sqrt{13}}{2}.$

These are the solutions of the given biquadratic equation.

Example 4. *Solve the equation $x^4 + 2x^3 - 7x^2 - 8x + 12 = 0$, by Ferrari's method.*

Solution. Since the given biquadratic is

$$x^4 + 2x^3 - 7x^2 - 8x + 12 = 0. \qquad \ldots(1)$$

Adding $(ax + b)^2$ of both sides of (1), we get

$$x^4 + 2x^3 - 7x^2 - 8x + 12 + (ax + b)^2 = (ax + b)^2$$

or

$$x^4 + 2x^3 + (a^2 - 7)x^2 + (2ab - 8)x + b^2 + 12 = (ax + b)^2. \qquad \ldots(2)$$

In order to determine a and b make the L.H.S. of (2) a perfect square. Let the perfect square be $(x^2 + a_1 x + k)^2$.

$$\therefore \quad x^4 + 2x^3 + (a^2 - 7)x^2 + (2ab - 8)x + b^2 + 12 \equiv (x^2 + a_1 x + k)^2$$

or

$$x^4 + 2x^3 + (a^2 - 7)x^2 + (2ab - 8)x + b^2 + 12 \equiv (x^2 + x + k)^2. \qquad \ldots(3)$$

$(\because \ a_1 = 1 \text{ from } (1))$

Equating the coefficients of like powers of x, we get

$$a^2 - 7 = 2k + 1, 2ab - 8 = 2k, b^2 + 12 = k^2$$

Eliminating a and b between above three equations, we get

$$(k + 4)^2 = (2k + 8)(k^2 - 12)$$

or

$$k^2 + 16 + 8k = 2k^3 + 8k^2 - 24k - 96 \quad \text{or} \quad 2k^3 + 7k^2 - 32k - 112 = 0.$$

This is a cubic in k so it must have one real root. By trial method, $k = -7/2$ satisfies above cubic.

Then $a^2 = 1, b^2 = \dfrac{1}{4}$

$\therefore$ $a = 1, b = \dfrac{1}{2}.$

Now substitute the values of k, a and b in (3) and using (2), we get

$$\left(x^2+x-\frac{7}{2}\right)^2=\left(x+\frac{1}{2}\right)^2 \quad \text{or} \quad x^2+x-\frac{7}{2}=\pm\left(x+\frac{1}{2}\right)$$

or $$x^2-4=0 \quad \text{and} \quad x^2+2x-3=0.$$

Solving these quadratics, we get

$$x=-2, 2, \text{ and } x=1, -3.$$

Hence the solution of given biquadratic are $x=-3, -2, 1, 2$.

• STUDENT ACTIVITY

1. Find roots of $x^4+8x^3+9x^2-9x-10=0$ by Descarte's method.

2. Find the roots of $x^4-8x^3-12x^2+60x+63=0$ by Ferrari's method.

• SUMMARY

- **Descarte's method** : The given biquadratic equation $f(x)=0$ is expressible as

$$f(x)=(x^2+kx+l)(x^2-kx+m).$$

- **Ferrari's Method :**

Step I : The given equation $f(x)=x^4+2a_1x^3+a_0x^2+2a_3x+a_4=0$ is expressible as

$$f(x)+(ax+b)^2=(ax+b)^2.$$

Now make L.H.S. a perfect square.

Step II : Let $f(x)+(ax+b)^2=(x^2+a_1x+k)^2$, where $2a_1$ is the coefficient of x^3 in $f(x)=0$.

On comparing the like terms both sides, we get

$$a_1^2+2k=a_2+a^2, a_1k=a_3+ab, \ k^2+a_4+b^2$$

liminating a and b between these three equations, we get

$$2k^3-a_2k^2+2(a_1a_3-a_4)k-a_1^2a_4+a_2a_4-a_3^2=0$$

Find the value of k by trial method and thus we obtain a and b.

Step III : Putting the values of k, a and b in the following equation

$$(x^2+a_1x+k)^2=(ax+b)^2$$

$$\Rightarrow \quad x^2+(a_1-a)x+k-b=0 \quad \text{and} \quad x^2+(a_1 \ a)x+k+b=0.$$

On solving these quadratics to find the roots of the given biquadratic.

• TEST YOURSELF

Solve the following biquadratic equation by Descarti's method

1. $x^4 - 6x^3 - 9x^2 + 66x - 22 = 0.$ 2. $x^4 - 8x^2 - 24x + 7 = 0.$
3. $x^4 - 10x^2 - 20x - 16 = 0.$ 4. $x^4 + 2x^3 - 7x^2 - 8x + 12 = 0.$
5. $x^4 - 8x^3 - 12x^2 + 60x + 63 = 0.$

Solve the following biquadratic equation by Ferrari's method.

6. $x^4 - 8x^3 - 12x^2 + 60x + 63 = 0.$ 7. $x^4 + 12x - 5 = 0.$
8. $x^4 - 2x^3 - 5x^2 + 10x - 3 = 0.$ 9. $x^4 - 3x^2 - 42x - 40 = 0.$
10. $x^4 + 9x^3 + 12x^2 - 80x - 192 = 0.$

ANSWERS

1. $\pm\sqrt{11}, 3 \pm \sqrt{7}.$ 2. $-2 \pm i\sqrt{3}, 2 \pm \sqrt{3}.$ 3. $4, -2, -1 \pm i.$
4. $\pm 2, -3, 1.$ 5. $-1, 3, 3 \pm \sqrt{30}.$ 6. $-1, 3, 3 \pm \sqrt{30}.$
7. $-1 \pm \sqrt{2}, -1 \pm 2i.$ 8. $\frac{3 \pm \sqrt{5}}{2}, \frac{-1 \pm \sqrt{13}}{2}.$ 9. $4, -1, -\frac{1}{2}(3 \pm i\sqrt{31}).$
10. $-4, -4, -4, 3$

OBJECTIVE EVALUATION

FILL IN THE BLANKS :

1. To solve the biquadratic equation $a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4 = 0$ by Descarte's method, we first remove its second term by diminishing its roots by $h = \ldots$
2. The biquadratic equation $a_0x^4 + 4a_1x^3 + 6a_2x^2 + 4a_3x + a_4 = 0$ reduces to the cubic $t^3 + 3Ht^2 + \left(3H^2 - \frac{a_0^2 I}{4}\right)t - \frac{G^2}{4} = 0$. Then this cubic is known as … .
3. If the two roots of $x^4 + 12x - 5 = 0$ are $-1 + \sqrt{2}$ and $1 - 2i$ then its other roots are … .
4. If $x^4 - 2x^2 + 8x - 3 \equiv (x^2 + kx + l)(x^2 - kx + m)$, then $l + m - k^2 = \ldots$ and $k(m - l) = \ldots$ and $lm = \ldots$.

TRUE OR FALSE :

Write 'T' for True and 'F' for False :

1. If the two roots of the equation $x^4 - 3x^2 - 6x - 2 = 0$ are $-1 + i$ and $1 + \sqrt{2}$, then its other roots are $1 + i$ and $1 - \sqrt{2}$. *(T/F)*
2. The equation $a_0x^4 + 4a_1x^3 + 6a_2x^2 + 4a_3x + a_4 = 0$ reduces to
$$z^4 + 6Hz^2 + 4Gz + (a_0^2 I - 3H^2) = 0 \text{ by } z = a_0x + a_1.$$ *(T/F)*
3. Solve the equation $x^4 - 6x^3 - 9x^2 + 66x - 22 = 0$ by Descarte's method we first remove the second term by diminishing its root by $h = \frac{3}{2}$. *(T/F)*

MULTIPLE CHOICE QUESTIONS :

Choose the most appropriate one :

1. If $x^4 - 2x^2 + 8x - 3 \equiv (x^2 + 2x + l)(x^2 - 2x + m)$, then the values of l and m are :
(a) $-1, -3$ (b) $-1, 3$ (c) $1, 3$ (d) $1, -3.$
2. If the two roots of $x^4 - 3x^2 - 6x - 2 = 0$ are $-1 + i$ and $1 + \sqrt{2}$ then its other two roots are :
(a) $-1 - i, -1 + \sqrt{2}$ (b) $-1 - i, -1 - \sqrt{2}$
(c) $-1 - i, 1 - \sqrt{2}$ (d) $1 + i, 1 - \sqrt{2}.$
3. The sum of all the four roots of $a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4 = 0$ is :
(a) a_1/a_0 (b) $-a_1/a_0$ (c) a_2/a_0 (d) $-a_1/a_2.$

ANSWERS

Fill in the Blanks : 1. $h = -\frac{a_1}{4a_0}$ 2. Euler cubic 3. $-1 - \sqrt{2}, 1 + 2i$ 4. $-2, 8, -3$

True of False : 1. F 2. T 3. T

Multiple Choice Questions : 1. (b) 2. (c) 3. (b).

UNIT 4

CIRCULAR AND HYPERBOLIC FUNCTIONS OF A COMPLEX VARIABLE

STRUCTURE

LEARNING OBJECTIVES

After going through this unit you will learn :

- About the circular and hypeerbolic functions of complex variable
- How to develop the relation between circular and hyperbolic function

4.1. EXPONENTIAL SERIES OF COMPLEX NUMBERS

We know that the exponential series for all real values of x is given by

$$e^x = 1 + x + \frac{x^2}{2\,!} + \frac{x^3}{3\,!} + \dots \text{ ad. inf.} \quad \dots(1)$$

But where x is complex, the expression e^x has no meaning at present. The series (1) is absolutely convergent for all finite values of x.

Now consider the series

$$E(z) = 1 + z + \frac{z^2}{2\,!} + \frac{z^3}{3\,!} + \dots \text{ ad. inf.} \quad \dots(2)$$

where $z = x + iy = r(\cos\theta + i\sin\theta)$ and therefore $|z| = r > 0$.

Let the series of the moduli be

$$1 + |z| + \frac{|z^2|}{2\,!} + \frac{|z^3|}{3\,!} + \dots + \frac{|z^n|}{n\,!} + \dots = 1 + r + \frac{r^2}{2\,!} + \frac{r^3}{3\,!} + \dots + \frac{r^n}{n\,!} + \dots$$

This is a series of positive numbers and convergent and hence the series (2) is absolutely convergent for all finite values of z.

In particular, if $z = x + i0$ which corresponding to the real number x, $E(z)$ assumes the value

$$1 + \frac{x}{1\,!} + \frac{x^2}{2\,!} + \dots + \frac{x^n}{n\,!} + \dots$$

which corresponds to exp. (x) or e^x, where e stands for,

$$\left(1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\ldots+\frac{1}{n!}+\ldots\right)$$

and e^x means $\left(1+\frac{1}{1!}+\frac{1}{2!}+\ldots+\frac{1}{n!}+\ldots\right)^x$

for all real values of x.

Hence the series (2) is usually written as exp. (z) or e^z in close analogy to the exponential series for real number, and also because the fact that

$$E(z)=1+z+\frac{z^2}{2!}+\ldots+\frac{z^n}{n!}+\ldots$$

where $z \equiv x+iy$ when $z \equiv x+i0$, corresponds to exp. (x) or e^x.

It should be clearly understood that the series

$$1+z+\frac{z^2}{2!}+\ldots+\frac{z^n}{n!}+\ldots$$

is written exp (z) *or* e^z *by definition only, and that it does not mean, unless it is so proved,* that the exp (z) or e^z stands for

$$\left(1+\frac{1}{1!}+\frac{1}{2!}+\ldots+\frac{1}{n!}+\ldots\right)^z$$

If z is complex.

Hence by definition, if $z=x+iy$,

$$\exp(z)=1+z+\frac{z^2}{2!}+\ldots+\frac{z^n}{n!}+\ldots$$

• 4.2. THEOREMS OF EXPONENTIAL FUNCTION OF COMPLEX NUMBERS

Theorem 1. *If z_1 and z_2 are any two complex numbers, then*

$$e^{z_1}\cdot e^{z_2}=e^{z_1+z_2} \quad or \quad \exp(z_1)\times\exp(z_2)=\exp(z_1+z_2).$$

Proof. By definition we have

$$\exp(z_1)\times\exp(z_2)=\left[1+z_1+\frac{z_1^2}{2!}+\ldots+\frac{z_1^n}{n!}+\ldots\right]\times\left[1+z_2+\frac{z_2^2}{2!}+\ldots+\frac{z_2^n}{n!}+\ldots\right]$$

$$=\left[1+(z_1+z_2)+\frac{1}{2!}+(z_1^2+2z_1z_2+z_2^2)+\ldots\right.$$
$$\left.+\frac{1}{n!}\left(z_1^n+nz_1^{n-1}z_2+\frac{n(n-1)}{2!}z_1^{n-2}z_2^2+\ldots+z_2^n\right)+\ldots\right]$$

$$=\left[1+(z_1+z_2)+\frac{(z_1+z_2)^2}{2!}+\frac{(z_1+z_2)^3}{3!}+\ldots+\frac{(z_1+z_2)^n}{n!}+\ldots\right]=\exp(z_1+z_2).$$

The series on the R.H.S. is absolutely convergent if exp (z_1) exp (z_2) are absolutely convergent.

Theorem 2. *If z is a complex number, then $(e^z)^m=e^{mz}$.*

Proof. If m is positive integer, we have by repeated application of theorem (1),

$$\exp(z_1)\exp(z_2)\ldots\exp(z_m)=\exp(z_1+z_2+\ldots+z_m).$$

If $z_1=z_2=\ldots=z_m=z$, we have

$$(\exp z)^m=\exp(mz).$$

Theorem 3. $E(z)\neq 0$, *for any value of z.*

Proof. By the addition theorem, we have

$$E(z)\,.\,E(-z)=E\{z+(-z)\}=E(0)=1$$

since $E(z)$ is well defined for all values of z, therefore, it follows that $E(z)\neq 0$, for any value of z.

REMARK

$\{E(z)\}^{-1}=E(-z)$.

• 4.3. CIRCULAR FUNCTION OF COMPLEX QUANTITIES

For real values of x,

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \ldots$$

and $$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + \frac{(-1)^n x^{2n}}{(2n)!} + \ldots .$$

These definitions are extended for the complex quantity $z = x + iy$, where x and y are real.

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots + \frac{(-1)^n z^{2n+1}}{(2n+1)!} + \ldots$$

and $$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \ldots + \frac{(-1)^n z^{2n}}{(2n)!} + \ldots$$

Similarly, $\tan z = \frac{\sin z}{\cos z}$, $\cot z = \frac{\cos z}{\sin z}$, $\sec z = \frac{1}{\cos z}$ and $\operatorname{cosec} z = \frac{1}{\sin z}$.

From these definitions we can deduce the fundamental properties of two functions.

(a) $\cos z + i \sin z = e^{iz}$; $\cos z - i \sin z = e^{-iz}$

$\therefore$ $\cos^2 z + \sin^2 z = e^{iz} e^{-iz} = e^{-0} = 1$.

(b) $\cos z = \frac{1}{2}[e^{iz} + e^{-iz}]$; $\sin z = \frac{1}{2i}[e^{iz} - e^{-iz}]$.

(c) To prove that

$$\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2.$$

$$\text{R.H.S.} = \frac{e^{iz_1} - e^{-iz_1}}{2i} \times \frac{e^{iz_2} + e^{iz_2}}{2} + \frac{e^{iz_1} + e^{-iz_1}}{2} \cdot \frac{e^{iz_2} - e^{-iz_2}}{2i}$$

$$= \frac{1}{4i}[2e^{i(z_1+z_2)} - e^{i(z_2-z_1)} - 2e^{-(z_1+z_2)} + e^{i(z_1-z_2)} - e^{i(z_1-z_2)} + e^{i(z_2-z_1)}]$$

$$= \frac{1}{2i}[e^{i(z_1+z_2)} - e^{-i(z_1+z_2)}] = \sin(z_1 + z_2).$$

(d) To prove that $\sin 3z = 3 \sin z - 4 \sin^3 z$.

$$\text{R.H.S.} = 3\left(\frac{e^{iz} - e^{-iz}}{2i}\right) - 4\left(\frac{e^{iz} - e^{-iz}}{2i}\right)^3$$

$$= \frac{1}{2i}(3e^{iz} - 3e^{-iz}) - 4 \cdot \left(\frac{e^{3iz} - 3e^{2iz} \cdot e^{-iz} + 3e^{iz} e^{-2iz} - e^{-3iz}}{8i^3}\right)$$

$$= \frac{1}{2i}(3e^{iz} - 3e^{-iz} + e^{3iz} - 3e^{iz} + 3e^{-iz} - e^{-3iz})$$

$$= \frac{1}{2i}(e^{3iz} - e^{-3iz}) = \sin 3z.$$

Similarly we can derive other results. These results show the generality of trigonometical formulae.

4.4. EULER'S EXPONENTIAL VALUE

To prove that $e^{i\theta} = \cos\theta + i \sin\theta$, for any real θ.

Proof. We have $e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots$.

Put $z = i\theta$, where θ is real

$$e^{i\theta} = 1 + i\theta + \frac{i^2\theta^2}{2!} + \frac{i^3\theta^3}{3!} + \ldots$$

$$= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \ldots + i\left[\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots\right] \qquad (\because\ i^2 = -1)$$

$$= \cos\theta + i \sin\theta \qquad \ldots(1)$$

$$\therefore \quad e^{i\theta} = \cos\theta + i \sin\theta \qquad \ldots(2)$$

and $$e^{-i\theta} = \cos\theta - i \sin\theta.$$

By adding of (1) and (2)

(a) $\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$.

By subtracting (2) from (1)

(b) $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$.

These results are known as **Euler's exponential values.**

(c) $\tan\theta = \dfrac{\sin\theta}{\cos\theta} = \dfrac{(e^{i\theta} - e^{-i\theta})}{i\,(e^{i\theta} + e^{-i\theta})}$.

(d) $\cot\theta = \dfrac{\cos\theta}{\sin\theta} = \dfrac{i\,(e^{i\theta} + e^{-i\theta})}{(e^{i\theta} - e^{-i\theta})}$.

REMARK

Since $e^{x+iy} = e^x . [\cos y + i \sin y]$, this method helps in breaking an exponential function into real and imaginary parts.

• 4.5. PERIODS OF COMPLEX CIRCULAR FUNCTIONS

$$\cos(z + 2n\pi) = \cos z \cos 2n\pi - \sin z \sin 2n\pi = \cos z \qquad [\text{if } n \text{ is an integer}]$$
$$\sin(z + 2n\pi) = \sin z \cos 2n\pi + \cos z \sin 2n\pi = \sin z \qquad [n \text{ being an integer}]$$
and
$$\tan(z + n\pi) = \frac{\sin(z + n\pi)}{\cos(n + n\pi)} = \frac{\pm \sin z}{\pm \cos z} = \tan z \ [\text{according as } n \text{ is even or odd integer}]$$

Hence the periods of cos z, sin z and tan z are real, and are the same (*i.e.*, 2π in case of cos z and sin z and π in case of tan z) as the periods of the circular functions of a real number.

• 4.6. TO PROVE THAT PERIOD OF e^z IS $2\pi i$

If $z = x + iy$, then
$$e^{z+2n\pi i} = e^x . e^{i(y+2\pi)} = e^x [\cos(y + 2\pi) + i \sin(y + 2\pi)]$$
$$= e^x . [\cos y + i \sin y] = e^x . e^{iy} = e^{x+iy} = e^z.$$
Hence $\quad e^{z+2n\pi i} = e^z$.

Thus the period of exp (z) is $2\pi i$.

• 4.7. SOME TRIGONOMETRICAL IDENTITIES FOR COMPLEX VARIABLE

For all x, y (Real or Complex) :

(*i*) $\cos^2 x + \sin^2 x = 1$

(*ii*) $\sin(-x) = -\sin x$

(*iii*) $\cos(-x) = \cos x$

(*iv*) $\sin 2x = 2 \sin x \cos x$

(*v*) $\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$

(*vi*) $\sin 3x = 3 \sin x - 4 \sin^3 x$

(*vii*) $\cos 3x = 4\cos^3 x - 3\cos x$

(*viii*) $\sin x + \sin y = 2 \sin \dfrac{x+y}{2} \cos \dfrac{x-y}{2}$

(*ix*) $\sin x - \sin y = 2 \cos \dfrac{x+y}{2} \sin \dfrac{x-y}{2}$

(*x*) $\cos x + \cos y = 2 \cos \dfrac{x+y}{2} \cos \dfrac{x-y}{2}$

(*xi*) $\cos x - \cos y = 2 \sin \dfrac{x+y}{2} \sin \dfrac{y-x}{2}$

(*xii*) $\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$

(*xiii*) $\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$.

• SOLVED EXAMPLES

Example 1. *Show that* $\exp\left(\pm i \dfrac{\pi}{2}\right) = \pm i$.

Solution. Since $\exp(\pm i\theta) = \cos\theta \pm i \sin\theta$, we have
$$\cos\left(\pm i \frac{\pi}{2}\right) = \cos\frac{\pi}{2} \pm i \sin\frac{\pi}{2} = \pm i.$$

Example 2. *Prove* $\sin(\alpha + n\theta) - e^{-i\alpha}\sin n\theta = e^{-in\theta}\sin\alpha$.

Solution. L.H.S. $= \sin(\alpha + n\theta) - (\cos\alpha + i\sin\alpha)\sin n\theta$

$$= \sin\alpha\cos n\theta + \cos\alpha\sin n\theta - \cos\alpha\sin n\theta - i\sin\alpha\sin n\theta$$

$$= \sin\alpha(\cos n\theta - i\sin n\theta) = \sin\alpha e^{-in\theta} = \text{R.H.S.}$$

Example 3. *Prove that*

$$\{\sin(\alpha - \theta) + e^{-\alpha i}\sin\theta\}^n = \sin^{n-1}\alpha\,\{\sin(\alpha - n\theta) + e^{-\alpha i}\sin n\theta\}.$$

Solution. L.H.S. $\{\sin(\alpha - \theta) + e^{-\alpha i}\sin\theta\}^n$

$$= \{\sin\alpha\cos\theta - \cos\alpha\sin\theta + (\cos\alpha - i\sin\alpha)\sin\theta\}^n$$

$$= \{\sin\alpha\cos\theta - i\sin\alpha\sin\theta\}^n = \sin^n\alpha\,\{\cos\theta - i\sin\theta\}^n$$

$$= \sin^n\alpha\,\{\cos n\theta - i\sin n\theta\} \qquad \text{[by De Moivre's theorem]}$$

Again R.H.S. $= \sin^{n-1}\alpha\,\{\sin(\alpha - n\theta) + e^{-i\alpha}\sin n\theta\}$

$$= \sin^{n-1}\alpha\,\{\sin\alpha\cos n\theta - \cos\alpha\sin n\theta + (\cos\alpha - i\sin\alpha)\sin n\theta\}$$

$$= \sin^{n-1}\alpha\,\{\sin\alpha\cos n\theta - i\sin\alpha\sin n\theta\}$$

$$= \sin^n\alpha\,\{\cos n\theta - i\sin n\theta\} = \text{L.H.S.}$$

$\therefore$ R.H.S. = L.H.S.

Example 4. *If* $\cos\theta + i\sin\theta = x$, $\sqrt{1 - c^2} = nc - 1$, *prove that*

$$1 + c\cos\theta = \frac{c}{2n}(1 + nx)\left(1 + \frac{n}{x}\right).$$

Solution. We have $x = \cos\theta + i\sin\theta$,

$\therefore$ $x^{-1} = \cos\theta - i\sin\theta$.

$\therefore$ R.H.S. $= \dfrac{c}{2n}(1 + nx)\left(1 + \dfrac{n}{x}\right)$

$$= \frac{c}{2n}[\{1 + n(\cos\theta + i\sin\theta)\}\{1 + n(\cos\theta - i\sin\theta)\}]$$

$$= \frac{c}{2n}[\{1 + ne^{i\theta}\}\{1 + ne^{-i\theta}\}] = \frac{c}{2n}[1 + ne^{-i\theta} + ne^{i\theta} + n^2]$$

$$= \frac{c}{2n}[1 + n(e^{i\theta} + e^{-i\theta}) + n^2] = \frac{c}{2n}[1 + 2n\cos\theta + n^2]$$

$$= \frac{c}{2n}(1 + n^2) + c\cos\theta \qquad \ldots(1)$$

But $\sqrt{1 - c^2} = nc - 1$ or $1 - c^2 = (nc - 1)^2$.

Hence from (1), we have

$\Rightarrow$ $c^2(1 + n^2) = 2nc$

$$\frac{c}{2n} \times 2n + c\cos\theta = 1 + c\cos\theta.$$

• TEST YOURSELF–1

1. $[\sin(\alpha + \theta) - e^{\alpha i}\sin\theta]^n = \sin^n\alpha e^{-n\theta i}$. 2. $\sin(\alpha + n\theta) - e^{i\alpha}\sin n\theta = e^{-in\theta}\sin\alpha$.
3. If $\tan^{-1}(e^{ix}) - \tan^{-1}(e^{-ix}) = \tan^{-1} i$, find x.

ANSWERS

3. $x = 2n\pi + \dfrac{\pi}{2}$ where n is an integer.

• 4.8. HYPERBOLIC FUNCTIONS

We have proved that for all values of the argument y (real or complex),

$$\cos y = 1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \frac{y^6}{6!} + \ldots \qquad \ldots(A)$$

$$\sin y = y - \frac{y^3}{3!} + \frac{y^5}{5!} - \frac{y^7}{7!} + \ldots \qquad \ldots(B)$$

We notice that in each of these series, the terms are alternatively positive and negative. If we place the positive sign before all the terms, we get two functions of y defined by indefinite series, which are related to the circular functions $\cos y$ and $\sin y$ by interesting properties. These functions are known as hyperbolic cosine and hyperbolic sine of y and are indicated for shortness by $\cosh y$ and $\sinh y$ respectively. Thus

$$\cosh y = 1 + \frac{y^2}{2!} + \frac{y^4}{4!} + \ldots \qquad \ldots(1)$$

$$\sinh y = y + \frac{y^3}{3!} + \frac{y^5}{5!} + \frac{y^7}{7!} + \ldots \qquad \ldots(2)$$

$$\therefore \quad \cosh y + \sinh y = 1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \ldots = e^y \qquad \ldots(3)$$

and

$$\cosh y - \sinh y = e^{-y} \qquad \ldots(4)$$

$$\therefore \quad \cosh y = \frac{1}{2}[e^y + e^{-y}] \text{ and } \sinh y = \frac{1}{2}[e^y - e^{-y}].$$

Now we give formal definition of these functions.

Definition. *The quantity* $\frac{e^y - e^{-y}}{2}$, *whether y be real or complex, is called the* **hyperbolic sine** *of y and is written as sinh y.*

Similarly $\frac{e^y + e^{-y}}{2}$ *is known as* ***hyperbolic cosine*** *of y and is written as cosh y.*

The hyperbolic tangent, secant, cosecant, and catangent can be obtained with the help of hyperbolic sine and cosine.

$$\tanh y = \frac{\sinh y}{\cosh y} = \frac{e^y - e^{-y}}{e^y + e^{-y}},$$

$$\operatorname{cosech} y = \frac{1}{\sinh y} = \frac{2}{e^y - e^{-y}},$$

$$\operatorname{sech} y = \frac{1}{\cosh y} = \frac{2}{e^y + e^{-y}},$$

$$\coth y = \frac{\cosh y}{\sinh y} = \frac{e^y + e^{-y}}{e^y - e^{-y}}.$$

• 4.9. RELATION BETWEEN HYPERBOLIC AND CIRCULAR FUNCTIONS

Hyperbolic functions can be expressed in terms of corresponding circular functions.

We know $\quad \sin x = \frac{e^{ix} - e^{-ix}}{2i}$, put $x = iy$

$$\sin iy = \frac{e^{i^2 y} - e^{i^2 y}}{2i} = \frac{i[e^{-y} - e^y]}{2i^2}$$

$$\sin iy = \frac{i[e^y - e^{-y}]}{2} = i \sinh y.$$

Similarly,

$$\cos iy = \frac{e^{i^2 y} + e^{-i^2 y}}{2} = \frac{e^y + e^{-y}}{2} = \cosh y$$

and

$$\tan iy = \frac{\sin iy}{\cos iy} = \frac{i \sinh y}{\cosh y} = i \tanh y.$$

From (3) and (4), we have

$$(\cosh y + \sinh y)^n = e^{ny} = \cosh ny + \sinh ny \qquad \ldots(5)$$

and

$$(\cosh y - \sinh y)^n = e^{-ny} = \cosh ny - \sinh ny. \qquad \ldots(6)$$

These results are analogous to De Moivre's Theorem.

• 4.10. SOME IMPORTANT RESULTS OF HYPERBOLIC FUNCTIONS

For any real x and y

(*i*) $\sinh 0 = 0$, $\cosh 0 = 1$, $\tanh 0 = 0$

(*ii*) $\cosh^2 x - \sinh^2 x = 1$

(*iii*) $1 - \tanh^2 x = \operatorname{sech}^2 x$

(iv) $coth^2 x - 1 = cosech^2 x$

(v) $sinh\ 2x = 2\ sinh\ x\ cosh\ x = \dfrac{2\ tanh\ x}{1 - tanh^2 x}$

(vi) $cosh\ 2x = cosh^2 x + sinh^2 x = 1 + 2\ sinh^2 x = 2\ cosh^2 x - 1 = \dfrac{1 + tanh^2 x}{1 - tanh^2 x}$

(vii) $tanh\ 2x = \dfrac{2\ tanh\ x}{1 + tanh^2 x}$

(viii) $sinh\ 3x = 3\ sinh\ x + 4\ sinh^3 x$

(ix) $cosh\ 3x = 4\ cosh^3 x - 3\ cosh\ x$

(x) $tanh\ 3x = \dfrac{3\ tanh\ x + tanh^3 x}{1 + 3\ tanh^2 x}$

(xi) $sinh\ (x + y) = sinh\ x\ cosh\ y + cosh\ x\ sinh\ y$

(xii) $cosh\ (x + y) = cosh\ x\ cosh\ y + sinh\ x\ sinh\ y$

(xiii) $e^x = cosh\ x + sinh\ x,\ e^{-x} = cosh\ x - sin\ hx.$

• 4.11. EXPANSIONS OF sinh *x* and cosh *x*

We know that

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

Then
$$\sinh x = \frac{1}{2}\left[\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots\right) - \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \ldots\right)\right]$$

$$= \frac{1}{2}\left[2x + 2\frac{x^3}{3!} + 2\frac{x^5}{5!} + \ldots\right]$$

$\therefore$
$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots \text{ to infinity}$$

Also,
$$\cosh x = \frac{1}{2}(e^x + e^{-x})$$

$$= \frac{1}{2}\left[\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots\right) + \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \ldots\right)\right]$$

$$= \frac{1}{2}\left[2x + 2\frac{x^2}{2!} + 2\frac{x^4}{4!} + \ldots\right]$$

$\therefore$
$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots \text{ to infinity.}$$

• 4.12. PERIODS OF HYPERBOLIC FUNCTIONS

We know $\cos i\theta = \cosh \theta$.

Therefore $\cosh (x + iy) = \cos [i (x + iy)] = \cos (xi - y) = \cos [-2\pi + ix - y]$
$= \cos [(2\pi i + x + iy)\, i] = \cosh [(2\pi i + x + iy)]$.

Similarly, $\cosh (x + iy) = \cosh [4\pi i + x + iy]$.

Hence the hyperbolic cosine is periodic, its period being imaginary and equal to $2\pi i$.

Similarly it can be shown for $\sinh (x + iy)$ that its period is $2\pi i$ and of $\tanh (x + iy)$ is πi.

It is to be noted here that hyperbolic functions differ from the circular functions in having imaginary periods.

After. These results can also be obtained in a simpler way.

Since $e^{2n\pi i} = \cos 2n\pi + i \sin 2n\pi = 1$.

$\therefore$ $e^{z + 2n\pi i} = e^z$ and $e^{-z - 2n\pi i} = e^{-z}$.

$\therefore$ $e^{z + 2n\pi i} + e^{-z - 2n\pi i} = e^z + e^{-z}$ and $e^{z + 2n\pi i} - e^{-z - n\pi i} = e^z - e^{-z}$.

$\therefore$ $\cosh (z + 2n\pi i) = \cosh (z)$ and $\sinh (z + 2n\pi i) = \sinh z$

Next $e^{n\pi i} = \cos n\pi + i \sin n\pi = (-1)^n$,

$e^{-n\pi i} = \cos n\pi - i \sin n\pi = (-1)^n$.

$$\therefore \quad \tanh(z + n\pi i) = \frac{e^{z+n\pi i} - e^{-z-n\pi i}}{e^{z+n\pi i} + e^{-z-n\pi i}} = \frac{e^z - e^{-z}}{e^z + e^{-z}} = \tanh z.$$

Hence $\cosh z$ and $\sinh z$ have an imaginary period of $2\pi i$ and $\tanh z$ an imaginary period of πi.

• SOLVED EXAMPLES

Example 1. *If* $\tan y = \tan\alpha \tanh\beta$, $\tan z = \cot\alpha \tanh\beta$, *prove that*

$$\tan(y+z) = \sinh 2\beta \operatorname{cosec} 2\alpha.$$

Solution. $\tan(y+z) = \dfrac{\tan y + \tan z}{1 - \tan y \tan z}$

$$= \frac{\tan\alpha\tanh\beta + \cot\alpha\tanh\beta}{1 - \tan\alpha\tanh\beta \times \cot\alpha\tanh\beta} = \frac{\tanh\beta\left[\dfrac{\sin\alpha}{\cos\alpha} + \dfrac{\cos\alpha}{\sin\alpha}\right]}{1 - \tanh^2\beta}$$

$$= \frac{\dfrac{\sinh\beta}{\cosh\beta}}{1 - \dfrac{\sinh^2\beta}{\cosh^2\beta}} \times \frac{1}{\sin\alpha\cos\alpha} = \frac{\sinh\beta\cosh\beta}{\sin\alpha\cos\alpha} \times \frac{2}{2} = \frac{\sinh 2\beta}{\sin 2\alpha}$$

$$= \sinh 2\beta \operatorname{cosec} 2\alpha.$$

Example 2. *If* $\cosh x = x \sec\theta$, *prove that* $\tanh^2 \dfrac{x}{2} = \tan^2 \dfrac{\theta}{2}$.

Solution. We know

$$\cosh x = \frac{1 + \tanh^2 x/2}{1 - \tanh^2 x/2} = \sec\theta = \frac{1}{\cos\theta}.$$

Apply componendo and dividendo, we have

$$\left[i.e., \text{ if } \frac{a}{b} = \frac{c}{d} \text{ then } \frac{a-b}{a+b} = \frac{c-d}{c+d}\right],$$

$$\frac{2\tanh^2 x/2}{2} = \frac{1-\cos\theta}{1+\cos\theta} = \tan^2\frac{\theta}{2}.$$

$$\therefore \quad \tanh^2\frac{x}{2} = \tan^2\frac{\theta}{2}.$$

Example 3. *If* θ *is acute and* $x = \log\tan\left(\dfrac{\pi}{4} + \dfrac{\theta}{2}\right)$, *show that* $\cos\theta\cosh x = 1$.

Solution. $x = \log\tan\left(\dfrac{\pi}{4} + \dfrac{\theta}{2}\right)$.

$$\therefore \quad e^x = \tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right) = \frac{1 + \tan\theta/2}{1 - \tan\theta/2}.$$

Now $\cosh x = \dfrac{1}{2}[e^x + e^{-x}] = \dfrac{1}{2}\left[\dfrac{1+\tan\theta/2}{1-\tan\theta/2} - \dfrac{1-\tan\theta/2}{1+\tan\theta/2}\right]$

$$= \frac{1}{2}\left[\frac{(1+\tan\theta/2)^2 + (1-\tan\theta/2)^2}{1-\tan^2\theta/2}\right] = \frac{1+\tan^2\theta/2}{1-\tan^2\theta/2} = \sec\theta.$$

$\therefore \cosh x \cos\theta = 1.$

Example 4. *If* $u = \log\tan\left(\dfrac{\pi}{4} + \dfrac{\theta}{2}\right)$, *prove that* $\tanh\dfrac{u}{2} = \tan\dfrac{\theta}{2}$.

Solution. Given $u = \log\tan\left(\dfrac{\pi}{2} + \dfrac{\theta}{2}\right)$

or $$e^u = \tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right) \quad \text{or} \quad \frac{e^{u/2}}{e^{-u/2}} = \frac{1+\tan\theta/2}{1-\tan\theta/2}.$$

By componendo and dividendo, we have

$$\frac{e^{u/2} - e^{-u/2}}{e^{u/2} + e^{-u/2}} = \frac{2\tan\theta/2}{2} = \tan\theta/2 \quad \text{or} \quad \tanh\frac{1}{2}u = \tan\frac{\theta}{2}.$$

Example 5. *If* $u = \log\tan\left(\dfrac{\pi}{4} + \dfrac{\theta}{2}\right)$, *prove that* (*i*) *sinh* $u = \tan\theta$ (*ii*) *tanh* $u = \sin\theta$

Solution. **(i)** We have $u = \log\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)$

or
$$e^u = \tan\left(\frac{\pi}{2}+\frac{\theta}{2}\right) = \frac{1+\tan\theta/2}{1-\tan\theta/2} \quad \ldots(1)$$

$\therefore$
$$e^{-u} = \frac{1-\tan\theta/2}{1+\tan\theta/2}. \quad \ldots(2)$$

Hence,
$$e^u - e^{-u} = \frac{1+\tan\theta/2}{1-\tan\theta/2} - \frac{1-\tan\theta/2}{1+\tan\theta/2}$$
$$= \frac{(1+\tan\theta/2)^2-(1-\tan\theta/2)^2}{1-\tan^2\theta/2} = \frac{4\tan\theta/2}{1-\tan\theta/2} = 2\tan\theta.$$

$\therefore$
$$\sinh u = \frac{1}{2}[e^u - e^{-u}] = \tan\theta.$$

(ii) From (1) and (2),
$$e^u + e^{-u} = \frac{1+\tan\theta/2}{1-\tan\theta/2} + \frac{1-\tan\theta/2}{1+\tan\theta/2}$$
$$= \frac{(1+\tan\theta/2)^2+(1-\tan\theta/2)^2}{1-\tan^2\theta}$$
$$= \frac{2(1+\tan^2\theta/2)}{1-\tan^2\theta/2} = \frac{2}{\cos\theta}.$$

$\therefore$
$$\cosh u = \sec\theta.$$

Hence
$$\tanh u = \frac{e^u-e^{-u}}{e^u+e^{-u}} = \frac{\tan\theta}{\sec\theta} = \sin\theta.$$

Example 6. *Prove that* $\tan\left(\frac{u+iv}{2}\right) = \frac{\sin u + i\sinh v}{\cos u + \cosh v}$.

Solution. L.H.S. $= \tan\left(\frac{u+iv}{2}\right)$
$$= \frac{\sin\left[\frac{u+iv}{2}\right]}{\cos\left[\frac{u+iv}{2}\right]} = \frac{2\cos\left(\frac{u+iv}{2}\right)\cos\left(\frac{u-iv}{2}\right)}{2\cos\left(\frac{u+iv}{2}\right)\cos\left(\frac{u-iv}{2}\right)}$$
$$= \frac{\sin u + \sin iv}{\cos u + \cos iv} = \frac{\sin u + i\sinh v}{\cos u + \cosh v} = \text{R.H.S.}$$

• STUDENT ACTIVITY

1. Show that $\exp\left(\pm\frac{i\pi}{2}\right) = \pm i$.

2. If $u = \log \tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right)$, then prove that $\tanh u = \sin \theta$.

• SUMMARY

- $\exp(z) = 1 + z + \frac{z^2}{2!} + \frac{3}{3!} + \ldots\ldots + \frac{z^n}{m!} + \ldots\ldots$
- Circular function :

$$\sin z = z \ \frac{z^3}{3!} + \frac{z^5}{5!} \ldots\ldots\ldots$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} \ldots\ldots$$

$$\tan z = \frac{\sin z}{\cos z} \text{ etc.}$$

- Period of $\exp(z)$ is $2\pi i$.
- **Hyprbolic function :** $\sinh x = \frac{e^x - e^{-x}}{2}$, $\cosh x = \frac{e^x + e^{-x}}{2}$
- **Relation between circular and hyperbolic functions :**
  $\sin(ix) = i \sin hx$, $\cos(ix) = os\, hx \tan(ix) = i \tan hx$ etc.
- Expansion of $\sinh x$ and **cos *hx*** :

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots\ldots$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots\ldots$$

- Priod of $\cos z$ is $2\pi i$.
- Period of $\sinh z$ is also $2\pi i$.

• *TEST YOURSELF–2*

Verify the following :

1. (a) $\sinh(x - y) = \sinh x \cosh y - \cosh x \sinh y$,
   (b) $\cosh(x - y) = \cosh x \cosh y - \sinh x \sinh y$,
2. $\tanh(x + y) = \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$
3. (a) $\sinh x - \sinh y = 2 \cosh \frac{x + y}{2} \sinh \frac{x - y}{2}$
   (b) $\cosh x + \cosh y = 2 \cosh \frac{x + y}{2} \cosh \frac{x - y}{2}$.

Prove that :

4. $\sinh(x + y) \cosh(x - y) = \frac{1}{2} [\sinh 2x + \sinh 2y]$

7. $\sinh \beta \sin \alpha + i \cosh \beta \cos \alpha = i \cos (\alpha + i\beta)$.

8. $\sin 2\alpha + i \sinh 2\beta = 2 \sin (\alpha + i\beta) \cos (\alpha - i\beta)$.

9. $\cos (\alpha + i\beta) + i \sin (\alpha + i\beta) = e^{-\beta} (\cos \alpha + i \sin \alpha)$.

10. $\dfrac{1 + \tanh x}{1 - \tanh x} = \cosh 2x + \sinh 2x$.

11. $\cos (\alpha - i\beta) + i \sin (\alpha - i\beta) = e^{-\beta} (\cos \alpha - i \sin \alpha)$.

12. If $\cosh \alpha = \sec \theta$, show that $\alpha = \log_e \tan (\pi/4 + \theta/2)$.

13. If $\tan \theta = \tanh x \cot y$ and $\tan \phi = \tanh x \tan y$, prove that

$$\frac{\sin 2\theta}{\sin 2\phi} = \frac{\cosh 2x + \cos 2y}{\cosh 2x - \cos 2y}.$$

OBJECTIVE EVALUATION

FILL IN THE BLANKS :

1. The series $\Sigma z_n = \Sigma x_n + i \Sigma y_n$ is convergent if Σx_n and Σy_n both are ....

2. Every absolutely convergent series is ......

3. If the series $| \Sigma z_n |$ is convergent, then it is said to be ......

4. $e^{z_1} . e^{z_2} =$ ....

TRUE OR FALSE :

Write T for True and F for False statement :

1. The hyperbolic function differ from the circular function in having imaginary periods. *(T/F)*

2. $\tanh z$ has an imaginary period of $2\pi i$. *(T/F)*

3. $\cosh z$ and $\sinh z$ have an imaginary period of $2\pi i$. *(T/F)*

4. If z is a complex number then $E(z) . E(-z)$ is one. *(T/F)*

5. The value of $\sin x$ is $\dfrac{e^x - e^{-x}}{2}$. *(T/F)*

MULTIPLE CHOICE QUESTIONS :

Choose the most appropriate one :

1. If z_1 and z_2 are two complex numbers then the value of $e^{z_1} . e^{z_2}$ is :

(a) $e^{z_1 - z_2}$ (b) $e^{z_1 \cdot z_2}$ (c) $e^{z_2 - z_1}$ (d) $e^{z_1 + z_2}$ (e) None of these.

2. If z is a complex number thus the value of $E(z) . E(-z)$:

(a) e^{-2z} (b) e^{2z} (c) 1 (d) -1 (e) None of these.

3. If $e^{i\theta} = \cos \theta + i \sin \theta$, then the value of $\cos \theta$ is :

(a) $\dfrac{e^{i\theta} - e^{-i\theta}}{2}$ (b) $\dfrac{e^{i\theta} + e^{-i\theta}}{2}$ (c) $\dfrac{e^{-i\theta} - e^{i\theta}}{2}$ (d) $\dfrac{e^{i\theta} + e^{-i\theta}}{2i}$ (e) None of these.

4. If $e^{i\theta} = \cos \theta + i \sin \theta$, then the value of $\sin \theta$ is :

(a) $\dfrac{e^{i\theta} + e^{-i\theta}}{2i}$ (b) $\dfrac{e^{i\theta} - e^{-i\theta}}{2i}$ (c) $\dfrac{e^{-i\theta} - e^{i\theta}}{2i}$ (d) $\dfrac{e^{i\theta} + e^{-i\theta}}{-2i}$ (e) None of these.

ANSWERS

Fill in the Blanks :

1. Convergent **2.** Convergent **3.** Absolutely convergent **4.** $e^{z_1 + z_2}$

True or False :

1. T **2.** F **3.** T **4.** T **5.** F.

Multiple Choice Questions :

1. (d) **2.** (c) **3.** (b) **4.** (b)

UNIT 5

LOGARITHMS OF COMPLEX NUMBERS

LEARNING OBJECTIVES

LEARNING OBJECTIVES

After going through this unit you will learn :

- How to find the value of th logarithms of complex numbers
- About the general exponential functions.

• 5.1. INTRODUCTION

We know that if x and y are real quantities and $e^x = y$, then x is said to be the logarithm of y to the base e and is written as

$$x = \log_e y.$$

Similarly if $e^{x+iy} = u + iv$, then $x + iy$ is called the logarithm (Napierian) of $u + iv$ to the base e and is written as

$$\log_e (u + iv) = x + iy. \qquad \ldots(1)$$

Since $\qquad e^{2n\pi i} = 1 \qquad$ (where n is an integer or zero), we have

$$e^{x+iy+2n\pi i} = e^{x+iy},$$

giving that

$$\text{Log}_e (u + iv) = 2n\pi i + x + iy \qquad \ldots(2)$$

$$= x + i(2n\pi + y).$$

This shows that the logarithm of a complex quantity has an infinite number of values and hence is ***many-valued function****. These values are called* ***general values*** *of* $\log_e (u + iv)$.

This is known as the general value of the logarithm, the principal value of the logarithm is obtained by putting $n = 0$ in (2).

In order to distinguish between the general value of the logarithm as given by (2) and the principal value as given by (1), we get by putting $n = 0$ in (2), general value is written as 'Log' and the principal values as 'log'.

Since n can take any integral values, there are an infinite number of logarithms of $x + iy$ and they differ from each other by $2\pi i$.

REMARK

The base of a logarithm will be e, unless or otherwise stated.

• 5.2. LOGARITHM OF A POSITIVE REAL NUMBER

Let x be a positive real number. Then

$$x = x + 0 \,.\, i = r(\cos\theta + i\sin\theta)$$

$$\Rightarrow \qquad r\cos\theta = x,\ r\sin\theta = 0$$
$$\therefore \qquad r = x \text{ and } \theta = 0.$$

Now
$$\begin{aligned}\text{Log } x &= \text{Log } r(\cos\theta + i\sin\theta)\\ &= 2n\pi i + \log r(\cos\theta + i\sin\theta)\\ &= 2n\pi i + \log r + i\theta\\ &= 2n\pi i + \log x + i\,.\,0.\end{aligned}$$
$$\therefore \qquad \text{Log } x = 2n\pi i + \log x$$

This is the general value of x.

• 5.3. LOGARITHM OF A NEGATIVE REAL NUMBER

Let x be a positive real number. Then
$$-x = -x + i\,.\,0 = r(\cos\theta + i\sin\theta)$$
$$\Rightarrow \qquad r\cos\theta = -x \text{ and } r\sin\theta = 0$$
$$\Rightarrow \qquad r = x \text{ and } \theta = \pi \text{ (not 0)}$$

Now
$$\begin{aligned}\text{Log }(-x) &= 2n\pi i + \log(-x)\\ &= 2n\pi i + \log r(\cos\theta + i\sin\theta)\\ &= 2n\pi i + \log r + i\theta\\ &= 2n\pi i + \log x + i\pi\end{aligned}$$
$$\text{Log }(-x) = (2n+1)\pi i + \log x.$$

This is the general value of $(-x)$.

• 5.4. LOGARITHM OF $x + iy$ IN THE FORM OF $A + iB$

Let $x + iy = r(\cos\theta + i\sin\theta)$
$$\Rightarrow \qquad x = r\cos\theta,\ y = r\sin\theta$$
$$\Rightarrow \qquad r = \sqrt{x^2+y^2} \text{ and } \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

Now
$$\begin{aligned}\text{Log }(x+iy) &= 2n\pi i + \log(x+iy)\\ &= 2n\pi i + \log r(\cos\theta + i\sin\theta)\\ &= 2n\pi i + \log r + i\theta\\ &= 2n\pi i + \log\sqrt{x^2+y^2} + i\tan^{-1}\tan^{-1}\left(\frac{y}{x}\right).\end{aligned}$$
$$\therefore \qquad \text{Log }(x+iy) = 2n\pi i + \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\left(\frac{y}{x}\right). \qquad \text{...(1)}$$

This equation gives the general logarithm of $x + iy$.

For the principal value put $n = 0$ in (1), we get
$$\log(x+iy) = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\left(\frac{y}{x}\right). \qquad \text{...(2)}$$

REMARK

In (2) if we put $-y$ for y, we get
$$\log(x-iy) = \frac{1}{2}\log(x^2+y^2) - i\tan^{-1}\left(\frac{y}{x}\right).$$

• 5.5. SOME IMPORTANT RESULTS

(i) $\log(z_1z_2) = \log z_1 + \log z_2$

(ii) $\log\dfrac{z_1}{z_2} = \log z_1 - \log z_2$.

Let $\qquad z_1 = r_1e^{i\theta_1},\ z_2 = r_2e^{i\theta_2}$.

Now
$$\begin{aligned}\text{Log } z_1 + \text{Log } z_2 &= [\log r_1 + i(2m_1\pi + \theta_1)] + [\log r_2 + i(2m_2\pi + \theta_2)]\\ &= (\log r_1 + \log r_2) + i(\theta_1 + \theta_2 + 2n\pi) \qquad \text{...(1)}\end{aligned}$$

where m_1 and m_2 are integers, and $n = m_1 + m_2$.

Also $\qquad \log z_1z_2 = \log r_1r_2e^{i(\theta_1+\theta_2)} = \log r_1r_2 + i(\theta_1 + \theta_2 + 2m\pi). \qquad \text{...(2)}$

Since n and m can take up any integral values, it is clear that every value of Log (z_1z_2) is equal to some value of $\log z_1 + \log z_2$ and that every value of latter is equal to some value of the former.
$$\therefore \qquad \log z_1z_2 = \log z_1 + \log z_2. \qquad \text{...(A)}$$

Similarly it can be proved that

$$\log \frac{z_1}{z_2} = \log z_1 - \log z_2. \qquad \ldots(B)$$

REMARK

It is important to note that,

$$\log z_1 z_2 = \log z_1 + \log z_2$$

and $$\log \frac{z_1}{z_2} = \log z_1 - \log z_2,$$

the principal values of the two sides of these equations need not necessarily be equal, for the simple reason that amp. $(z_1) \pm$ amp. (z_2) need not necessarily lie between $-\pi$ and $+\pi$, whereas amp. $(z_1 z_2)$ and amp. $\left(\frac{z_1}{z_2}\right)$ must lie between $-\pi$ and $+\pi$.

• SOLVED EXAMPLES

Example 1. *Find the general value of Log* (-3).

Solution. Let $-3 = r(\cos\theta + i\sin\theta)$

$\Rightarrow \quad r\cos\theta = -3, \ \ r\sin\theta = 0$

$\Rightarrow \quad r = 3$ and $\theta = \pi$.

$$\begin{aligned}\therefore \quad \text{Log}(-3) &= 2n\pi i + \log(-3)\\ &= 2n\pi i + \log r(\cos\theta + i\sin\theta)\\ &= 2n\pi i + \log re^{i\theta} \qquad (\because \ e^{i\theta} = \cos\theta + i\sin\theta)\\ &= 2n\pi i + \log r + i\theta\\ &= 2n\pi i + \log 3 + i\pi.\end{aligned}$$

$$\therefore \quad \text{Log}(-3) = (2n+1)i\pi + \log 3.$$

Example 2. *Prove that* $\text{Log}(1+i) = \frac{1}{2}\log 2 + i\left(2n\pi + \frac{\pi}{4}\right)$.

Solution.
$$\begin{aligned}\text{Log}(1+i) &= 2n\pi i + \log(1+i)\\ &= 2n\pi i + \frac{1}{2}\log(1^2+1^2) + i\tan^{-1}\left(\frac{1}{1}\right)\\ &= 2n\pi i + \frac{1}{2}\log 2 + i\tan^{-1}(1)\\ &= 2n\pi i + \frac{1}{2}\log 2 + \frac{i\pi}{4}\\ &= \frac{1}{2}\log 2 + i\left(2n\pi + \frac{\pi}{4}\right).\end{aligned}$$

Example 3. *Show that* $i\log\left(\frac{x-i}{x+i}\right) = \pi - 2\tan^{-1}x$.

Solution.
$$\begin{aligned}i\log\left(\frac{x-i}{x+i}\right) &= i\left[\log\frac{i(-1-xi)}{i(1-xi)}\right]\\ &= i\log\left(\frac{-1-xi}{1-xi}\right)\\ &= i\{\log(-1-xi) - \log(1-xi)]\\ &= i[\log(-1)(1+xi) - \log(1-ix)]\\ &= i[\log(-1) + \log(1+xi) - \log(1-xi)]\\ &= i\left[-i\pi + \frac{1}{2}\log(1+x^2) + i\tan^{-1}x - \frac{1}{2}\log(1+x^2) - i\tan^{-1}(-x)\right]\\ &= i[-i\pi + i\tan^{-1}x - i\tan^{-1}(-x)]\\ &= i[-i\pi + i\tan^{-1}x + i\tan^{-1}x\} \qquad [\because \ \tan^{-1}(-x) = -\tan^{-1}x]\\ &= i[-i\pi + 2i\tan^{-1}x]\\ &= \pi - 2\tan^{-1}x.\end{aligned}$$

Example 4. *If* $\tan\log(x+iy) = a+ib$, *where* $a^2+b^2 \neq 1$, *prove that*

$$\tan\{\log(x^2+y^2)\} = \frac{2a}{1-a^2-b^2}.$$

Solution. $\tan\{\log(x+iy)\} = a+ib. \qquad \ldots(1)$

Then $\tan\{\log(x - iy)\} = a - ib.$ ...(2)

Adding (1) and (2), we get

$$\tan\{\log(x + iy)\} + \tan\{\log(x - iy)\} = 2a. \quad ...(3)$$

Multiplying (1) and (2), we get

$$\tan\{\log(x + iy)\}\tan\{\log(x - iy)\} = a^2 + b^2$$

Now $$\frac{2a}{1 - a^2 - b^2} = \frac{\tan\{\log(x + iy)\} + \tan\{\log(x - iy)\}}{1 - \tan\{\log(x + iy)\}\tan\{\log(x - iy)\}}$$

$$= \tan[\log(x + iy) + \log(x - iy)]$$

$$= \tan[\log(x + iy)(x - iy)]$$

$$= \tan\{\log(x^2 + y^2)\}.$$

Example 5. *Prove that* $\log_e \tan\left(\frac{\pi}{4} + \frac{x}{2}i\right) = i\tan^{-1}\sinh x.$

Solution. $$\text{L.H.S.} = \log_e \tan\left(\frac{\pi}{4} + \frac{x}{2}i\right) = \log_e \frac{\sin\left(\frac{\pi}{4} + \frac{x}{2}i\right)}{\cos\left(\frac{\pi}{4} + \frac{x}{2}i\right)}$$

$$= \log_e \left\{\frac{2\sin\left(\frac{x}{4} + \frac{x}{2}i\right)\cos\left(\frac{\pi}{4} - \frac{x}{2}i\right)}{2\cos\left(\frac{\pi}{4} + \frac{x}{2}i\right)\cos\left(\frac{\pi}{4} - \frac{x}{2}i\right)}\right\}$$

$$= \log_e \frac{\sin\frac{\pi}{2} + \sin xi}{\cos\frac{\pi}{2} + \cos xi} = \log_e \frac{1 + i\sinh x}{\cosh x} \qquad (\text{as } \sin ix = i\sinh x)$$

$$= \log_e \frac{1 + \sinh^2 x}{\cosh^2 x} + i\tan^{-1}(\sinh x)$$

$$= \frac{1}{2}\log_e \frac{\cosh^2 x}{\cosh^2 x} + i\tan^{-1}(\sinh x)$$

$$= \frac{1}{2}\log 1 + i\tan^{-1}(\sinh x) = i\tan^{-1}(\sinh x).$$

Exampl 6. *Show that* $\tan\left(i\log\frac{a - ib}{a + ib}\right) = \frac{2ab}{a^2 - b^2}.$

Solution. Let $a = r\cos\theta,\ b = r\sin\theta.$

$$\therefore \quad \frac{a - ib}{a + ib} = \frac{r(\cos\theta - i\sin\theta)}{r(\cos\theta + i\sin\theta)} = \frac{e^{-i\theta}}{e^{i\theta}} = e^{-2i\theta}$$

$$\therefore \quad \tan\left[i\log\frac{a - ib}{a + ib}\right] = \tan[i(-2i\theta)]$$

$$= \tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta} = \frac{2\frac{b}{2}}{1 - \frac{b^2}{a^2}} = \frac{2ab}{a^2 - b^2}.$$

Example 7. *If* $a + ib = e^{x + iy}$ *prove that* $\frac{y}{x} = \frac{2\tan^{-1} b/a}{\log(a^2 + b^2)}.$

Solution. $a + ib = e^{x + iy}.$

This gives $x + iy = \log(a + ib)$

$$= \frac{1}{2}\log(a^2 + b^2) - i\tan^{-1}\frac{b}{a}.$$

Separating real and imaginary parts, we have

$$x = \frac{1}{2}\log(a^2 + b^2)$$

$$y = \tan^{-1}\frac{b}{a}.$$

Therefore, $$\frac{y}{x} = \frac{2\tan^{-1} b/a}{\log(a^2+b^2)}.$$

• TEST YOURSELF–1

Prove that :

1. (a) $\text{Log}\, i = \frac{1}{2}(4n+1)\pi i.$

(c) $\text{Log}\, 3i = \log 3 + \left(2n\pi + \frac{1}{2}\pi\right) i.$

(d) $\text{Log}\left(\frac{a+ib}{a-ib}\right) = 2i\tan^{-1}\left(\frac{b}{a}\right).$

2. (a) $\text{Log}(-i) = \frac{1}{2}(4n-1)\pi i.$ (b) $\text{Log}\sqrt{i} = \frac{1}{4}(8n+1)\pi i.$

3. $\log(1+i\tan\theta) = \log_e \sec\theta + i\theta.$

4. Show that $\log_e \frac{1}{1-e^{i\alpha}} = \log_e\left[\frac{1}{2}\operatorname{cosec}\frac{\alpha}{2}\right] + i\left(\frac{\pi}{2}-\frac{\alpha}{2}\right).$

5. Show that $\log\log(x+iy) = \frac{1}{2}\log(\alpha^2+\beta^2) + \tan^{-1}\frac{\beta}{a},$

where $2\alpha = \log_e(x^2+y^2)$ and $\beta = \tan^{-1}\frac{y}{x}.$

6. (a) If $(a_1+ib_1)(a_2+ib_2)\ldots(a_n+ib_n) = A+iB$, prove that

$$\tan^{-1}\frac{b_1}{a_1} + \tan^{-1}\frac{b_2}{a_2} + \ldots + \tan^{-1}\frac{b_n}{a_n} = \tan^{-1}\frac{B}{A}$$

and $$(a_1^2+b_1^2)(a_2^2+b_2^2)\ldots(a_n^2+b_n^2) = A^2+B^2.$$

(b) If $(1+i)(1+2i)(1+3i)\ldots(1+ni) = A+iB.$

Show that $2 \,.\, 5 \,.\, 10 \ldots (1+n^2) = A^2+B^2.$

7. Prove that the value of log log sin $(x+iy)$ is $\frac{1}{2}\log(u^2+v^2) + i\tan^{-1}\frac{v}{u},$

where $$u = \frac{1}{2}\log\frac{\cosh 2y - \cos 2x}{2}$$

and $$v = \tan^{-1}(\cot x \tanh y).$$

• 5.6. GENERAL EXPONENTIAL FUNCTION

The general exponential function is defined as

$$a^z = e^{z\,\text{Log}\,a} \qquad \ldots(1)$$

where a and z are any two complete numbers.

The function a^z is many valued function as Log a is many-valued.

(*i*) *General value of a^z :*

From (1), we have

$$a^z = e^{z\,\text{Log}\,a}$$
$$= \exp[z\,\text{Log}\,a].$$

$$\therefore \quad a^z = \exp[z(\log a + 2n\pi i)] \qquad \ldots(2)$$

[$\because$ $\text{Log}\, a = \log a + 2n\pi i$]

(*ii*) *Principal value of a^z :*

Putting $n = 0$ in (2), we get

$$a^z = \exp[z\log a]$$

which is the principal value of a^z.

• 5.7. LOGARITHMS TO ANY BASE

Definition. If z, w and σ be any three complex numbers, and if

$$\sigma^w = z, \qquad \ldots(1)$$

then we define that w is a logarithm of z to the base σ, and we write

$$\log_\sigma z = w. \qquad \ldots(2)$$

But we have already defined σ^w as $e^{w \log_e \sigma}$

$\therefore$ $$e^{w \log_e \sigma} = z \quad \text{or} \quad w \log_e \sigma = \log_e z$$

or $$w = \frac{\log_e z}{\log_e \sigma} \qquad \ldots(3)$$

From (2) and (3), we have

$$\log_\sigma z = \frac{\log_e z}{\log_e \sigma}. \qquad \ldots(A)$$

With the help of formula (A), we can write logarithm of any base to base 'e'.

The principal value of $\text{Log}_\sigma z$ is defined by

$$\log_\sigma z = \frac{\log_e z}{\log_e \sigma}. \qquad \ldots(B)$$

• SOLVED EXAMPLES

Example 1. *Prove that* $Log_i\, i = \frac{4m+1}{4n+1}$, *where m and n are integers.*

Solution. We know that

$$\text{Log}_a b = \frac{\text{Log}\, b}{\text{Log}\, a}$$

Then $$\text{Log}_i i = \frac{\text{Log}\, i}{\text{Log}\, i}$$

$$= \frac{\text{Log}\, i + 2m\pi i}{\text{Log}\, i + 2n\pi i}, \quad m, n \in \mathbf{I}$$

$$= \frac{i\pi/2 + 2m\pi i}{i\pi/2 + 2n\pi i} = \frac{4m+1}{4n+1}.$$

Example 2. *Find the general and principal value of* $(i)^i$.

Solution. We know that

$$a^z = e^{z \,\text{Log}\, a}$$

So, $$(i)^i = e^{i \,\text{Log}\, i}$$

$$= e^{i\,[\text{Log}\, i + 2n\pi i]}$$

$$= e^{i\,[i\pi/2 + 2n\pi i]} \qquad \left[\because \ \text{Log}\, i = \frac{i\pi}{2}\right]$$

$$= e^{-(\pi/2 + 2n\pi)}.$$

$\therefore$ $$(i)^i = e^{-(4n+1)\pi/2}. \qquad \ldots(1)$$

For principal value, put $n = 0$ in (1), we get

$$i^i = e^{-\pi/2}.$$

Also, putting $n = 0, 1, 2, 3, \ldots$ in (1), the various values i^i are $e^{-\pi/2}, e^{-5\pi/2}, e^{-9\pi/2}, e^{-13\pi/2}, \ldots$, which form a geometric progression with common ratio $e^{-\pi/2}$.

Example 3. *If* $i^{\alpha + i\beta} = e^x (\cos y + i \sin y)$, *then prove that*

$$x = -\frac{1}{2}(4n+1)\pi\beta \ \ \textit{and} \ \ y = \frac{1}{2}(4n+1)\pi\alpha.$$

Solution. We know that

$$a^z = e^{z \,\text{Log}\, a}$$

So, $$i^{\alpha+i\beta} = e^{(\alpha+i\beta)\,\text{Log}\, i}$$

$$= e^{(\alpha+i\beta)\,[\log i + 2n\pi i]}$$

$$= e^{(\alpha+i\beta)\,[i\pi/2 + 2n\pi i]}$$

$$= e^{(\alpha+i\beta)\frac{i(4n+1)\pi}{2}}$$

$$= e^{i(4n+1)\pi\alpha/2}\, e^{-\left(\frac{4n+1}{2}\right)\pi\beta}.$$

$\therefore$ $$i^{\alpha+i\beta} = e^{-\frac{1}{2}(4n+1)\pi\beta}\left[\cos\left\{\frac{1}{2}(4n+1)\pi\alpha\right\} + i \sin\left\{\frac{1}{2}(4n+1)\pi\alpha\right\}\right].$$

But $\quad i^{\alpha + i\beta} = e^x (\cos y + i \sin y)$

$\Rightarrow e^x (\cos y + i \sin y) = e^{-\frac{1}{2}(4n+1)\pi\beta}\left[\cos\left\{\frac{1}{2}(4n+1)\pi\alpha\right\} + i \sin\left\{\frac{1}{2}(4n+1)\pi\alpha\right\}\right]$

$\Rightarrow \quad x = -\frac{1}{2}(4n+1)\pi\beta$ and $y = \frac{1}{2}(4n+1)\pi\alpha.$

Example 4. *If $\sin(\log i^i) = a + ib$, find a and b. Hence find $\cos(\log i^i)$.*

Solution. $\quad \log i^i = i \log i$

$$= i\,[i \tan^{-1}\infty]$$
$$= i\,[i\pi/2] = -\pi/2.$$

$\therefore \quad \sin(\log i^i) = \sin(-\pi/2) = -1.$

But $\quad \sin(\log i^i) = a + ib$

$\Rightarrow \quad a = -1, b = 0.$

Also, $\quad \cos(\log i^i) = \sqrt{1 - \sin^2(\log i^i)}$

$$= \sqrt{1-(-1)^2} = \sqrt{1-1} = 0.$$

Example 5. *If $i^{i^{\cdots \text{ad. inf}}} = A + iB$, principal values only being considered, prove that*

(*i*) $\tan\frac{1}{2}\pi A = \frac{B}{A}$ $\qquad$ (*ii*) $A^2 + B^2 = e^{-\pi B}.$

Solution. We have

$$i^{i^{\cdots \text{ad inf.}}} = A + iB$$

$\Rightarrow \quad i^{(A+iB)} = A + iB$

$\Rightarrow \quad e^{(A+iB)\,\mathrm{Log}\, i} = A + iB$

$\Rightarrow \quad e^{(A+iB)\log i} = A + iB \qquad$ [$\because$ principal value being taken]

$\Rightarrow \quad e^{(A+iB)\left(\frac{i\pi}{2}\right)} = A + iB$

$\Rightarrow \quad e^{-\frac{\pi B}{2}}\, e^{\frac{i\pi A}{2}} = A + iB$

$\Rightarrow \quad e^{-\frac{\pi B}{2}}\left(\cos\frac{\pi A}{2} + i \sin\frac{\pi A}{2}\right) = A + iB$

Separating real and imaginary parts, we get

$$e^{-\frac{\pi B}{2}} \cos\frac{\pi A}{2} = A \qquad ...(1)$$

and
$$e^{-\frac{\pi B}{2}} \sin\frac{\pi A}{2} = B. \qquad ...(2)$$

(i) Dividing (2) by (1), we get

$$\frac{\sin\frac{\pi A}{2}}{\cos\frac{\pi A}{2}} = \frac{B}{A}.$$

$\therefore \quad \tan\frac{1}{2}\pi A = \frac{B}{A}.$ **Proved.**

(ii) Squaring (1) and (2) and adding, we get

$$e^{-\pi B}\left(\cos^2\frac{\pi A}{2} + \sin^2\frac{\pi A}{2}\right) = A^2 + B^2$$

$\therefore \quad A^2 + B^2 = e^{-\pi B}.$ **Proved.**

Example 6. *If $i^{x+iy} = x + iy$, prove that $x^2 + y^2 = e^{-(4n+1)\pi y}$.*

Solution. We have

$$i^{(x+iy)} = x + iy$$

$\Rightarrow \quad e^{(x+yi)\,\mathrm{Log}\, i} = x + iy$

$\Rightarrow \quad e^{(x+iy)(2n\pi i + \log i)} = x + iy$

$\Rightarrow \quad e^{(x+iy)\left(2n\pi i + \frac{i\pi}{2}\right)} = x + iy$

$\Rightarrow \quad e^{-(4n+1)\pi y/2}\, e^{(4n+1)i\pi x/2} = x + iy$

$$\Rightarrow \qquad e^{-(4n+1)\pi y/2}\left[\cos\left\{(4n+1)\frac{\pi x}{2}\right\} + i\sin\left\{(4n+1)\frac{\pi x}{2}\right\}\right] = x + iy.$$

Separating the real and imaginary parts, we get

$$e^{-(4n+1)\pi y/2}\cos\left\{(4n+1)\frac{\pi x}{2}\right\} = x$$

and

$$e^{-(4n+1)\pi y/2}\sin\left\{(4n+1)\frac{\pi x}{2}\right\} = y.$$

Squaring and adding, we get

$$x^2 + y^2 = e^{-(4n+1)\pi y}.$$ **Proved.**

Example 7. *Prove that the real part of the principal value of* $(i)^{\log(1+i)}$ *is*

$$e^{-\pi^2/8}\cos\left(\frac{1}{4}\pi\log 2\right).$$

Solution. Let $\qquad (i)^{\log(1+i)} = A + iB$

$\Rightarrow \qquad e^{\log(1+i)\log i} = A + iB \qquad$ [Principal values are considered]

$\Rightarrow \qquad e^{\log(1+i)\left(\frac{i\pi}{2}\right)} = A + iB$

$\Rightarrow \qquad e^{\frac{i\pi}{2}\left[\frac{1}{2}\log 2 + \frac{i\pi}{4}\right]} = A + iB$

$\Rightarrow \qquad e^{-\pi^2/8}\, e^{\frac{i\pi}{4}\log 2} = A + iB$

$\Rightarrow \qquad e^{-\pi^2/8}\left[\cos\left(\frac{\pi}{4}\log 2\right) + i\sin\left(\frac{\pi}{4}\log 2\right)\right] = A + iB.$

Equating real parts, we get

$$A = e^{-\pi^2/8}\cos\left(\frac{1}{4}\pi\log 2\right).$$ **Proved.**

Example 8. *Find the general value of* $Log_4(-2)$.

Solution. We know that $\mathrm{Log}_a\, b = \dfrac{\mathrm{Log}\, b}{\mathrm{Log}\, a}$

Then $\qquad \mathrm{Log}_4(-2) = \dfrac{\mathrm{Log}(-2)}{\mathrm{Log}\, 4}$

$$= \frac{\log(-2) + 2m\pi i}{\log 4 + 2n\pi i}$$

$$= \frac{\log(2e^{i\pi}) + 2m\pi i}{\log 2^2 + 2n\pi i}$$

$$= \frac{\log 2 + i\pi + 2m\pi i}{2\log 2 + 2n\pi i}$$

$$= \frac{\log 2 + (2m+1)\pi i}{2\log 2 + 2n\pi i}$$

$$= \frac{[\log 2 + (2m+1)\pi i]\,[\log 2 - n\pi i]}{2(\log 2)^2 + 2n^2\pi^2}$$

$$\therefore \qquad \mathrm{Log}_4(-2) = \left[\frac{(\log 2)^2 + (2m+1)n\pi^2}{2(\log 2)^2 + 2n^2\pi^2}\right] + i\left[\frac{(2m+1-n)\pi\log 2}{2(\log 2)^2 + 2n^2\pi^2}\right].$$

- **STUDENT ACTIVITY**

1. Find the value of Log (1 – i).

2. Find the real part of $(i)^{\log(1+i)}$.

• SUMMARY

- $\text{Log } x = 2n\pi i + \log x$
- $\text{Log}(-x) = (2n+1)\pi i + \log x.$
- $\text{Log}(x+iy) = 2\pi i + \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\left(\frac{y}{x}\right)$
- For any complex numbers a and z,
  $$a^z = e^{z\log a}.$$
- Principal value of $\text{Log}_\sigma z = \frac{\log_e z}{\log_e \sigma}.$

• TEST YOURSELF–2

1. If $(i^i)^i = \cos\theta - i\sin\theta$, prove that $\theta = \frac{1}{2}\pi(4n+1)$.
2. If $i^{i'} = \cos\theta + i\sin\theta$, prove that $\theta = \left(2m+\frac{1}{2}\right)\pi\exp\left[-\left(2n+\frac{1}{2}\right)\pi\right]$.
3. Separate $(1-i)^i$ into real and imaginary parts.
4. Prove that $i^a = \cos\left[\left(2m+\frac{1}{2}\right)\pi a\right] + i\sin\left[\left(2m+\frac{1}{2}\right)\pi a\right]$.

ANSWERS

3. Real part $= e^{\pi/4}\cos\left(\frac{1}{2}\log 2\right)$. Imaginary $= e^{\pi/4}\sin\left(\frac{1}{2}\log 2\right)$.

OBJECTIVE EVALUATION

FILL IN THE BLANKS

1. The principal value of a logarithm the coefficient of i should lie between ......
2. The principal value of a logarithm of a negative quantity is logarithm of the positive quantity added with ......
3. $\log(1 + i\tan\theta) = \log_e \sec\theta + ...$
4. The value of $\log\tan\left(\frac{\pi}{4}+\frac{i\alpha}{2}\right) = i\tan^{-1}(......)$.

TRUE OR FALSE :

Write T for True and F for False statemeant :

1. The logarithm function is a one-one function. *(T/F)*
2. The logarithm function is a many-one function. *(T/F)*
3. The principal value of $\log z_1z_2$ is always equal to the $\log z_1 + \log z_2$. *(T/F)*

4. $\log(1 + i\tan\theta) = \log_e \sec\theta + i\theta$. (T/F)

MULTIPLE CHOICE QUESTIONS :

Choose the most appropriate one :

1. If z_1 and z_2 are complex numbers then $\log(z_1 . z_2)$ is :
   (a) $\log z_1 - \log z_2$ (b) $\log z_2 - \log z_1$ (c) $\log z_1 + \log z_2$
   (d) $\log(z_1 - z_2)$ (e) None of these.
2. If z_1 and z_2 are complex numbers then $\log\left(\frac{z_1}{z_2}\right)$ is :
   (a) $\log z_1 - \log z_2$ (b) $\log z_2 - \log z_1$ (c) $\log z_1 + \log z_2$
   (d) $\log(z_1 + z_2)$ (e) None of these.
3. Value of $\tan\left[i \log\left(\frac{a - ib}{a + ib}\right)\right]$ is :
   (a) $\frac{2ab}{b^2 - a^2}$ (b) $\frac{2ab}{a^2 + b^2}$ (c) $\frac{-2ab}{a^2 + b^2}$ (d) $\frac{2ab}{a^2 - b^2}$
   (e) None of these.
4. Principal value of $\log(-1)$ is :
   (a) π (b) πi (c) $\frac{1}{i}\pi$ (d) $\frac{i}{\pi}$ (e) None of these.

ANSWERS

Fill in the Blanks :

1. $-\pi$ to π **2.** πi **3.** $i\theta$ **4.** $\sin h\alpha$

True or False :

1. F **2.** T **3.** F **4.** T

Multiple Choice Questions :

1. (c) **2.** (a) **3.** (d) **4.** (b).

$\therefore \qquad n = -1.$ $\qquad\left[\because\ n\pi - \frac{\pi}{4} \le \theta \le n\pi + \frac{\pi}{4}\right]$

(iii) Here θ lies between

$$-\frac{15\pi}{4} \text{ and } -\frac{17\pi}{4}$$

or $$-\frac{17\pi}{4} \text{ and } -\frac{15\pi}{4}$$

or $$\left(-4\pi - \frac{\pi}{4}\right) \text{ and } \left(-4\pi + \frac{\pi}{4}\right)$$

$\therefore \qquad n = -4$ $\qquad\left[\because\ n\pi - \frac{\pi}{4} \le \theta \le n\pi + \frac{\pi}{4}\right]$

(iv) Here θ lies between

$$\frac{19\pi}{4} \text{ and } \frac{21\pi}{4}$$

or $$\left(5\pi - \frac{\pi}{4}\right) \text{ and } \left(5\pi + \frac{\pi}{4}\right)$$

$\therefore \qquad n = 5$ $\qquad\left[\because\ n\pi - \frac{\pi}{4} \le \theta \le n\pi + \frac{\pi}{4}\right]$

Example 2. *Sum to infinite the series :*

(*i*) $\frac{1}{2^3} - \frac{1}{3 \cdot 2^7} + \frac{1}{5 \cdot 2^{11}} - \dots$ *ad. inf.* $\qquad$ (*ii*) $1 - \frac{1}{3 \cdot 4^2} + \frac{1}{5 \cdot 4^4} - \dots$ *ad. inf.*

Solution. (i) $\frac{1}{2^3} - \frac{1}{3 \cdot 2^7} + \frac{1}{5 \cdot 2^{11}} - \dots$ ad. inf.

$$= \frac{1}{2}\left[\left(\frac{1}{2^2}\right) - \frac{1}{3}\left(\frac{1}{2^2}\right)^3 + \frac{1}{5}\left(\frac{1}{2^2}\right)^5 - \dots\right]$$

$$= \frac{1}{2}\left[\tan^{-1}\left(\frac{1}{2^2}\right)\right] \qquad \left[\text{using Gregory's series as } \frac{1}{4} < 1\right]$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{1}{4}\right).$$

(ii) $1 - \frac{1}{3 \cdot 4^2} + \frac{1}{5 \cdot 4^4} - \dots$ ad. inf.

$$= 4\left[\frac{1}{4} - \frac{1}{3}\left(\frac{1}{4}\right)^3 + \frac{1}{5}\left(\frac{1}{4}\right)^5 - \dots\right]$$

$$= 4\left[\tan^{-1}\frac{1}{4}\right] \qquad \left[\text{using Gregory's series as } \frac{1}{4} < 1\right]$$

$$= 4\tan^{-1}\left(\frac{1}{4}\right).$$

Example 3. *Prove that* $\pi = 2\sqrt{3}\left[1 - \frac{1}{3^2} + \frac{1}{5 \cdot 3^2} - \frac{1}{7 \cdot 3^3} + \dots \textit{ ad. inf.}\right].$

Solution. $1 - \frac{1}{3^2} + \frac{1}{5 \cdot 3^2} - \frac{1}{7 \cdot 3^3} + \dots$ ad. inf.

$$= 1 - \frac{1}{3 \cdot 3} + \frac{1}{5 \cdot 3^2} - \frac{1}{7 \cdot 3^3} + \dots \text{ ad. inf.}$$

$$= \sqrt{3}\left[\frac{1}{\sqrt{3}} - \frac{1}{3}\left(\frac{1}{\sqrt{3}}\right)^3 + \frac{1}{5}\left(\frac{1}{\sqrt{3}}\right)^5 - \frac{1}{7}\left(\frac{1}{\sqrt{3}}\right)^7 + \dots\right]$$

$$= \sqrt{3}\left[\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)\right] \qquad \left[\text{By Gregory's series, as } \frac{1}{\sqrt{3}} < 1\right]$$

$$= \sqrt{3}\left[\frac{\pi}{6}\right]$$

$$= \frac{\pi}{2\sqrt{3}}.$$

Hence $$\pi = 2\sqrt{3}\left[1 - \frac{1}{3^2} + \frac{1}{5.3^2} - \frac{1}{7.3^3} + \dots \text{ ad. inf.}\right].$$

Example 4. *If $x > 0$, prove that*

$$\tan^{-1} x = \frac{\pi}{4} + \left(\frac{x-1}{x+1}\right) - \frac{1}{3}\left(\frac{x-1}{x+1}\right)^3 + \frac{1}{5}\left(\frac{x-1}{x+1}\right)^5 - \dots \textit{ ad. inf.}$$

Solution. We have

$$x > 0 \Rightarrow \frac{x-1}{x+1} < 1.$$

Then, by Gregory's series

$$\left(\frac{x-1}{x+1}\right) - \frac{1}{3}\left(\frac{x-1}{x+1}\right)^3 + \frac{1}{5}\left(\frac{x-1}{x+1}\right)^5 - \dots \text{ ad. inf.}$$

$$= \tan^{-1}\left(\frac{x-1}{x+1}\right)$$

$$= \tan^{-1}\left(\frac{\tan\phi - \tan\pi/4}{1 + \tan\phi\tan\pi/4}\right), \text{ where } x = \tan\phi$$

$$= \tan^{-1}[\tan(\phi - \pi/4)]$$

$$= \phi - \pi/4$$

$$= \tan^{-1} x - \pi/4 \qquad [\because x = \tan\phi]$$

$$\therefore \quad \tan^{-1} x = \frac{\pi}{4} + \left(\frac{x-1}{x+1}\right) - \frac{1}{3}\left(\frac{x-1}{x+1}\right)^3 + \frac{1}{5}\left(\frac{x-1}{x+1}\right)^5 - \dots \text{ ad. inf.}$$

• TEST YOURSELF–1

Subjective Questions :

1. Assuming that $\theta - n\pi = \tan\theta - \frac{1}{3}\tan^3\theta + \frac{1}{5}\tan^5\theta - \dots$, write down the value of n when θ lies between

(i) $\frac{7\pi}{4}$ and $\frac{9\pi}{4}$ (ii) $\frac{3\pi}{4}$ and $\frac{5\pi}{4}$ (iii) $\frac{-13\pi}{4}$ and $\frac{-11\pi}{4}$.

2. Prove that $\frac{\pi}{8} = \frac{1}{1.3} + \frac{1}{5.7} + \frac{1}{9.11} + \dots$

3. If θ lies between $\frac{\pi}{4}$ and $\frac{3\pi}{4}$, show that $\theta = \frac{\pi}{2} - \cot\theta + \frac{1}{3}\cot^3\theta - \frac{1}{5}\cot^5\theta + \dots$ ad. inf.

4. When both θ and $\tan^{-1}(\sec\theta)$ lies between 0 and $\frac{\pi}{2}$, prove that

$$\tan^{-1}(\sec\theta) = \frac{\pi}{4} + \tan^2\frac{\theta}{2} - \frac{1}{3}\tan^6\frac{\theta}{2} + \frac{1}{5}\tan^{10}\frac{\theta}{2} - \dots \infty.$$

5. If $\tan x < 1$, show that

$$\tan^2 x - \frac{1}{2}\tan^4 x + \frac{1}{3}\tan^6 x - \dots = \sin^2 x + \frac{1}{2}\sin^4 x + \frac{1}{3}\sin^6 x + \dots$$

ANSWERS

1. (i) $n = 2$ (ii) $n = 1$ (iii) $n = -3$

OBJECTIVE EVALUATION

FILL IN THE BLANKS :

1. The series $\theta = \tan\theta - \frac{\tan^3\theta}{3} + \frac{\tan^5\theta}{5} + \dots$ is valid only when θ lies between ......

2. The main use of Gregory's series is to find the value of ......

3. The value of $1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \dots\dots$ is = ...

4. The value of $\frac{1}{2}\tan^2\theta - \frac{1}{4}\tan^4\theta + \frac{1}{6}\tan^6\theta$ is ....

UNIT

6

GREGORY'S SERIES AND SUMMATION OF SERIES

LEARNING OBJECTIVES

LEARNING OBJECTIVES

After going through this unit you will leam :

- About the Geoggory's and trigonometric series.
- How to to obtain the sum of the given series.

6.1. GREGORY'S SERIES

To prove that, if θ *lies within the closed interval* $[-\pi/4, \pi/4]$, *i.e., if* $-\pi/4 \le \theta \le \pi/4$,

$$\theta = \tan\theta - \frac{1}{3}\tan^3\theta + \frac{1}{5}\tan^5\theta - \frac{1}{7}\tan^7\theta + \ldots \text{ ad. inf.}$$

Proof. We have $(1 + i\tan\theta) = \left(1 + i\frac{\sin\theta}{\cos\theta}\right) = \frac{1}{\cos\theta}(\cos\theta + i\sin\theta) = \sec\theta \cdot e^{i\theta}$.

Trigonometry and Algebra

Taking logarithm of both sides (considering only principal values)

$$\log(1 + i\tan\theta) = \log\sec\theta + \log e^{i\theta}.$$

$$\therefore \quad \log\sec\theta + i\theta = \log(1 + i\tan\theta). \quad \ldots(1)$$

Now since $-\pi/4 \le \theta \le \pi/4$, $\tan\theta$ is numerically less than unity.

Expanding R.H.S. of (1), we have

$$\log\sec\theta + i\theta = i\tan\theta - \frac{i^2\tan^2\theta}{2} + \frac{i^3\tan^3\theta}{3} - \frac{i^4\tan^4\theta}{4} + \ldots \quad \ldots(2)$$

$$= i\tan\theta + \frac{\tan^2\theta}{2} - \frac{i\tan^3\theta}{3} - \frac{\tan^4\theta}{4} + \frac{i\tan^5\theta}{5} \ldots$$

Equating imaginary parts on both sides,

$$\theta = \tan\theta - \frac{\tan^3\theta}{3} + \frac{\tan^5\theta}{5} \dots \text{ ad. inf.} \qquad \dots(3)$$

This series is called Gregory's series after the name of **James Gregory.**

If we put $\tan\theta = x$, so that $\theta = \tan^{-1} x$, we have another form of the Gregory's series.

$$\mathbf{\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots \text{ ad. inf.}} \qquad \dots(4)$$

when x lies between -1 and 1 and $-\pi/4 \le \tan^{-1} x \le \pi/4$.

Equating real parts on both sides of (2), we have

$$\mathbf{\log\sec\theta = \frac{1}{2}\tan^2\theta - \frac{1}{4}\tan^4\theta + \frac{1}{6}\tan^6\theta - \dots}.$$

• 6.2. GENERAL GREGORY'S SERIES

Gregory's series may be considered as particular cases of the theorem.

To prove that if θ lies between $n\pi - \frac{\pi}{4}$ and $n\pi + \frac{\pi}{4}$, both limits being inclusive

i.e., $$n\pi - \frac{\pi}{4} \le \theta \le n\pi + \frac{\pi}{4},$$

then $$\theta - n\pi = \tan\theta - \frac{\tan^3\theta}{3} + \frac{\tan^5\theta}{5} - \frac{\tan^7\theta}{7} + \dots.$$

• 6.3. VALUE OF π

Gregory's series has been used for evaluating the value of π. We have seen that

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$$

Putting $x = 1$, we get $$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

From this series the values of π can be calculated; but as the successive terms do not rapidly become small, a very large number of terms would have to be taken to obtain the value of π correct to a certain decimal place. On account of this other series have been found out.

• SOLVED EXAMPLES

Example 1. *Assuming that $\theta - n\pi = \tan\theta - \frac{\tan^3\theta}{3} + \frac{\tan^5\theta}{5} - \dots$ ad inf. when θ lies between $n\pi - \frac{\pi}{4}$ and $n\pi + \frac{\pi}{4}$, write down the values of n when θ lies between*

(*i*) $\frac{11\pi}{4}$ *and* $\frac{13\pi}{4}$ (*ii*) $-\frac{3\pi}{4}$ *and* $-\frac{5\pi}{4}$

(*iii*) $-\frac{15\pi}{4}$ *and* $-\frac{17\pi}{4}$ (*iv*) $\frac{19\pi}{4}$ *and* $\frac{21\pi}{4}$.

Solution. (i) Here θ lies between

$$\frac{11\pi}{4} \text{ and } \frac{13\pi}{4}$$

or $$\left(3\pi - \frac{\pi}{4}\right) \text{ and } \left(3\pi + \frac{\pi}{4}\right)$$

$\therefore$ $$n = 3 \qquad \left[\because\ n\pi - \frac{\pi}{4} \le \theta \le n\pi + \frac{\pi}{4}\right]$$

(ii) Here θ lies between

$$-\frac{3\pi}{4} \text{ and } -\frac{5\pi}{4}$$

or $$\left(-\pi + \frac{\pi}{4}\right) \text{ and } \left(-\pi - \frac{\pi}{4}\right)$$

or $$\left(-\pi - \frac{\pi}{4}\right) \text{ and } \left(-\pi + \frac{\pi}{4}\right)$$

TRUE OR FALSE :

Write T for True and F for False statement :

1. The Gregory's series is valid for all value of θ. *(T/F)*
2. If θ lies between $\frac{-9\pi}{4}$ and $\frac{-7\pi}{4}$ then value of the Gregory's series is $\pi + \frac{\theta}{2}$. **(T/F)**
3. $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots\ldots$ **(T/F)**
4. James Gregory discovered the Gregory Series *(T/F)*

MULTIPLE CHOICE QUESTIONS :

***Choose the most appropriate one* :**

1. If $\theta = \tan\theta - \frac{1}{3}\tan^3\theta + \frac{1}{5}\tan^5\theta - \frac{1}{7}\tan^7\theta + \ldots \infty$ then limits of θ are :

   (a) $0 \le \theta \le \frac{\pi}{4}$ (b) $-\frac{\pi}{4} < \theta \le \frac{\pi}{4}$ (c) $-\frac{\pi}{4} \le \theta < \frac{\pi}{4}$

   (d) $-\frac{\pi}{4} \le \theta \le \frac{\pi}{4}$ (e) None of these.

2. The main use of Gregory's series is to find :
   (a) trigonometrical expansions
   (b) to find the value of π
   (c) to find the value of θ in form of $\tan\theta$ in any limits of θ
   (d) None of these.
3. Gregory's series is :

   (a) $\theta = \tan\theta + \frac{1}{2}\tan^2\theta + \frac{1}{3}\tan^3\theta + \ldots$ (b) $\theta = \tan\theta - \frac{1}{2}\tan^2\theta + \frac{1}{3}\tan^3\theta - \frac{1}{4}\tan^{-4}\theta + \ldots$

   (c) $\theta = \tan\theta - \frac{1}{3}\tan^3\theta + \frac{1}{5}\tan^5\theta - \ldots$ (d) None of these.

4. If θ lies between $n\pi - \frac{\pi}{4}$, and $n\pi + \frac{\pi}{4}$ then $\theta - n\pi = \tan\theta - \frac{\tan^3\theta}{3} + \frac{\tan^5\theta}{5} - \ldots$ then value of n, if θ lies between $\frac{19\pi}{4}$ and $\frac{21\pi}{4}$ is :

   (a) 1 (b) 2 (c) 3 (d) 5.

ANSWERS

Fill in the Blanks :

1. $-\pi/4$ to $\pi/4$ **2.** π **3.** $\frac{\pi}{2\sqrt{2}}$ **4.** $\log\sec\theta$.

True or False :

1. F **2.** T **3.** T **4.** T.

Multiple Choice Questions :

1. (d) **2.** (b) **3.** (c) **4.** (d)

• SUMMATION OF SERIES

General (C + iS) Method

A general method for the summation of series of the type

$$c_1\cos\theta + c_2\cos(\theta+\alpha) + c_3\cos(\theta+2\alpha) + \ldots \quad \ldots(1)$$

or

$$c_1\sin\theta + c_2\sin(\theta+\alpha) + c_3\sin(\theta+2\alpha) + \ldots \quad \ldots(2)$$

is to denote the series (1) by C, since the coefficients of C's in this series are cosines of the angles increasing in arithmetic progression and the series (2) by S, being a sine series. Multiplying (2) by i and adding,

$$\begin{aligned} C + iS &= c_1(\cos\theta + i\sin\theta) + c_2\{\cos(\theta+\alpha) + i\sin(\theta+\alpha)\} \\ &\quad + c_3\{\cos(\theta+2\alpha) + i\sin(\theta+2\alpha)\} + \ldots \\ &= c_1e^{i\theta} + c_2e^{i(\theta+2\alpha)} + c_3e^{i(\theta+2\alpha)} + \ldots \\ &= e^{i\theta}\{c_1 + c_2e^{i\alpha} + c_3e^{2i\alpha} + \ldots\}. \quad \ldots(3) \end{aligned}$$

The real part of (3) gives C and the imaginary part gives S. *If either of the series C or S is given, the other series, known as auxiliary series, can be formed and the sum of $C + iS$ is found. The real part of the sum so found is equal to C and the imaginary part equal to S.*

Use of Geometric Series

Trigonometric series is given by $\alpha + \alpha z + \alpha z^2 + \alpha z^3 + \ldots$ inf.

Sums of n terms and infinity of the above series are given by

$$S_n = \alpha \frac{(z^n - 1)}{z - 1} \text{ if } |z| > 1, \qquad \ldots(1)$$

$$= \alpha \frac{(1 - z^n)}{1 - z} \text{ if } |z| < 1. \qquad \ldots(2)$$

$$S_\infty = \frac{\alpha}{1 - z} \text{ provided } |z| < 1. \qquad \ldots(3)$$

• SOLVED EXAMPLES

Example 1. *Sum the following series to n terms and to infinity*

$$1 + x \cos\theta + x^2 \cos 2\theta + \ldots \textit{ ad. inf.}$$

where x is less than unity.

Let $\quad C = 1 + x\cos\theta + x^2\cos 2\theta + \ldots$

and $\quad S = x\sin\theta + x^2 \sin 2\theta + \ldots$

$\therefore \quad C + iS = 1 + x(\cos\theta + i\sin\theta) + x^2(\cos 2\theta + i\sin 2\theta) + \ldots$

$$= 1 + xe^{i\theta} + x^2 e^{i\theta} + \ldots .$$

This is a geometric series with common ratio $xe^{i\theta}$, modulus of which is less than 1; hence

$$C_n + iS_n = \frac{1 - (xe^{i\theta})^n}{1 - xe^{i\theta}} = \frac{1 - x^n e^{in\theta}}{1 - xe^{i\theta}} = \frac{(1 - x^n e^{in\theta})(1 - xe^{-i\theta})}{(1 - xe^{i\theta})(1 - xe^{-i\theta})},$$

multiplying the numerator and denominator by complex conjugate of denominator

$$= \frac{1 - xe^{-i\theta} - x^n e^{in\theta} + x^{n+1} e^{i(n-1)\theta}}{1 + x^2 - x(e^{i\theta} + e^{-i\theta})}$$

$$= \frac{1 - x(\cos\theta - i\sin\theta) - x^n(\cos n\theta + i\sin n\theta) + x^{n+1}\{\cos(n-1)\theta + i\sin(n-1)\theta\}}{1 - 2x\cos\theta + x^2}$$

Equating real and imaginary parts, we get

$$C_n = \frac{1 - x\cos\theta - x^n\cos n\theta + x^{n+1}\cos(n-1)\theta}{1 - 2x\cos\theta + x^2},$$

$$S_n = \frac{x\sin\theta - x^n \sin n\theta + x^{n+1}\sin(n-1)\theta}{1 - 2x\cos\theta + x^2}.$$

Since $x < 1$, $x^n, x^{n+1} \to 0$ as $n \to \infty$,

$$\therefore \quad C_\infty = \frac{1 - x\cos\theta}{1 - 2x\cos\theta + x^2} \text{ and } S_\infty = \frac{x\sin\theta}{1 - 2\cos\theta + x^2}.$$

Example 2. *Sum the series*

(*i*) $\cos\alpha + c\cos(\alpha + \beta) + c^2\cos(\alpha + 2\beta) + \ldots$ *to n terms*

(*ii*) $\sin\alpha + c\sin(\alpha + \beta) + c^2\sin(\alpha + 2\beta) + \ldots$ *to n terms..*

If $c < 1$, deduce the sum to infinity.

Solution. Let

$$C_n = \cos\alpha + c\cos(\alpha + \beta) + c^2\cos(\alpha + 2\beta) + \ldots \text{ to } n \text{ terms},$$

$$S_n = \sin\alpha + c\sin(\alpha + \beta) + c^2\sin(\alpha + 2\beta) + \ldots \text{ to } n \text{ terms},$$

$$\therefore \quad C_n + iS_n = e^{i\alpha} + ce^{i(\alpha+\beta)} + c^2 e^{i(\alpha+2\beta)} + \ldots \text{ to n terms.}$$

This is a geometric series with first term $e^{i\alpha}$ and common ratio $ce^{i\beta}$; hence

$$C_n + iS_n = \frac{e^{i\alpha}(1 - c^n e^{i\beta n})}{(1 - ce^{i\beta})} = \frac{e^{i\alpha}(1 - c^n e^{i\beta n})(1 - ce^{-i\beta})}{(1 - ce^{i\beta})(1 - ce^{-i\beta})}$$

$$\therefore \quad = \frac{e^{i\alpha} - ce^{i(\alpha-\beta)} - c^n e^{i(\alpha+n\beta)} + c^{n+1} e^{i(\alpha+(n-1)\beta)}}{1 - c(e^{i\beta} + e^{-i\beta}) + c^2}.$$

Equating real and imaginary parts, we get

$$C_n = \frac{\cos\alpha - c\cos(\alpha-\beta) - c^n\cos(\alpha+n\beta) + c^{n+1}\cos[\alpha+(n-1)\beta]}{1 - 2c\cos\beta + c^2}$$

$$S_n = \frac{\sin\alpha - c\sin(\alpha-\beta) - c^n\sin(\alpha+n\beta) + c^{n+1}\sin[\alpha+(n-1)\beta]}{1 - 2c\cos\beta + c^2}.$$

Now for sum to infinity, we have since

$$|ce^{i\beta}| = c < 1, \text{ hence } c^n \text{ and } c^{n+1} \to 0 \text{ as } n \to \infty.$$

$$\therefore \quad C_\infty = \frac{\cos\alpha - c\cos(\alpha-\beta)}{1 - 2c\cos\beta + c^2} \quad \text{and} \quad S_\infty = \frac{\sin\alpha - c\sin(\alpha-\beta)}{1 - 2c\cos\beta + c^2}.$$

Example 3. *Sum the series* $1 + \frac{\cos\theta}{\cos\theta} + \frac{\cos 2\theta}{\cos^2\theta} + \frac{\cos 3\theta}{\cos^3\theta} + \ldots$ *n terms.*

Solution. Let $C_n = 1 + \frac{\cos\theta}{\cos\theta} + \frac{\cos 2\theta}{\cos^2\theta} + \frac{\cos 3\theta}{\cos^3\theta} + \ldots$ to n terms

and $S_n = \frac{\sin\theta}{\cos\theta} + \frac{\sin 2\theta}{\cos^2\theta} + \frac{\sin 3\theta}{\cos^3\theta} + \ldots$ to n terms.

$$\therefore \quad C_n + iS_n = 1 + \frac{\cos\theta + i\sin\theta}{\cos\theta} + \frac{\cos 2\theta + i\sin 2\theta}{\cos^2\theta} + \frac{\cos 3\theta + i\sin 3\theta}{\cos^3\theta}$$

$$= 1 + \sec\theta\, e^{i\theta} + \sec^2\theta\, e^{2i\theta} + \ldots \text{ to } n \text{ terms}$$

$$= \frac{(\sec\theta\, e^{i\theta})^n - 1}{\sec\theta\, e^{i\theta} - 1}.$$

Since the series on R.H.S. in G.P. with common ratio $e^{i\theta}\sec\theta$..., the modulus of which is greater than unity, hence

$$C_n + iS_n = \frac{\sec^n\theta\, e^{in\theta} - 1}{\sec\theta(\cos\theta + i\sin\theta) - 1} = \frac{e^{in\theta} - \cos^n\theta}{i\sin\theta \,.\, \cos^{n-1}\theta}$$

$$= \frac{\cos n\theta + i\sin n\theta - \cos^n\theta}{i\sin\theta \,.\, \cos^{n-1}\theta} = \frac{\sin n\theta + i(\cos^n\theta - \cos n\theta)}{\sin\theta \,.\, \cos^{n-1}\theta}$$

$$\therefore \quad C_n = \frac{\sin n\theta}{\sin\theta\cos^{n-1}\theta}.$$

• 6.5. SUM OF FINITE SERIES OF SINE AND COSINE WHOSE ANGLES AR IN AN A.P.

(i) $\sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \ldots\ldots\ldots + \sin(\alpha+(n-1)\beta)$

$$= \frac{\sin\left[\alpha + \left(\frac{n-1}{2}\right)\beta\right]\sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}$$

(ii) $\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \ldots\ldots + \cos(\alpha+(n-1)\beta)$

$$= \frac{\cos\left[\alpha + \left(\frac{n-1}{2}\right)\beta\right]\sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}$$

• 6.6. SUMMATION DEPENDING ON ARITHMETICO-GEOMETRIC SERIES

Example 4. *Sum the series* $3\sin\alpha + 5\sin 2\alpha + 7\sin 3\alpha + \ldots$ *to n terms.*

Solution. Let

$$S = 3\sin\alpha + 5\sin 2\alpha + 7\sin 3\alpha + \ldots \text{ to } n \text{ terms},$$

$$C = 3\cos\alpha + 5\cos 2\alpha + 7\cos 3\alpha + \ldots \text{ to } n \text{ terms}.$$

$$\therefore \quad C + iS = 3e^{i\alpha} + 5e^{2i\alpha} + 7e^{3i\alpha} + \ldots + (2n+1)e^{in\alpha}$$

and $e^{i\alpha}[C + iS] = 3e^{2i\alpha} + 5e^{3i\alpha} + \ldots + (2n-1)e^{in\alpha} + (2n+1)e^{i(n+1)\alpha}$.

Multiplying both sides by the common ratio and shifting by one term.

$$\therefore \quad (1 - e^{i\alpha})[C + iS] = 3e^{i\alpha} + 2[e^{2i\alpha} + e^{3i\alpha} + \ldots + e^{in\alpha}] - (2n+1)e^{i(n+1)\alpha}$$

$$= [e^{i\alpha} - (2n+1)e^{i(n+1)\alpha}] + \frac{2e^{i\alpha}[1 - e^{in\alpha}]}{1 - e^{i\alpha}}$$

$$= [e^{i\alpha} - (2n+1)e^{i(n+1)\alpha}] + \frac{2[1 - e^{in\alpha}]}{e^{-i\alpha} - 1}$$

$$\therefore \quad C + iS = \frac{[e^{i\alpha} - (2n+1)e^{i(n+1)\alpha}]}{1 - e^{i\alpha}} - \frac{2[1 - e^{in\alpha}]}{(1 - e^{-i\alpha})(1 - e^{i\alpha})}$$

$$= \frac{[e^{i\alpha} - (2n+1)e^{i(n+1)\alpha}][1 - e^{-i\alpha}]}{(1 - e^{i\alpha})(1 - e^{-i\alpha})} - \frac{2(1 - e^{in\alpha})}{(1 - e^{-i\alpha})(1 - e^{i\alpha})}$$

$$= \frac{e^{i\alpha} - (2n+1)e^{i(n+1)\alpha} + (2n+3)e^{in\alpha} - 3}{2 - 2\cos\alpha}.$$

Hence equating imaginary parts,

$$S = \frac{\sin\alpha - (2n+1)\sin(n+1)\alpha + (2n+3)\sin n\alpha}{2(1 - \cos\alpha)}.$$

Example 5. *If S_n denotes the sum of n terms of the series $\sin x + \sin 2x + \ldots$ n terms show that*

$$\lim_{n \to \infty} \frac{(S_1 + S_2 + \ldots S_n)}{n} = \frac{1}{2}\cot\frac{x}{2}.$$

Solution. We have

$$S_n = \sin x + \sin 2x + \ldots \sin nx.$$

$$\therefore \quad C_n + iS_n = e^{ix} + e^{2ix} + \ldots e^{inx} = \frac{e^{ix}[1 - e^{inx}]}{1 - e^{ix}}$$

$$= \frac{e^{ix}[1 - e^{inx}][1 - e^{-ix}]}{[1 - e^{ix}][1 - e^{-ix}]} = \frac{e^{ix} - 1 - e^{i(n+1)x} + e^{inx}}{2[1 - \cos x]}$$

So $$S_n = \frac{\sin x - \sin(n+1)x + \sin nx}{2[1 - \cos x]} = \frac{\sin x}{2[1 - \cos x]} - \frac{\cos\left(n + \frac{1}{2}\right)x \sin\frac{x}{2}}{[1 - \cos x]}$$

Now $$\frac{S_1 + S_2 + \ldots S_n}{n} = \frac{\sum_{k=1}^{n} S_k}{n} = \frac{\sin x}{2[1 - \cos x]} - \frac{\sin x/2}{(1 - \cos x)} \sum_{k=1}^{n} \frac{\cos\left(k + \frac{1}{2}\right)x}{n}.$$

Also $$\sum_{k=1}^{n} \cos\left(k + \frac{1}{2}\right)x = \cos\frac{3}{2}x + \cos\frac{5}{2}x + \ldots \cos\left(n + \frac{1}{2}\right)x.$$

Let $$C = \cos\frac{3}{2}x + \cos\frac{5}{2}x + \ldots \cos\left(n + \frac{1}{2}\right)x.$$

Then $C + iS = e^{\frac{3}{2}ix} + e^{\frac{5}{2}ix} + \ldots e^{\left(n + \frac{1}{2}\right)ix}$

$$= e^{\frac{3}{2}ix}\frac{[1 - e^{inx}]}{1 - e^{ix}} = \frac{e^{\frac{3}{2}ix}[1 - e^{inx}][1 - e^{-ix}]}{[1 - e^{ix}][1 - e^{-ix}]}$$

$$= \frac{e^{\frac{3}{2}ix} - e^{i\left(n + \frac{3}{2}\right)x} - e^{i\frac{x}{2}} + e^{i\left(n + \frac{1}{2}\right)x}}{2[1 - \cos x]}$$

$$\therefore \quad C = \frac{\cos\frac{3}{2}x - \cos\left(n + \frac{3}{2}\right)x - \cos\frac{x}{2} + \cos\left(n + \frac{1}{2}\right)x}{2[1 - \cos x]}$$

Now $$\lim_{n \to \infty} \frac{S_1 + S_2 + \ldots S_n}{n} = \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{2 \times 2\sin^2\frac{x}{2}} - \lim_{n \to \infty} \frac{1}{n}[C]$$

$$= \frac{1}{2}\cot\frac{x}{2} - \lim_{n\to\infty}\frac{1}{n}\left[\frac{\sin\frac{x}{2}}{2[1-\cos x]^2}\left\{\cos\frac{3}{2}x - \cos\left(n+\frac{3}{2}\right)x - \cos\frac{x}{2} + \cos\left(n+\frac{1}{2}\right)x\right\}\right]$$

$$= \frac{1}{2}\cot\frac{x}{2} - \lim_{n\to\infty}\frac{1}{n}\,[\text{finite quantity}] = \frac{1}{2}\cot\frac{x}{2}.$$

• TEST YOURSELF–2

Sum the following series to n terms :

1. $\sin A + \sin 3A + \sin 5A + \ldots$ and deduce the sum of $1 + 3 + 5 \ldots$ to n terms.
2. $\cos^2\theta + \cos^2 2\theta + \cos^2 3\theta + \ldots$.
3. Prove that the two series $\sin\frac{\pi}{14} + \sin\frac{2\pi}{14} + \sin\frac{3\pi}{14} + \ldots$ to 28 terms,

   and $\cos\frac{\pi}{14} + \cos\frac{2\pi}{14} + \cos 3\frac{\pi}{14} + \ldots$ to 28 terms,

   have the same sum. What is the magnitude of the sum ?
4. Sum the series $\cos\theta\cos 2\theta + \cos 2\theta\cos 3\theta + \cos 3\theta\cos 4\theta + \ldots$ to 20 terms.
5. Sum the following series :

   $\cos x - \sin 2x - \cos 3x + \sin 4x + \ldots$ to n terms.
6. Sum to infinity the following series :

   (i) $\sin\alpha + \frac{1}{2}\sin 2\alpha + \frac{1}{2^2}\sin 3\alpha + \ldots$ ad. inf.

   (ii) $\frac{\sin\alpha}{\tan\beta} - \frac{\sin 2\alpha}{\tan^2\beta} + \frac{\sin 3\alpha}{\tan^3\beta} - \ldots$ ad inf.

   where $\tan\beta > 1$ numerically.
7. Find the sum of the following series to n terms and hence deduce the sum to infinity given $\alpha \neq \frac{\pi}{2}$

   (i) $\cos\alpha\sin\alpha + \cos^2\alpha\sin 2\alpha + \cos^3\alpha\sin 3\alpha + \ldots$

   (ii) $\cos\alpha\cos\alpha + \cos^2\alpha\cos 2\alpha + \cos^3\alpha\cos 3\alpha + \ldots$

   (iii) $\cos\alpha\sin\alpha + \cos 2\alpha\sin^2\alpha + \cos 3\alpha\sin^3\alpha + \ldots$

ANSWERS

1. $(\sin^2 nA)/\sin A$; n^2.
2. $\frac{1}{2}\{n + \cos(n+1)\theta\sin n\theta\,\mathrm{cosec}\,\theta\}$
3. Magnitude of the sum = 0.
4. $\frac{1}{2}\left[2\theta\cos\theta + \frac{\cos 12\theta\sin 10\theta}{\sin\theta}\right]$.
5. $\dfrac{\sin\left[\frac{n+1}{2}\left(\frac{\pi}{2}+x\right)\right]\sin\left(\frac{n\pi}{4}+\frac{nx}{2}\right)}{\sin\left(\frac{\pi}{4}+\frac{x}{2}\right)}$

   (i) $s_n = \dfrac{\cos^{n+1}\alpha\sin n\alpha}{\sin\alpha}$ (ii) $s_n = \cos\alpha\,(1 - \cos^n\alpha\cos n\alpha)$
7. (i) $\dfrac{1+\cos n\alpha}{2(1-\cos\alpha)} + \dfrac{n\sin\left(\frac{2\pi\alpha+\alpha}{2}\right)}{2\sin\alpha/2}$

   (ii) $C_n = \dfrac{1+\cos 2\alpha - 2\cos\alpha + (-1)^n\cos n\alpha + (-1)^n\cos(n\alpha - 2\alpha)}{(2+2\cos\alpha)^2} - \dfrac{(-1)^n n\,[\cos n\alpha + \cos(n\alpha-\alpha)]}{2(1+\cos\alpha)}$

   (iii) $C_\infty = \dfrac{1+\frac{1}{4}\cos 2\alpha - \cos\alpha}{\left(\frac{1}{4}-\cos\alpha\right)^2}$

• 6.7. USE OF BINOMIAL SERIES

Following are the binomial expansions when z is complex.

(i) If n is positive integer,

$$(1+z)^n = 1 + {}^nC_1 z + {}^nC_2 z^2 + \ldots + z^n.$$

(ii) If n is any quantity (say negative integer) or a positive or negative fraction,

$$(1+z)^n = 1 + nz + \frac{n(n-1)}{2!} z^2 + \frac{n(n-1)(n-2)}{3!} z^3 + \ldots \text{ ad. inf.}$$

Provided $|z| < 1$, when $|z| = 1$, this result is still true if (i) $n > 0$ or if (ii) $-1 < n < 0$ and $z \neq -1$.

With the help of results (i) and (ii) we can recognise all binomial expansions without remembering any more fromula. The method is illustrated in the following solved examples.

Example 1. *Sum the series*

$$1 + \frac{1}{2} \cos 2\alpha - \frac{1}{2.4} \cos 4\alpha + \frac{1.3}{2.4.6} \cos 6\alpha - \ldots \text{ ad. inf.}$$

where α *lies between* $-\frac{\pi}{2}$ *and* $\frac{\pi}{2}$.

Solution. Let $\quad C = 1 + \frac{1}{2} \cos 2\alpha - \frac{1}{2.4} \cos 4\alpha + \frac{1.3}{2.4.6} \cos 6\alpha - \ldots$

$$S = \frac{1}{2} \sin 2\alpha - \frac{1}{2.4} \sin 4\alpha + \frac{1.2}{2.4.6} \sin 6\alpha - \ldots$$

$$\therefore \quad C + iS = 1 + \frac{1}{2} e^{2i\alpha} - \frac{1}{2.4} e^{4i\alpha} + \frac{1.3}{2.4.5} e^{6i\alpha} - \ldots .$$

Comparing this with

$$(1+z)^n = 1 + nz + \frac{n(n-1)}{2!} z^2 + \ldots$$

we get
$$nz = \frac{1}{2} e^{2i\alpha}, \quad \ldots(1) \qquad \frac{n(n-1)}{2} z^2 = -\frac{1}{8} e^{4i\alpha}. \quad \ldots(2)$$

$$\therefore \quad \frac{n(n-1)}{2n^2} = -\frac{1}{8} \times \frac{4}{1} \qquad \text{[dividing (2) by square of (1)]}$$

$$\therefore \quad \frac{n-1}{n} = -1 \quad \text{or} \quad n = \frac{1}{2},$$

$$\therefore \quad z = e^{2i\alpha}.$$

Hence $\quad C + iS = (1 + e^{2i\alpha})^{1/2} \qquad$ [Note that $n > 0$ and $|e^{2i\alpha}| = 1$]

$$= (1 + \cos 2\alpha + i \sin 2\alpha)^{1/2} = \sqrt{2 \cos \alpha}\, [\cos \alpha + i \sin \alpha]^{1/2}$$

$$= \sqrt{2 \cos \alpha} \left[\cos \frac{\alpha}{2} + i \sin \frac{\alpha}{2} \right].$$

Equating real and imaginary parts, we have

$$C = \sqrt{2 \cos \alpha} \,.\, \cos \frac{\alpha}{2} = \sqrt{\cos \alpha (1 + \cos \alpha)},$$

$$S = \sqrt{2 \cos \alpha} \,.\, \sin \frac{\alpha}{2} = \sqrt{\cos \alpha (1 - \cos \alpha)}.$$

Example 2. *Sum the series*

$$\sin \alpha + \frac{1}{2} \sin 3\alpha + \frac{1.3}{2.4} \sin 5\alpha + \ldots \text{ ad. inf.}$$

Solution. Let

$$S = \sin \alpha + \frac{1}{2} \sin 3\alpha + \frac{1.3}{2.4} \cos 5\alpha + \ldots$$

$$C = \cos \alpha + \frac{1}{2} \cos 3\alpha + \frac{1.3}{2.4} \sin 5\alpha + \ldots$$

$$C + iS = e^{i\alpha} + \frac{1}{2} e^{3i\alpha} + \frac{1.3}{2.4} e^{5i\alpha} + \ldots = e^{i\alpha} \left[1 + \frac{1}{2} e^{i\alpha} + \frac{1.3}{2.4} e^{4i\alpha} + \ldots \right].$$

Comparing the series within square brackets with

$$(1+z)^n = 1 + nz + \frac{n(n-1)}{2!} z^2 + \ldots,$$

we have $nz = \frac{1}{2} e^{2i\alpha}$, $\frac{n(n-1)}{2!} z^2 = \frac{3}{8} z^{4i\alpha}$.

Solving, we get $n = -\frac{1}{2}$, $z = -e^{2i\alpha}$; hence

$$C + iS = e^{i\alpha}(1 - e^{2i\alpha})^{-1/2}, \qquad \text{Provided } \alpha \neq m\pi,\ m \text{ being an integer}$$

$$= [\cos\alpha + i\sin\alpha]\,[1 - \cos 2\alpha - i\sin 2\alpha]^{-1/2}$$

$$= (\sqrt{2\sin\alpha})^{-1/2}[\cos\alpha + i\sin\alpha]\,[\sin\alpha - i\cos\alpha]^{-1/2}$$

$$= (\sqrt{2\sin\alpha})^{-1/2}[\cos\alpha + i\sin\alpha]\left[\cos\left(\frac{\pi}{2} - \alpha\right) - i\sin\left(\frac{\pi}{2} - \alpha\right)\right]^{-1/2}$$

$$= (\sqrt{2\sin\alpha})^{-1/2}[\cos\alpha + i\sin\alpha]\left[\cos\left(\frac{\pi}{4} - \frac{\alpha}{2}\right) + i\sin\left(\frac{\pi}{4} - \frac{\alpha}{2}\right)\right].$$

Equating imaginary parts, we get

$$S = (\sqrt{2\sin\alpha})^{-1/2}\left[\cos\alpha\sin\left(\frac{\pi}{4} - \frac{\alpha}{2}\right) + \sin\alpha\cos\left(\frac{\pi}{4} - \frac{\alpha}{2}\right)\right]$$

$$= (\sqrt{2\sin\alpha})^{-1/2}\left[\sin\left(\alpha + \frac{\pi}{4} - \frac{\alpha}{2}\right)\right] = (\sqrt{2\sin\alpha})^{-1/2}\sin\left(\frac{\pi}{4} + \frac{\alpha}{2}\right).$$

• 6.8. USE OF EXPONENTIAL SERIES

The following are important results for complex z :

(i) $e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} \ldots$ad inf. (ii) $e^{-z} = 1 - z + \frac{z^2}{2!} - \frac{z^3}{3!} + \ldots$ad inf.

(iii) $\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots$ad inf. (iv) $\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \ldots$ad inf.

(v) $\cosh z = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \ldots$ad inf. (vi) $\sinh z = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \ldots$ad inf.

Method of summing up such series which reduce to either of the forms given above is illustrated below.

Example 3. *Sum the series*

$$1 - \cos\alpha\cos\beta + \frac{\cos^2\alpha\cos 2\beta}{2!} - \frac{\cos^3\alpha\cos 3\beta}{3!} + \ldots \textit{ ad inf.}$$

Solution. Let

$$C = 1 - \cos\alpha\cos\beta + \frac{\cos^2\alpha\cos 2\beta}{2!} - \frac{\cos^3\alpha\cos 3\beta}{3!} + \ldots$$

$$S = -\cos\alpha\sin\beta + \frac{\cos^2\alpha\sin 2\beta}{2!} - \frac{\cos^3\sin 3\beta}{3!}.$$

$$\therefore \quad C + iS = 1 - \cos\alpha\, e^{i\beta} + \frac{\cos^2\alpha e^{2i\beta}}{2!} - \frac{\cos^3\alpha e^{3i\beta}}{3!} + \ldots$$

$$= \exp\{-\cos\alpha e^{i\beta}\} = \exp\{-\cos\alpha(\cos\beta + i\sin\beta)\}$$

$$= \exp\{-\cos\alpha\cos\beta\}\,[\cos(\cos\alpha\sin\beta) - i\sin(\cos\alpha\sin\beta)]$$

$$\therefore \quad C = \exp\{-\cos\alpha\cos\beta\}\cos(\cos\alpha\sin\beta).$$

• 6.9. USE OF LOGARITHMIC AND GREGORY'S SERIES

If $|z| \le 1$, but $z \neq -1$,

$$\log(1 + z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \ldots \text{ ad. inf.} \qquad \ldots\text{(i)}$$

If $|z| \le 1$, but $z \neq 1$,

$$\log(1 - z) = -\left(z + \frac{z^2}{2} + \frac{z^3}{3} \ldots\text{ad. inf.}\right) \qquad \ldots\text{(ii)}$$

From (i) and (ii), we have

$$\log(1 + z) + \log(1 - z) = -2\left[\frac{z^3}{2} + \frac{z^4}{4} + \frac{z^6}{6} + \ldots\right]$$

and $$\log(1+z)-\log(1-z)=2\left[z+\frac{z^3}{3}+\frac{z^5}{5}+\ldots\right].$$

Gregory's series is, if $-1\le|z|\le 1$, then

$$\tan^{-1}z=z-\frac{z^3}{3}+\frac{z^5}{5}\ldots \text{ ad. inf.}$$

The method of summing such series is given below.

Example 4. *Sum the following series;*

(*i*) $\cos\alpha-\frac{1}{2}\cos 2\alpha+\frac{1}{3}\cos 3\alpha-\ldots ad.\ inf.$

(*ii*) $\sin\alpha-\frac{1}{2}\sin 2\alpha+\frac{1}{3}\sin 3\alpha-\ldots ad.\ inf.$

Solution. Let series (i) and (ii) be denoted by C and S respectively, then

$$C+iS=e^{i\alpha}-\frac{1}{2}e^{2i\alpha}+\frac{1}{3}e^{3i\alpha}-\ldots$$

$$=\log(1+e^{i\alpha}) \text{ provided } e^{i\alpha}\ne 1 \text{ or } \alpha\ne(2n+1)\pi$$

$$=\log(1+\cos\alpha+i\sin\alpha)=\log\left[2\cos\frac{\alpha}{2}\left(\cos\frac{\alpha}{2}+i\sin\frac{\alpha}{2}\right)\right]$$

$$=\log\left[2\cos\frac{\alpha}{2}e^{i(\alpha/2)}\right]=\log\left(2\cos\frac{\alpha}{2}\right)+i\frac{\alpha}{2}.$$

$$\therefore\qquad C=\log\left(2\cos\frac{\alpha}{2}\right),\quad S=\frac{\alpha}{2}.$$

Condition is that $\alpha\ne(2n+1)\pi$.

Example 5. *Find the sum of the series*

(*i*) $a\cos\alpha-\frac{1}{2}a^2\cos 2\alpha+\frac{1}{3}a^3\cos 3\alpha-\ldots ad.\ inf.$

(*ii*) $a\sin\alpha-\frac{1}{2}a^2\sin 2\alpha+\frac{1}{3}a^3\sin 3\alpha-\ldots ad.\ inf.$

Solution. Let series (i) and (ii) be denoted by C and S respectively : then

$$C+iS=ae^{i\alpha}-\frac{1}{2}a^2e^{2i\alpha}+\frac{1}{3}a^3e^{3i\alpha}-\ldots$$

$$=\log(1+ae^{i\alpha}) \text{ provided } |a|\le 1, \text{ and } ae^{i\alpha}\ne -1$$

$$=\log(1+a\cos\alpha+ai\sin\alpha)$$

$$=\frac{1}{2}\log[1+a^2+2a\cos\alpha]+i\tan^{-1}\frac{a\sin\alpha}{1+a\cos\alpha}.$$

$$\left[\text{since } \log(x+iy)=\frac{1}{2}\log(x^2+y^2)+i\tan^{-1}\frac{y}{x}\right]$$

Equating real and imaginary parts, we get

$$C=\frac{1}{2}\log[1+a^2+2a\cos\alpha],\ S=\tan^{-1}\frac{a\sin\alpha}{1+a\cos\alpha}.$$

Conditions of validity are

(i) $|ae^{i\alpha}|\le 1$, *i.e.* $|a|\le 1$ numerically

and (ii)α is not odd multiple of π when $a=1$ and even multiple of π when $a=-\alpha$, *i.e.*,

$$ae^{i\alpha}\ne -1.$$

• TEST YOURSELF–3

Sum the following series :

1. $\cos\alpha+\frac{\sin\alpha\cos 2\alpha}{1!}+\frac{\sin^2\alpha\cos 3\alpha}{2!}+\ldots$ ad inf.

2. $\cos\alpha+\frac{\cos\alpha\cos 2\alpha}{1!}+\frac{\cos^2\alpha\cos 3\alpha}{2!}+\ldots$ ad inf.

3. $\sin\alpha\sin\alpha-\frac{1}{2}\sin^2\alpha\sin 2\alpha+\frac{1}{3}\sin^3\alpha\sin 3\alpha-\ldots$

4. $\sin\alpha\sin\beta+\frac{1}{2}\sin 2\alpha\sin 2\beta+\frac{1}{3}\sin 3\alpha\sin 3\beta+\ldots$ ad inf.

5. $\cos\frac{\pi}{3}+\frac{1}{2}\cos\frac{2\pi}{3}+\frac{1}{3}\cos\frac{3\pi}{3}+\ldots$ ad inf.

6. (i) $c\cos\alpha+\frac{1}{2}c^2\cos 2\alpha+\frac{1}{3}c^3\cos 3\alpha+\ldots$ ad inf.

(ii) $c\cos\alpha-\frac{1}{2}c^2\cos 2\alpha+\ \ldots$ ad inf.

7. $c\sin\alpha+\frac{c^2}{2}\sin 2\alpha+\frac{c^3}{3}\sin 3\alpha+\ldots$ ad inf.

ANSWERS

1. $e^{\sin\alpha\cos\alpha}\,.\cos(\alpha+\sin^2\alpha)$.
2. $e^{\cos^2\alpha}\,.\cos(\alpha+\sin\alpha\cos\alpha)$.
3. $\tan^{-1}\left[\frac{\sin^2\alpha}{1+\sin\alpha\cos\alpha}\right]$.
4. $\frac{1}{2}\left[\log\operatorname{cosec}\frac{\alpha-\beta}{2}-\log\operatorname{cosec}\frac{\alpha+\beta}{2}\right]$.
5. 0.
6. (i) $-\frac{1}{2}\log(1-2c\cos\alpha+c^2)$ (ii) $\frac{1}{2}\log[1+c^2+2c\cos\alpha]$.
7. $\tan^{-1}\frac{c\sin\alpha}{1-c\cos\alpha}$.

• 6.10. THE DIFFERENCE METHOD

Sometimes it is easier to sum the series by expressing each term as the difference of two terms, so that the expressions into successive terms cancel out, leaving only one or two terms. No particular method can be given for splitting the terms and it generally depends upon the practice and chance in many cases.

Let T_n be the nth term of a series and let it be expressed in the form

$$T_n=C\,[f(n+1)-f(n)]$$

Then

$$\begin{aligned}S_n&=T_1+T_2+\ldots+T_n\\&=C\,[f(2)-f(1)+f(3)-f(2)+\ldots+f(n+1)-f(n)]\\&=C\,[f(n+1)-f(1)],\end{aligned}$$

since the intermediate terms all cancel out.

If series is convergent and sum to infinity is required, we deduce the sum as below :

$$\lim_{n\to\infty}S_n=\lim_{n\to\infty}[f(n+1)-f(1)].$$

REMARK

If a series is such that its n^{th} term is of the form $\tan^{-1}\left(\frac{a}{b}\right)$ then we put

$$T_n=\tan^{-1}\left(\frac{a}{b}\right)=\tan^{-1}x-\tan^{-1}y=\tan^{-1}\frac{x-y}{1+xy},$$

i.e., $\left.\begin{aligned}x-y&=a\\xy&=b-1.\end{aligned}\right\}$ $\therefore\ (x+y)=\sqrt{a^2+4(b-1)}$.

Solving, we get x and y. Then puting $x=1,2,3,\ldots$ and adding, we get the required sum.

• SOLVED EXAMPLES

Example 1. *Sum the series to n terms*

$$\tan^{-1}\frac{4}{1+3.4}+\tan^{-1}\frac{6}{1+8.9}+\tan^{-1}\frac{8}{1+15.16}+\ldots.$$

Solution. Here $T_n=\tan^{-1}\frac{2(n+1)}{1+(n+2)\,n\,(n+1)^2}$

$$=\tan^{-1}x-\tan^{-1}y=\tan^{-1}\frac{x-y}{1+xy}$$

$\therefore\quad x-y=2(n+1),\ xy=n(n+2)(n+1)^2.$

$\therefore\quad x+y=2(n+1)^2.$

$\therefore\quad x=(n+2)(n+1),\ y=n(n+1).$

$\therefore \qquad T_n = \tan^{-1}[(n+2)(n+1)] - \tan^{-1}[(n+1)n].$

Now giving values of n as 1, 2, 3, ... n, we have

$$T_1 = \tan^{-1} 3.2 - \tan^{-1} 2.1,$$
$$T_2 = \tan^{-1} 4.3 - \tan^{-1} 3.2,$$
$$T_3 = \tan^{-1} 5.4 - \tan^{-1} 4.3,$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$
$$T_n = \tan^{-1}[(n+2)(n+1)] - \tan^{-1}[(n+1)n].$$

Adding, we get

$$S_n = \tan^{-1}\{(n+2)(n+1)\} - \tan^{-1} 2.1 = \tan^{-1}(n^2+3n+2) - \tan^{-1} 2$$
$$= \tan^{-1}\frac{n^2+3n}{1+2(n^2+3n+2)} = \tan^{-1}\frac{n^2+3n}{2n^2+6n+5}.$$

Example 2. *Sum the series*

$$\tan^{-1}\frac{1}{3+3.1+1^2} + \tan^{-1}\frac{1}{3+3.2+2^2} + \tan^{-1}\frac{1}{3+3.3+3^2} + \ldots + \tan^{-1}\frac{1}{3+3n+n^2}.$$

Solution. Let $\quad T_n = \tan^{-1}\frac{1}{3+3n+n^2} = \tan^{-1}\frac{1}{1+2+3n+n^2}$

$$= \tan^{-1} x - \tan^{-1} y = \tan^{-1}\frac{x-y}{1+xy}.$$

$$\therefore \qquad x - y = 1$$
$$xy = n^2+3n+2,$$
$$x + y = \sqrt{1+4(n^2+3n+2)} = 2n+3.$$
$$\therefore \qquad x = n+2, y = n+1.$$
$$\therefore \qquad T_n = \tan^{-1}(n+2) - \tan^{-1}(n+1).$$

Putting $n = 1, 2, 3, \ldots n$ and adding, we get

$$S_n = \tan^{-1}(n+2) - \tan^{-1} 2.$$

Example 3. *Sum the series* $\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{13} + \ldots$ *n terms.*

Solution. Method I. The given series can be put in the form

$$S_n = \tan^{-1}\frac{2-1}{1+2.1} + \tan^{-1}\frac{3-2}{1+3.2} + \tan^{-1}\frac{4-3}{1+4.3} + \ldots + \tan^{-1}\frac{(n+1)-n}{1+(n+1)n}$$
$$= (\tan^{-1} 2 - \tan^{-1} 1) + (\tan^{-1} 3 - \tan^{-1} 2) + (\tan^{-1} 4 - \tan^{-1} 3) + \ldots$$
$$+ (\tan^{-1}(n+1) = \tan^{-1} n) - \tan^{-1}(n+1) - \tan^{-1} 1$$
$$= \tan^{-1}\frac{n}{n+2}.$$

Example 4. *Sum the series*

$$\sec\theta \sec 2\theta + \sec 2\theta \sec 3\theta + \sec 3\theta \sec 4\theta + \ldots \text{ to } n \text{ terms.}$$

Solution. $\qquad T_n = \sec n\theta \sec(n+1)\theta = \frac{1}{\cos n\theta \cos(n+1)\theta}$

$$= \frac{1}{\sin\theta}\left[\frac{\sin\theta}{\cos n\theta\cos(n+1)\theta}\right] = \frac{1}{\sin\theta}\left[\frac{\sin(n\theta+\theta-n\theta)}{\cos n\theta\cos(n+1)\theta}\right]$$
$$= \frac{1}{\sin\theta}\left[\frac{\sin(n+1)\theta\cos n\theta - \cos(n+1)\theta\sin n\theta}{\cos n\theta\cos(n+1)\theta}\right]$$
$$= \operatorname{cosec}\theta\,[\tan(n+1)\theta - \tan n\theta].$$

Putting $n = 1, 2, 3, \ldots, n$ and adding.

$$S_n = \operatorname{cosec}\theta\,[\tan(n+1)\theta - \tan\theta].$$

• STUDENT ACTIVITY

1. Sum the series $1 - \frac{1}{3 \cdot 4^2} + \frac{1}{5 \cdot 4^4} - \ldots\ldots \infty$.

2. Sum the series $\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{13} + \ldots\ldots n$ terms

• TEST YOURSELF–4

Sum the following series :

1. $\operatorname{cosec}\theta + \operatorname{cosec}2\theta + \operatorname{cosec}2^2\theta + \ldots$ *n terms.*
2. $\tan\alpha\tan(\alpha+\beta) + \tan(\alpha+\beta)\tan(\alpha+2\beta) + \tan(\alpha+2\beta)\tan(\alpha+3\beta) + \ldots$ to n terms.
3. $\tan\theta\tan^2\frac{\theta}{2} + 2\tan\frac{\theta}{2}\tan^2\frac{\theta}{2^2} + 2^2\tan\frac{\theta}{2^2}\tan^2\frac{\theta}{2^3} + \ldots$ to n terms.
4. $\tan^2\theta\tan2\theta + \frac{1}{2}\tan^2 2\theta\tan4\theta + 1/2^2\tan^2 4\theta\tan8\theta + \ldots$ to n terms.
5. $\tan^{-1}\frac{2}{4} + \tan^{-1}\frac{2}{9} + \tan^{-1}\frac{2}{16} + \ldots$ to n terms.
6. $\tan^{-1}x + \tan^{-1}\frac{x}{1+1.2 \cdot x^2} + \tan^{-1}\frac{x}{1+2.3 \cdot x^2} + \ldots$ to n terms.
7. $\cot^{-1}3 + \cot^{-1}7 + \cot^{-1}13 + \ldots + \cot^{-1}(1+n+n^2)$.
8. Sum the series $\sum_{1}^{n} \tan^{-1}\frac{4}{4n^2+3}$.
9. $\tan^{-1}\frac{1}{1+1+1^2} + \tan^{-1}\frac{1}{1+2+2^2} + \tan^{-1}\frac{1}{1+3+3^2} + \ldots$ to n terms.
10. (a) $\cot^{-1}(2.1)^2 + \cot^{-1}(2.2^2) + \cot^{-1}(2.3^2) + \ldots$ ad. inf.
    Deduce its sum to infinity.

(b) $\tan^{-1}\frac{1}{2.1^2}+\tan^{-1}\left(\frac{1}{2.2^2}\right)+\tan^{-1}\left(\frac{1}{2.3^2}\right)+\ldots\tan^{-1}\frac{1}{2n^2}$.

Deduce its sum to infinity.

11. $\cot^{-1}\left(1^2+\frac{3}{4}\right)+\cot^{-1}\left(2^2+\frac{3}{4}\right)+\cot^{-1}\left(3^2+\frac{3}{4}\right)+\ldots$.

12. $\tan\theta\sec 2\theta+\tan 2\theta\sec 4\theta+\tan 4\theta\sec 8\theta+\ldots n$ terms.

ANSWERS

1. $\cot\frac{\theta}{2}-\cot 2^{n-1}\theta$.

2. $\cot\beta\,[\tan(\alpha+n\beta)-\tan\alpha]-n$.

3. $\tan\theta-2^n\tan\frac{\theta}{2^n}$.

4. $\frac{1}{2^{n-1}}\tan 2^n\theta-2\tan\theta$.

5. $\tan^{-1}(n+2)+\tan^{-1}(n+1)-\tan^{-1}2-\tan^{-1}1$.

6. $\tan^{-1}nx$.

7. $\tan^{-1}\frac{n}{n+2}$.

8. $\tan^{-1}\frac{4n}{2n+5}$.

9. $\tan^{-1}(n+1)-\tan^{-1}1$.

10. (a) $\frac{\pi}{4}$.

11. $\tan^{-1}\frac{4n}{4n+5}$.

12. $\tan 2^n\theta-\tan\theta$.

OBJECTIVE EVALUATION

FILL IN THE BLANKS :

1. In $C+iS$ method C is a cosine and S is a ...... series.
2. The sum of the series $\cos\alpha+\frac{1}{2}\cos 2\alpha+\frac{1}{2^2}\cos 3\alpha+\ldots\infty$ is ...
3. The sum of $\operatorname{cosec}\theta\operatorname{cosec}2\theta+\operatorname{cosec}2\theta\operatorname{cosec}3\theta+\operatorname{cosec}3\theta\operatorname{cosec}4\theta+\ldots$ upto n terms is ...

TRUE OR FALSE :

Write T for True and F for False statement :

1. The exponential and binomial series is not convergent. *(T/F)*
2. The sum of the series is $\frac{\sinh\alpha}{1!}+\frac{\sinh 2\alpha}{2!}+\frac{\sinh 3\alpha}{3!}+\ldots$ is equal to $\frac{1}{2}e^{\cosh\alpha}.\sinh(\sinh\alpha)$ *(T/F)*
3. The sum of the infinite series $\frac{\sin\theta}{1!}+\frac{\sin 2\theta}{2!}+\frac{\sin 3\theta}{3!}+\ldots$ is $e^{\cos\theta}\sin(\sin\theta)$. *(T/F)*
4. The sum of infinite series $1+\frac{\cos\theta}{\cos\theta}+\frac{\cos 2\theta}{\cos^2\theta}+\frac{\cos 3\theta}{\cos^3\theta}+\ldots$ is 0. *(T/F)*

MULTIPLE CHOICE QUESTIONS :

Choose the most appropriate one :

1. $C+iS$ method of finding the sum of these series which involve :
   (a) sine and tangents of multiple of angles
   (b) cosine and cotangent of multiple of angles
   (c) sine and cosine of multiple of angles
   (d) tangent and cotangent of multiple of angles
   (e) None of these.
2. If $C+iS$ method of finding the sum, the resulting series is $a+ar+ar^2+\ldots$to n terms then we use the formula :
   (a) $S_n=\frac{a(r^n-1)}{1-r}$ (b) $S_n=\frac{a(1-r^n)}{r-1}$ (c) $S_n=\frac{a(1-r^n)}{1-r}$
   (d) $S_n=\frac{a}{1-r}$ (e) None of these.
3. If in above case the resulting series is $a+ar+ar^2+ar^3+\ldots\infty$, then we use the following formula to find its sum :

(a) $S = \frac{a(r^n - 1)}{1 - r}$ (b) $S = \frac{a(1 - r^n)}{r - 1}$ (c) $S = \frac{a(1 - r^n)}{1 - r}$

(d) $S = \frac{a}{1 - r}$ (e) None of these.

4. Sum to infinite terms of the series $1 + \frac{\cos\theta}{\cos\theta} + \frac{\cos 2\theta}{\cos^2\theta} + \frac{\cos 3\theta}{\cos^3\theta} + \ldots \infty$ is :

(a) $\tan\theta$ (b) 0 (c) $\cot\theta$ (d) $\sec\theta$ (e) None of these.

ANSWERS

Fill in the Blanks :

1. sine 2. $\frac{4\cos\alpha - 2}{5 - 4\cos\alpha}$ 3. $\operatorname{cosec}\theta\,[\cot\theta - \cot(n+1)\theta]$

True or False :

1. F **2.** T **3.** T **4.** F.

Multiple Choice Questions :

1. (c) **2.** (c) **3.** (d) **4.** (b)

UNIT

7

RANK OF A MATRIX

LEARNING OBJECTIVES

- Marices
- Sub-Matrix of a Matrix
- Rank of a Matrix
- Elementary ransformations of a Matrix
- Elemenary Matrices
- Invariance of Rank Under *E*-Transformation
- Normal Form
- Equivalence of Matrices
- Solved Examples
  - Student Activity
  - Summary
  - Test Yourself

LEARNING OBJECTIVES

After going through this unit you will learn :

- About the Matrices in detail.
- How to calculate the rank of a given matrices.
- About the equivalent matrices.

• 7.1. MATRICES

1. Definition :

A set having mn numbers either real or complex, arranged in the form of a rectangular array in which there are m rows and n columns. This rectangular arrangement is called a **matrix** of **order** $m \times n$ which is denoted by $[a_{ij}]_{m \times n}$ where $i = 1, 2, 3, \ldots m$ and $j = 1, 2, 3, \ldots n$ and a matrix of order $m \times n$ is usually written as

$$[a_{ij}]_{m \times n} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}_{m \times n}$$

• 7.2. SUB-MATRIX OF A MATRIX

Definition. Let A be a matrix of order $m \times n$, then a matrix which is obtained by leaving some rows and columns from the given matrix A, is called a **submatrix** of a matrix A. **For example**

Let $\quad A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}_{3 \times 4}$

Then the matrix $B = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

is a submatrix of A, which is obtained by leaving second row and fourth column.

If the given matrix A is a square matrix, then a square submatrix of the given matrix is called **Principal submatrix.**

Minors of a Matrix :

Definition. Let A be a matrix of order $m \times n$, then the determinant of every square submatrix of A is called a minor of A.

• 7.3. RANK OF A MATRIX

Definition. A positive integer r is said to be the rank of a matrix A if it contains at least one square submatrix of order $r \times r$ whose determinant is non-zero while any square submatrix of A of order $(r+1) \times (r+1)$ or greater is singular *i.e.*, having determinant zero. The rank of a matrix A is denoted by $\rho(A)$.

It is obvious that the rank r of a matrix of order $m \times n$ may at most be equal to the smaller of the numbers m and n, but it may be less.

If the rank of a square matrix A of order $n \times n$ is r and $r < n$, then the matrix A is said to be *singular*, on the other hand if $r = n$, then the matrix is said to be *non-singular.*

REMARKS

If the rank of a matrix is zero, then matrix is a **null matrix**.

The rank of every non-zero matrix must be greater than or equal to 1.

The rank of a unit matrix is equal to the order of the unit matrix.

Echelon form of a Matrix :

Definition. A matrix A *is said to be in* **Echelon form** *if it satisfies following conditions :*

(i) Every row of A has all its entries zero which occurs below the every row having a non-zero entry.

(ii) The number of zeros before the first non-zero entry in the same row is less than the number *of zeros in the next row.*

REMARK

The rank of a matrix is equal to the number of non-zero rows in **Echelon form** of the given matrix.

For example :

Let $$A = \begin{bmatrix} 0 & 2 & 3 & 5 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

This *matrix* A *is in* Echelon form and it has two non-zero rows since rank of A is equal to the number of non-zero rows. Hence rank of $A = 2$.

Theorem 1. *The rank of the transpose of a matrix is the same as that of the original matrix.*

Proof. Let us suppose A is any matrix and A' is its transpose and let rank of $A = r$. This implies that A contains at least one r-rowed square matrix whose determinant is non-zero, let it be B. Obviously B' is a submatrix of A' but we know that det B' = det B and since det $B \neq 0 \Rightarrow$ det $B' \neq 0$. Thus the rank of $A' \geq r$. Now if A contains a $(r+1)$-rowed square sub matrix C, then det $C = 0$ because rank of $A = r$. Obviously C' is a submatrix of A' and det C' = det $C = 0$, it follows that A' does not contain $(r+1)$-rowed square submatrix with non-zero determinant. Hence rank of $A' \leq r$ and consequently we obtained rank of A' = rank of A.

• 7.4. ELEMENTARY TRANSFORMATIONS OF A MATRIX

Definition. *A transformation is said to be elementary transformation if it is one of the followings :*

(*i*) *Interchanging of any two rows (or columns).*

(*ii*) *Multiplying any row (or column) by any non-zero number.*

(*iii*) *Addition of any row to K times the other row, where K is any non-zero number.*

REMARKS

If the elementary transformation (or E-transformation) is performed on rows, then it is called **row-transformation.**

If the E-transformation is performed on column, it is called **column- transformation.**

• 7.5. ELEMENTARY MATRICES

Definition. *A matrix which is obtained by a single E-transformation is called an* ***elementary matrix.*** **For example**

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{etc.}$$

Here first E-matrix is obtained from I_3 by interchanging C_1 and C_3 columns and the second E-matrix is obtained by $R_1 \to R_1 + 2R_2$.

REMARKS

All the elementary matrices are non-singular.

Each elementary matrix possesses its inverse.

• 7.6. INVARIANCE OF RANK UNDER *E*-TRANSFORMATION

Theorem 2. *Elementary transformations (E-transformation) do not change the rank of a matrix.*

Proof. Since we know that E-transformations are of three types. Therefore, we shall prove this theorem for three cases.

Case I. *Interchanging the rows (or columns) does not change the rank.*

Let A be a matrix of order $m \times n$ of rank r and let B be a matrix obtained from A by interchanging the rows R_i and R_j *i.e.*, by E-transformation $R_i \leftrightarrow R_j$. Let the rank of B be s. Then we shall prove $r = s$.

Since rank of $A = r$. This implies A contains at least one. r-rowed square submatrix with non-zero determinant let it be R *i.e.*, $\det R \neq 0$. Let us suppose S be the r-rowed square submatrix of B having the same rows as are in R though these rows may be in different positions. Then either

$$\det S = \det R \quad \text{or} \quad \det S = -\det R$$

But $\qquad \det R \neq 0 \;\Rightarrow\; \det S \neq 0$

$\therefore \qquad$ Rank of $B \geq r \;\Rightarrow\; s \geq r. \qquad$ ...(1)

Further since the matrix A can also be obtained from B by E-transformation $R_i \leftrightarrow R_j$. Then we have

$$r \geq s. \qquad \text{...(2)}$$

Hence from (1) and (2) we conclude that $r = s$.

Case II. *Multiplication of the elements of a row by a non-zero number does not change the rank.*

Let A be a matrix of order $m \times n$ of rank r and let B be a matrix which obtained from A by E-transformation $R_i \to KR_i$ where $K \neq 0$ and let rank of B be s. Therefore we shall prove that $s = r$. Suppose B_0 is an $(r+1)$-rowed square submatrix of B, then these exists A_0 of $(r+1)$-rowed square submatrix of A such that either

$$\det B_0 = \det A_0 \;\text{ or }\; \det B_0 = K \det A_0.$$

But rank of $A = r$ this means that every $(r+1)$-rowed square submatrix of A has zero determinant

$\therefore \qquad |A_0| = 0 \;\Rightarrow\; |B_0| = 0$

$\Rightarrow$ Every $(r+1)$-rowed square submatrix will have zero determinant

$\Rightarrow$ Rank of B can not exceed the rank of A

$\Rightarrow \qquad s \leq r. \qquad$ ...(1)

Further since the matrix A can also be obtained from B by E-transformation $R_i \to \left(\frac{1}{K}\right) R_i$. Thus we have

$$r \leq s. \qquad \text{...(2)}$$

Hence from (1) and (2) we conclude that

$$r = s.$$

Case III. *Addition of any row to the product of any number K and other row does not change the rank.*

Let the rank of a matrix A of order $m \times n$ be r and let B is obtained by the E-transformation $R_i \to R_i + KR_j$ and let rank of B be s. Then we shall prove $s = r$.

Now if B_0 is an $(r+1)$-rowed square submatrix of B, there exists uniqually A_0 an $(r+1)$-rowed square submatrix of A.

Since we know that any E-transformation does not change the determinant value. Therefore if no row of A_0 is a part of i^{th} row of A, or if two rows of A_0 are the parts of the i^{th} and j^{th} rows of A, then $\det B_0 = \det A_0$.

But the rank of $\; A = r \;\Rightarrow\; \det A_0 = 0 \;\Rightarrow\; \det B_0 = 0.$

Now suppose if a row of A_0 is a part of i^{th} row of A and no row is a part of j^{th} row, then

$$\det B_0 = \det A_0 + K \det C_0$$

where C_0 is an $(r+1)$-rowed square submatrix which is obtained from A_0 by E-transformation $R_i \to R_i + KR_j$.

Clearly, all the $(r+1)$ rows of C_0 are exactly same as the rows of some $(r+1)$-rowed square submatrix of A, though in some different position. Therefore det C_0 is ± 1 times det of some $(r+1)$-rowed square submatrix A. But the rank of A is r. This implies every $(r+1)$-rowed square submatrix will have zero determinant.

$$\therefore \qquad \det A_0 = 0, \det C_0 = 0 \;\Rightarrow\; \det B_0 = 0$$

hence rank of B can not exceed the rank of A

$$\therefore \qquad s \le r. \qquad \ldots(1)$$

Further since A can also be obtained from B by E-transformation $R_i \to R_i - KR_j$, therefore we have

$$r \le s. \qquad \ldots(2)$$

From (1) and (2) we conclude that

$$r = s.$$

• 7.7. NORMAL FORM

Definition. *If a matrix is reduced to the form* $\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$. *Then this form is called* **normal form** *of the given matrix.*

Theorem 3. *Every matrix of order* $m \times n$ *of rank* r *can be reduced to the form* $\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$ *by a finite number of E-transformations, where* I_r *is the unit matrix of order* $r \times r$.

Proof. Let $A = [a_{ij}]_{m \times n}$ be a matrix of order $m \times n$ and of rank r. If A is a zero matrix, then its rank is zero and thus A can be written as $\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$.

Let us suppose A is a non-zero matrix it means that it has at least one of its element non-zero. Let this non-zero element be $a_{ij} = K \neq 0$.

Let B be a matrix which is obtained from A by E-transformations $R_1 \leftrightarrow R_i$ and $C_1 \leftrightarrow C_j$ and whose leading element is K. Again using the E-transformation $R_1 \to \frac{1}{K} R_1$ on B and we get a matrix C whose leading element becomes 1. Let this matrix C be

$$C = \begin{bmatrix} 1 & C_{12} & C_{13} & \ldots & C_{1n} \\ C_{21} & C_{22} & C_{23} & \ldots & C_{2n} \\ C_{31} & C_{32} & C_{33} & \ldots & C_{3n} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ C_{m1} & C_{m2} & C_{m3} & \ldots & C_{mn} \end{bmatrix}_{m \times n}.$$

Now subtracting first column after multiplying by suitable number from remaining columns of C and subtracting first row after multiplying by suitable number from remaining rows of C. We therefore obtain a matrix D whose elements of the first row and first column are zero except the leading element. Let D be given as

$$D = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \\ 0 & & & & \\ 0 & & A_1 & & \\ \vdots & & & & \\ 0 & & & & \end{bmatrix}_{m \times n}.$$

where A_1 is a matrix of order $(m-1) \times (n-1)$.

If this matrix A_1 is non-zero matrix, then we shall apply above process on A_1. Since we know that E-transformation will not effect the first row and first column of D, so that we shall apply E-transformations on D *and there no need to take* A_1 *separately. Continuing this process finitely we* obtain a matrix M such that

$$M = \begin{pmatrix} I_K & O \\ O & O \end{pmatrix}.$$

This implies the matrix M has a rank K. But M is obtained from A by a finite number of E-transformations and we know that E-transformations do not change the rank, therefore K must be equal to r.

Hence the matrix A of order $m \times n$ of rank r can be reduced to the form $\begin{pmatrix} I_r & O \\ O & O \end{pmatrix}$ by a finite number of E-transformations.

REMARK

The form $\begin{pmatrix} I_r & O \\ O & O \end{pmatrix}$ of A is also called first canonical form.

Corollary 1. *The rank of matrix of order $m \times n$ is r if it can be reduced to $\begin{pmatrix} I_r & O \\ O & O \end{pmatrix}$ by a finite number of E-transformations.*

Corollary 2. *If A is a matrix of order $m \times n$ of rank r, then there exist non-singular matrices P and Q such that*

$$PAQ = \begin{pmatrix} I_r & O \\ O & O \end{pmatrix}.$$

• 7.8. EQUIVALENCE OF MATRICES

Definition. *Let A be a matrix of order $m \times n$. If a matrix B of order $m \times n$ is obtained from A by a finite number of E-transformations, then A is called* **equivalent** *to B. It is denoted by $A \sim B$ (Read as A is equivalent to B).*

Theorem. *The relation "~" in the set of all $m \times n$ matrices is an equivalence relation.*

Proof.

(i) Reflexivity. If A is a matrix of order $m \times n$, then A is equivalent to A *i.e.*, $A \sim A$.

(ii) Symmetry. Let A and B be two matrix of order $m \times n$ and $A \sim B$. This implies if B is obtained from A by a finite number of E-transformation, the A can also be obtained from B by a finite E-transformations. Hence $A \sim A$.

(iii) Transitivity. Let A, B, C be three matrices of order $m \times n$ and $A \sim B, B \sim C$. This implies that of B is obtained from A by a finite number of E-transformations and C is obtained from B by a finite number of E-transformations, then C can also be obtained from A by a finite number of E-transformations. Hence $A \sim C$.

Hence the relation "~" is an equivalence relation.

• SOLVED EXAMPLES

Examples 1. *Determine the rank of the following matrices*

$$(i) \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 9 & 12 \end{bmatrix} \qquad (ii) \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix}.$$

Solution. **(i)** The square submatrices of the given matrix are

$$A_1 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix} A_2 = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \\ 3 & 6 & 12 \end{bmatrix} A_3 = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 6 & 8 \\ 3 & 9 & 12 \end{bmatrix} A_4 = \begin{bmatrix} 2 & 3 & 4 \\ 4 & 6 & 8 \\ 6 & 9 & 12 \end{bmatrix}$$

$$\det A_1 = 1\,(36-36) + 2\,(18-18) + 3\,(12-12) = 0$$
$$\det A_2 = 1\,(48-48) + 2\,(24-24) + 4\,(12-12) = 0$$
$$\det A_3 = 1\,(72-72) + 3\,(24-24) + 4\,(18-18) = 0$$
$$\det A_4 = 2\,(72-72) + 3\,(48-48) + 4\,(36-36) = 0.$$

Therefore determinant of all square submatrices of the given matrix of order 3×3 are zero so the rank of the given matrix is less than 3. Now the square submatrices of the given matrix of order 2×2 are

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}, \begin{bmatrix} 1 & 4 \\ 2 & 8 \end{bmatrix}, \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}, \begin{bmatrix} 2 & 4 \\ 4 & 8 \end{bmatrix}, \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix}$$
$$\begin{bmatrix} 2 & 6 \\ 3 & 9 \end{bmatrix}, \begin{bmatrix} 2 & 8 \\ 3 & 12 \end{bmatrix}, \begin{bmatrix} 4 & 6 \\ 6 & 9 \end{bmatrix}, \begin{bmatrix} 4 & 8 \\ 6 & 12 \end{bmatrix}, \begin{bmatrix} 6 & 8 \\ 9 & 12 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}, \begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 4 \\ 3 & 12 \end{bmatrix}, \begin{bmatrix} 2 & 3 \\ 6 & 9 \end{bmatrix}, \begin{bmatrix} 2 & 4 \\ 6 & 12 \end{bmatrix}, \begin{bmatrix} 3 & 4 \\ 9 & 12 \end{bmatrix}.$$

Obviously the determinant of all square submatrices of order 2×2 are zero. Thus the rank of the given matrix is less than 2. Since the given matrix is non-zero matrix. Hence the rank of the given matrix is 1.

$$\text{(ii)} \quad A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix}$$

$$\det A = 1\,(24-25)+2\,(20-18)+3\,(15-16) = -1+4-3 = -2+2 = 0.$$

Therefore the rank $A \neq 3$

Now the square submatrices of A of order 2×2 are $A_1 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $A_2 = \begin{bmatrix} 1 & 3 \\ 3 & 5 \end{bmatrix}$ etc.

$$\det A_1 = 4-6 = -2 \neq 0$$

Hence the rank $A = 2$.

Example 2. *If* $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, *find the rank of A and A^2.*

Solution. Since the matrix A is an *Echelon form* and there are three non-zero rows. Therefore rank of A is equal to the number of non-zero rows. Hence rank of $A = 3$.

Next find A^2

$$A^2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Obviously A^2 is an *Echelon form* and having two non-zero rows. Hence the rank of $A^2 = 2$.

Example 3. *Use E-transformation to reduce the following matrix A to triangular form and hence find the rank of A.*

$$A = \begin{bmatrix} 8 & 1 & 3 & 6 \\ 0 & 3 & 2 & 2 \\ -8 & -1 & -3 & 4 \end{bmatrix}.$$

Solution. Since we have

$$A = \begin{bmatrix} 8 & 1 & 3 & 6 \\ 0 & 3 & 2 & 2 \\ -8 & -1 & -3 & 4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 3 & 6 \\ 0 & 3 & 2 & 2 \\ -1 & -1 & -3 & 4 \end{bmatrix} \text{by } C_1 \to \frac{1}{8}C_1 \sim \begin{bmatrix} 1 & 1 & 3 & 6 \\ 0 & 3 & 2 & 2 \\ 0 & 0 & 0 & 10 \end{bmatrix} \text{by } R_3 \to R_3 + R_1.$$

This matrix is a triangular matrix (Echelon form) and it contains three non-zero rows. Hence the rank of $A = 3$.

Example 4. *Find two non-singular matrices **P** and **Q** such that **PAQ** is in the normal form where*

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 3 & 1 & 1 \end{bmatrix}.$$

Solution. Since we have

$A = I_3 A I_3$

i.e.,
$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \qquad \ldots(1)$$

Now applying E-transformation on the matrix A on the L.H.S. of (1) until A reduced to the normal form. In this process we apply E-row transformation to pre-factor I_3 of R.H.S. of (1) and E-column transformation to post-factor I_3 of R.H.S. of (1). Now performing $R_2 \to R_2 - R_1$, $R_3 \to R_3 - 3R_1$, we get

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & -2 \\ 0 & -2 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

performing $C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & -2 \\ 0 & -2 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

performing $R_2 \to -\frac{1}{2} R_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -2 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ -3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

performing $R_3 \to R_3 + 2R_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ -2 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

performing $C_3 \to C_3 - C_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ -2 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$\therefore \qquad \begin{pmatrix} I_2 & O \\ O & O \end{pmatrix} = PAQ$$

where $$P = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ -2 & -1 & 1 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Hence rank of $A = 2$.

• STUDENT ACTIVITY

1. Determine the rank of the matrix

$$A = \begin{bmatrix} 8 & 1 & 3 & 6 \\ 0 & 3 & 2 & 2 \\ -8 & -1 & -3 & 4 \end{bmatrix}$$

2. Convert the following matrix into normal form and hence find its rank $\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 3 & 1 & 1 \end{bmatrix}$

• SUMMARY

- **Sub-matrix :** A matrix obtained by deleting some rows and columns from the iven matrix is called sub-matrix.
- **Minor of a Matrix :** The determinant of every square sub-matrix of a given matrix is called a minor of the given matrix.
- **Rank of matrix :** A positive integer r is said to the rank of a matrix A, if it contains at least one square submatrix of order $r \times r$ whose determinant is non-zero while any square submatrix of A of order $(r\ \ 1) \times (r + 1)$ or greater is sinular.
- **Normal form :** If the given matrix is reduced to the form $\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$, then this form is called normal form of the given matrix.

• TEST YOURSELF

Determine the rank of the following matrices :

1. $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}$. **2.** $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix}$. **3.** $\begin{bmatrix} 1 & 2 & -7 & 5 \\ 0 & 5 & 0 & 8 \\ 0 & 0 & 0 & -8 \end{bmatrix}$.

4. $\begin{bmatrix} 1 & -1 & 3 & 6 \\ 1 & 3 & -3 & -4 \\ 5 & 3 & 3 & 11 \end{bmatrix}$. **5.** $\begin{bmatrix} 1 & -3 & 4 & 7 \\ 9 & 1 & 2 & 0 \end{bmatrix}$.

6. $\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -2 & 1 \\ 1 & -1 & 4 & 0 \\ -2 & 2 & 8 & 0 \end{bmatrix}$. **7.** $\begin{bmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 10 & 3 & 9 & 7 \\ 16 & 4 & 12 & 15 \end{bmatrix}$. **8.** $\begin{bmatrix} 8 & 0 & 0 & 1 \\ 1 & 0 & 8 & 1 \\ 0 & 0 & 1 & 8 \\ 0 & 1 & 1 & 8 \end{bmatrix}$.

10. Reduce the following matrix to its Echelon form and find its rank :

$$\begin{bmatrix} 1 & 3 & 4 & 5 \\ 3 & 9 & 12 & 9 \\ -1 & -3 & -4 & -3 \end{bmatrix}.$$

10. Reduce the following matrix to normal from and find its rank :

$$\begin{bmatrix} 0 & 1 & 3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}.$$

11. Change the following matrix A into normal form and find its rank

$$A = \begin{bmatrix} 1 & 2 & -1 & 4 \\ 2 & 4 & 3 & 5 \\ -1 & -2 & 6 & -7 \end{bmatrix}.$$

ANSWERS

1. 2. **2.** 1. **3.** 2. **4.** 3. **5.** 3. **6.** 2. **7.** 3. **8.** 2.
9. 3 **10.** 2. **11.** 3

OBJECTIVE EVALUATION

➤ FILL IN THE BLANKS :

1. The rank of A and A^T are ....
2. The rank of $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \end{bmatrix}$ is ....
3. If A is a non-zero column matrix and B a non-zero row matrix, then rank of (AB) = ...
4. The rank of two equivalent matrix are ....
5. The rank of a matrix $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix}$ is ....

➤ TRUE OR FALSE :

Write 'T' for True and 'F' for False statement :

1. If rank $A = 3$, then the rank of its transpose is 3. *(T/F)*
2. The rank of the matrix $\begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ is 3. (T/F)
3. If $A = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}$ and $B = [b_{11}\ b_{12} \ldots b_{nn}]$, then the rank of $AB = 1$. (T/F)
4. If the rank of a square matrix of order n is $n - 1$, then Adj $A \neq 0$. *(T/F)*
5. The rank of a matrix is always greater than or equal to the rank of its every submatrix. *(T/F)*
6. The rank of $(AB) \geq$ rank of A. *(T/F)*

MULTIPLE CHOICE QUESTIONS :

Choose the most appropriate one :

1. If the rank of $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is 2, then rank of $\begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}$ is :

   (a) 3 (b) 2 (c) 1 (d) None of these.
2. If A is a null matrix, then its rank is :

   (a) 0 (b) 1 (c) 2 (d) None of these.
3. The rank of I_6 is :

   (a) 2 (b) 3 (c) 5 (d) 6.
4. If A and B are equivalent, and rank $A = r$, then rank of B is :

   (a) $r - 1$ (b) $r + 1$ (c) r (d) 0.

ANSWERS

Fill in the Blanks :

1. Singular **2.** 1 **3.** n **4.** Vanish **5.** Same

True or False :

1. F **2.** T **3.** T **4.** F **5.** T **6.** F

Multiple Choice Questions :

1. (b) **2.** (a) **3.** (d) **4.** (c)

❑❑❑

UNIT 8

INVERSE OF A MATRIX

LEARNING OBJECTIVES

- Inverse of a Marices
- Adjoint of a Matrix
- Solved Examples
  - Test Yourself–1
- Some Important Theorems on Inverse of a Matrix
- Solved Examples
  - Student Activity
  - Summary
  - Test Yourself–2

LEARNING OBJECTIVES

After going through this unit you will learn :

- How to adjoint of a matrix
- How to to find the inverse of the given matrix

8.1. INVERSE OF A MATRIX

Let A be any square and non-singular matrix. If there exist a matrix B such that $AB = BA = I$, where I is the unit matrix, then B is called the inverse of A.

i.e., $A^{-1} = B$.

Some Important Theorems :

Theorem 1. *The inverse of a matrix, if exist is unique.*

Proof. Let A be any given square matrix.

Let if possible there exist two inverses B and C of A.

Then, by definition of inverse of a matrix, we have

$$AB = BA = I \quad \ldots(1)$$

and $$AC = CA = I \quad \ldots(2)$$

Now from (1) and (2), we have

$$AB = AC \Rightarrow B(AB) = B(AC)$$

$\Rightarrow$ $(BA)B = (BA)C$ (by associativity)

$\Rightarrow$ $IB = IC$ [using (1)]

$\Rightarrow$ $B = C$.

Theorem 2. *If A and B be two non-singular or invertible matrices of the same order then AB is also non-singular and*

$$(AB)^{-1} = B^{-1}A^{-1}.$$

Proof. Let A and B be two non-singular matrix.

$\Rightarrow$ A^{-1} and B^{-1} must exist *i.e.*, $AA^{-1} = I$ and $BB^{-1} = I$

Therefore $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1}$ (by associativity)

$= A(I)A^{-1}$ $(\because BB^{-1} = I)$

$= AA^{-1} = I$

$\Rightarrow$ $(AB)(B^{-1}A^{-1}) = I$ ...(i)

Also $(B^{-1}A^{-1})(AB) = A^{-1}(A^{-1}A)B$ (by associativity)

$= B^{-1}(IB)$ $(\because AA^{-1} = I)$

$= B^{-1}B = I.$ ...(ii)

Now from (i) and (ii), we conclude that

$$(AB)(B^{-1}A^{-1}) = (B^{-1}A^{-1})(AB) = I$$

$\Rightarrow$ $B^{-1}A^{-1}$ is the inverse of AB

i.e., $$(AB)^{-1} = B^{-1}A^{-1}.$$

REMARK

- The result, discussed in theorem 2 is also known as "Reciprocal law for the inverse of the product".

• 8.2. ADJOINT OF A MATRIX

Let $A = [a_{ij}]$ be a square matrix of order $n \times n$ and C_{ij} be the cofactors of the element a_{ij} in the determinant of A *i.e.* $|a_{ij}|$

then adjoint of $A = \begin{bmatrix} C_{11} & C_{21} & \dots & C_{n1} \\ C_{12} & C_{22} & \dots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \dots & C_{nm} \end{bmatrix}$.

REMARKS

- Adjoint of a given square matrix can be obtained by transpose the matrix of cofactors.
- Here, it is clear that the cofactors of the elements of the first row of $|a_{ij}|$ are the elements of the first column of Adj A.

Some Important Theorems :

Theorem 1. *If $A = (a_{ij})$ be a non-singular matrix of order an $n \times n$, then*

$$A \,.\, (Adj.\, A) = (Adj.\, A) \,.\, A = |A| \,.\, I_n$$

where I_n is an $n \times n$ identity matrix.

Proof. Since, we know that Adj $A = [C'_{jk}]$

where C_{kj} is the cofactors of a_{kj} in $|a_{ij}|$ such that $C'_{jk} = C_{kj}$.

Now $$A \,.\, (\text{Adj } A) = (a_{ij})(C'_{jk}) = [B_{ik}] \qquad \text{(say)}$$

where $$B_{ik} = \sum_{j=1}^{n} a_{ij} C'_{jk} = \sum_{j=1}^{n} a_{ij} C_{kj} \qquad (\text{by using } C'_{jk} = C_{kj})$$

$$= |A|, \text{ if } i = k$$
$$= 0 \text{ if } i \neq k.$$

Therefore, all diagonal terms of A . (Adj. A) are $|A|$ and all non-diagonal elements are zero.

Now $$A \,.\, (\text{Adj. } A) = \begin{bmatrix} |A| & 0 & 0 & \dots 0 \\ 0 & |A| & 0 & \dots 0 \\ 0 & 0 & |A| & \dots 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & |A| \end{bmatrix}$$

$$= |A| \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} = |A| \,.\, I. \qquad \dots(\text{i})$$

Similarly, we can easily prove that

$$(\text{Adj. } A) \,.\, A = |A| \,.\, I. \qquad \dots(\text{ii})$$

Now, from (i) and (ii) we conclude that

$$A \,.\, (\textbf{Adj } A) = (\textbf{Adj } A) \,.\, A = |A| \,.\, I$$

$$\Rightarrow \quad A \,.\, \frac{(\text{Adj } A)}{|A|} = \frac{(\text{Adj} \,.\, A)}{|A|} \,.\, A = I$$

$$\Rightarrow \quad A^{-1} = \frac{\text{Adj} \,.\, A}{|A|} \qquad (\text{by using } I = AA^{-1} = A^{-1}A)$$

$\Rightarrow$ The inverse of $A = \dfrac{\text{Adj} \,.\, A}{|A|}$.

Theorem 2. *If $A = [a_{ij}]$ be a non-signals matrix of order $n \times n$ matrix, then*

$$|Adj \,.\, A| = |A|^{n-1}.$$

Proof. Since, we know that

$$|A|.|B| = |AB|.$$

Therefore $\quad |A|.|\text{Adj } A| = |A . \text{adj } A|$

$$= \begin{bmatrix} |A| & 0 & 0 & 0 \\ 0 & |A| & 0 & 0 \\ 0 & 0 & |A| & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & |A| \end{bmatrix}$$

$\Rightarrow \quad |A|.|\text{Adj } A| = [|A|]^n$

Since $|A| \neq 0$, therefore divide both sides by $|A|$, we get.

$$|\text{Adj } A| = |A|^{n-1}.$$

Theorem 3. *If A and B are two non-singular $n \times n$ matrices, then*

$$\text{Adj } (AB) = (\text{Adj } B) . (\text{Adj } A).$$

Proof. Let A and B be two $n \times n$ matrices.

We know that $A . (\text{adj } A) = |A| . I$

$\Rightarrow \quad (AB) . (\text{Adj } AB) = |AB| . I. \quad \ldots(1)$

Now consider $(AB) . (\text{Adj } B) . (\text{Adj } A)$

$= A . B . \text{Adj } B . \text{Adj } A$

$= A . (B \text{ Adj } B) . (\text{Adj } A)$

$= A . |B| . I . \text{Adj } A \quad (\because B . \text{Adj } B = |B| . I)$

$= A . |B| . \text{Adj} . A$

$= |B| . A . \text{Adj } A$

$= |B| . |A| . I = |A| . |B| . I$

$= |AB| . I \quad \ldots(\text{ii})$

$(\because |AB| = |A| . |B|)$

Now, from (i) and (ii), we conclude that

$$(AB) . (\text{Adj } AB) = (AB) . (\text{Adj } B) . (\text{Adj } A)$$

$\Rightarrow \quad \text{Adj} . (AB) = (\text{Adj} . B) . (\text{Adj} . A).$

• SOLVED EXAMPLES

Example 1. *Find the adjoint of the matrix A if* $A = \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & -1 \\ 2 & 0 & 4 \end{bmatrix}$.

Solution. Here, first we find the cofactors of A such that

$$C_{11} = \begin{vmatrix} 1 & -1 \\ 0 & 4 \end{vmatrix} = 4 \qquad C_{12} = -\begin{vmatrix} 0 & -1 \\ 2 & 4 \end{vmatrix} = -2$$

$$C_{13} = \begin{vmatrix} 0 & 1 \\ 2 & 0 \end{vmatrix} = -2 \qquad C_{21} = -\begin{vmatrix} 1 & 3 \\ 0 & 4 \end{vmatrix} = -4$$

$$C_{22} = \begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} = -2 \qquad C_{23} = -\begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix} = 2$$

$$C_{31} = \begin{vmatrix} 1 & 3 \\ 1 & -1 \end{vmatrix} = -4 \qquad C_{32} = -\begin{vmatrix} 1 & 3 \\ 0 & -1 \end{vmatrix} = 1$$

and $\quad C_{33} = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1.$

Therefore, the matrix of the co-factors of A is given by

$$C = \begin{bmatrix} 4 & -2 & -2 \\ -4 & -2 & 2 \\ -4 & 1 & 1 \end{bmatrix}$$

$\Rightarrow \quad \text{Adj } (A) = \text{transpose of } C = C' = \begin{bmatrix} 4 & -4 & -4 \\ -2 & -2 & 1 \\ -2 & 2 & 1 \end{bmatrix}.$

Example 2. *Find the adjoint of the matrix* $A = \begin{bmatrix} 1 & 2 \\ 3 & -5 \end{bmatrix}$ *and verify the following result*

$$A\,.\,(\text{Adj}.\,A) = (\text{Adj}.\,A)\,.\,A = |A|\,.\,I.$$

Solution. For the given matrix A, we have

$$C_{11} = -5,\ C_{12} = -3,\ C_{21} = -2, \text{ and } C_{22} = 1.$$

Therefore, the matrix of cofactors is given by

$$C = \begin{bmatrix} -5 & -3 \\ -2 & 1 \end{bmatrix}$$

$$\Rightarrow \qquad \text{Adj}\,(A) = C' = \begin{bmatrix} -5 & -2 \\ -3 & 1 \end{bmatrix}$$

Now $\quad A\,.\,(\text{Adj}\,A) = \begin{bmatrix} 1 & 2 \\ 3 & -5 \end{bmatrix} . \begin{bmatrix} -5 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -11 & 0 \\ 0 & -11 \end{bmatrix}.$

and $\quad (\text{Adj}.\,A)\,.\,A = \begin{bmatrix} -5 & -2 \\ -3 & 1 \end{bmatrix} . \begin{bmatrix} 1 & 2 \\ 3 & -5 \end{bmatrix} = \begin{bmatrix} -11 & 0 \\ 0 & -11 \end{bmatrix}.$

Also, we have

$$|A| = \begin{vmatrix} 1 & 2 \\ 3 & -5 \end{vmatrix} = -5 - 6 = -11.$$

Therefore, we conclude that

$$A\,.\,(\text{Adj}\,A) = (\text{Adj}\,A)\,.\,A = \begin{bmatrix} -11 & 0 \\ 0 & -11 \end{bmatrix} = -11 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = |A|\,.\,I.$$

• TEST YOURSELF–1

1. Find the adjoint of the following matrices :

(i) $\begin{bmatrix} 1 & 2 & 4 \\ 5 & 7 & 8 \\ 9 & 10 & 12 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & -6 & -7 \end{bmatrix}$

(iii) $\begin{bmatrix} -1 & -2 & 3 \\ -2 & 2 & 1 \\ 4 & -5 & 2 \end{bmatrix}$ (iv) $\begin{bmatrix} 1 & 5 & 7 \\ 2 & 3 & 1 \\ 4 & 3 & 2 \end{bmatrix}$

(v) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 0 \\ 2 & 4 & 3 \end{bmatrix}$ (vi) $\begin{bmatrix} -1 & -2 & 3 \\ -2 & 1 & 1 \\ -4 & -5 & 2 \end{bmatrix}$

(vii) $\begin{bmatrix} -4 & -3 & -3 \\ 1 & 0 & 1 \\ 4 & 4 & 3 \end{bmatrix}$ (viii) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 6 & 7 & 9 \end{bmatrix}$

2. Verify that the adjoint of a diagonal matrix of order 3 is a diagonal matrix.

3. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 0 \\ 2 & 4 & 3 \end{bmatrix}$, find $A^2 - 2A + \text{Adj}\,A$.

4. If $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$, then verify that $A\,(\text{Adj}\,.\,A) = (\text{Adj}\,A)\,.\,A = |A|\,.\,I$.

ANSWERS

1. (i) $\begin{bmatrix} 4 & 16 & -12 \\ 12 & -24 & 12 \\ -13 & 8 & -3 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & -6 & 4 \\ 21 & -7 & -8 \\ -18 & 6 & 4 \end{bmatrix}$

(iii) $\begin{bmatrix} 9 & -11 & -8 \\ 8 & -14 & -5 \\ 2 & -13 & -6 \end{bmatrix}$ (iv) $\begin{bmatrix} 3 & 11 & -16 \\ 0 & -26 & 13 \\ -6 & 17 & 17 \end{bmatrix}$

(v) $\begin{bmatrix} 15 & 6 & -15 \\ 0 & -3 & 0 \\ -10 & 0 & 5 \end{bmatrix}$ (vi) $\begin{bmatrix} 7 & -11 & -5 \\ 0 & 10 & -5 \\ 14 & 3 & -5 \end{bmatrix}$

(vii) $\begin{bmatrix} -4 & -3 & -3 \\ 1 & 0 & 1 \\ 4 & 4 & 3 \end{bmatrix}$ (viii) $\begin{bmatrix} 3 & 3 & -3 \\ 0 & -9 & 6 \\ -2 & 5 & -3 \end{bmatrix}$

3. $\begin{bmatrix} 20 & 26 & -9 \\ 0 & 12 & 0 \\ -6 & 28 & 14 \end{bmatrix}$.

• 8.3. SOME IMPORTANT THEOREMS ON INVERSE OF A MATRIX

Theorem 1. *The necessary and sufficient condition that a square matrix may possess an inverse is that it be non-singular.*

Proof. (i) Necessary Condition. Let A be a $n \times n$ matrix, and B is the inverse of A.

Then, we have

$$AB = I \Rightarrow |AB| = |I|$$

$\Rightarrow$ $$|A|.|B| = |I| = 1 \Rightarrow |A| \neq 0$$

$\Rightarrow$ A is non-singular.

(ii) Sufficient Condition. Let A be an $n \times n$ matrix, which is non-singular and there be another matrix B defined by

$$B = \frac{1}{|A|}(\text{Adj. } A).$$

Then, we have $$AB = A\left[\frac{1}{|A|}.(\text{Adj. } A)\right] = \frac{1}{|A|}(A . \text{Adj. } A)$$

$$= \frac{1}{|A|}.|A|.I = I.$$

Similarly, we can find

$$BA = \frac{1}{|A|}(\text{Adj. } A).A = \frac{1}{|A|}[(\text{Adj. } A).A]$$

$$= \frac{1}{|A|}.|A|.I = I.$$

Hence, B is the inverse of A.

Theorem 2. *The inverse of transpose of a matrix is the transpose of the inverse.*

Proof. Let A be the given matrix, whose inverse is A^{-1}.

Then, we have

$$AA^{-1} = A^{-1}A = I. \quad \ldots(i)$$

Taking transpose of (i), we get

$$(AA^{-1})' = (A^{-1}A)' = (I)'$$

$\Rightarrow$ $$(A^{-1})'A' = A'(A^{-1})' = I \quad \text{(by using } (AB)' = B'A' \text{ and } I' = I)$$

$\Rightarrow$ A' is invertiable

and $$(A')^{-1} = (A^{-1})'.$$

Hence, the inverse of a transpose of a matrix is the transpose of the inverse.

Theorem 3. *The inverse of the inverse of a matrix is the matrix itself.*

Proof. Let A be the given matrix and A^{-1} is the inverse of A.

Also we have

$$AA^{-1} = A^{-1}A = I. \quad \ldots(i)$$

Taking inverse of (i), we get

$$(AA^{-1})^{-1} = (A^{-1}A)^{-1} = (I)^{-1}$$

$\Rightarrow$ $$(A^{-1})^{-1}A^{-1} = (A^{-1})(A^{-1})^{-1} = (I^{-1}) = I$$

(by using $(AB)^{-1} = B^{-1}A^{-1}$ and $I^{-1} = I$)

$\Rightarrow$ $$(A^{-1})^{-1}A^{-1} = (A^{-1})(A^{-1})^{-1} = I$$

$\Rightarrow$ A^{-1} is invertiable.

and $$(A^{-1})^{-1} = A$$

$\Rightarrow$ the inverse of the inverse of A is A itself.

Theorem 4. *If a non-singular matrix* A *is symmetric, then* A^{-1} *is also symmetric.*

Proof. Let A be a non-singular matrix.
Also, let A is symmetric. To show A^{-1} is symmetric.
Since A is non-singular $\Rightarrow A^{-1}$ exist.

$\Rightarrow \quad A^{-1}A = I = I' = (AA^{-1})' = (A^{-1})'A' \quad (\because \ (AB)' = B'A')$

$\Rightarrow \quad A^{-1}A = (A^{-1})'A \quad (\because \ A$ is symmetric $\Rightarrow A^1 = A)$

$\Rightarrow \quad A^{-1} = (A^{-1})'$

$\Rightarrow \quad A^{-1}$ is symmetric.

• SOLVED EXAMPLES

Example 1. *Find the inverse of the matrix* $A = \begin{bmatrix} 3 & -2 & -1 \\ -4 & 1 & -1 \\ 2 & 0 & 1 \end{bmatrix}$.

Solution. Here, we have $|A| = \begin{vmatrix} 3 & -2 & -1 \\ -4 & 1 & -1 \\ 2 & 0 & 1 \end{vmatrix} = 1 \neq 0$

$\Rightarrow$ A is non-singular

$\Rightarrow$ A^{-1} must exist

Now, find the cofactors of A

$$C_{11} = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1; \quad C_{12} = -\begin{vmatrix} -4 & -1 \\ 2 & 1 \end{vmatrix} = 2; \quad C_{13} = \begin{vmatrix} -4 & 1 \\ 2 & 0 \end{vmatrix} = -2$$

$$C_{21} = -\begin{vmatrix} -2 & -1 \\ 0 & 1 \end{vmatrix} = 2; \quad C_{22} = \begin{vmatrix} 3 & -1 \\ 2 & 1 \end{vmatrix} = 5; \quad C_{23} = -\begin{vmatrix} 3 & -2 \\ 2 & 0 \end{vmatrix} = -4$$

$$C_{31} = \begin{vmatrix} -2 & -1 \\ 1 & -1 \end{vmatrix} = 3; \quad C_{32} = \begin{vmatrix} 3 & -1 \\ -4 & -1 \end{vmatrix} = 7; \quad C_{33} = \begin{vmatrix} 3 & -2 \\ -4 & 1 \end{vmatrix} = -5.$$

Therefore, the matrix of the cofactors is given by

$$C = \begin{bmatrix} 1 & 2 & -2 \\ 2 & 5 & -4 \\ 3 & 7 & -5 \end{bmatrix}$$

$$\Rightarrow \quad \text{Adj.}\, A = \text{Transpose of } C = C' = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ -2 & -4 & -5 \end{bmatrix}$$

Therefore, $\quad A^{-1} = \dfrac{\textbf{Adj. } A}{|A|} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ -2 & -4 & -5 \end{bmatrix}$.

Example 2. *Find inverse of the matrix* $A = \begin{bmatrix} 4 & 3 & 3 \\ -1 & 0 & -1 \\ -4 & -4 & -3 \end{bmatrix}$.

Solution. Here, we have

$$|A| = \begin{vmatrix} 4 & 3 & 3 \\ -1 & 0 & -1 \\ -4 & -4 & -3 \end{vmatrix} = -1 \neq 0.$$

$\Rightarrow$ A is non-singular and therefore invertible.

Now we find the cofactors of A.

$$C_{11} = \begin{vmatrix} 0 & -1 \\ -4 & -3 \end{vmatrix} = -4, \quad C_{12} = -\begin{vmatrix} -1 & -1 \\ -4 & -3 \end{vmatrix} = 1, \quad C_{13} = \begin{vmatrix} -1 & 0 \\ -4 & -4 \end{vmatrix} = 4$$

$$C_{21} = -\begin{vmatrix} 3 & 3 \\ -4 & -3 \end{vmatrix} = -3, \quad C_{22} = -\begin{vmatrix} 4 & 3 \\ -4 & -3 \end{vmatrix} = 0, \quad C_{23} = -\begin{vmatrix} 4 & 3 \\ -4 & -4 \end{vmatrix} = 4$$

$$C_{31} = -\begin{vmatrix} 3 & 3 \\ 0 & -1 \end{vmatrix} = -3, \quad C_{32} = -\begin{vmatrix} 4 & 3 \\ -1 & -1 \end{vmatrix} = 1, \quad C_{33} = -\begin{vmatrix} 4 & 3 \\ -1 & 0 \end{vmatrix} = 3.$$

Therefore, the matrix of the cofactors is given by

$$C = \begin{bmatrix} -4 & 1 & 4 \\ -3 & 0 & 4 \\ -3 & 1 & 3 \end{bmatrix}$$

$$\Rightarrow \text{ Adj. } A = \text{Transpose of } \quad C = \begin{bmatrix} -4 & -3 & -3 \\ 1 & 0 & 1 \\ 4 & 4 & 3 \end{bmatrix}$$

Now $\qquad A^{-1} = \dfrac{\text{Adj. } A}{|A|} = \dfrac{-1}{1}\begin{bmatrix} -4 & -3 & -3 \\ 1 & 0 & 1 \\ 4 & 4 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 3 & 3 \\ -1 & 0 & -1 \\ -4 & -4 & -3 \end{bmatrix}$

Solved Examples Based on Second Method :

Example 3. *Find the inverse of the matrix* $A = \begin{bmatrix} 1 & -3 & 2 \\ 2 & 0 & 0 \\ 1 & 4 & 1 \end{bmatrix}$.

Solution. Consider $AB = I$, where B is the inverse of A.

$$= \begin{bmatrix} 1 & -3 & 2 \\ 2 & 0 & 0 \\ 1 & 4 & 1 \end{bmatrix}[B] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 \to R_{2/2}$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 1 & -3 & 2 \\ 1 & 4 & 1 \end{bmatrix}[B] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_1 \leftrightarrow R_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 1 & -3 & 2 \\ 1 & 4 & 1 \end{bmatrix}[B] = \begin{bmatrix} 0 & 1/2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we get

$$\sim \begin{bmatrix} -1 & 0 & 0 \\ 0 & -3 & 2 \\ 0 & 4 & 1 \end{bmatrix}[B] = \begin{bmatrix} 0 & 1/2 & 0 \\ 1 & -1/2 & 0 \\ 0 & -1/2 & 1 \end{bmatrix}$$

$R_2 \to R_2 - 2R_3$, we get

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & -11 & 0 \\ 0 & 4 & 1 \end{bmatrix}[B] = \begin{bmatrix} 0 & 1/2 & 0 \\ 1 & 1/2 & -2 \\ 0 & -1/2 & 1 \end{bmatrix}$$

$R_2 \to -\dfrac{1}{11} R_2$, we get

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix}[B] = \begin{bmatrix} 0 & 1/2 & 0 \\ -1/11 & -1/22 & 2/11 \\ 0 & -1/2 & 1 \end{bmatrix}$$

$R_3 \to R_3 - 4R_2$, we get

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}[B] = \begin{bmatrix} 0 & 1/2 & 0 \\ -1/11 & -1/22 & 2/11 \\ 4/11 & -7/22 & 3/11 \end{bmatrix}$$

$$\Rightarrow \qquad B = \begin{bmatrix} 0 & 1/2 & 0 \\ -1/11 & -1/22 & 2/11 \\ 4/11 & -7/22 & 3/11 \end{bmatrix}$$

which is the required inverse of A.

• STUDENT'S ACTIVITY

1. If A be a non-singular matrix of order $n \times n$, then

$$A \cdot (Adj.\ A) = Adj.\ A)\ A = |A| \cdot I_n.$$

where I_n is the unit matrix of order $n \times n$.

2. Find the inverse of the matrix $A = \begin{bmatrix} 1 & -3 & 2 \\ 2 & 0 & 0 \\ 1 & 4 & 1 \end{bmatrix}$.

• SUMMARY

- For square non-singular matrix A,

$$A\ (\text{Adj}.\ A)\ (\text{Adj}.\ A) = |A|^{n-1}$$

- For any non-singular matrix of order $n \times n$,

$$|\text{Adj}.\ A\ | = |A|^{n-1}$$

- Adj (AB) = (Adj. B) (Adj. A)
- Inverse of a matrix : $A^{-1} = \dfrac{\text{Adj}.\ (A)}{|A|}$
- $(AB)^{-1} = B^{-1} A^{-1}$
- $(A^{-1})^{-1} = A.$

• TEST YOURSELF–2

1. Find the inverse of the following matrices :

(i) $\begin{bmatrix} 2 & 4 & 3 \\ 0 & 1 & 1 \\ 2 & 2 & -1 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 2 & 1 \\ 3 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & 2 & 1 \\ 3 & 2 & 3 \\ 1 & 1 & 2 \end{bmatrix}$ (iv) $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 3 \\ 2 & 4 & 9 \end{bmatrix}$

(v) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$ (vi) $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 5 & 12 \end{bmatrix}$ (vii) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 0 \\ 2 & 0 & 3 \end{bmatrix}$ (viii) $\begin{bmatrix} 2 & 2 & 2 \\ 2 & 5 & 5 \\ 2 & 5 & 11 \end{bmatrix}$

(ix) $\begin{bmatrix} 1 & 2 & 4 \\ 5 & 7 & 8 \\ 9 & 10 & 12 \end{bmatrix}$ (x) $\begin{bmatrix} 1 & 2 & -1 \\ -1 & 1 & 2 \\ 2 & -1 & 1 \end{bmatrix}$

2. If $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ then show that $A^2 = A^{-1}$.

3. Find the inverse of

(i) $A = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (ii) $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & w & w^2 \\ 1 & w^2 & w \end{bmatrix}$, where w is the cube roots of unity.

4. Show that the matrix $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is its own inverse.

5. If $\alpha + i\beta = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$, show that $(\alpha + i\beta)^{-1} = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}^{1}$.

6. If $A = \begin{bmatrix} a+ib & c+id \\ -c+id & a-ib \end{bmatrix}$ if $a^2 + b^2 + c^3 + d^2 = 1$.

Then show that $A^{-1} = \begin{bmatrix} a-ib & -c-id \\ c-id & a+ib \end{bmatrix}$.

7. Prove that $|\text{Adj}\,(\text{Adj}\,A)| = |A|^{(n-1)^2}$, if $|A| \neq 0$ is any $n \times n$ matrix.

ANSWERS

1. (i) $\frac{1}{4}\begin{bmatrix} 3 & -10 & -1 \\ -2 & 8 & 2 \\ 2 & -4 & -2 \end{bmatrix}$ (ii) $\frac{1}{9}\begin{bmatrix} 0 & -3 & -3 \\ 6 & -2 & -1 \\ -3 & 1 & 5 \end{bmatrix}$

(iii) $\begin{bmatrix} -1/4 & 3/4 & -1 \\ 3/4 & -1/4 & 0 \\ -1/4 & -1/4 & 1 \end{bmatrix}$ (iv) $\frac{1}{3}\begin{bmatrix} -6 & 5 & -1 \\ 15 & -8 & 1 \\ -6 & 3 & 0 \end{bmatrix}$

(v) $\begin{bmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{bmatrix}$ (vi) $\begin{bmatrix} 11/3 & -3 & 1/3 \\ -7/3 & 3 & -2/3 \\ 2/3 & -1 & 1/3 \end{bmatrix}$

(vii) $-\frac{1}{15}\begin{bmatrix} 15 & -6 & -15 \\ 0 & -3 & 0 \\ -10 & 4 & 5 \end{bmatrix}$ (viii) $\begin{bmatrix} 5/6 & -1/3 & 0 \\ -1/3 & 1/2 & -1/6 \\ 0 & -1/6 & 1/6 \end{bmatrix}$

(ix) $\begin{bmatrix} -1/6 & -2/3 & 1/2 \\ -1/2 & 1 & -1/2 \\ 13/24 & -1/3 & 1/8 \end{bmatrix}$ (x) $\frac{1}{14}\begin{bmatrix} 3 & -1 & 5 \\ 5 & 3 & -1 \\ -1 & 5 & 3 \end{bmatrix}$

3. (i) $\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} 0 & 1/4 & -1/2 \\ -1 & \frac{3}{4}i & i/2 \\ 0 & 1/4 & 1/2 \end{bmatrix}$

OBJECTIVE EVALUATION

➤ FILL IN THE BLANKS :

1. Non-square matrix has .... inverse.
2. A matrix A is said to be singular if $|A| =$ ...
3. A matrix is said to be .... if it is square and non-singular.
4. If $|A| \neq 0$, then matrix is said to be ......

➤ TRUE OR FALSE :

Write T for true and F for false statement :

1. If A, B are any two $n \times n$ matrices such that $BA = 0$, where $\mathbf{0}$ is the null matrix, then at least one of them is non-singular. *(T/F)*
2. The inverse of an orthogonal matrix is not necessarily orthogonal. *(T/F)*
3. Adj. $(AB) =$ Adj (A). Adj. (B) *(T/F)*
4. The inverse of matrix A exist if A is singular *(T/F)*

➤ MULTIPLE CHOICE QUESTIONS :

Choose the most appropriate one :

1. The transpose of the matrix of cofactors is known as :
   (a) Inverse (b) Adjoint (c) Transpose (d) None of these.
2. For the inverse of a matrix A it is necessary that A must be :
   (a) Singular (b) Non-singular (c) Diagonal (d) None of these.
3. The (Adj. A)/$|A|$ is known as :
   (a) A^{-1} (b) A^2 (c) A (d) None of these.

ANSWERS

Fill in the Blanks :

1. no **2.** 0 **3.** Invertible **4.** Non-singular

True or False :

1. T **2.** F **3.** T **4.** F

Multiple Choice Questions :

1. (b) **2.** (b) **3.** (a)

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UNIT 9

APPLICATION OF MATRIX

LEARNING OBJECTIVES

- Homogeneous Linear Equations
- Nature of the Solution of the Equation $AX = 0$
- Solved Example
  - Test Yourself–1
- Non-Homogeneous Equations
- Condition for Consistency
- Solved Examples
  - Student Activity
  - Summary
  - Test Yourself–2

LEARNING OBJECTIVES

After going through this unit you will learn :

- How to find the solution of the system of linear equations using matrix method.

• 9.1. HOMOGENEOUS LINEAR EQUATIONS

Let us consider a system of linear homogeneous equations as follows :

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = 0 \end{array}\right\} \qquad \ldots(1)$$

These equations are m equations in n unknowns. Any set of numbers $x_1, x_2, \ldots x_n$ that satisfies all the equations (1) is called a solution of (1).

Trivial solution. The solution $x_1 = 0, x_2 = 0, \ldots x_n = 0$ of the equations (1) are called **trivial** solution.

Non-trival solution. Any other solution, if it exists, is called a **non-trivial** solution of equations (1).

Let the coefficient matrix be

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m \times n}$$

and $$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}_{n \times 1}, \quad 0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{m \times 1}$$

Then the equations (1) can also be written as

$$\mathbf{AX = 0}. \qquad \ldots(2)$$

This equation (2) is called a ***matrix equation.***

Theorem. *If X_1 and X_2 are two non-trivial solutions of (2), then $k_1X_1 + k_2X_2$ is also a solution of (2), where k_1 and k_2 are any arbitrary numbers.*

Proof. Since the equation (2) is

$$AX = 0 \text{ and } AX_1 = 0, AX_2 = 0 \text{ are given}$$

Now consider,

$$A(k_1X_1 + k_2X_2) = k_1(AX_1) + k_2(AX_2) = k_1(0) + k_2(0) = 0.$$

Hence $k_1X_1 + k_2X_2$ is the solution of (2).

• 9.2. NATURE OF THE SOLUTION OF THE EQUATION $AX = 0$

Since $AX = 0$ is a matrix equation of a system of m homogeneous linear equations in n unknowns and A is a coefficients matrix of order $m \times n$. Let the rank of A be r. Then obviously r can not be greater than n. So that either r is n or r is less than n. Therefore these are some cases.

Case I. If $r = n$, then the equation $AX = 0$ will have no linearly independent solution. So in this case only trivial solution will exist.

(Meerut 2002)

Case II. If $r < n$, then there will be $(n - r)$ linearly independent solution of $AX = 0$ and thus in this case we shall have infinite solutions.

Case III. Suppose the number of equations are less than number of unknowns *i.e.*, $m < n$ and since $r \leq m$, then obviously $r < n$ thus in this case a non-zero solution will exist. Therefore the equation $AX = 0$ will have infinite solution.

• SOLVED EXAMPLES

Example 1. *Find all the solution of the follwoing system of linear homogenous equations*

$$x - 2y + z - w = 0$$
$$x + y - 2z + 3w = 0$$
$$4x + y - 5z + 8w = 0$$
$$5x - 7y + 2z - w = 0.$$

Solution. The coefficients matrix is given by

$$A = \begin{bmatrix} 1 & -2 & 1 & -1 \\ 1 & 1 & -2 & 3 \\ 4 & 1 & -5 & 8 \\ 5 & -7 & 2 & -1 \end{bmatrix}.$$

Change this matrix into Echelon form as follow :

performing $R_2 \to R_2 - R_1$, $R_3 \to R_3 - 4R_1$ and $R_4 \to R_4 - 5R_1$

$$\sim \begin{bmatrix} 1 & -2 & 1 & -1 \\ 0 & 3 & -3 & 4 \\ 0 & 9 & -9 & 12 \\ 0 & 3 & -3 & 4 \end{bmatrix}$$

performing $R_2 \to \frac{1}{3} R_2$

$$\sim \begin{bmatrix} 1 & -2 & 1 & -1 \\ 0 & 1 & -1 & \frac{4}{3} \\ 0 & 9 & -9 & 12 \\ 0 & 3 & -3 & 4 \end{bmatrix}$$

performing $R_3 \to R_3 - 9R_2$, $R_4 \to R_4 - 3R_2$

$$\sim \begin{bmatrix} 1 & -2 & 1 & -1 \\ 0 & 1 & -1 & \frac{4}{3} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

This is an Echelon form and having two non-zero rows. Hence rank of $A = 2$. Therefore the given system of equation is equivalent to

$$\begin{bmatrix} 1 & -2 & 1 & -1 \\ 0 & 1 & -1 & \frac{4}{3} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = 0$$

or $$x - 2y + z - w = 0 \quad \ldots(1)$$

$$y - z + \frac{4}{3} w = 0. \quad \ldots(2)$$

Let $z = c_1,\ w = c_2$

From (2) $y = c_1 - \frac{4}{3} c_2$

and from (1) $x = c_1 - \frac{5}{3} c_2$

Hence solution is $x = c_1 - \frac{5}{3} c_2,\ y = c_1 - \frac{4}{3} c_2,\ z = c_1,\ w = c_2$

where c_1 and c_2 are arbitrary numbers.

• TEST YOURSELF–1

Find the solutions of the following system of linear homogeneous equations :

1. $x + 2y + 3z = 0$
   $3x + 4y + 4z = 0$
   $7x + 10y + 12z = 0.$
2. $x + y - 3z + 2w = 0$
   $3x - 2y + z - 4w = 0$
   $-4x + y - 3z + w = 0.$
3. $x + y + z = 0$
   $2x + 5y + 7z = 0$
   $2x - 5y + 3z = 0.$
4. $2x - 3y + z = 0$
   $x + 2y - 3z = 0$
   $4x - y - 2z = 0.$

Show that the only real values of λ for which the following equations have non-zero solution is 6 : $x + 2y + 3z = \lambda x,\ 3x + 1y + 2z = \lambda y,\ 2x + 3y + z = \lambda z.$

ANSWERS

1. $x = 0 = y = z$ **2.** $x = 0 = y = z = w$ **3.** $x = 0 = y = z$ **4.** $x = 0 = y = z$

• 9.3. NON-HOMOGENEOUS EQUATIONS

Let us consider a system of equations which are non-homogeneous as follows :

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m \end{array}\right\} \quad \ldots(1)$$

These are m equations in n unknowns. Let

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}_{m \times n}$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n \times 1},\ B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m \times 1}.$$

Then the system of equations (1) can also be written as

$$AX = B. \quad \ldots(2)$$

This equation is called matrix equation. If $x_1, x_2, \ldots x_n$ simultaneously satisfy the equation (2), then $(x_1, x_2, \ldots x_n)$ is called the solution of (2).

Consistency and inconsistency. When there will exist one or more than one solution of the equation $AX = B$. Then the equations are said to be **consistent** otherwise said to be **inconsistent**.

Augmented matrix. The matrix of the type

$$[A \mid B] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}$$

is called the Augmented matrix of the equations.

• 9.4. CONDITION FOR CONSISTENCY

Theorem. *The equation $AX = B$ is consistent if and only if the rank of A and the rank of the augmented matrix $[A \mid B]$ are same.*

Proof. Since the equation is

$$AX = B. \qquad \ldots(1)$$

The matrix A can be written as

$$A = [C_1, C_2, \ldots C_n]$$

where $C_1, C_2, \ldots C_n$ are column vectors. Then the equation (1) can be written as

$$[C_1, C_2 \ldots C_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = B$$

or
$$x_1C_1 + x_2C_2 + \ldots + x_nC_n = B \qquad \ldots(2)$$

Suppose the rank of A is r, then A has r linearly independent columns. Let these columns be $C_1, C_2, \ldots C_r$ and these $C_1, C_2, \ldots C_r$ are linearly independent and remaining $(n - r)$ columns are linear combination of $C_1, C_2, \ldots C_r$.

Necessary condition. Suppose the equations are consistent, there must exist $k_1, k_2, \ldots k_n$ such that

$$k_1C_1 + k_2C_2 + \ldots + k_nC_n = B. \qquad \ldots(3)$$

But $C_{r+1}, C_{r+2}, \ldots C_n$ is a linear combination of $C_1, C_2, \ldots C_r$, then from (2) it is obvious that B is also a linear combination of $C_1, C_2, \ldots C_r$ and thus $[A \mid B]$ has the rank r. Hence the rank of A is same as the rank of $[A \mid B]$.

Sufficient condition. Suppose rank A = rank $[A \mid B] = r$. This implies that $[A \mid B]$ has r linearly independent columns. But $C_1, C_2, \ldots C_r$ of $[A \mid B]$ are already linearly independent. Thus B can be expressed as

$$B = k_1C_1 + k_2C_2 + \ldots + k_rC_r \qquad \ldots(4)$$

where $k_1, k_2, \ldots k_r$ are scalars.

Now, equation (4) becomes

$$B = k_1C_1 + k_2C_2 + \ldots + k_rC_r + 0 \,.\, C_{r+1} + \ldots + 0 \,.\, C_n. \qquad \ldots(5)$$

Comparing (2) and (5), we get $x_1 = k_1, x_2 = k_2, \ldots x_r = k_r, x_{r+1} = 0, \ldots x_n = 0$ and these values of $x_1, x_2, \ldots x_n$ are the solution of $AX = B$. Hence the equations are consistent.

REMARKS

- The n equations in n unknowns have a unique solution.
- If rank of $A <$ rank of $[A \mid B]$, then there is no solution.
- If $r = n$, then there will be a unique solution.
- If $r < n$, then $(n - r)$ variables can be assigned arbitrary values. Thus there will be infinite solution and $(n - r + 1)$ solutions will be lineraly independent.
- If $m < n$ and $r \le m < n$, then equations will have infinite solutions.

• SOLVED EXAMPLES

Examples 1. *Show that the equations*

$$x + 2y - z = 3,\ 3x - y + 2z = 1,\ 2x - 2y + 3z = 2,\ x - y + z = -1$$

are consistent and solve them.

Solution. The given equations can be written as

$$\begin{bmatrix} 1 & 2 & -1 \\ 3 & -1 & 2 \\ 2 & -2 & 3 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \\ -1 \end{bmatrix} \quad i.e., \quad \boldsymbol{AX} = \boldsymbol{B}.$$

Therefore Augmented matrix is

$$[A \mid B] = \left[\begin{array}{ccc:c} 1 & 2 & -1 & 3 \\ 3 & -1 & 2 & 1 \\ 2 & -2 & 3 & 2 \\ 1 & -1 & 1 & -1 \end{array}\right]$$

performing $R_2 \to R_2 - 3R_1$, $R_3 \to R_3 - 2R_1$, $R_4 \to R_4 - R_1$

we get $$[A \mid B] \sim \left[\begin{array}{ccc:c} 1 & 2 & -1 & 3 \\ 0 & -7 & 5 & -8 \\ 0 & -6 & 5 & -4 \\ 0 & -3 & 2 & -4 \end{array}\right]$$

performing $R_2 \to R_2 - R_3$

$$\sim \left[\begin{array}{ccc:c} 1 & 2 & -1 & 3 \\ 0 & -1 & 0 & -4 \\ 0 & -6 & 5 & -4 \\ 0 & -3 & 2 & -4 \end{array}\right]$$

performing $R_3 \to R_3 - 6R_2$, $R_4 \to R_4 - 3R_2$

$$\sim \left[\begin{array}{ccc:c} 1 & 2 & -1 & 3 \\ 0 & -1 & 0 & -4 \\ 0 & 0 & 5 & 20 \\ 0 & 0 & 2 & 8 \end{array}\right]$$

performing $R_3 \to \frac{1}{5} R_3$, $R_4 \to \frac{1}{2} R_4$

$$\sim \left[\begin{array}{ccc:c} 1 & 2 & -1 & 3 \\ 0 & -1 & 0 & -4 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

performing $R_4 \to R_4 - R_3$

$$\sim \left[\begin{array}{ccc:c} 1 & 2 & -1 & 3 \\ 0 & -1 & 0 & -4 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

This is an Echelon form and having three non-zero rows. Thus rank $\boldsymbol{A}$ = rank of $[\boldsymbol{A} \mid \boldsymbol{B}] = 3$

Therefore the equations are consistent

and $$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \\ 4 \\ 0 \end{bmatrix}.$$

$\therefore \quad x + 2y - z = 3,\ -y = -4,\ z = 4$

Hence the solution is

$x = -1,\ y = 4,\ z = 4.$

Example 2. *Solve the following equations by matrix method* :

$$x - 2y + 3z = 6$$
$$3x + y - 4z = -7$$
$$5x - 3y + 2z = 5.$$

Solution. The given equations can be written as

$$\begin{bmatrix} 1 & -2 & 3 \\ 3 & 1 & -4 \\ 5 & -3 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ -7 \\ 5 \end{bmatrix}$$

i.e., $\boldsymbol{AX} = \boldsymbol{B}.$

$\therefore$ Augmented matrix is

$$[A \mid B] = \begin{bmatrix} 1 & -2 & 3 & \vdots & 6 \\ 3 & 1 & -4 & \vdots & -7 \\ 5 & -3 & 2 & \vdots & 5 \end{bmatrix}.$$

performing $R_2 \to R_2 - 3R_1$, $R_3 \to R_3 - 5R_1$, we get

$$[A \mid B] \sim \begin{bmatrix} 1 & -2 & 3 & \vdots & 6 \\ 0 & 7 & -13 & \vdots & -25 \\ 0 & 7 & -13 & \vdots & -25 \end{bmatrix}$$

performing $R_3 \to R_3 - R_2$

$$\sim \begin{bmatrix} 1 & -2 & 3 & \vdots & 6 \\ 0 & 7 & -13 & \vdots & -25 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}.$$

This is an Echelon form and having two non-zero rows and rank A = rank $[A \mid B] = 2$. Thus the equations are consistant.

$$\begin{bmatrix} 1 & -2 & 3 \\ 0 & 7 & -13 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ -25 \\ 5 \end{bmatrix}$$

i.e.,

$$x - 2y + 3z = 6$$
$$7y - 13z = -25$$

Let $z = c$, then

$$y = -\frac{25}{7} + \frac{13}{7}c$$
$$x = -\frac{8}{7} + \frac{5}{7}c.$$

Hence the solution is

$$x = -\frac{8}{7} + \frac{5}{7}c,\ y = -\frac{25}{7} + \frac{13}{7}c,\ z = c$$

where c is an arbitrary constant.

Example 3. *Investigate for what values of λ, μ the simultaneous equations*

$$x + y + z = 6,\ x + 2y + 3z = 10,\ x + 2y + \lambda z = \mu$$

have (i) No solution (ii) a unique solution (iii) an infinite solutions.

Solution. The given equations can be written as

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ \mu \end{bmatrix}$$

i.e.,

$$AX = B$$

Therefore Augmented matrix is

$$(A \mid B) = \begin{bmatrix} 1 & 1 & 1 & \vdots & 6 \\ 1 & 2 & 3 & \vdots & 10 \\ 1 & 2 & \lambda & \vdots & \mu \end{bmatrix}$$

performing $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$, we get

$$\sim \begin{bmatrix} 1 & 1 & 1 & \vdots & 6 \\ 0 & 1 & 2 & \vdots & 4 \\ 0 & 1 & \lambda - 1 & \vdots & \mu - 6 \end{bmatrix}$$

performing $R_3 \to R_3 - R_2$

$$\sim \begin{bmatrix} 1 & 1 & 1 & \vdots & 6 \\ 0 & 1 & 2 & \vdots & 4 \\ 0 & 0 & \lambda - 3 & \vdots & \mu - 10 \end{bmatrix}$$

If $\lambda \neq 3$, then rank A = rank $(A \mid B) = 3$. Thus in this case a unique solution exists.

If $\lambda = 3$ and $\mu_0 \neq 10$, then rank $A = 2$, rank $(A \mid B)$ is 3. Thus rank $A \neq$ rank $(A \mid B)$. Hence in this case equations are inconsistent.

If $\lambda = 3$ and $\mu = 10$, then rank A = rank $(A \mid B) = 2$. Thus in this case infinite solutions exist.

3. Examine if the system of equations $x+y+4z=6, 3x+2y-2z=9, 5x+y+2z=13$ is consistent. Find also the solutions if exists.
4. For what values of λ will the following equations fail to have a unique solution
$$3x-y+\lambda z=1, 2x+y+z=2, x+2y-\lambda z=-1.$$
Will the equations have any solutions for these values of λ ?
5. Solve $2x+3y+z=9, x+2y+3z=6, 3x+y+2z=8$.

Solve the following equations by matrix method :

6. Show that the following equations are inconsistent
$$2x-y+z=4, 3x-y+z=6, 4x-y+2z=7, -x+y-z=9.$$
7. Show that the equations are inconsistent
$$x-4y+7z=14, 3x+8y-2z=13, 7x-8y+26z=5.$$
8. Prove that the following system of equations have a unique solution
$$5x+3y+14z=4, y+2z=1, x-y+2z=0.$$
9. Solve the equations by matrix method
$$x+y+z=9, 2x+5y+7z=52, 2x+y-z=0.$$

ANSWERS

1. $x=1, y=2, z=3$ 2. $x=2c-1, y=3-2c, z=c$

3. Consistent; $x=2, y=2, z=\frac{1}{2}$. 4. $\lambda \neq -\frac{7}{2}$ solution is unique; $\lambda=-\frac{7}{2}$, no solution.

5. $x=\frac{35}{18}, y=\frac{29}{18}, z=\frac{5}{18}$. 9. $x=1, y=3, z=4$.

OBJECTIVE EVALUATION

➤ FILL IN THE BLANKS :

1. The matrix equation $\boldsymbol{AX}=\mathbf{0}$ is a system of linear ...... equations.
2. If the rank of $\boldsymbol{A}=r$, then the number of linearly independent solutions of m homogeneous equation in n variables is ....
3. If X_1 and X_2 are the solutions of $\boldsymbol{AX}=\mathbf{0}$, then ...... is also a solution of $\boldsymbol{AX}=\mathbf{0}$.

➤ TRUE OR FALSE :

Write 'T' for True and 'F' for False statement :

1. If X_1 is a solution of $\boldsymbol{AX}=\mathbf{0}$, then $2X_1$ is not a solution of $\boldsymbol{AX}=\mathbf{0}$. ***(T/F)***
2. If the rank $\boldsymbol{A}$ is less than the number of unknowns, then there will infinite solution of $\boldsymbol{AX}=\mathbf{0}$. ***(T/F)***
3. The equation $\boldsymbol{AX}=\boldsymbol{B}$ are inconsistent if rank $\boldsymbol{A} \neq$ rank $(\boldsymbol{A} \mid \boldsymbol{B})$. ***(T/F)***

➤ MULTIPLE CHOICE QUESTIONS :

Choose the most appropriate one :

1. The matrix equation $\boldsymbol{AX}=\mathbf{0}$ represents :
(a) non-homogeneous linear equations (b) homogeneous linear equations
(c) homogeneous-non linear equations (d) None of these.
2. If X_1 and X_2 are the solution of $\boldsymbol{AX}=\mathbf{0}$, then which one is also the solution of $\boldsymbol{AX}=\mathbf{0}$:
(a) $X_1^2+X_2^2$ (b) $(X_1+X_2)^2$ (c) X_1+X_2 (d) X_1/X_2.
3. If the equations $\boldsymbol{AX}=\boldsymbol{B}$ are consistent and rank of $(\boldsymbol{A} \mid \boldsymbol{B})=4$, then rank of $\boldsymbol{A}$ is :
(a) 4 (b) 8 (c) 3 (d) 2.

ANSWERS

Fill in the Blanks :

1. $C_1X_1+C_2X_2$ 2. rank $(A \mid B)$ 3. Inconsistent

True or False :

1. F 2. T 3. T

Multiple Choice Questions :

1. (b) 2. (c) 3. (a)

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• STUDENT ACTIVITY

1. Find the solution of

$$x + y + z = 0$$
$$2x + 5y + 7z = 0$$
$$2x - 5y + 3z = 0$$

2. Solve the following equations by matrix method :

$$x + 2y + 3z + 11$$
$$3x + y \;\; 2z \;\; 11$$
$$2x + 3y + z = 11$$

• SUMMARY

- $AX = 0$ is the matrix equation of system of homogeneous equations.
- If X_1 and X_2 are two non-trivial solutionsof $AX = 0$, then $k_1X_1 + k_2X_2$ is also its solution, where k_1 and k_2 are any arbitrary numbers.
- If r is the rank of the matrix A in $AX = 0$, then we have following conclusions :
  (i) If $r = n$, then $AX = 0$ has only trivial solution.
  (ii) If $r < n$, then $AX = 0$ has infinitely many solutions.
- $AX = B$ is the metrix equation of the system of non-homogeneous equations.
- If r is the rank of the matrix A of order $n \times n$ in $AX = B$, then we have following conclusions .
  (i) If $r = n$, then $AX = B$ will have unique solution.
  (ii) If $r < n$, then $AX = B$ will have infinitely many solutions.
  (iii) If A is the matrix of order $m \times n$ with $m < n$ and $r \leq m < n$, then $AX = B$ will have infinitely many solutions.

• TEST YOURSELF–2

1. Use matrix method to solve the equations

$$2x - y + 3z = 9, \; x + y + z = 6, \; x - y + z = 2.$$

2. Show that the equations $x - 3y - 8z + 10 = 0$, $3x + y - 4z = 0$, $2x + 5y + 6z - 13 = 0$ are consistent and solve them.

$$\left.\begin{array}{l}(a_{11}-\lambda)\,x_1+a_{12}x_2+\ldots+a_{1n}\,x_n=0\\ a_{21}\,x_1+(a_{22}-\lambda)\,x_2+\ldots+a_{2n}\,x_n=0\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ a_{n1}\,x_1+a_{n2}\,x_2+\ldots+(a_{nn}-\lambda)\,x_n=0\end{array}\right\} \quad \ldots(3)$$

Therefore, the coefficient matrix is

$$(A-\lambda I)=\begin{bmatrix}a_{11}-\lambda & a_{12} & \ldots & a_{1n}\\ a_{21} & a_{22}-\lambda & \ldots & a_{2n}\\ \ldots & \ldots & \ldots & \ldots\\ a_{n1} & a_{n2} & \ldots & a_{nn}-\lambda\end{bmatrix}_{n\times n}$$

For non-zero solution, the rank of the matrix $(A-\lambda I)$ must be less than the number of unknowns *i.e.*, the rank of $(A-\lambda I)$ must be less than n and for this the matrix $(A-\lambda I)$ must be singular *i.e.*, det $(A-\lambda I)=0$.

i.e., $$|A-\lambda I|=0. \quad \ldots(4)$$

This equation (4) is called the **characteristic equation** of the matrix A. Infact this equation is an equation of degree n in λ so it has n roots. These roots are called **characteristic roots** or **eigen values** of A and the set of eigen values are called **spectrum** of A.

Some Important Theorems :

Theorem 1. *λ is a characteristic root of a matrix A if and only if there exists a non-zero vector X such that*

$$AX=\lambda X.$$

Proof. Suppose λ is a characteristic root of the matrix A. This implies λ is a root of the equation

$$|A-\lambda I|=0. \quad \ldots(1)$$

Thus from (1) it is concluded that the matrix $A-\lambda I$ is a singular matrix. Therefore the rank of the matrix $(A-\lambda I)$ is less than the number of unknowns so there must exists a non-zero solutions of the equation.

$$(A-\lambda I)\,X=0 \quad i.e., \quad AX=\lambda X.$$

Conversely, suppose there exists a non-zero solution $X\neq 0$ such that

$$AX=\lambda X \quad i.e., \quad (A-\lambda I)\,X=0. \quad \ldots(2)$$

Since the matrix equation (2) has a non-zero solution thus the matrix $A-\lambda I$ is singular. That is

$$|A-\lambda I|=0.$$

Hence, λ is a characteristic root of a matrix A.

Theorem 2. *If X is a characteristic vector of a matrix A, then X cannot correspond to more than one characteristic values of A.*

Proof. Let us suppose X is a characteristic vector of a matrix A corresponding to two characteristic root λ_1 and λ_2 of A. We have to prove that $\lambda_1=\lambda_2$. Since X is characteristic vector corresponding to λ_1 and λ_2, then

$$AX=\lambda_1 X \quad \ldots(1)$$

and $$AX=\lambda_2 X. \quad \ldots(2)$$

From (1) and (2), we get

$$\lambda_1 X=\lambda_2 X \quad \text{or} \quad (\lambda_1-\lambda_2)\,X=0$$

or $$\lambda_1-\lambda_2=0 \quad (\because X\neq 0)$$

or $$\lambda_1=\lambda_2.$$

Hence proved.

Theorem 3. *If X is an eigenvector of A corresponding to eigenvalue λ, then KX is also an eigenvector of A corresponding to same eigenvalue λ, where K is any non-zero number.*

Proof. Since X is an eigenvector corresponding to the eigenvalue λ of A, then

$$AX=\lambda X \quad \ldots(1)$$

suppose K is any non-zero number, then $KX\neq 0$

$$\therefore \quad A\,(KX)=K\,(AX)=K\,(\lambda X) \quad \text{[from (1)]}$$
$$=\lambda(KX)$$

i.e., $$A\,(KX)=\lambda(KX).$$

This implies that KX is also an eigenvector corresponding to the same eigenvalue λ of A.

• 10.3. CAYLEY-HAMILTON'S THEOREM

Statement. *Every square matrix satisfies its characteristic equation*

UNIT

10

EIGENVALUES AND EIGENVECTORS

LEARNING OBJECTIVES

- Linear Dependence and Independence of Vectors
- Eigenvalues and Eigenvectors
- Cayley-Hamilton's Theorem
- Solved Examples
  - Student Activity
  - Summary
  - Test Yourself

LEARNING OBJECTIVES

After going through this unit you will learn :

- How to find the eigen values and eigen vectors of a given vectors.
- How to verify Cayley-Hamilton's Theorem.

10.1. LINEAR DEPENDENCE AND INDEPENDENCE OF VECTORS

Some General Definitions :

Definition. Any ordered n-tuples of numbers is called an **n-vector**. Let $x_1, x_2, \dots x_n$ be n numbers and be placed in fixed position. Then the ordered n-tuples $(x_1, x_2, \dots x_n)$ is called an n-vector. It is denoted by $X = (x_1, x_2, \dots x_n)$. These n numbers $x_1, x_2, \dots x_n$ are also called the components of X.

Definition. A set of r vectors $X_1, X_2, \dots X_r$ is said to be linearly dependent if there must exist r scalars $a_1, a_2 \dots a_r$ not all zero such that

$$a_1X_1 + a_2X_2 + \dots + a_rX_r = 0$$

where $X_1, X_2, \dots X_r$ are all n-vectors and O is also n-vector whose components are all zero.

Definition. A set of r vectors $X_1, X_2, \dots X_r$ is said to be **linearly independent** if we have a relation

$$a_1X_1 + a_2X_2 + \dots + a_rX_r = 0$$

for which $a_1 = 0 = a_2 = \dots = a_r$.

Definition. A vector X is said to be a **linear combination** of $X_1, X_2, \dots X_r$ if X can be expressed as

$$X = a_1X_1 + a_2X_2 + \dots + a_rX_r$$

where $a_1, a_2, a_3, \dots a_r$ are any numbers.

10.2. EIGENVALUES AND EIGENVECTORS

Let $A = [a_{ij}]_{n \times n}$ be a given square matrix of order $n \times n$ and let

$$AX = \lambda X \qquad \dots(1)$$

be a vector equation. It is obvious that $X = 0$ is a trival solution of (1) for all values of λ. A value of λ for which the vector equation (1) has a non-zero solution *i.e.*, $X \neq 0$, is called **eigen value** of the matrix A. This eigen value is also known as **characteristic** value and the corresponding non-zero solution $X \neq 0$ is called a **eigen vector** or **characteristic vector.**

The equation (1) can also be written as

$$AX = \lambda IX \quad \text{or} \quad (A - \lambda I)\,X = 0 \qquad \dots(2)$$

where I is a unit matrix of order $n \times n$. Now the equation (2) represents a matrix equation of n homogeneous linear equation in n unknowns. Let these homogeneous equations be given as

Example 2. *Find the characteristic roots and the corresponding characteristic vectors of the matrix* $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$.

Solution. The characteristic equation of the matrix A is given by

$$|A - \lambda I| = 0$$

$$\therefore \quad \begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0$$

or $\quad (8-\lambda)[(7-\lambda)(3-\lambda)-16] - 6[-8+6(3-\lambda)] + 2[24-2(7-\lambda)] = 0$

or $\quad (8-\lambda)(7-\lambda)(3-\lambda) - 16(8-\lambda) + 48 - 36(3-\lambda) + 48 - 4(7-\lambda) = 0$

or $\quad 168 - 101\lambda + 18\lambda^2 - \lambda^3 - 128 + 16\lambda + 48 - 108 + 36\lambda + 48 - 28 + 4\lambda = 0$

or $\quad \lambda^3 - 18\lambda^2 + 45\lambda = 0 \quad$ or $\quad \lambda(\lambda-3)(\lambda-15) = 0$

or $\quad \lambda = 0, 3, 15.$

Hence the characteristic roots are 0, 3, 15.

Determination of Eigenvectors :

Let $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be an eigenvector corresponding to $\lambda = 0$, then we have

$$AX = \lambda X$$

or $$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 . \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

performing $R_3 \longleftrightarrow R_1$

$$\begin{bmatrix} 2 & -4 & 3 \\ -6 & 7 & -4 \\ 8 & -6 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

performing $R_2 \to R_2 + 3R_1, R_3 \to R_3 - 4R_1$

$$\begin{bmatrix} 2 & -4 & 3 \\ 0 & -5 & 5 \\ 0 & 10 & -10 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

performing $R_3 \to R_3 + 2R_2$

$$\begin{bmatrix} 2 & -4 & 3 \\ 0 & -5 & 5 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

This is an *Echelon form*. Therefore rank of this matrix is 2. Thus there will be $3 - 2 = 1$, non-zero linearly independent solution obtained by the equations

$$2x_1 - 4x_2 + 3x_3 = 0 \quad \text{and} \quad -5x_2 + 5x_3 = 0.$$

From these equations, we have $x_2 = x_3$ and assume $x_2 = x_3 = 1$ then from $2x_1 - 4x_2 + 3x_3 = 0$, we get $x_1 = \frac{1}{2}$.

Thus $X = \begin{bmatrix} \frac{1}{2} \\ 1 \\ 1 \end{bmatrix}$ is a characteristic vector corresponding to $\lambda = 0$.

Characteristic Vector Corresponding to $\lambda = 3$.

The characteristic vector corresponding $\lambda = 3$ is given by the non-zero solution of the equation

$$(A - 3I)X = 0$$

or $$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 3\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

or *Let A be a square matrix of order n and the characteristic equation of A is*

$$|A - \lambda I| = (-1)^n [\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \ldots + a_{n-1}\lambda + a_n] = 0$$

then its matrix equation

$$X^n + a_1X^{n-1} + a_2X^{n-2} + \ldots + a_{n-1}X + a_nI = 0$$

is satisfied by the matrix $X = A$

i.e., $$A^n + a_1A^{n-1} + a_2A^{n-2} + \ldots + a_{n-1}A + a_nI = 0$$

where I is a unit matrix of order n and 0 is null matrix of order n.

Proof. Since A and I are two square matrix of order n and λ is any characteristic root of A, then the matrix $(A - \lambda I)$ is also a square matrix of order n and whose elements are at most of degree one in λ. Therefore Adj $(A - \lambda I)$ will have its elements a polynomials in λ of degree $n - 1$ or less and thus Adj $(A - \lambda I)$ can be expressed as a matrix polynomial in λ as follows :

$$\text{Adj}\,(A - \lambda I) = B_0\lambda^{n-1} + B_1\lambda^{n-2} + \ldots + B_{n-2}\lambda + B_{n-1} \quad \ldots(1)$$

where $B_0, B_1, \ldots B_{n-1}$ are the square matrices of order n.

Since we know that $A\,(\text{Adj}\,A) = |A|\,I_n$

$$\therefore \quad (A - \lambda I)\,\text{Adj}\,(A - \lambda I) = |A - \lambda I|\,I$$

or $$(A - \lambda I)\,\text{Adj}\,(A - \lambda I) = (-1)^n [\lambda^n + a_1\lambda^{n-1} + \ldots + a_{n-1}\lambda + a_n]\,I. \quad \ldots(2)$$

Multiplying both sides of (1) by $(A - \lambda I)$, we get

$$(A - \lambda I)\,\text{Adj}\,(A - \lambda I) = (A - \lambda I)\,[B_0\lambda^{n-1} + B_1\lambda^{n-2} + \ldots + B_{n-2}\lambda + B_{n-1}]. \quad \ldots(3)$$

From (2) and (3), we get

$$(A - \lambda I)\,[B_0\lambda^{n-1} + B_1\lambda^{n-2} + \ldots + B_{n-2}\lambda + B_{n-1}]$$
$$= (-1)^n [\lambda^n + a_1\lambda^{n-1} + \ldots + a_{n-1}\lambda + a_n]\,I.$$

Now comparing the coefficients of like power of λ, we get

$$\left.\begin{aligned} -IB_0 &= (-1)^n I \\ AB_0 - IB_1 &= (-1)^n a_1 I \\ AB_1 - IB_2 &= (-1)^n a_2 I \\ &\ldots\ldots\ldots\ldots \\ AB_{n-2} - IB_{n-3} &= (-1)^n a_{n-1} I \\ AB_{n-1} &= (-1)^n a_n I \end{aligned}\right\} \quad \ldots(4)$$

premultiplying first, second, third etc. equations of (4) by A^n, A^{n-1}, A^{n-2} etc. respectively and then adding, we get

$$-A^nB_0 + A^nB_0 - A^{n-1}B_1 + A^{n-1}B_1 + \ldots = (-1)^n [A^n + a_1A^{n-1} + \ldots + a_nI]$$

or $$0 = (-1)^n [A^n + a_1A^{n-1} + \ldots + a_nI]$$

or $$A^n + a_1A^{n-1} + \ldots + a_nI = 0. \quad \ldots(5)$$

Determination of A^{-1}. If the matrix A is non-singular *i.e.*, $|A| \neq 0$ and provided $a_n \neq 0$ because $|A| = (-1)^n a_n$. Then A^{-1} exists. Now from (5), we have

$$A^n + a_1A^{n-1} + \ldots + a_nI = 0$$

premultiplying this equation by A^{-1}, we get

$$A^{n-1} + a_1A^{n-2} + \ldots + a_{n-1}I + a_nA^{-1} = 0$$

or $$A^{-1} = -\frac{1}{a_n}[A^{n-1} + a_1A^{n-2} + \ldots + a_{n-1}I.$$

• SOLVED EXAMPLES

Example 1. *Determine the characteristic roots of the matrix* $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & 2 \\ 0 & 0 & 7 \end{bmatrix}$.

Solution. The characteristic equation of A is given by

$$|A - \lambda I| = 0$$

$$\therefore \quad \begin{vmatrix} 1-\lambda & 2 & 3 \\ 0 & -4-\lambda & 2 \\ 0 & 0 & 7-\lambda \end{vmatrix} = 0$$

or $$(1-\lambda)(-4-\lambda)(7-\lambda) = 0 \text{ or } \lambda = -4, 1, 7.$$

$$\begin{bmatrix} 1 & -2 & -6 \\ 0 & -20 & -40 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The rank of above coefficient matrix is 2. Therefore there will be $3 - 2 = 1$ non-zero solution which is given by

$$x_1 - 2x_2 - 6x_3 = 0$$
$$-20x_2 - 40x_3 = 0.$$

From second equation we get $x_2 = -2x_3$. Let us assume $x_3 = -1$, $x_2 = 2$. Then from first equation, we get

$$x_1 = -2.$$

Hence the eigenvector $X = \begin{bmatrix} -2 \\ 2 \\ -1 \end{bmatrix}$.

Example 3. *Obtain the characteristic equation of the matrix* $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$ *and verify that it is satisfied by* A *and hence find its inverse.*

Solution The characteristic equation of A is given by

$$|A - \lambda I| = 0$$

or $$\begin{vmatrix} 1-\lambda & 0 & 2 \\ 0 & 2-\lambda & 1 \\ 2 & 0 & 3-\lambda \end{vmatrix} = 0$$

or $$(1-\lambda)[(2-\lambda)(3-\lambda) - 0] + 2[0 - 2(2-\lambda)] = 0$$

or $$(1-\lambda)(2-\lambda)(3-\lambda) - 4(2-\lambda) = 0$$

or $$(2-\lambda)[(1-\lambda)(3-\lambda) - 4] = 0 \quad \text{or} \quad (2-\lambda)(\lambda^2 - 4\lambda - 1) = 0$$

or $$2\lambda^2 - 8\lambda - 2 - \lambda^3 + 4\lambda^2 + \lambda = 0 \quad \text{or} \quad \lambda^3 - 6\lambda^2 + 7\lambda + 2 = 0.$$

This is the required characteristic equation of A.

Next we have to show that

$\boldsymbol{A^3 - 6A^2 + 7A + 2I = 0}$

$$\therefore \quad A^2 = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix}$$

and $$A^3 = A^2 . A = \begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix}\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 21 & 0 & 34 \\ 12 & 8 & 23 \\ 34 & 0 & 55 \end{bmatrix}.$$

$\therefore \quad \boldsymbol{A^3 - 6A^2 + 7A + 2I}$

$$= \begin{bmatrix} 21 & 0 & 34 \\ 12 & 8 & 23 \\ 34 & 0 & 55 \end{bmatrix} - 6\begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 21-30 & 0-0 & 34-48 \\ 12-12 & 8-24 & 23-30 \\ 34-48 & 0-0 & 55-78 \end{bmatrix} + \begin{bmatrix} 9 & 0 & 14 \\ 0 & 16 & 7 \\ 14 & 0 & 23 \end{bmatrix}$$

$$= \begin{bmatrix} -9+9 & 0 & -14+14 \\ 0 & -16+16 & -7+7 \\ -14+14 & 0 & -23+23 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0. \qquad \ldots(1)$$

Hence $\boldsymbol{A^3 - 6A^2 + 7A + 2I = 0}$.

Determination of A^{-1}.

Since $|A| = -2 \neq 0$.

Premultiplying (1) by A^{-1}, we get

$$\boldsymbol{A^2 - 6A + 7I + 2A^{-1} = 0}$$

or $$\boldsymbol{A^{-1} = -\frac{1}{2}[A^2 - 6A + 7I]}$$

$$= -\frac{1}{2}\left\{\begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix} - 6\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right\} = -\frac{1}{2}\left\{\begin{bmatrix} 6 & 0 & -4 \\ 2 & -1 & -1 \\ -4 & 0 & 2 \end{bmatrix}\right\}$$

or $$\begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

performing $R_3 \longleftrightarrow R_1$, we get

$$\begin{bmatrix} 2 & -4 & 0 \\ -6 & 4 & -4 \\ 5 & -6 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

performing $R_1 \to \frac{1}{2} R_1$

$$\begin{bmatrix} 1 & -2 & 0 \\ -6 & 4 & -4 \\ 5 & -6 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

performing $R_2 \to R_2 + 6R_1$, $R_3 \to R_3 - 5R_1$, we get

$$\begin{bmatrix} 1 & -2 & 0 \\ 0 & -8 & -4 \\ 0 & 4 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

performing $R_3 \to R_3 + \frac{1}{2} R_2$

$$\begin{bmatrix} 1 & -2 & 0 \\ 0 & -8 & -4 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Thus the rank of this above matrix is 2. Therefore these above equation will have $3 - 2 = 1$ non-zero solution given by

$$x_1 - 2x_2 = 0 \quad \text{and} \quad -8x_2 - 4x_3 = 0.$$

From second equation we get $x_2 = -\frac{1}{2} x_3$. Let us assume $x_2 = -2$, $x_3 = 4$. Then from first equation, we get

$$x_1 = 2x_2 = 2\,(-2) = -4.$$

Hence the eigenvector $X = \begin{bmatrix} -4 \\ -2 \\ 4 \end{bmatrix}$.

Eigenvector Corresponding to $\lambda = 15$.

The eigenvector X of A corresponding to $\lambda = 15$ is given by the solution of the equation

$$(A - 15I)\,X = 0$$

or $$\begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

performing $R_3 \leftrightarrow R_1$, we get

$$\begin{bmatrix} 2 & -4 & -12 \\ -6 & -8 & -4 \\ -7 & -6 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

performing $R_1 \to \frac{1}{2} R_1$

$$\begin{bmatrix} 1 & -2 & -6 \\ -6 & -8 & -4 \\ -7 & -6 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

performing $R_2 \to R_2 + 6R_1$, $R_3 \to R_3 + 7R_1$

$$\begin{bmatrix} 1 & -2 & -6 \\ 0 & -20 & -40 \\ 0 & -20 & -40 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

performing $R_3 \to R_3 - R_2$

2. Find the eigenvalues and corresponding eigenvectors of the matrix $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$.

3. Verify the Cayley-Hamilton theorem for the matrix $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

4. Verify that the matrix A satisfies its characteristic equation and find A^{-1}, where
$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 3 & -1 & 1 \end{bmatrix}.$$

5. Show that the matrix $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ satisfies Cayley-Hamilton theorem.

6. Find the eigenvalues and corresponding vectors of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{bmatrix}$.

ANSWERS

1. $2, -1 \pm \sqrt{3}$

2. 2, 2, 8; [2 −1 1]′, [−1 0 2]′, [1 2 0]′

4. $A^{-1} = \frac{1}{9}\begin{bmatrix} 0 & 3 & 3 \\ 3 & 2 & -1 \\ 3 & -7 & -1 \end{bmatrix}$

6. 1, 2 ; [1, 0, 0]′; [2, 1, 0]′

OBJECTIVE EVALUATION

➤ FILL IN THE BLANKS :

1. The characteristic roots of a Hermitian matrix are ...
2. The characteristic roots of a unitary matrix are of ...
3. The characteristic roots of the matrix $\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$ are .... .
4. The eigenvalues of the matrix A and A^T are .... .

➤ TRUE OR FALSE :

Write 'T' for True and 'F' for False statement :

1. If $X \neq 0$ is an eigenvector corresponding to the eigenvalue 3 of A, then $(A - 3I)X = 0$. *(T/F)*
2. The characteristic roots of a skew Hermitian matrix are always real. *(T/F)*
3. The eigenvalues of $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ are 1, 0, 1. *(T/F)*

➤ MULTIPLE CHOICE QUESTIONS :

Choose the most appropriate one :

1. If $\lambda = 0$ is an eigenvalues of A, then det (A) is :
   (a) 0 (b) 1 (c) λ (d) None of these.
2. If $|A| \neq 0$ and λ is an eigenvalue of A, then the eigenvalue of A^{-1} is :
   (a) λ (b) λ^2 (c) $\frac{1}{\lambda}$ (d) 0.
3. If $|A| \neq 0$ and λ is an eigenvalue of A, then the eigenvalue of Adj (A) is :
   (a) $\lambda |A|$ (b) $\frac{1}{\lambda}$ (c) λ^2 (d) $\frac{|A|}{\lambda}$.

ANSWERS

Fill in the Blanks :

1. real **2.** unit modulus **3.** 6, 1 **4.** same

True or False :

1. T **2.** F **3.** F

Multiple Choice Questions :

1. (a) **2.** (c) **3.** (d)

❑❑❑

$$A^{-1} = \begin{bmatrix} -3 & 0 & 2 \\ -1 & \frac{1}{2} & \frac{1}{2} \\ 2 & 0 & -1 \end{bmatrix}.$$

• STUDENT ACTIVITY

1. Prove that the characteristic roots of a Hermitian matrix are real.

2. Verify Cayley-Hamilton's theorem for the matrix

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$$

• SUMMARY

- Let A be a square matrix of order $n \times n$ and λ be its one of eigen-value, then
$$AX = \lambda X \quad \text{for} \quad X \neq 0$$
The characteristic equation of A is given by
$$|A - \lambda I| = 0.$$
- If X be an eigen vector of A corresponding to eigen value λ, then kX is also an eigen vector of A corresponding to the sam eigen value λ, k being a non-zero number.
- The characteristic roots of a Hermitian matrix are all real.
- The characteristic roots of a unitary matrix are of unit modulus.
- **Cayley-Hemilton's Theorem :** Every square matrix satisfies its characteristic equation.

• TEST YOURSELF

1. Find the characteristic roots of the matrix $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & -1 \\ 2 & -1 & 0 \end{bmatrix}$.

UNIT

11

GROUPS

LEARNING OBJECTIVES

- Binary Operations
- Groups
- Solved Examples
- Some Properties of Groups
- Composition Table for Finite Set
  - Test Yourself–1
- Integral Power of an Element
- Order of an Element of a Group
- Some Important Theorems
- Solved Examples
  - Test Yourself–2
- Permutations and Permutations Groups
- Cyclic Permutation
- Even and Odd Permutations
- Solved Examples
  - Test Yourself–3
- Homomorphism and Isomorphism of Groups
- Solved Examples
  - Test Yourself–4
- Subgroups of a Groups
- Union and Intersection of Subgroups
- Solved Examples
- Cosets
- Solved Examples
  - Test Yourself–5
- Cyclic Groups
- Solved Examples
  - Student Activity
  - Summary
  - Test Yourself–6

LEARNING OBJECTIVES

After going through this unit you will learn :

- About the groups, properties of the groups, permutation groups etc.
- Homomorphism and isomorphism of groups
- Some important theorems related to groups.

• 11.1. BINARY COMPOSITIONS

Let S be a non-empty set. Any function from $S \times S$ to S is called a **binary composition** (or a **binary operation**) in S.

If $f : S \times S \to S$ be a binary composition in S and $x, y \in S$ then $f(x, y)$ is called the **composite** of x and y under the composition f. It is usually denote by any of the following symbols.

$*, T, \perp \oplus, \odot, +, .,$ Juxtaposition

If we denote a binary composition in a set $*$ and $x, y \in S$, then the composite of x and y under this composition is denoted by $x * y$.

REMARKS

➤ The number of total binary operations defined on S, where $n(S) = m$, is $(m)^{m^2}$.

Algebraic structure. A non-empty set with one or more binary operations defined over it and satisfying certain laws of binary operation is called algebraic structure or an algebraic system.

• 11.2. GROUPS

"Let G be a non-empty set and $*$ be a binary operation defined on it, then the structure $(G, *)$ is said to be a **group** if the following axioms are satisfied.

(i) **Closure property.** $a * b \in G \;\; \forall \; a, b \in G$.

(ii) **Associativity.** The operation $*$ is associative on G. *i.e.*,

$$a * (b * c) = (a * b) * c \;\; \forall \; a, b, c \in G.$$

(iii) **Existence of identity.** There exists an element $e \in G$ such that

$$a * e = e * a = a \;\; \forall \; a \in G.$$

The element e is called identity of $*$ in G.

(iv) **Existence of inverse.** For each element $a \in G$, there exist an element $b \in G$ such that

$$a * b = b * a = e.$$

The element b is called the **inverse** of element a with respect to $*$ and we write $b = a^{-1}$.

ABELIAN OR COMMUTATIVE GROUP

A group $(G, *)$ is said to be **abelian** or **commutative** if $a * b = b * a \;\; \forall \; a, b \in G$.

The group which are not abelian are called non-abelian or non-commutative.

FINITE AND INFINITE GROUP

If a group contains a finite number of elements, it is called a **finite group.**

If the number of elements in a group is infinite, it is called an **infinite group.**

Order of a group. The number of elements in a finite group is called the order of the group. It is denoted by $o(G)$.

An infinite group is called a group of infinite order.

• SOLVED EXAMPLES

Example 1. *Show that the set* **Z** *of integers (positive or negative including* 0) *with additive binary operation is an infinite ablian group.*

Solution. Let us apply the group-axioms to all integers.

(i) Closure property. Closure property is satisfied because the sum of any two integers is an integers.

(ii) Associativity. The associative property is satisfied, because of a, b, c are any three integers, then $(a + b) + c = a + (b + c)$.

(iii) Existence of identity. The axiom on identity is satisfied, because 0 is the identity element in the set Z such that $a + 0 = a \;\; \forall \; a \in \mathbf{Z}$.

(iv) Existence of inverse. The axiom on inverse is satisfied, because the inverse of any integer a is the integer $-a$ such that $a + (-a) = (-a) + a = 0$, the identity element.

(v) Commutativity. Since, we know that $a + b = b + a \;\; \forall \; a, b \in \mathbf{Z}$, the cummutative law is satisfied.

Also, the number of elements in **Z** is infinite.

Hence, the set **Z** is an infinite abelian group with additive binary operation.

Example 2. *Show that the set* $\{1, -1, i, -i\}$ *is an abelian finite group of order* 4 *under multiplication.*

Solution. (i) Closure property. Closure property is satisfied as

$$1(-1) = -1,\; 1 \,.\, i = i,\; i(-i) = 1,\; i(-i) = -i \text{ etc.}$$

(ii) Associativity. Associative property is satisfied as

$$(1 \,.\, i)(-i) = 1 \,.\, \{i(-i)\} = 1,\; \{1 \,.\, i\} \,.\, (-1) = 1,\; \{i(-1)\} = -i \text{ etc.}$$

(iii) Existence of identity. Axioms on identity is satisfied, 1 being the multiplicative identity.

(iv) Existence of inverse. Axiom an inverse in satisfied since the inverse of each element of the set exists $1 \cdot 1 = e = 1,\; (-1)(-1) = e = 1,\; i(-i) = e = 1,\; (-i)(i) = e = 1$

(v) Commutativity. The commutative law is also satisfied as $1\,(-1) = (-1)\,.\,1$, $(-1)\,i = i\,(-1)$ etc.

Since, there are four elements in the given set, hence it is a group of order 4.

Example 3. *Show that the set of all positive rational numbers forms an abelian group under the composition defined by* $a * b = \frac{(ab)}{2}$.

Solution. Let Q^+ denote the set of all positive rational numbers to show $(Q^+, *)$ is a group.

(i) Closure property. For every $a, b \in Q^+$, $ab/2 \in Q^+$

$\Rightarrow$ Q^+ is a closed under the composition*.

(ii) Associativity. Let $a, b, c \in Q^+$, then

$$(a * b) * c = \left(\frac{ab}{2}\right) * c = \frac{[(ab)/2]\,.\,c}{2} = \frac{a[(bc)/2]}{2} = a * \left(\frac{bc}{2}\right) = a * (b * c).$$

(iii) Existence of identity. An element e will be the identity element if $e \in Q^+$ and if

$$e * a = a = a * e \;\; \forall \;\; a \in Q^+.$$

Now, $\quad e * a = a\frac{(ea)}{2} = a \;\Rightarrow\; \left(\frac{a}{2}\right)(e-2) = 0 \;\;\Rightarrow e = 2$

Since, $\quad a \in Q^+ \;\Rightarrow\; a \neq 0$

$2 \in Q^+$ and we have $2 * a = (2a)/2 = a = a * 2 \;\; \forall\, a \in Q^+$.

$\Rightarrow$ 2 is the identity element.

(iv) Existence of inverse. Let $a \in Q^+$, b is the inverse of a, then we must have

$$b * a = e = 2 \;\Rightarrow \frac{(ba)}{2} = 2 \Rightarrow b = \frac{4}{a}$$

Now, $\quad a \in Q^+ \;\Rightarrow\; 4/a \in Q^+$.

We have $(4/a) * a = \{(4/a)\,.\,a\}/2 = 2 = a * (4/a)$

$\Rightarrow$ $4/a$ is the inverse of a

$\Rightarrow$ inverse of each element of Q^+ exist.

(v) Commutativity. Let $a, b \in Q^+ \;\Rightarrow a * b = (ab)/2 = (ba)/2 = b * a$

Hence $(Q^+, *)$ is an abelian group.

Example 4. *Show that the set* **Z** *of all integers form a group with respect to binary operation* * *defined by* $a * b = a + b + 1 \;\; \forall \;\; a, b \in Z$ *is an abelian group.*

Solution. (i) Closure property. Let $a, b \in Z$

$\Rightarrow \;\; a + b + 1 \in Z \;\Rightarrow\; a * b \in Z \qquad \Rightarrow \;\; Z$ is closed with respect to *.

(ii) Associativity. If $a, b, c \in Z$, then

$$(a * b) * c = (a + b + 1) * c = (a + b + 1) + c + 1 = a + b + c + 2.$$

Also, $\quad a * (b * c) = a * (b + c + 1) = a + (b + (+1) + 1) = a + b + c + 2$

$\Rightarrow \quad (a * b) * c = a * (b * c) \;\; \forall \;\; a, b, c \in Z.$

(iii) Existence of identity. An element $e \in Z$ will be the identity if $e * a = a \;\forall\;\; a \in Z$.

Now, $\quad e * a = e + a + 1$

$\qquad e + a + 1 = a \;\Rightarrow e = -1.$

Since, $-1 \in Z$ and we have for any $a \in Z$

$\qquad (-1) * a = -1 + a + 1 = a \qquad \Rightarrow -1$ is the identity element.

(iv) Existence of inverse. If $a \in Z$, then $b \in Z$ will be the inverse of a if $b * a = -1$ ($\because$ -1 is the identity element)

Now, $\quad b * a = -1 \;\Rightarrow\; b + a + 1 = -1 \;\Rightarrow\; b = -2 - a.$

Also $\quad a \in Z \;\Rightarrow -2 - a \in Z$

and $\quad (-2 - a) * a = (-2 - a) + a + 1 = -1$, identity element

$\therefore \quad (-2 - a)$ is the inverse of a.

(v) Commutativity. Since, $a * b = a + b + 1 = b + a + 1 = b * a \;\Rightarrow$ commutativity satisfied.

Hence, **Z** is an infinite abelian group under the given composition.

• 11.3. SOME PROPERTIES OF GROUPS

Theorem 1. (Uniqueness of Identity). *Let* $(G, *)$ *be a group, then the identity element in* G *is unique.*

Proof : Let e_1 and e_2 be two identities of a group G. Then by the definition of identity, we have if e_1 is identity, then $\quad e_1 * e_2 = e_2 \quad \ldots (1)$

and if e_2 is identity, then $\quad e_1 * e_2 = e_1 \quad \ldots (2)$

Since $e_1 * e_2$ is the unique element of G, then from (1) and (2), we obtain

$$e_1 = e_2$$

Hence, the identity element in a group is unique.

Theorem 2. (Uniqueness of inverse) : *Let $(G, *)$ be a group, then the inverse of each element of G is unique.*

Proof : Let b and c be two inverses of any element a of G. Then by the definition of inverse, we have

$$a * b = e = b * a \quad \ldots (1)$$

and
$$a * c = e = c * a \quad \ldots (2)$$

where e is the identity element of G.

$\therefore \quad b = b * e = b * (a * c)$ [using (2)]

$= (b * a) * c$ (By associativity)

$= e * c$ [using (1)]

$= c$ (By the definition of inverse)

$\Rightarrow \quad b = c$

Hence, each element of G has unique inverse in G.

Theorem 3. *If $(G, *)$ is a group, then $(a^{-1})^{-1} = a \ \forall \ a \in G$.*

Proof : If e is the identity element of G, then for each element a of G there exists an element b of G such that

$a * b = e = b * a$

$\Rightarrow \quad b = a^{-1}$ and $a = b^{-1}$

Now $\quad a * b = e$

$\Rightarrow \quad a * a^{-1} = e \quad [\because b = a^{-1}]$

$\Rightarrow \quad (a * a^{-1}) \times (a^{-1})^{-1} = e * (a^{-1})^{-1} \quad [\because a^{-1} \in G \Rightarrow (a^{-1})^{-1} \in G)]$

$\Rightarrow \quad a * (a^{-1} * (a^{-1})^{-1}) = (a^{-1})^{-1}$ [$\because$ $*$ in G is associativ and e is the idntity of G]

$\Rightarrow \quad a * e = (a^{-1})^{-1} \quad [\because (a^{-1})^{-1}$ is the inverse of $a^{-1}]$

$\Rightarrow \quad a = (a^{-1})^{-1}$

Hence, $\quad (a^{-1})^{-1} = a \ \forall \ a \in G.$

Note : If $(G, +)$ is a group then $-(-a) = a \ \forall \ a \in G$.

Theorem 4. (Reversal law) : *If $(G, *)$ is a group, then $(a * b)^{-1} = b^{-1} * a^{-1} \forall \ a, b \in G$.*

Proof : For all $a, b \in G$, we have $a * b \in G$. If a^{-1} and b^{-1} are the inverses of a and b respectively, then

$$a * a^{-1} = e = a^{-1} * a \quad \ldots (1)$$

and
$$b * b^{-1} = e = b^{-1} * b \quad \ldots (2)$$

Now $\forall \ a, b \in G$,

$(a * b) * (b^{-1} * a^{-1}) = a * (b * b^{-1}) * a^{-1}$ [$\because$ $*$ is G is associative]

$= (a * e) * a^{-1}$ [using (2)]

$= a * a^{-1}$ [$\because a * e = a$]

$= e$ [using (1)]

Also,

$(b^{-1} * a^{-1}) * (a * b) = b^{-1} * (a^{-1} * (a * b))$ (By associativity)

$= b^{-1} * ((a^{-1} * a) * b)$ (By associativity)

$= b^{-1} * (e * b)$ [Using (1)]

$= b^{-1} * b$ [$\because e * b = b$]

$= e$ [Using (2)]

$\therefore \quad (a * b) * (b^{-1} * a^{-1}) = e = (b^{-1} * a^{-1}) * (a * b)$

$\Rightarrow \quad (a * b)^{-1} = b^{-1} * a^{-1} \forall \ a, b \in G.$

Note : If $(G, *)$ is an abelian group, then $(a * b)^{-1} = a^{-1} * b^{-1}$.

Note : If $(G, +)$ is a group then $-(a + b) = (-b) + (-a)$.

Note : If $a_1, a_2, \ldots, a_n$ are elements of a group G, then

$$(a_1 * a_2 * \dots a_n)^{-1} = a_n^{-1} * a_{n-1}^{-1} * \dots * a_2^{-1} * a_1^{-1}.$$

Theorem 5. (Cancellation laws hold good in a group) : *If a, b, c are three elements of a group $(G, *)$, then*

(i) $a * b = a * b \Rightarrow b = c$ *[Left cancellation law]*

(ii) $b * a = c * a \Rightarrow b = c$ *[Right cancellation law]*

Proof : If e is the identity element of G, then we have

For each $a \in G f a^{-1} \in G$ such that

$$a * a^{-1} = e = a^{-1} * a \quad \dots (1)$$

Now,

$$a * b = a * c$$

$\Rightarrow \quad a^{-1} * (a * b) = a^{-1} * (a * c)$

$\Rightarrow \quad (a^{-1} * a) * b = (a^{-1} * a) * c$ (By associativity)

$\Rightarrow \quad e * b = e * c$ [Using (1)]

$\Rightarrow \quad b = c$ [$\because$ e is the identity]

Also, $\quad b * a = c * a$

$\Rightarrow \quad (b * a) * a^{-1} = (c * a) * a^{-1}$

$\Rightarrow \quad b * (a * a^{-1}) = c * (a * a^{-1})$ [By associativity]

$\Rightarrow \quad b * e = c * e$ [Using (1)]

$\Rightarrow \quad b = c$

Note : If $(G, +)$ is group then

$$a + b = a + c \Rightarrow b = c$$

and $\quad b + a = c + a \Rightarrow b = c \ \forall a, b, c, \in G$

Theorem 6. *In a group $(G, *)$, the equations $a * x = b$ and $y * a = b$, where $a, b \in G$ have unique solutions in G.*

Proof : If e is the identity element of G, then for each $a \in G \ \exists \ a^{-1} \in G$ such that

$$a * a^{-1} = e = a^{-1} * a \quad \dots (1)$$

Now $\quad a * x = b$

$\Rightarrow \quad a^{-1} * (a * r) = a^{-1} * b$ [By associativity]

$\Rightarrow \quad e * x = a^{-1} * b$ [Using (1)]

$\Rightarrow \quad x - a^{-1} * b$ [$\because$ e is the identity]

Also $\quad y * a = b$

$\Rightarrow \quad (y * a) * a^{-1} = b * a^{-1}$

$\Rightarrow \quad y * (a * a^{-1}) = b * a^{-1}$ [By associativity]

$\Rightarrow \quad y * e = b * a^{-1}$ [Using (1)]

$\Rightarrow \quad y = b * a^{-1}$ [$\because$ e is identity]

Now, $\forall a, b \in G \Rightarrow a^{-1} * b \in G, b * a^{-1} \in G.$

Hence, the equations $a * x = b$ and $y * a = b$ have solution in G.

Next, we prove that these solutions are unique.

Uniqueness : Let, if possible x_1 and x_2 be two solutions of the euation $a * x = b$, then

$$a * x_1 = b \quad \text{and} \quad a * x_2 = b$$

$\Rightarrow \quad a * x_1 = a * x_2$

$\Rightarrow \quad x_1 = x_2$ [By left cancellation law]

$\Rightarrow$ the equation $a * x = b$ law has unique solution.

Similarly, if y_1 and y_2 be two solutions of $y * a = b$, then

$$y_1 * a = b = y_2 * a$$

$\Rightarrow \quad y_1 = y_2$ [By right cancellation law]

$\Rightarrow$ the euation $y * a = b$ has unique solution.

• COMPOSITION TABLE FOR FINITE SETS

Let $S = \{a_1, a_2, a_3, \ldots\ldots a_n\}$ be a finite set and * be a binary operation on S, then the table shown below is called the **composition table for** S.

*	a_1	a_2	a_3	......	a_n
a_1	$a_1 * a_1$	$a_2 * a_2$	$a_1 * a_3$	......	$a_1 * a_n$
a_2	$a_2 * a_1$	$a_2 * a_2$	$a_2 * a_3$	......	$a_2 * a_n$
a_3	$a_3 * a_1$	$a_3 * a_2$	$a_3 * a_3$	......	$a_3 * a_n$
: : a_n	....	....	....	....	....
a_n	$a_n * a_1$	$a_n * a_2$	$a_n * a_3$	......	$a_n * a_n$

• TEST YOURSELF–1

1. Show that the following are groups :
   (i) Set of all even integers (including zero) under addition.
   (ii) Set of all non-zero rational numbers with respect to binary operation of multiplication.
   (iii) The set of all real numbers with respect to addition.
   (iv) The set C of all non-zero complex numbers with respect to multiplication.
2. Show that the set of positive rational numbers does not form a group with respect to the binary operation * defined by $a * b = \frac{a}{b}$.
3. Show that the four metrices $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ forms a group with respect to matrix multiplication.
4. Show that the set of all n, n^{th} roots of unity forms a finite abelian group of order n with respect to multiplication.
5. Show that the set Z of all integers is an abelian group with operation defined by $a * b = a + b + 2$.
6. Show that the set Q of all rational number, other than 1 with operation *, defined by $a * b = a + b - ab$ from a group under binary operation *.
7. Show that the set $G = \{1, \omega, \omega^2\}$. where ω is an imaginary cube root of unity is a group with respect to multiplication.

ANSWER

2. No.

• 11.4. INTEGRAL POWER OF AN ELEMENT

Let G be a group with respect to multiplication. If $a \in G$, then aa is denoted by a^2, aaa is denoted by a^3 and so on. We have

$$aaa \ldots n \text{ times} = a^n \; n \in Z^+$$

But closure property $a^2 . a^3 \ldots a^n \in G$.

Also, if e is the identity element in G, we define $a^\circ = e$.

If n is a positive integer, we define

$$a^{-n} = (a^n)^{-1} \in G \text{ since } a^n = a \,.\, a \,.\, a \,.\ldots n \text{ times} \in G.$$

Further, $\quad (a^n)^{-1} = (aa \ldots n \text{ times})^{-1} = a^{-1} a^{-1} \ldots n \text{ times} = (a^{-1})^n$

Thus, $\quad a^{-n} = (a^n)^{-1} = (a^{-1})^n$.

REMARKS

➤ For additive groups we write na instead of a^n. Thus, if n is a positive integer, we write

$$na = a + a + \ldots \text{ upto } n \text{ terms.}$$

- For an arbitrary element a of a multiplicative group G and for arbitrary constant m and n, it is easy to verify that

  (i) $a^m a^n = a^{m+n}$

  (ii) $(a^m)^n = a^{mn}$

  (iii) $e^n = e$, where e is the identity of G.

• 11.5. ORDER OF AN ELEMENT OF A GROUP

Let G be a group under multiplication. Let e be the identity element in G. Suppose a is any element of G then the least positive integer n, if exist, such that $a^n = e$ is said to be order of an element $a \in G$, and can be written as

$$o(a) = n.$$

In case, such a positive integer n does not exist, we say that the element a is of **infinite** or **zero** order.

REMARKS

- If G is an additive group, we write na in place of a^n.
- If m is a positive integer such that $a^m = e$ the $o(a) \leq m$.
- Identity element e in a group G, is the **only** element whose order is one.
- The order of an element of an infinite group may be finite or infinite.

• SOLVED EXAMPLES

Example 1. *Consider the multiplicative group $G = \{1 - 1, i, - i\}$ of cube roots of unity. Find the order of each element of G.*

Solution. Since 1 is the identity element, therefore

$$0(1) = 1$$

Also, $(-1)^2 = 1 \Rightarrow 0(-1) = 2$

$$(i)^4 = 1 \Rightarrow 0(i) = 4$$

$$(-i)^4 = 1 \Rightarrow 0(-i) = 4.$$

Eample 2. *Consider the additive group $Z = \{\ldots -3, -2, -1, 0, 1, 2, 3, \ldots\}$ of all integers. Show that 0 is the only element of finite order.*

Solution. If a be any non-zero integers, then there exists no positive integer n such that

$$na = (a + a + \ldots + n \text{ times}) = 0$$

$\Rightarrow$ $0(a)$ is infinite.

Hence, in Z the identity 0 is the only element of finite order.

• 11.6. Some Important Theorems :

Theorem 1. *The order of every element of a finite group is finite.*

Proof. Let G be a finite multiplicative group and $a \in G$.

Consider all positive integral powers of a, *i.e.*,

$$a, a^2, a^3, \ldots a^s, \ldots a^r, \ldots .$$

By closure property, these all are element of G.

Since, G is finite, therefore, all the integral power of a can not be distinct element of G.

Suppose that $a^r = a^s$, where $r > s$ ...(1)

Then, $a^r = a^s \Rightarrow a^r a^{-s} = a^s . a^{-s}$

$$\Rightarrow a^{r-s} = a^0 = e$$

$$\Rightarrow a^m = e, \text{ where } m = r - s > 0$$

Thus, then exist a positive integer m such that $a^m = e$. Now since every set of positive integers has a least member it follows that the set of all positive integes m such that $a^m = e$ has a least member say n.

Thus, $o(a) = n$, which is finite.

Hence, the order of every element of the finite group G is finite.

Theorem 2. *If the element a of a group G is of order n, then $a^m = e$ iff n is a divisor of m.*

Proof. Let $o(a) = n$ and $a^m = e$ for some positive integer m then $m \geq n$.

If $m = n$ then n is a divisor of m.

If $m > n$, then by divison algorithm, there exists two integers q and r such that

$$m = nq + r, \text{ where } 0 \leq r < n.$$

Therefore, $a^m = e \Rightarrow a^{nq+r} = e$

$$\Rightarrow a^{nq}. a^r = e$$

$$\Rightarrow a^r = e \qquad (\because \; a^{nq} = (a^n)^q = e^q = e)$$

Thus, $a^r = e$, where $0 < r < n$ or $r = 0$ (By divison algorithm)

Now, since $0(a) = n$, n is the least positive integer such that $a^n = e$. Hence, it follows that $r = 0$.

Therefore, $m = nq \Rightarrow n$ is a divisor of m.

Conversely. Let n be a divisor of m, so that

$$m = nq, \text{ for } q \in \mathbf{Z}^+.$$

Hence, $a^m = a^{nq} = (a^n)^q = e^q = e.$

Theorem 3. *The order of an element of a group is the same as that of its inverse.*

Proof. Let G be a group under multiplication and a is any element of G.

Suppose that $o(a) = m$ and $o(a^{-1}) = n$.

Now, $0(a) = m \Rightarrow a^m = e \Rightarrow (a^m)^{-1} = e^{-1} = e$

$$\Rightarrow (a^{-1})^m = e, \text{ since } (a^m)^{-1} = (a^{-1})^m$$

$$\Rightarrow o(a^{-1}) \leq m \Rightarrow n \leq m. \qquad \ldots(1)$$

Again $o(a^{-1}) = n \Rightarrow (a^{-1})^n = e \Rightarrow (a^n)^{-1} = e$

$$\Rightarrow ((a^n)^{-1})^{-1} = e^{-1} \Rightarrow a^n = e$$

$$\Rightarrow o(a) \leq n \Rightarrow m \leq n. \qquad \ldots(2)$$

Now, From (1) and (2), we conclude that

$$m = n, \text{ i.e., } o(a) = o(a^{-1}).$$

Theorem 4. *The order of any integral power of an element a cannot exceed the order of a.*

Proof. Let a^r be any integral power of a and let $o(a) = n$.

Now, $o(a) = n \Rightarrow a^n = e$

$\Rightarrow$ $(a^n)^r = e^r \Rightarrow a^{nr} = e$

$\Rightarrow$ $(a^r)^n = e \Rightarrow o(a^r) \leq n.$

$\Rightarrow$ $o(a^r)$ cannot exceeds the $o(a)$.

Theorem 5. *If a and b are any two elements of a group G, then*

$$o(a) = o(b^{-1}ab).$$

Proof. Let $o(a) = m$, hence m is the least positive integer, such that $a^m = e$.

Now $(b^{-1}ab)^2 = (b^{-1}ab)(b^{-1}ab) = b^{-1}a(bb^1)ab$ (By associativity)

$$= b^{-1}aeab \qquad (\because \; bb^{-1} = e)$$

$$= b^{-1}a^2b \qquad (\because \; ae = a)$$

Similarly $(b^{-1}ab)^3 = b^{-1}a^3b$

$$\vdots \qquad \ldots \text{ and so on}$$

$$(b^{-1}ab)^m = (b^{-1}ab)(b^{-1}ab) \ldots \text{to } m \text{ factors}$$

$$= b^{-1}abb^{-1}ab \ldots b^{-1}ab \qquad \text{(By associativity)}$$

$$= b^{-1}a(bb^{-1})a(bb^{-1}) \ldots (bb^{-1})ab \qquad \text{(By associativity)}$$

$$= b^{-1}a^mb = b^{-1}eb = b^{-1}b = e \qquad (\because \; a^m = e)$$

Thus, we have $(b^{-1}ab)^m = b^{-1}a^mb = e.$

Now, since, m is the least positive integer such that $a^m = e$, it follows that m is the least positive integer such that

$$(b^{-1}ab)^m = e$$

Hence, $o(b^{-1}ab) = m.$

Theorem 6. *For any two elements a, b of a group G,*

$$o(ab) = o(ba).$$

Proof. We have

$$ba = e(ba) = (a^{-1}a)(ba) = a^{-1}(ab)(a).$$

Hence, by theorem 5

$$o\,[a^{-1}(ab)\,a] = o(ab) \Rightarrow o(ba) = o(ab).$$

• SOLVED EXAMPLES

Example 1. *For any two elements a and b of a group G, show that G is abelian iff* $(ab)^2 = a^2b^2$.

Solution. Let us first suppose, G be abelian.

So that $ab = ba \;\; \forall\; a, b \in G$

Consider $(ab)^2 = (ab)\,(ab) = a(ba)\,b$ (By associativity)

$= a(ab)\,b$ (By commutativity)

$= (aa)(bb)$ (By associativity)

$= a^2 \,.\, b^2.$

Thus, $(ab)^2 = a^2b^2 \;\; \forall\; a, b \in G.$

Convesely, Let $(ab)^2 = a^2b^2 \;\; \forall\; a, b \in G.$

To show $ab = ba.$

Consider $(ab)^2 = a^2b^2 \Rightarrow (ab)\,(ab) = (aa)\,(bb)$

$\Rightarrow$ $a(ba)\,b = a(ab)\,b,$ (By associativity)

$\Rightarrow$ $ba = ab$ (By left and right cancellation law)

Thus, we have $ab = ba \;\; \forall\, a, b \in G.$

Hence, G is abelian.

Example 2. *Show that if G is an abelian group than for all $a, b \in G$ and all integers n* $(ab)^n = a^nb^n$.

Solution. (i) Let $n = 0.$

Then $(ab)^o = e.$

Also, $a^ob^o = e\cdot e = e$

$\therefore$ $(ab)^o = a^ob^o.$

(ii) Let $n > 0.$

If $n = 1$, then $(ab)^1 = ab = a^1b^1.$

Let us suppose our result is true for $n = r$

i.e., $(ab)^r = a^rb^r.$

Then $(ab)^{r+1} = (ab)^r \,.\, ab = a^rb^rab = a^rab^rb$ $(\because\;\; ab^r = b^ra)$

$= a^{r+1}\, b^{r+1}.$

Then, by mathematical induction for all $n > 0$, $(ab)^n = a^n \,.\, b^n.$

(iii) Let $n < 0.$

Let $n = -r$, where r is a positive integer.

Then $(ab)^n = (ab)^{-r} = [(ab)^r]^{-1} = [a^rb^r]^{-1} = [b^ra^r]^{-1}$ $(\because\;\; a^mb^m = b^ma^m)$

$= [a^r]^{-1}\,[b^r]^{-1}$ $[\because\;\; (ab)^{-1} = b^{-1}a^{-1}]$

$= a^{-r}\,b^{-r} = a^nb^n.$

Example 3. *If G is a group of even order, then show that there exist an element a, other than the identity 'e', such that* $a^2 = e$.

Solution. Let $o(G) = 2r$, where r is any positive integer.

Since, we know that, in a group every element possesses a unique inverse and $e^{-1} = e$. The remaining $(2r - 1)$ elements should, therefore, be divided into pairs in such away that each pair consists of two distinct elements, which are inverse of each other. But this is not possible, since $(2r - 1)$ is odd.

Hence, $\exists$ an element $a \in G$, such that

$$a^{-1} = a, \text{ where } a \neq e$$

But $a = a^{-1} \Rightarrow a^2 = a^{-1}a = e.$

Thus, there exists $a \in G$ such that $a \neq e$ and $a^2 = e$.

• TEST YOURSELF–2

1. If a and b any elements of a group G, then show that $(bab^{-1})^n = ba^nb^{-1}$ for any $n \in \mathbf{Z}$.
2. Show that if for every element a in a group G, $a^2 = e$, then G is abelian.

3. Show that if every element of a group G has its own inverse, then G is abelian.
4. If G is group such that $(ab)^p = a^p b^p$ for three consecutive integers p $\forall$ $a, b \in G$, show that G is abelian.
5. Show that a group G is abelian if every element of G except the identity element is of order 2.
6. Find the order of each element in the multiplicative group $G = \{1, \omega, \omega^2\}$ where ω is the cube root of unity.
7. If the element a, b and ab of a group are each of order 2 show that $ab = ba$.
8. Show that in a group G, we have
   (i) $ab = e \Rightarrow a = b^{-1}$ and $b = a^{-1}$(ii) $ab = a$ or $ba = a \Rightarrow b = e$.
9. If a is an element of a group, prove that the integral powers of a form a multiplicative group.
10. Show that a group G is abelian iff $(ab)^{-1} = a^{-1}b^{-1}$ $\forall$ $a, b \in G$.
11. If in the group G, $a^5 = e$, $aba^{-1} = b^2$ for $a, b \in G$.
    Show that $0(b) = 1$ if $b = e$ and $0(b) = 31$ if $b \neq e$.
12. If in a group G, the elements a and b commutes, then prove that
    (i) a^{-1} and b^{-1} also commute (ii) a^{-1} and b also commute
    (iii) a and b^{-1} also commute.

ANSWER

6. $o(1) = 1$, $o(\omega) = 3$, $o(\omega^2) = 3$.

• 11.7. PERMUTATIONS AND PERMUTATIONS GROUPS

(i) Permutations. A one-one mapping f of **a finite** non-empty set S onto itself is called a **permutations.**

If the set S consists of n distinct elements, then a one-one mapping of S onto itself is called a permutation of **degree** n or a permutations of n-symbols.

Notation. Let $S = \{a_1, a_2, \dots a_n\}$.

Then, we denote a permutation f on the set S in a two-rowed notation

$$f = \begin{pmatrix} a_1, a_2, \dots, a_i, \dots, a_n \\ b_1, b_2, \dots, b_i, \dots, b_n \end{pmatrix}.$$

So that in the first row all the elements of S are written in a certain order and

$$f(a_1) = b_1, f(a_2) = b_2 \dots, f(a_i) = b_i, \dots, f(a_n) = b_n.$$

REMARKS

➤ It is clear that each $b_i \in S$, $i = 1, 2, \dots, n$.

➤ It is immaterial in which order, the elements of S are written in the first row, but the image of element a_i must be written under a_i. Thus, the interchange of columns does not change the permutation.

For example. $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 4 & 3 & 2 \\ 2 & 1 & 4 & 3 \end{pmatrix}$.

Equality of permutations. Two permutations f and g of a set S are said to be **equal** if $f(a) = g(a)$ $\forall$ $a \in S$.

For example. If $f = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$ and $g = \begin{pmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \end{pmatrix}$ are two permutation of degree 3 then we have

$$f = g \text{ since } f(1) = g(1) = 3, f(2) = g(2) = 1, f(3) = g(3) = 2.$$

Identity permutations. A permutations on the set S is called the **identity permutations** if it maps each element of S onto itself.

It is usually denoted by the symbol I.

Thus, $I(a) = a$ $\forall$ $a \in S$.

For example. $I = \begin{pmatrix} 1 & 2 & 3 \dots n \\ 1 & 2 & 3 \dots n \end{pmatrix}$ is the identity permutaiton of degree n.

Total number of distinct permutations. Let S be a set consisting of n distinct elements. Then the elements of S can be permuted in n ! distinct ways, *i.e.*, n ! distinct arrangement of the elements belonging to S are possible. If P_n be the set consisting of all permutations of degree n, then the set P_n will have n ! distinct permutations of degree n.

This set P_n is called the **symmetric set** of permutations of degree n ! and is denoted by

$$P_n = \{f : f \text{ is a permutation of degree } n\}.$$

Inverse permutations. Since a permutation is a one-one onto mapping and hence it is inversible, *i.e.*, every permutation f on a set $P = \{a_1, a_2, ..., a_n\}$ has a unique inverse permutation denoted by f^{-1}.

For example. If $\quad f = \begin{pmatrix} a_1 & a_2 \dots a_n \\ b_1 & b_2 \dots b_n \end{pmatrix}$.

Then $\quad f^{-1} = \begin{pmatrix} b_1 & b_2 \dots b_n \\ a_1 & a_2 \dots a_n \end{pmatrix}$.

Product or composite of permutations. The product or composite of two permutations f and g of the same degree, denoted by $f \cdot g$ is obtained by first carrying out the operation defined by mapping f and they by mapping g.

Let, $f, g \in P_n$, so that

$$f = \begin{pmatrix} a_1 & a_2 \dots a_n \\ b_1 & b_2 \dots b_n \end{pmatrix}, \quad g = \begin{pmatrix} b_1 & b_2 \dots b_n \\ c_1 & c_2 \dots c_n \end{pmatrix}$$

where $\quad a_i, b_i, c_i \in S, \; i = 1, 2, ..., n.$

Then, $\quad fg = \begin{pmatrix} a_1 & a_2 \dots a_n \\ b_1 & b_2 \dots b_n \end{pmatrix}\begin{pmatrix} b_1 & b_2 \dots b_n \\ c_1 & c_2 \dots c_n \end{pmatrix} = \begin{pmatrix} a_1 & a_2 \dots a_n \\ c_1 & c_2 \dots c_n \end{pmatrix} \in P_n.$

For example. Let $f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$, $g = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix}$ by two permutations of degree 4. Then

$$fg = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}\begin{pmatrix} 2 & 3 & 4 & 1 \\ 4 & 2 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 3 \end{pmatrix}$$

and

$$gf = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix}\begin{pmatrix} 3 & 4 & 3 & 1 \\ 4 & 1 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 3 & 2 \end{pmatrix}.$$

Here, we observe that $fg \neq gf$. Thus, **the product of permutations is not commutative.**

• 11.8. CYCLIC PERMUTATION

Let f be a permutation of degree n on a set having n distinct elements and let it be possible to arrange m elements of the set S in a row in such a way that the f-image of each element in the row is the element which follows, the f-image of the last element is the first element and the remaining $n - m$ elements of the set S are left unchanged by f. Then f is called a cyclic permutation or a cycle of length m or an m-cycle.

REMARKS

- The following have the same meaning **cyclic permutation, circular permutations** or **cycles.**
- *The number of distinct objects permuted by a cycle is known as the* **length** of the cycle.

For example

(i) $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$ is a cyclic permutation of length 3.

(ii) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \end{pmatrix}$ is a cyclic permutation of length 5.

Symbol for cyclic permutations. We denote the cyclic permutations

$$\begin{pmatrix} a_1 & a_2 & a_3 \dots a_n & a_n \\ a_2 & a_3 & a_4 \dots a_{n-1} & a_1 \end{pmatrix}$$

by the symbols $(a_1, a_2, ..., a_n)$, which means that each member in the bracket is replaced by its successor on the right and the last member is replaced by the first one. Thus the cyclic permutation $(1, 4, 2, 6)$ is expressible as $\begin{pmatrix} 1 & 4 & 2 & 6 \\ 4 & 2 & 6 & 1 \end{pmatrix}$.

It is interpreted at 1 replaced by 4, 4 replaced by 2, 2 replaced by 6 and 6 replaced by 1.

REMARKS

- The length of the cycle (4 5 6) is 3, where as the degree of the permutation $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 5 & 6 & 4 \end{pmatrix}$ is 6.
- A cycle does not change by changing the places of its elements in cyclic order. Thus, $(1\ 2\ 3\ 4\ 5) = (2\ 3\ 4\ 5\ 1) = (3\ 4\ 5\ 1\ 2)$.

Cycle of length 1. *A cycle of length 1 means that the image of the element involved is the same element and the missing elements are unchanged. Thus, all the elements are unchanged. Hence, every cycle of length one represents the* ***identity permutations.***

Transposition. A cycle of length two is called a transposition. Thus, the cycle (9 6) or (1 2) is a **transposition.** If the cycle (1 2) is a permutation of degree 3 on three symbols 1, 2, 3, then the corresponding permutation is $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$.

Disjoint cycles. Two cycles are said to be disjoint, if they have no elements in common.

Examples. (i) (1 2) and (3 4) are disjoint cycles

(ii) (1 5 3) and (8 9 10) are disjoint cycles

Multiplication of cycles. We multiply cycles by multiplying the permutations represented by them.

For example. Let us suppose that (2 3 4) and (5 3 1 2) represents permutations of degree 6 on six symbols 1, 2, 3, 4, 5, 6. Then, we have

$$\begin{aligned}\begin{pmatrix} 2 & 3 & 4 \\ 3 & 4 & 2 \end{pmatrix}\begin{pmatrix} 5 & 3 & 1 & 2 \\ 3 & 1 & 2 & 5 \end{pmatrix} &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 3 & 4 & 2 & 5 & 6 \end{pmatrix}\begin{pmatrix} 5 & 3 & 1 & 2 & 4 & 6 \\ 3 & 1 & 2 & 5 & 4 & 6 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 3 & 4 & 2 & 5 & 6 \end{pmatrix}\begin{pmatrix} 1 & 3 & 4 & 2 & 5 & 6 \\ 2 & 1 & 4 & 5 & 3 & 6 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 4 & 5 & 3 & 6 \end{pmatrix} = (1\ \ 2)(3\ \ 4\ \ 5)(6) \\ &= (1\ \ 2)(3\ \ 4\ \ 5)\end{aligned}$$

($\because$ a cycle of length 1 represents the identity permutations)

Some Important Theorems :

Theorem 1. *Every permutation can be expressed as a product of disjoint cycles.*

Proof. Let f be a given permutation of degree n, defined n the set $S = \{a_1, a_2, ..., a_n\}$. First select all the cycles of length 1 each given by the invarient element.

Now select an element, which is non-invarient and construct a row, starting with this element and writing after writing each element its image under f.

As the number of elements in S is finite, after a finite number of steps, we get an element whose image under f is the one with which we started. This row is a cycle.

Now, we choose an element of S which is not contained in the above cycles and get another cycle, as above.

Proceeding in the same way, each and every element of S is included in one or the other cycle.

Obviously, these cycles have no element in common and hence they are disjoint.

Hence, the permutation f can be expressed as a product of disjoint cycles.

Theorem 2. *Every permutations can be expressed as a product of transpositions.*

Proof. By theorem-1 we have that every permutation can be expressed as product of disjoint cycle.

Consider the cycle $(a_1, a_2, ..., a_n)$ of length n, where $n > 1$ then, we see.

$$(a_1, a_2, ..., a_n) = (a_1a_2)(a_1a_3)(a_1a_4) \dots (a_1a_n).$$

For example. $(a_1\, a_2\, a_3) = (a_1\, a_2)(a_1\, a_3)$.

Also for $n = 1$

$$(a_1) = (a_1\, a_2)(a_2\, a_1)$$

i.e., the identity permutation can also be expressed as a product of transpositions.

$\Rightarrow$ Every cycle can be expressed as a product of transpositions. Hence, it follows that every permutations can be expressed as a product of transpositions.

REMARK

- For any manner of expressing a given permutations as a product of transpositions, the number of transpositions is either even or a odd.

• 11.9. EVEN AND ODD PERMUTATIONS

A permutation is said to be even or odd according as it can be expressed as a product of even or odd number of transpositions.

There is another easy way determining whether a permutation is even or odd.

Let $$P = \begin{pmatrix} 1 & 2 & 3 \ldots n \\ a_1 & a_2 & a_3 \ldots a_n \end{pmatrix}$$

be a permutation of degree n. *The pair (i, j) is said to be* ***regular*** *if $i - j$ and $a_i - a_j$ both have the same sign. Otherwise* ***irregular.*** Thus for irregularity of any pair (i, j), $(i - j)$ and $(a_i - a_j)$ are of opposite signs. The number of irregular pairs denotes number of inversions.

A permutation of a set of integers onto itself is ***even*** *or* ***odd*** *according as it contains an even or odd number of inversions.*

For examples

(i) $\begin{pmatrix} 2 & 3 & 4 \\ 2 & 3 & 4 \end{pmatrix}$ $\Rightarrow$ No inversion

(ii) $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$ $\Rightarrow$ Two invesions

(iii) $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$ $\Rightarrow$ Three inversions

(iv) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 3 & 2 & 4 & 1 \end{pmatrix}$ $\Rightarrow$ Eight inversions

(v) $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$ $\Rightarrow$ One inversion.

Hence, (i), (ii) (iv) $\Rightarrow$ Even permutations
(iii) and (v) $\Rightarrow$ odd permutation.

Theorem 1. *Of the n ! permutations of n symbols, $\frac{n!}{2}$ are even permutation and $\frac{n!}{2}$ are odd permutations.*

Proof. Let P_n be the set of all permutations on n distinct symbols. The P_n contains n ! distinct permutations of degree n. Also P_n is a group with respect to permutation multiplication as composition.

Out of those n ! permutations, let the even permutations be $E_1, E_2, \ldots, E_p$ and the odd permutation be $o_1, o_2, \ldots, o_q$ so that $p + q = n$!

Let $t \in P_n$ be arbitrary such that t is a transposition so that t is an odd permutation. Let t be operated on each of E_i $(i = 1, 2, \ldots, p)$ and similarly on each o_i $(i = 1, 2, \ldots, q)$.

Now, since $(P_n, \cdot)$ is a group and therefore $tE_i \in P_m$ and $to_j \in P_n$ for $1 \le j \le p$, $1 \le j \le q$.

The permutations tE_i for $i = 1, 2, .., p$ are all odd permutation ($\because$ transposition is an odd permutation and product of even and odd permutation is an odd permutation). Also, these permutation are all distinct.

For $\quad tE_i = tE_j \Rightarrow E_i = E_j$

$\therefore \quad tE_i \neq tE_j$ if $E_i \neq E_j$.

Similarly, the permutations to_j $(j = 1, 2, \ldots, q)$ are all distinct even permutations.

Since, a permutation can not be both even and odd, we have that the even permutations $E_1, E_2, \ldots, E_p$ are equal to q even permutations $t0_1, t0_2, \ldots, t0_q$.

Similarly, q odd permutations $0_1, 0_2, .., 0_q$ are equal to p odd permutation $tE_1, tE_2, \ldots, tE_p$.

Consequently $p = q$. Also $p + q = n$!.

$$\therefore \quad p = q = \frac{n!}{2}.$$

REMARK

- If A_n is the set of all even permutations of degree n then $A_n \subset P_n$ and A_n contains $\frac{n!}{2}$ elements. The set A_n is called an alternating set of permutations.

Theorem 2. *The set A_n of all even permutations of degree n forms a finite non-abelian group of order $\frac{n!}{2}$ with respect to permutation multiplication as composition.*

Proof. Let A_n be set of all even permutation of degree n. Let $f, g, h \in A_n$ be arbitrary.

To show that $(A_n, \cdot)$ is a finite non-abelian group of order $\frac{n!}{2}$.

(i) Closure property. Let $f, g \in A_n \Rightarrow f$ and g are even permutation

$\Rightarrow$ $f \cdot g$ is an even permutation

($\because$ product of two even permutation is even permutation)

$\Rightarrow$ $f \cdot g \in A_n$.

(ii) Associativity. Let $f, g, h \in A_n \Rightarrow f, g, h$ are expressible as

$$f = \begin{pmatrix} a_1 \ldots a_n \\ b_1 \ldots b_n \end{pmatrix}, g = \begin{pmatrix} b_1 \ldots b_n \\ c_1 \ldots c_n \end{pmatrix}, h = \begin{pmatrix} c_1 \ldots c_n \\ d_1 \ldots d_n \end{pmatrix}$$

where the elements $b_1, b_2, ..., b_n, c_1, c_2 \ldots c_n, d_1, d_2, ..., d_n$ are simply different arrangement of the same n elements $a_1, \ldots a_n$.

Then
$$fg = \begin{pmatrix} a_1 \ldots a_n \\ b_1 \ldots b_n \end{pmatrix}\begin{pmatrix} b_1 \ldots b_n \\ c_1 \ldots c_n \end{pmatrix} = \begin{pmatrix} a_1 \ldots a_n \\ c_1 \ldots c_n \end{pmatrix}$$

$$(fg)h = \begin{pmatrix} a_1 \ldots a_n \\ c_1 \ldots c_n \end{pmatrix}\begin{pmatrix} c_1 \ldots c_n \\ d_1 \ldots d_n \end{pmatrix} = \begin{pmatrix} a_1 \ldots a_n \\ d_1 \ldots d_n \end{pmatrix} \quad \ldots(1)$$

Now,
$$gh = \begin{pmatrix} b_1 \ldots b_n \\ c_1 \ldots c_n \end{pmatrix}\begin{pmatrix} c_1 \ldots c_n \\ d_1 \ldots d_n \end{pmatrix} = \begin{pmatrix} b_1 \ldots b_n \\ d_1 \ldots d_n \end{pmatrix}$$

$\therefore$
$$f(gh) = \begin{pmatrix} a_1 \ldots a_n \\ b_1 \ldots b_n \end{pmatrix}\begin{pmatrix} b_1 \ldots b_n \\ d_1 \ldots d_n \end{pmatrix} = \begin{pmatrix} a_1 \ldots a_n \\ d_1 \ldots d_n \end{pmatrix}. \quad \ldots(2)$$

Now, from (1) and (2), we have $(fg)h = f(gh)$.

(iii) Existence of identity. Let I be the identity permutation of degree n.

Then $\quad fI = If = f \quad \forall\ f \in A_n$.

(iv) Existence of inverse. Let f^{-1} denote the inverse of f. Then $f^{-1}f = I$ = an identity permutation.

Also f is an even permutation

$\Rightarrow$ f^{-1} is an even permutation

$\therefore$ $f^{-1} \in A_n$.

Hence, every element of A_n is inversible.

(v) Commutativity. The product of permutation is not commutative. Also the set of permutation of degree n contains n ! permutations out of which $\frac{n!}{2}$ are even and $\frac{n!}{2}$ are odd.

$\Rightarrow$ $(A_n, \cdot)$ is a non-abelian group of order $\frac{n!}{2}$.

REMARKS

- The set of all odd permutations is not a group with respect to permutation multiplication as composition. Because closure property is not satisfied, since product of two odd permutations is an even permutations.

• SOLVED EXAMPLES

Example 1. *Express the permutation* $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 6 & 5 & 3 & 4 & 2 \end{pmatrix}$ *as a product of disjoint cycles.*

Solution. Clearly

$$f=\begin{pmatrix}1&2&3&4&5&6\\1&6&5&3&4&2\end{pmatrix}=\begin{pmatrix}1&2&6&3&5&4\\1&6&2&5&4&3\end{pmatrix}$$

$$=(1)(2\ \ 6)(3\ \ 5\ \ 4)$$

Example 2. *Express the permutations* $f=\begin{pmatrix}1&2&3&4&5&6&7\\1&3&2&6&4&5&7\end{pmatrix}$ *as a product of transposition.*

Solution. $f=\begin{pmatrix}1&2&3&4&5&6&7\\1&3&2&6&4&5&7\end{pmatrix}=(1)(2\ \ 3)(4\ \ 6\ \ 5)(7)$

$$=(1\ \ 2)(2\ \ 1)(2\ \ 3)(4\ \ 6)(4\ \ 5)(7\ \ 1)(1\ \ 7).$$

Since, $(1)=(1\ \ 2)(2\ \ 1)$ and $(4\ \ 6\ \ 5)=(4\ \ 6)(4\ \ 5)$, $(7)=(7\ \ 1)(1\ \ 7)$.

Example 3. *Decompose the permutation* $f=\begin{pmatrix}1&2&3&4&5&6&7&8\\3&1&4&7&2&5&8&6\end{pmatrix}$ *into transposition. Hence, show that f is an odd permutations.*

Solution. Here, we have

$$f=\begin{pmatrix}1&3&4&7&8&6&5&2\\3&4&7&8&6&5&2&1\end{pmatrix}=(1\ \ 3\ \ 4\ \ 7\ \ 8\ \ 6\ \ 5\ \ 2)$$

$$=(1\ \ 3)(1\ \ 4)(1\ \ 7)(1\ \ 8)(1\ \ 6)(1\ \ 5)(1\ \ 2).$$

Hence, f is an odd permutation.

• TEST YOURSELF–3

1. Find the order of each element of the group $[\{0, 1, 2, 3, 4\} +_5]$.
2. Show that the set $G = \{0, 1, 2, 3, 4, 5\}$ is a finite abelian group of order 6 with respect to addition modulo 6.
3. Show that $G = \{1\ 5, 7, 11\}$ is a group under multiplication modulo 12.
4. Show that the set P_3 of all permutation on three symbols 1, 2, 3, is a finite non-abelian group of order 6 with respect to permutation multiplication as composition.
5. Show that the set A_3 of three permutations (a) $(a\ b\ c)$, $(a\ c\ b)$ on three symbols a, b, c forms a finite abelian group with respect to the permutation multiplication.
6. If $f=\begin{pmatrix}1&2&3\\3&1&2\end{pmatrix}$ and $g=\begin{pmatrix}1&2&3\\1&3&2\end{pmatrix}$, then find fg and gf.
7. Find the inverse of the following permutation :-

   (a) $\begin{pmatrix}1&2&3&4\\1&3&4&2\end{pmatrix}$ (b) $\begin{pmatrix}1&2&3&4\\3&4&1&2\end{pmatrix}$
8. Show that if S has more than two element of G. Then the symmetric group S_n is not abelian.
9. Show that a cycle containing an odd number of symbols is an even permutation where as a cycle containing an even number of symbols is an odd permutation.
10. If $f=(1\ \ 2\ \ 3\ \ 4\ \ 5\ \ 6)$, then show that

$$f^3=(1\ \ 4)(2\ \ 5)(3\ \ 6).$$

11. Examine whether the following permutation to even or odd

$$\begin{pmatrix}1&2&3&4&5&6&7\\6&5&2&4&3&1&7\end{pmatrix}.$$

ANSWERS

1. $O(o) = 1$ and order of other element is 5.

6. $fg=\begin{pmatrix}1&2&3\\2&1&3\end{pmatrix}$ and $gf=\begin{pmatrix}1&2&3\\3&2&1\end{pmatrix}$

7. (a) $\begin{pmatrix}1&2&3&4\\1&4&2&3\end{pmatrix}$ (b) $\begin{pmatrix}1&2&3&4\\3&4&1&2\end{pmatrix}$.

• 11.10. HOMOMORPHISM AND ISOMORPHISM OF GROUPS

Homomorphism. *Let* $(G, 0)$ *and* $(G', *)$ *be two groups. Then a mapping* $f : G \to G'$ *is called a homomorphism if*

$$f(x \circ y) = f(x) * f(y)\ \ \forall\ x, y \in G.$$

Here, we say that f preserves compositions in G and G'.

Some Examples of Subgroup :

(i) $[\{1, -1\}, \cdot]$ is a subgroup of $[\{1, -1, -i, -i\}, \cdot]$

(ii) $(Z, +)$ is a subgroup of $(Q, +)$

(iii) $(Q, +)$ is a subgroup of $(R, +)$

(iv) The set of all non-singular matrices with real elements whose determinents are 1, is a subgroup of multiplicative group of all non-singular $n \times n$ metrices.

(v) The multiplicative group of positive rational numbers is a subgroup of the multiplicative group of all non-zero rational numbers.

Some Important Theorems :

Theorem 1. *If H is any subgroup of G, then $H^{-1} = H$. Also, show that converse is not true.*

Proof. Let $h^{-1} \in H$. Then $h \in H$.

Since, H is a subgroup of G, therefore $h \in H \Rightarrow h^{-1} \in H$.

Thus, $\quad h^{-1} \in H^{-1} \Rightarrow h^{-1} \in H$

$$\Rightarrow H^{-1} \subseteq H \qquad \text{...(1)}$$

Again $\quad h \in H \Rightarrow h^{-1} \in H$

$$\Rightarrow (h^{-1})^{-1} \in H^{-1} \Rightarrow h \in H^{-1}.$$

$$\therefore \quad H \subseteq H^{-1}. \qquad \text{...(2)}$$

Now, from (1) and (2) we have $H = H^{-1}$.

Now, to show converse is not true.

i.e., If H is a complex of a group G and $H^{-1} = H$, then it is not necessary that H is a subgroup of G. **For example :**

$$H = \{-1\} \text{ is a complex of the multiplicative group } G = \{-1, 1\}.$$

Also $H^{-1} = \{-1\}$ ($\because -1$ is the inverse of -1). But $H = \{-1\}$ is not a subgroup of G. We have $(-1)(-1) = 1 \notin H$, *i.e.*, H is not closed with respect to multiplication.

Theorem 2. *A non-empty subset H of a group G is a subgroup of G if and only if*

(*i*) $a, b \in H \Rightarrow ab \in H$

(*ii*) $a \in H \Rightarrow a^{-1} \in H$, *where a^{-1} is the inverse of $a \in G$.*

Proof. Let H be a subgroup of G, then H must be closed with respect to multiplication, *i.e.*, the composition in G.

Therefore $\quad a \in H, b \in H \Rightarrow ab \in H$.

Conversely. Suppose H is a subset of a group G such that (i) and (ii), the given conditions, holds. In order to show that H is a subgroup, all that is needed is to verify that the identity element $e \in H$ and that the associative law holds for elements of H.

If $a \in H$, then by (2), $a^{-1} \in H$ and so by (1) we see that $e = aa^{-1} \in H$, again since associative law does holds in G, it holds all the more so for H, which is a subset of G. Hence, is a subgroup of G.

Theorem 3. *Let H be a non-empty subset of a group G. Then H is a subgroup of G iff*

$$a, b \in H \Rightarrow ab^{-1} \in H, \text{ where } b^{-1} \text{ is the inverse of } b \text{ in } G.$$

Proof. Necessary conditions. Let us first suppose H is a subgroup of G and $a, b \in H$. Since H is a group, each element of H must have its inverse in H. Thus if $b \in H \Rightarrow b^{-1} \in H$ and then by closure property $ab^{-1} \in H$. This proves the necessary condition.

Condition is sufficient. Conversly, let H is a subset of G for which $a, b \in H$ implies $ab^{-1} \in H$. To show that H is a subgroup of G, we must verify that H is closed, the identity element $e \in H$, every element of H has an inverse in H and the associative law holds for elements of H.

Let $b = a$, then we see that $a \in H \Rightarrow aa^{-1} \in H \Rightarrow e \in H$

$\Rightarrow$ identity element of G also belongs to H.

Now, for the elements e and b of H, we have $eb^{-1} \in H$ and so $b^{-1} \in H$, since b is an arbitrary element of H, we see that for any $b \in H$, $b^{-1} \in H$.

Now, $\quad a, b \in H \Rightarrow a, b^{-1} \in H$

$$\Rightarrow a(b^{-1})^{-1} \in H \qquad \text{(By hypothesis)}$$

$$\Rightarrow ab \in H$$

$$\Rightarrow H \text{ is closed.}$$

Finally, since the associative law does hold for H, it also holds for H which is a subset of G

$\Rightarrow$ $(H, \cdot)$ is a subgroup

Theorem 4. *A necessary and sufficient condition of a non-empty subset H of a group G to be a subgroup is $HH^{-1} \subset H$.*

Proof. Let H be a non-empty of a group G such that H is a subgroup of G.

To show that $HH^{-1} \subset H$

Let $x \in HH^{-1} \Rightarrow \exists\, a, b \in H$ s.t. $x = ab^{-1}$

$\Rightarrow a, b^{-1} \in H$ s.t. $x = ab^{-1}$ ($\because$ H is a subgroup)

$\Rightarrow ab^{-1} \in H$

$\Rightarrow x \in H$

$\Rightarrow HH^{-1} \subset H$.

Conversely. Suppose that H is a non-empty subset of a group G such that $HH^{-1} \subset H$.

To show that H is a subgroup of G. For this we shall show that $a, b \in H \Rightarrow ab^{-1} \in H$ and $a, b \in H \Rightarrow ab^{-1} \in HH^{-1}$ (By definition of HH^{-1})

$\Rightarrow ab^{-1} \in H$ for $HH^{-1} \subset H$.

Theorem 5. *A necessary and sufficient condition for a non-empty subset H of a group G to be a subgroup is that $HH^{-1} = H$.*

Proof. Let H be a non-empty subset of a group G such that H is a subgroup of H so that $(H, \cdot)$ is a group.

To show $HH^{-1} = H$.

Let $x \in HH^{-1} \Rightarrow \exists\, a \in H, b^{-1} \in H^{-1}$ s.t. $x = ab^{-1}$

$\Rightarrow a, b \in H : x = ab^{-1}$

$\Rightarrow a, b^{-1} \in H : x = ab^{-1}$

$\Rightarrow ab^{-1} \in H : x = ab^{-1} \Rightarrow x \in H$.

Therefore $HH^{-1} \subset H$. ...(1)

Now, let $x \in H \Rightarrow x \in H, e \in H$.

For $(H, \cdot)$ is a group and e is the identity for G

$\Rightarrow xe^{-1} \in HH^{-1} \Rightarrow xe \in HH^{-1}$ ($\because$ $e^{-1} = e$)

$\Rightarrow x \in HH^{-1}$

$\Rightarrow H \subset HH^{-1}$. ...(2)

Now, from (1) and (2), we get

$$HH^{-1} = H.$$

Conversely. Let H be a non-empty subset of a group G such that $HH^{-1} = H$.

To show that H is a subgroup of G, it is sufficient to show that

$$a, b \in H \Rightarrow ab^{-1} \in H$$

$$a, b \in H \Rightarrow a \in H,\ b^{-1} \in H^{-1} \Rightarrow ab^{-1} \in HH^{-1} = H \Rightarrow ab^{-1} \in H.$$

Theorem 6. *If H, K are subgroups of a group G, then HK is a subgroup of G iff $HK = KH$.*

Proof. Let H and K subgroups of a group G so that

$$HH^{-1} = H,\ KK^{-1} = K \quad \text{...(1)}$$

and

$$K^{-1} = K,\ H^{-1} = H. \quad \text{...(2)}$$

Step I. Let HK be a subgrop of G so that

$$(HK)^{-1} = HK. \quad \text{...(3)}$$

To show $HK = KH$.

(3) $\Rightarrow$ $K^{-1}H^{-1} = HK$.

Using (2), we have $HK = KH$. ...(4)

Step II. Let $HK = KH$

To show that HK is a subgroup of G. For this we have to prove

$$(HK)(HK)^{-1} = HK. \quad \text{...(5)}$$

Consider $(HK)(HK)^{-1} = (HK)(K^{-1}H^{-1}) = H(KK^{-1})H^{-1}$ (By associativity)

$= HKH^{-1}$ [by (1)]

$= KHH^{-1}$ [by (4)]

$= K(HH^{-1}) = KH = HK$ [by (4)]

Hence, the theorem is proved.

• 11.12. UNION AND INTERSECTION OF SUBGROUPS

Theorem 8. *The intersection of any two subgroups of a group G is a subgroup of G.*

Proof. Let H_1 and H_2 be two subgroup of a group G. To show that $H_1 \cap H_2$ is a subgroup of G. For this we have to show that $a, b \in H_1 \cap H_2 \Rightarrow ab^{-1} \in H_1 \cap H_2$.

Let $a, b \in H_1 \cap H_2 \Rightarrow a, b \in H_1$ and $a, b \in H_2$.

Now, since H_1 and H_2 are subgroups, then

$$a, b \in H_1 \Rightarrow ab^{-1} \in H_1$$

and
$$a, b \in H_2 \Rightarrow ab^{-1} \in H_2$$

$$ab^{-1} \in H_1 \text{ and } ab^{-1} \in H_2 \Rightarrow ab^{-1} \in H_1 \cap H_2.$$

Theorem 9. *An arbitrary intersection of subgroups of a group G is a subgroup of G.*

Proof. Let H_r be the collection of subgroups for $r \in N$.

Let
$$H = \bigcap_{r=1}^{\infty} H_r.$$

To show that H is a subgroup of G

$$a, b \in H \Rightarrow a \in \bigcap_{r=1}^{\infty} H_r \text{ and } b \in \bigcap_{r=1}^{\infty} H_r$$

$$\Rightarrow a \in H_r, b \in H_r \;\; \forall \; r \in N$$

$$\Rightarrow ab^{-1} \in H_r \;\; \forall \; r \in N \Rightarrow ab^{-1} \in \bigcap_{r=1}^{\infty} H_r = H \Rightarrow ab^{-1} \in H.$$

Thus, we have proved that $a, b \in H \Rightarrow ab^{-1} \in H$.

This declares that H is a subgroup of G.

Theorem 10. *The union of two subgroups of a group G is a subgroup of G iff one is contained in the other.*

Proof. Let H_1 and H_2 be subgroups of a group G. Let $H_1 \subset H_2$ or $H_2 \subset H_1$.

To show that $H_1 \cup H_2$ is a subgroup of G

$$H_1 \subset H_2 \Rightarrow H_1 \cup H_2 = H_2.$$

Also, H_2 is a subgroup of $G \Rightarrow H_1 \cup H_2$ is a subgroup of G. Again $H_2 \subset H_1 \Rightarrow H_1 \cup H_2 = H_1$.

Also, H_1 is a subgroup of $G \Rightarrow H_1 \cup H_2$ is a subgroup of G.

Hence, $H_1 \cup H_2$ is a subgroup of G, in both cases.

Conversely. Suppose that H_1 and H_2 are subgroups of a group G such that $H_1 \cup H_2$ is a subgroup of G.

To show $\quad H_1 \subset H_2 \;$ or $\; H_2 \subset H_1$.

Suppose the contrary. Then $H_1 \not\subset H_2$ or $H_2 \not\subset H_1$

$$H_1 \not\subset H_2 \Rightarrow \exists\, a \in H_1 \text{ s.t. } a \notin H_2$$

and
$$H_2 \not\subset H_1 \Rightarrow \exists\, b \in H_2 \text{ s.t. } b \notin H_1.$$

Now, $a, b \in H_1, \cup H_2$, and $H_1 \cup H_2$ is a subgroup of G

$\Rightarrow \quad ab \in H_1 \cup H_2$.

This implies $\quad ab \in H_1 \;$ or $\; ab \in H_2$

$$a \in H_1, ab \in H_1 \Rightarrow a^{-1}(ab) \in H_1 \qquad (\because \; H_1 \text{ is a subgroup})$$

$$\Rightarrow \quad (a^{-1}a)\, b \in H_1 \Rightarrow eb \in H_1 \Rightarrow b \in H_1$$

which is a contradiction $\qquad (\because \; b \notin H_1)$

$$b \in H_2, ab \in H_2 \Rightarrow (ab)\, b^{-1} \in H_2$$

$$\Rightarrow \quad a \in H_2 \qquad (\text{For } (ab)\, b^{-1} = a\,(bb^{-1}) = ae = a)$$

Again, we get a contradiction $\qquad (\because \; a \notin H_2)$

Hence, our initial assumption is wrong.

Consequently $\quad H_1 \subset H_2 \;$ or $\; H_2 \subset H_1$.

• SOLVED EXAMPLES

Example 1. *Is Z a subgroup of $(Q, +)$?*

Solution. For $Z \subset Q$ and the inverse of $b \in Q$ is $-b$ $a, b \in Z \Rightarrow a + (-b) = a - b \in Z$.

Therefore **Z** is a sugbroup fo Q, under addition.

Example 2. *Let G be the additive group of integers and* $H = \{nl : n$ *is a fixed integer and* $l \in \mathbf{Z}\}$. *Show that H is a subgroup of G.*

Solution. Here, we have that

$$H \subseteq G.$$

Let $a = nh$ and $b = nk$, be any two elements of H, with $h, k \in \mathbf{Z}$. Then $a + b = n(h + k)$ certainly $\in H$.

Thus, $a, b \in H$ implies that $a + b \in H$.

Also, $-a = n(-h)$, the additive inverse of a, is in H. Thus $a \in H$ implies that $-a \in H$. Hence, H is a subgroup of G.

Example 3. *If G is a group, then the set* **Z**, *defined by*

$$Z = \{z \in G : zx = xz \ \ \forall x \in G\}.$$

Prove that **Z** *is a subgroup of* **G**.

Solution. Let $z_1, z_2 \in Z$, then

$$z_1x = xz_1, \ z_2x = xz_2 \ \ \forall \ x \in G \qquad \ldots(1)$$

Now, $$xz_1 = z_1x = z_1(z_2^{-1} z_2x) \ \ \forall \ x \in G$$

$$= z_1z_2^{-1}(z_2x) = z_1z_2^{-1}(xz_2) \qquad \text{[from (1)]}$$

$\therefore$ $$(xz_1)\, z_2^{-1} = z_1z_2^{-1}\,(xz_2)\, z_2^{-1}$$

or $$x\,(z_1z_2^{-1}) = z_1z_2^{-1}x\,(z_2z_2^{-1}) = (z_1z_2^{-1})\,x \ \ \forall\, x \in G.$$

Therefore, $z_1z_2^{-1} \in Z \Rightarrow Z$ is a subgroup of G.

Example 4. *If a is a fixed element of a group G, then prove that the set*

$$N(a) = \{x \in G : xa = ax\}$$

is a subgroup of G.

Solution. Let $x, y \in N(a)$, then $xa = ax$, $ya = ay$.

Now $$ya = ay \Rightarrow y^{-1}\,(ya)\,y^{-1} = y^{-1}(ay)\,y^{-1}$$

$$\Rightarrow ay^{-1} = y^{-1}a \qquad \ldots(1)$$

$$\Rightarrow y^{-1} \in N(a)$$

Also $$(xy^{-1})\,a = x(y^{-1}a) = x(ay^{-1}) \qquad \text{[by (i)]}$$

$$= (xa)\,y^{-1} = (ax)\,y^{-1} = a(xy^{-1})$$

$\Rightarrow$ $xy^{-1} \in N(a)$, whenever $x, y \in N(a)$

Hence, $N(a)$ is a subgroup of G.

• 11.13. COSETS

Let H be a subgroup of a group $(G, \cdot)$. Let $a \in G$ be arbitrary. We define

$$aH = \{ah : h \in H\} \ \text{ and } \quad Ha = \{ha : h \in H\}$$

then aH is called **left coset** of H in G generated by a, and Ha is called **right coset** of H in G generated by a.

REMARKS

- If e is the identity of G, then $e \in H$ is also identity for H

  $$a = ae \in aH, \ a = ea \in Ha$$

  This gives that any left or right cosets of H in G is non-empty.
- Since $He = H = eH$, hence H itself is right as well as left cosets.
- If the group $(G, \cdot)$ is abelian, then $ah = ha \ \forall \ h \in H$ so that

  $$aH = Ha \ \ \forall \ a \in H.$$
- If the composition in G is additive, then the right coset of H in G generated by a is defined as

  $$H + a = \{h + a : h \in H\}$$

  and $$a + H = \{a + h : h \in H\}.$$

Index of a subgroup in a group. If H is a subgroup of a group G, then number of distinct right (or left) cosets of H in G is called index of H in G and is denoted by $[G : H]$ or by $i_G(H) = o(G)/o(H)$.

Relation of Congruence modulo a subgroup in a group. Let H be a subgroup of a group G. Let $a, b \in G$ be arbitrary, we define $a \equiv b \bmod H$ iff $ab^{-1} \in H$.

The symbol $a \equiv b \pmod H$ is read as a is congruent to b modulo H.

REMARKS

- $a \equiv b \pmod H$ iff $ab^{-1} \in H$ or $Ha = Hb$.
- $a \in Hb \Leftrightarrow ab^{-1} \in H \Leftrightarrow Ha = Hb$.

Some Important Theorems :

Theorem 1. *Let $a \in G$ be arbitrary and let H be a subgroup of a group G. Then $Ha = H = aH \Leftrightarrow a \in H$.*

Proof. Let H be a subgroup of G and let $a \in G$ be arbitrary.

Step I. To show that

$$Ha = H \Leftrightarrow a \in H.$$

Let us first suppose $Ha = H$, to show $a \in H$

$$e \in H, a \in H \Rightarrow ea \in Ha \Rightarrow a \in Ha$$

$$\Rightarrow \quad a \in H. \text{ For } H = Ha.$$

Now, let $a \in H$, to show $Ha = H$.

Let $xa \in Ha \Rightarrow x \in H$

$$\Rightarrow x \in H, \ a \in H, \quad \text{For } a \in H$$

$$\Rightarrow xa \in H \qquad (\because \ H \text{ is a subgroup})$$

Thus, any $xa \in Ha \Rightarrow xa \in H$.

This prove that $Ha \subset H$ ...(1)

$$a \in H \Rightarrow a^{-1} \in H \qquad (\because \ H \text{ is a subgroup})$$

For any $y \in H, a^{-1} \in H \Rightarrow ya^{-1} \in H$

$$\Rightarrow (ya^{-1})\, a \in Ha$$

$$\Rightarrow y \in Ha \text{ for } ya^{-1}a = ye = y$$

Thus any $y \in H \Rightarrow y \in Ha$

$$\Rightarrow H \subset Ha. \qquad \text{...(2)}$$

Now from (i) and (ii) we get

$$H = Ha.$$

Step II. To show $aH = H \Leftrightarrow a \in H$.

We can prove step II by making the parallel arguments as in I.

Theorem 2. *If a and b are arbitrary distinct elements of a group G and H is any subgroup of G, then*

$$Ha = Hb \Leftrightarrow ab^{-1} \in H$$

$$aH = bH \Leftrightarrow b^{-1}a \in H.$$

Proof. Let a and b be arbitrary elements of a group G such that $a \neq b$.

Let e be the identity of $G \Rightarrow e \in H$.

Firstly, we shall show that

$$Ha = Hb \Leftrightarrow ab^{-1} \in H$$

$$Ha = Hb \Rightarrow (Ha)\, b^{-1} = (Hb)(b^{-1})$$

$$\Rightarrow \quad H(ab^{-1}) = H(bb^{-1}) = He = H$$

$$\Rightarrow \quad H(ab^{-1}) = H \Rightarrow ab^{-1} \in H \qquad \text{(By previous theorem)}$$

Conversely. $ab^{-1} \in H \Rightarrow H(ab^{-1}) = H$

$$\Rightarrow (Hab^{-1})\,(b) = Hb$$

$$\Rightarrow (Ha)\,(b^{-1}b) = Hb$$

$$\Rightarrow (Ha)e = Hb \Rightarrow Ha = Hb.$$

Therefore, we have

$$Ha = Hb \Leftrightarrow ab^{-1} \in H.$$

Similarly, we can prove $aH = bH \Leftrightarrow b^{-1}a \in H$.

Theorem 3. *Any two left cosets of a sugroups are either disjoint or identical.*

Proof. Let aH and bH be any two left cosets of H. To show if aH and bH have an element in common, *i.e.*, If $aH \cap bH$ is not the empty set, then they are identical, *i.e.*, $aH = bH$.

Let $aH \cap bH \neq \phi$ and let c be any element of $aH \cap bH$ then there exist elements $h_1, h_2 \in H$ such that $c = ah_1$ and $c = bh_2$ it follows that

$$ah_1 = bh_2 \text{ so } a = bh_2(h_1)^{-1}. \quad \ldots(1)$$

Now, let ah be any element of aH. Then

$$ah = bh_2(h_1)^{-1}h. \quad \text{[Using (1)]}$$

Now, since H is a subgroup, $h_2(h_1)^{-1}h \in H$ and so $ah \in bH$.

This shows that every $ah \in aH$ is also in bH. Therefore

$$aH \subseteq bH.$$

Similarly, we can show that

$$bH \subseteq aH.$$

Therefore, we have

$$aH = bH.$$

Hence, we have shown that any two left cosets which are not disjoint are identical.

Theorem 4. (Lagrange's Theorem)

The order of each subgroup of a finite group is a divisor (factor) of the group.

Proof. Let H be a subgroup of a finite group G and let

$$o(G) = n \quad \text{and} \quad o(H) = m.$$

To show m is a divisor of n

For this we have to show that $n = mp$ for same $p \in N$.

Let Ha be any right coset of H in G.

Then $o(H) = m \Rightarrow \exists\ m$ distinct elements $h_1, h_2, ..., h_m \in H$

$\Rightarrow \exists\ m$ distinct elements $h_1a, h_2a, \ldots, h_ma \in Ha$. For any map form H into Ha is one-one onto

$\Rightarrow o(Ha) = m = o(H)\ \forall\ a \in G.$

$\Rightarrow$ every right coset of H in G has m distinct elements. Since, G is finite and therefore, number of distinct right cosets of H in G will be finite say p. Also, any two right cosets of H in G will be either identical or disjoint. Hence p disjoint right cosets of H in G will contain mp distinct elements.

$\therefore \quad G = H \cup Ha \cup Hb \cup Hc \cup \ldots$ where $a, b, c, ..., \in G$.

$$o(G) = o(H) + o(Ha) + o(Hb) + \ldots = m + m + \ldots p \text{ times} = mp$$

$$o(G) = mp \Rightarrow n = mp$$

$\Rightarrow$ Order of the subgroup of a finite group is a divisor of the order of the group.

REMARKS

- The converse of the Lagrange's theorem is need **not be true**, *i.e.*, if G is a finite group of order n and m is any divisor of n, then it is not necessary that G must have a subgroup of order m.

  For example. Consider the symmetric group P_4 of permutation of degree 4. Then $o(P_4) = 4\,! = 24$. Let A_4 be the alternating group of even permutation of degree 4. Then $o(A_4) = \frac{24}{2} = 12$. There exist no subgroup H of A_4 such that $O(H) = 6$, though 6 is a divisor of 12.

- The Lagrange's theorem has important applications in group theory. If G is a group of order 8, then there will not exist subgroup of G of order 3, 5, 6, 7. The only subgroup of G **may be** of order 2 and 4. Since, 2 and 4 are divisors of 8.

Theorem 5. *The order of every element of a finite group G is a divisor of the order of the group, i.e., $o(a) \mid o(G)$.*

Proof. Let G be a finite group of order n and let $a \in G$ be arbitrary, such that $o(a) = m$.

To show m is a dvisor of n.

Define $\quad H = \{a^p : p \in Z\}$

$$o(a) = m$$

$\Rightarrow\ m$ is the least positive integer in such that $a^m = e$.

$$H = \{..., a^{-2}, a^{-1}, a^0, a^1, a^2, ..., a^m = e\}$$

Let $x, y \in H \Rightarrow \exists, p, q, \in \mathbf{Z}$ such that $a^p = x$, $a^q = y$

$\Rightarrow\ xy^{-1} = a^{p-q} = a^r$ where $p - q = r \in \mathbf{Z}$

$\Rightarrow\ xy^{-1} = a^r \in H$

$\Rightarrow\ xy^{-1} \in H$

$\Rightarrow H$ is a subgroup of G.

Now, to show $o(H) = m$, *i.e.*, H contains m distinct element

$$a, a^2, a^3, ..., a^m = e = a^0.$$

Let $r, s \in Z^+$ such that

$$r > 0,\ 1 \le r \le m,\ 1 \le s \le m$$

Now, $a^r = a^s \Rightarrow a^{r-s} = e \Rightarrow o(a) \le r - s < m \Rightarrow o(a) < m$

which is a contradiction($\because\ o(a) = m$)

$\therefore\ \ a^r \ne a^s$ if $r \ne s$

$\Rightarrow\ \ a, a^2, a^3, ..., a^m$ are distinct elements of H

$\Rightarrow\ \ o(H) = m = o(a)$

Then, by Lagrange's theorem, we have m is a divisor of n

$\Rightarrow\ \ o(a)$ is a divisor of $o(G)$.

Theorem 6. *Let G be a finite group of order n and $a \in G$ then $a^n = e$.*

Proof. Let G be a finite group of order n and let $a \in G$ be an element of order m so that $a^m = e$.

To show $\qquad a^m = e$

Let $\qquad H = \{a^p : p \in Z\}$.

Then, by previous theorem, H is a subgroup of order m.

Using Lagrange's theorem, we have m is a divisor of n

$\Rightarrow\ \ \exists\, p \in N$ such that $\dfrac{n}{m} = p \Rightarrow n = mp$

Now, $\quad a^n = a^{mp} = (a^m)^p = (e)^p = e \ \Rightarrow\ a^n = e.$

Theorem 7. (Cayley Theorem). *Every finite group G is isomorphic to a permutation group.*

Proof. Let G be a finite group of order n such that $G = \{a_1, a_2, ..., a_n\}$

Let $a \in G$. Define a map $f_a : G \to G$ given by

$$f_a(x) = ax \quad \forall\ x \in G. \qquad ...(1)$$

f_a is one-one.

Let $\qquad f_a(x_1) = f_a(x_2) : x_1, x_2 \in G$

$\Rightarrow \qquad ax_1 = ax_2 \ \Rightarrow\ x_1 = x_2.$

f_a is onto.

$f_a : G \to G$ is one-one and G is finite $\Rightarrow f_a$ is onto. Thus, f_a is one-one map of a finite set G onto itself. It means that f_a is a permutation of degree n. Here

$$f_a = \begin{pmatrix} a_1 & a_2 \ldots a_n \\ aa_1 & aa_2 \ldots aa_n \end{pmatrix}.$$

The elements $aa_1, aa_2, ..., aa_n$ are all distinct elements of G.

Write $G' = \{f_a : a \in G\}$ then G' is a set of permutations of degre n.

Now, we claim that $(G', \cdot)$ is a group, where $(\cdot)$ denotes permutations multiplication.

Let $a, b, c \in G$ be arbitrary and e be the identity in G.

Let a^{-1} denote inverse of a in G.

So that $\qquad a^{-1}a = aa^{-1} = e.$

(i) Closure property : $f_a, f_b \in G' \Rightarrow f_a f_b \in G'$

Consider $\qquad (f_a f_b)(x) = f_a[f_b(x)] = f_a(bx)$

$$= a(bx) = (ab)x, \text{ by associativity in } G.$$

$$= f_{ab}(x)$$

$\Rightarrow \qquad f_a f_b = f_{ab} \qquad ...(2)$

$$a, b \in G \Rightarrow ab \in G \Rightarrow f_{ab} \in G' \Rightarrow f_a f_b \in G'.$$

(ii) Associativity : Let $a, b, c \in G$

$\Rightarrow \qquad (ab)c = a(bc) \ \Rightarrow\ f_{(ab)c} = f_{a(bc)}$

$$\Rightarrow\ f_{(ab)} f_c = f_a f_{bc} \ \Rightarrow\ (f_a f_b) f_c = f_a (f_b f_c).$$

(iii) Existence of identity : $a, e \in G \Rightarrow f_e \in G'$ and $ae = ea = a$

$$\Rightarrow\ f_{ae} = f_{ea} = f_a \ \Rightarrow\ f_a f_e = f_e f_a = f_a$$

$\Rightarrow\ \ f_e \in G'$ is identity element of G'.

(iv) Existence of inverse : $a \in G \Rightarrow a, a^{-1} \in G \Rightarrow f_a, f_a^{-1} \in G'$.

Also, $\qquad aa^{-1} = a^{-1}a = e$

$$f_{aa^{-1}} = f_{(a^{-1}a)} = f_e \text{ or } f_a f_{a^{-1}} = f_{a^{-1}} f_a = f_e$$

$\Rightarrow$ $f_{a^{-1}} \in G'$ is the inverse of $f_a \in G$.

(v) Order of G' :

For $\quad 0(G) = n, G' = \{f_a : a \in G\} \Rightarrow 0(G') = H.$

Therefore, we have $(G', \cdot)$ is a finite group of order n.

Now we claim that $(G, \cdot) \equiv (G', \cdot)$.

Now define a map $g : G \to G'$ such that $g(x) = f_x \ \forall x \in G$.

(i) g is one-one :

For $g(x_1) = g(x_2) : x_1, x_2 \in G$

$\Rightarrow \quad f_{x_1} = f_{x_2}$

$\Rightarrow \quad f_{x_1}(x) = f_{x_2}(x) \ \forall \ x \in G$

$\Rightarrow \quad x_1 x = x_2 x$ [by (i)]

$\Rightarrow \quad x_1 = x_2.$

(ii) g is onto :

For any $f_a \in G' \Rightarrow a \in G$ such that $g(a) = f_a$.

(iii) g preserves comparision in G and G' :

For $\quad g(x_1 x_2) = f_{x_1 x_2}$ (where $x_1, x_2 \in G \Rightarrow x_1 x_2 \in G$)

$$= f_{x_1} f_{x_2} = g(x_1) \cdot g(x_2).$$

Hence, G is an isomorphism of G onto G' and hence $G \equiv G'$.

• SOLVED EXAMPLES

Example 1. *If G is a group and $a \in G$, then show that the set $H = \{a^n : n \in Z\}$ is a subgroup of G and it is the smallest subgroup of G which contains the element a.*

Solution. Clearly, H is non-empty subset of G.

Let $x, y \in H$, then $x = a^p, y = a^q$, where $p, q \in Z$.

Therefore, $\quad xy^{-1} = (a^p)(a^q)^{-1} = a^p a^{-q} = a^{p-q} \in H \qquad (\because \ p - q \in Z)$

$\Rightarrow H$ is a non-empty subset of G and $x, y \in H \Rightarrow xy^{-1} \in H$. Therefore, H is a subgroup of G.

Now, if K is any subgroup of G which contain a, they by closure property in K, $a^n \in K$ for every integer n. Also

$$a^0 = e \in K \quad \text{and} \quad a^{-n} = (a^n)^{-1} \in K$$

$\Rightarrow$ every integral power of a belong to K, *i.e.*, $H \subseteq K$. Hence, H is the smallest subgroup of G which contain a.

Example 2. *If H be a subgroup of group G and*

$$T = \{x \in G : xH = Hx\}.$$

Show that T is a subgroup of G.

Solution. Since $e \in G$ and $eH = He \Rightarrow e \in T$

$\Rightarrow$ T is a non-empty subset of G.

Let $x_1, x_2 \in T$ so that $x_1 H = Hx_1$, $x_2 H = Hx_2$

Now $\quad x_2 \in T \Rightarrow x_2 H = Hx_2 \Rightarrow x_2^{-1}(x_2 H)x_2^{-1} = x_2^{-1}(Hx_2)x_2^{-1}$

$$\Rightarrow x_2^{-1}x_2(Hx_2^{-1}) = (x_2^{-1}H)(x_2 x_2^{-1}) \Rightarrow e(Hx_2^{-1}) = (x_2^{-1}H)e$$

$$\Rightarrow Hx_2^{-1} = x_2^{-1}H \ \Rightarrow \ x_2^{-1} \in T.$$

Thus, $\quad x_2 \in T \Rightarrow x_2^{-1} \in T.$

Also, $\quad (x_1 x_2^{-1})H = x_1(x_2^{-1}H) = x_1(Hx_2^{-1}) = (x_1 H)x_2^{-1} = (Hx_1)x_2^{-1} = H(x_1 x_2^{-1})$

$$\Rightarrow x_1 x_2^{-1} \in T.$$

Thus, T is a non-empty subset of G and $x_1, x_2 \in T \Rightarrow x_1 x_2^{-1} \in T$ therefore, T is a subgroup of G.

• TEST YOURSELF–5

1. Let G be the additive group of integers. Then show that the set of all multiples of integers by a mixed integer m is a subgroup of G.
2. Show that the integral multiples of 5 from a subgroup of the additive group of integers.

3. Show that the 24 permutations on 4 symbols from a group with respect to permutation multiplication.
4. Use Lagrange's theorem to show that any group of prime order can have no proper subgroups.
5. If a finite group G contains an element of even order, show that G must also be of even order.
6. If a finite group possesses an element of order 2, show that it possesses an odd number of such elements.

• 11.14. CYCLIC GROUPS

If a group G contain an element a such that every element $x \in G$ is of the form a^m, where $m \in Z$, then G is said to by cyclic group and G is generated by a *i.e.*, a is the generator of G, and we write $G = \{a\}$.

REMARK

➤ If G is a cyclic group generated by a then, since G is closed under multiplication, then $a^k \in G \ \forall \ k \in Z^+$ Also, since the inverse of a^k is a^{-k} we see that $a^{-k} \in G \ \forall \ k \in Z^+$. Also a^0 is the identity e of G. Then

$$G = \{a\} = \{a^k : k \in Z\}$$

Thus the cyclic group G generated by **a**, consist of all elements of the form a^k.

Example on Cyclic Groups :

The multiplicative group $G = \{1, -1, i, -i]$ as a cyclic group with generator i, Because

$$(i)^1 = i, (i)^2 = -1, (i)^3 = -i \text{ and } (i)^4 = 1$$

$\Rightarrow$ each element of G can be expressed as some integral power of i

$\Rightarrow$ G is cyclic, generated by i.

REMARK

➤ A cyclic group always at least have two generator. For example if a is the generator of G then a^{-1} is also the generator of G.

2. The multiplicative group of n n^{th} roots of unity is cyclic with generated $e^{2\pi i/n}$.
3. Let n be a positive integer. We construct a group G of order n as follows : Suppose that G consists of all symbols a^i, $i = 0, 1, 2, ..., n-1$, where we insists that $a^0 = a^n = e$, $a^i a^j = a^{i+j}$ if $i + j \le n$ and $a^i a_{ij} = a_{i+j-n}$. If $i + j > n$. Then we may easily verify that this is a cyclic group of order n.

$$G = \{a\} = \{e, a, a^2, ..., a^{n-1}\}.$$

4. The additive group of integers $\{... -3, -2, -1, 0, 1, 2, 3 ...\}$ is a cyclic group with generators 1 and -1.

Properties of Cyclic Group :

Theorem 1. *Every cyclic group is necessarily abelian but the converse is not necessarily true.*

Proof. Let $G = \{a\}$ is a cyclic group generated by an element $a \in G$.

Let x and y be any two elements of G.

Then $x = a^m$ and $y = a^n$, for some integer m and n.

Now, $\quad xy = a^m a^n = a^{m+n} = a^{n+m} = a^n \cdot a^m = yx$

$\Rightarrow$ $xy = yx \ \forall \ x, y \in G$

$\Rightarrow$ G is abelian.

Conversly. An abelian group is not always a cyclic groups. It is illusted by the following example.

The set R_0 of all non-zero real numbers is an abelian group with respect to multiplication.

If $a \in R_0$, then $H = \{a^n : n \in Z\}$ is a countable subset of R_0 and so it can not be equal to the uncountable set R_0

$\Rightarrow$ All the elements of R_0 cannot be expressed as some integral power of a single element of R_0

$\Rightarrow$ $(R_0, \cdot)$ is not a cyclic group.

Theorem 2. *If the generator of a cyclic gorup G is of infinite order (or of zero order), then G is isomorphic to the additive group of integers.*

Proof. Let a is the generator of the cyclic group $G = \{a\}$ let $o(a) = \infty \Rightarrow a^n \ne e$

To show $\qquad (G, \cdot) \cong (Z, +)$.

Firstly we shall show that any two powers of a can not be equal let if possible, $a^m = a^n$ for $m \neq n$

$$a^m = a^n \Rightarrow a^m = ea^n \Rightarrow a^{m-n} = e$$
$$\Rightarrow o(a) \leq m - n = a \text{ finite number}$$
$$\Rightarrow o(a) \text{ is finite.}$$

A contradiction

$$\Rightarrow a^m \neq a^n \text{ for } m \neq n$$
$$\Rightarrow G \text{ contains an inifinite number of distinct elements}$$
$$G = \{a^0 = e, a^{\pm 1}, a^{\pm 2}, a^{\pm 3} \ldots\}$$
$$\mathbf{Z} = \{(0), \pm 1, \pm 2, \pm 3, \pm \ldots\}.$$

Now, define a map $f : G \to \mathbf{Z}$ such that $f(a^n) = n \;\; \forall \; a^n \in G$.

f is one-one.

Let $\quad f(a^m) = f(a^n)\,;\, a^m, a^n \in G \;\Rightarrow\; m = n \Rightarrow a^m = a^n$

i.e., $\quad f(a^m) = f(a^n) \;\Rightarrow\; a^m = a^n \;\Rightarrow\; f$ is one-one.

f is onto.

Since $o(G) = \infty = o(\mathbf{Z})$ and f is one-one $\Rightarrow f$ is onto.

f preserves compositions in G and $\mathbf{Z}$.

Let $a^m, a^n \in G$, then

$$f(a^m \,.\, a^n) = f(a^{m+n}) = m + n = f(a^m) + f(a^n)$$

i.e., $$f(a^m . a^n) = f(a^m) + f(a^n)$$

$\Rightarrow$ f preserves compositions in G and $\mathbf{Z}$.

Hence, f is an isomorphism and $(G, \cdot) \equiv (\mathbf{Z}, +)$.

Theorem 3. *The order of a cyclic group is equal to the order of any generator of the group.*

Proof. Let a be the generator of a group $G = \{a\}$. Let $o(a) = \text{finite} = n$

$\Rightarrow \quad a^n = e,\; a^r \neq e$ for $0 < r < n$.

To show $\quad o(a) = o(G) = n$.

Step I. Firstly we show G contains n elements.

The elements of the cyclic group G is given below :

$$a, a^2, a^3, \ldots, a^n = e = a^0.$$

Let if possible, G contains an element a^m besides these elements where $m > n$. Then by divison algorithm

$$m = nq + r,\; 0 \leq r < n \text{ and } q, r \in N$$
$$a^m = a^{nq+r} = a^{nq} . a^r = (a^n)^q . a^r = e^q . a^r = e \,.\, a^r = a^r$$

$\therefore \quad a^m = a^r,\; 0 \leq r < n.$

$\because$ a^r is already contained in the set of n elements and so a^m is also contained

$\Rightarrow$ G contains n elements.

Step II. Now to show that any two elements of G are not equal. For this we have to show that $a^r \neq a^s$ where $r \neq s$, $0 < r < n$, $0 < s < n$.

Let $\quad r < s < n$

Then $\quad s - r > 0$

$$a^r = a^s \Rightarrow ea^r = a^s \Rightarrow a^{s-r} = e$$
$$\Rightarrow o(a) \leq s - r \quad \text{and} \quad s - r < n \quad \Rightarrow o(a) < n.$$

Which is a contradiction. Hence, $a^r \neq a^s$ where $r \neq s$.

Thus, we have shown that G contains n distinct elements and hence $o(G) = n$.

Theorem 4. *A finite group of order n containing an element of order n must be cyclic.*

Proof. Let G be a finite group of order n and let $a \in G$ such that $o(a) = n$. Then $o(G) = n = o(a)$.

To show G is cyclic

Let H be a cyclic group generated by a, then

$$o(H) = o(a) = n$$ ($\because$ order of a cyclic group is equal to the order of its generator)

$\Rightarrow$ H can be expressed as

$$H = \{a^r : r = 1, 2, 3, \ldots, n\}.$$

Since, G is a group

then $\Rightarrow$ $a \in G \Rightarrow a^r \in G$ for every integral value of r.

Thus $H \subseteq G$.

Moreover $o(G) = n = o(H)$.

$\therefore$ $H = G$, but H is cyclic $\Rightarrow G$ is cyclic.

Theorem 5. *Every group of prime order is cyclic.*

Proof. Let G be a finite group of order p, with p is prime. To show G is cyclic.

Since G is a group of prime order $\Rightarrow G$ must contains at least 2 elements.

($\because$ 2 is the least positive prime integer)

$\Rightarrow$ There must exist an element $a \in G : a \neq e$

$\because$ $a \neq e \Rightarrow o(a) \geq 2$.

Let $o(a) = m$. Then $H = \{a\}$ is a cyclic group of G and $o(H) = o(a) = m$. Then, by Lagrange's theorem m must be a divisor of p. But p is prime and $m \geq 2$. Hence $m = p$.

Therefore, $H = G$.

Since, H is cyclic, therefore G is cyclic with generator a.

Theorem 6. *Every subgroup of a cyclic group is cyclic.*

Proof. Let $G = \{a\}$ be a cyclic group generated by a. If $H = G$ or $\{e\}$, then obviously H is cyclic.

Now let H is a proper subgroup of G.

H contains the element of integral power of a.

If $a^s \in H \Rightarrow a^{-s} \in H$. Therefore, H contains elements which are positive as well as negative integral of a. Let k be the least positive integer such that $a^k \in H$. To show $H = \{a^m\}$.

Now let $a^t \in H$. Then, by division algorithm

Then exist $q + r \in Z$ such that

$$t = kq + r, \ 0 \leq r < K$$

Now $a^k \in H \Rightarrow (a^k)^q \in H \Rightarrow a^{kq} \in H \Rightarrow (a^{kq})^{-1} \in H \Rightarrow a^{kq} \in H$.

Also, $a^t \in H, a^{-kq} \in H \Rightarrow a^t a^{-kq} \in H \Rightarrow a^{t-kq} \in H \Rightarrow a^r \in H$.

$\because$ k is the least positive integer such that $a^k \in H$ and $0 \leq r < k$.

$\Rightarrow$ r must be equal to 0

$\Rightarrow$ $t = kq$

$\therefore$ $a^t = a^{kq} = (a^k)^q$

$\Rightarrow$ every element $a^t \in H$ is of the form $(a^k)^q$

$\Rightarrow$ H is cyclic with generator a^m.

• SOLVED EXAMPLES

Example 1. *How many generators are there of the cyclic group of order* 8.

Solution. Suppose that the cyclic group G of order 8 is generated by an element a then $o(a) = 8$.

Clearly, $G = \{a, a^2, a^3, a^4, a^5, a^6, a^7, a^8 = e\}$.

Now from theorem 7, an element a^m is also a generator of G, if m is less than 8 and relatively prime to 8.

Such numbers are 1, 3, 5 and 7.

Hence, a, a^3, a^5 and a^7 are generators of G.

$\Rightarrow$ There are four generators of G.

Example 2. *Show that the group* $G = [\{1, -1, i, -i\}, \bullet]$ *is cyclic.*

Solution. Let $G = \{1, -1, i, -i\}$.

To show G is cyclic.

If there exist an element $a \in G$ such that $o(a) = 4 = o(G)$. Then G will be cyclic group with its generator a.

Evidently $i^1 = i, i^2 = -1, i^3 = -i, i^4 = 1$.

Here identity element e of G is 1.

Thus $i^4 = 1, i^r = 1$ for any $r < 4$

$\Rightarrow$ $o(i) = 4 = o(G)$

$\Rightarrow$ i is the generator of G.

Now G is expressible as $G = \{i, i^2, i^3, i^4\}$.

• STUDENT ACTIVITY

1. Show that the set Z of all integer form a group with respect to binary operation $*$ defined by $a * b = a + b + 1 \forall a, b \in Z$ is an abelien group.

2. State and prove Lagrange's theorem.

• SUMMARY

- A structure $(G, *)$ is said to be a group :
  (i) $a * b \in G \forall a, b \in G$
  (ii) $(a * b) * c = a * (b * c) \forall a, b \in a, S, C \in G$
  (iii) Identity e exists in G *i.e.*, $a * e = a = e * a \forall a \in G$
  (iv) Inverse of each element of G exists in G.
- **Order of group :** No. of distinct elements of G gives the order of G it is denoted by $O(G)$ or $|G|$.
- **Order of an element of a group :** Let $a \in G$ and e be its identity element, then a least positive integer n is said to be the order of a it $a^n = e$ *i.e.*, $O(a) = n$.
- **Modulato system :**
  (i) Addition modulo n : $a +_n b = r,\ 0 \le r < n$ where r is the remainder obtained after dividing $a + b$ by x.
  (ii) Multiplication modulo n : $a \times_n b = r, 0 \le r < n$, where r is the remainder obtained after dividing ab by n.

- **Permutation :** A one-one mapping f of a finite non-empty set S onto itself is called a permutation.
- **Transposition :** A cycle of length two is called a transposition.
- **Even and odd permutations :** A permutation is said to be even or odd according as it can be expressed as a product of even or odd number of transpositions.
- **Homomorphism and isomorphism of groups :** A mapping $f : (G, 0) \to (G', *)$ is said to be a homomorphism if $f(x, y) = f(x) * f(y) \forall x, y \in G$. A homomorphism f is called an isomorphism if f is one-one and onto.
- **Subgroups of a Group :** A non-empty subset H of G is called a subgroup of G is closed under the sam binary operation and defined on G and H itself forms a group.
- **Cosets :** Let H be a subgroup of a group G. $aH = \{ah : h \in H\}\}$ and $Ha = \{ha : h \in H\}$ The aH is called a left coset of H in G and Ha is called a right coset of H in G.
- **Lagrange's theorem :** The order of each subgroup of a finite group is a divisor of the group.
- **Cayley Theorem :** Every finite group is isomorphic to its permutation group.
- **Cyclic Group :** If a group is generated by a single element, then it is called a cyclic group and that single element is called a generator of that group.

• TEST YOURSELF–6

1. If w is the cube roots of unity, show that the set $\{1, w, w^2\}$ is a cyclic group of order 3 with respect to multiplication.
2. Show that the group $[\{1, 2, 3, 4, 5, 6\}, \times_7]$ is cyclic. How many generators are there ?
3. Show that the two cyclic groups of same order are isomorphic.
4. Show that, every finite group of composite order possesses proper subgroups.
5. Show that the set U_n of n, n^{th} complex roots of unity forms a cyclic group with respect to multiplication.
6. Show that every finite group of order 6 must be abelian.
7. Show that the group $[\{1, 2, 3, 4\}, \times_5]$ is cyclic.
8. How many generators are there of the cyclic group of order 10.
9. Show that the residue classes [1], [2], [3], [4], [5] [6] and mod 7 from a multiplicative cyclic group. Find the number of generators.
10. Let G be the set of four matrices $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

    where 0 and 1 are the elements of the set $Z = \{0, 1\}$ modulo 2. Show that G is an abelian, non-cyclic group under matrix addition.
11. Find the order of each element in the cyclic group $G = \{a, a^2, a^3, a^4, a^5, a^6, = e\}$.

ANSWERS

2. Two, *i.e.*, 3 and 5   8. Four, *i.e.*, a, a^3, a^7, a^9.

11. $o(a) = 6, o(a^2) = 3, o(a^3) = 2, o(a^4) = 3, o(a^5) = 6$ and $o(a^6) = 1$.

OBJECTIVE EVALUATION

➤ FILL IN THE BLANKS :

1. A group G is said to be abelian if a ....
2. The number of element in a finite group G is said to .... of the group.
3. The identity of a group is ....
4. The invese of each element of a group G is ....

➤ TRUE OR FALSE :

Write 'T' for True and 'F' for False statement :

1. The product of two odd permutation is an odd permutation. *(T/F)*
2. If for every element a in a group G, $a^2 = e$, than G is an abelian group. *(T/F)*
3. If each elements having its own inverse then group is said to be abelian. *(T/F)*
4. If $*$ be a commutative composition in a set S, then

   $a * (b * c) = (c * b) * a \;\; \forall \;\; a, b, c \in S.$ *(T/F)*

➤ MULTIPLE CHOICE QUESTIONS :

Choose the most appropriate one :

1. A subgroup of an abelian group is :

(a) not abelian (b) necessarily abelian
(c) may be abelian (d) none of these.

2. Let $G = \{1, w, w^2\}$ is a cyclic, then the generators of G are :
(a) 1 and w (b) w and w^2 (c) 1 and w^2 (d) none of these.

3. If H, K are two subgroups of a group G, than HK is a subgroup of G if :
(a) $HK = KH$ (b) $HK = e'$ (c) $KH = e$ (d) $HK \neq KH$.

4. Every group of prime order is :
(a) Abelian (b) Cyclic (c) both (a) and (b) (d) none of these.

ANSWERS

Fill in the Blanks :

1. $ab = ba \ \forall \ ab \in G$ 2. order 3. unique 4. unique

True or False :

1. F 2. T 3. T 4. T

Multiple Choice Questions :

1. (b) 2. (b) 3. (a) 4. (c)

UNIT

12

MULTIPLE PRODUCTS OF VECTORS

LEARNING OBJECTIVES

LEARNING OBJECTIVES

After going through this unit you will learn :

- How to find the scalar and vector products of vectors.
- How to find the vectors reciprocal of the given vectors.

• 12.1. VECTOR AND SCALAR PRODUCTS

Let **a** *and* **b** *be two non-zero vectors. Then the product of the type* $\mathbf{a} \times \mathbf{b}$ *which gives a vectors quantity, is called* ***vector product*** *of* **a** *and* **b** *whereas the product* $\mathbf{a} \cdot \mathbf{b}$ *which gives a scalar quantity is called* ***scalar product*** *of the vectors* **a** *and* **b**. The vector $\mathbf{a} \times \mathbf{b}$ is perpendicular to both the vectors **a** and **b**.

• 12.2. TRIPLE PRODUCTS

Definition. *A product involving three non-zero vectors is known as a* ***triple product*** *of these three vectors.*

There are **two** type of triple products :

(i) Scalar Triple Product.

(ii) Vector Triple Product.

• 12.3. SCALAR TRIPLE PRODUCT

Definition. *If* **a, b, c** *are three non zero vectors, then the triple product* $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$ *which gives a scalar quantity is known as* ***scalar triple product*** *of the vectors* **a, b** *and* **c**.

Geometrical Interpretation of Scalar Triple Product:

Let **a, b, c** be three non-zero vectors and consider a parallelopiped whose coterminous edges are along these vectors and lengths of these edges are magnitudes of **a, b** and **c** respectively.

Since we have that $OA = |\mathbf{a}|$, $OB = |\mathbf{b}|$ and $OC = |\mathbf{c}|$. Let V be the volume of this parallelopiped

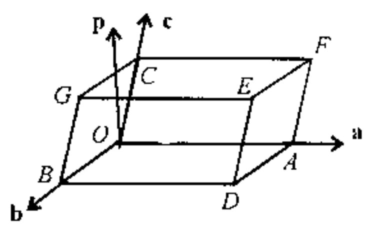

Fig. 1

which is taken as positive. Let **p** be a vector which is perpendicular to both the vectors **a** and **b** such that $\mathbf{p} = \mathbf{a} \times \mathbf{b}$. Thus vector **p** is perpendicular to the face *OBDA* of the parallelopiped.

Therefore the area of the face $OBDA = |\mathbf{a} \times \mathbf{b}|$. Suppose the vector **p** makes an angle θ with vector **c**. Then the perpendicular distance from C to the face *OBDA* is $OC \cos\theta$ *which takes the positive and negative signs according to the angle* θ *is acute or obtuse.* Thus the vectors **a, b, c** form a right hand triad if θ is acute otherwise will form a left hand triad.

$\therefore$ The volume of the prallelopiped

= (area of face *OBDA*)

× (perpendicular distance from C to the face *OBDA*)

or $$V = |\mathbf{a} \times \mathbf{b}|\, OC \cos\theta$$

$$V = |\mathbf{p}|\,|\mathbf{c}| \cos\theta. \qquad ...(1)$$

Now consider

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = |\mathbf{a} \times \mathbf{b}|\,|\mathbf{c}| \cos\theta$$

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = |\mathbf{p}|\,|\mathbf{c}| \cos\theta. \qquad ...(2)$$

From (1) and (2), we have

$$V = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}.$$

Hence, we conclude that the scalar triple product of three vectors gives the volume of the parallelopiped whose coterminous edges are along these three vectors.

If θ ***is acute,*** *the volume V will be positive and thus the triad* **a, b, c** *form right handed triad whereas* ***if*** θ ***is obtuse,*** *the volume V will be negative and the vectors* **a, b, c** *thus form left handed triad.*

If **a, b, c** form right handed triad, then **b, c, a** and **c, a, b** will also form right handed triad and hence we obtain that

$$V = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = (\mathbf{b} \times \mathbf{c}) \cdot \mathbf{a} = (\mathbf{c} \times \mathbf{a}) \cdot \mathbf{b}.$$

Hence we conclude that the scalar triple product is independent of the position of the ***dot*** *and* ***cross*** *but depends on the cyclic order of the factors and therefore* ***dot*** *and* ***cross*** *may be interchanged.*

• 12.4. PROPERTIES OF SCALAR TRIPLE PRODUCT

Property I. *If any two vector of* **a, b. c** *are equal, then the value of their scalar triple product will be zero.*

Since **a, b, c** are three non-zero vectors and we know that the scalar triple product is

$$[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}.$$

If $\mathbf{a} = \mathbf{b} = \mathbf{k}$ (say), then

$$[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = [\mathbf{k}\ \mathbf{k}\ \mathbf{c}] = (\mathbf{k} \times \mathbf{k}) \cdot \mathbf{c} = \mathbf{0} \cdot \mathbf{c} = 0.$$

REMARK

➤ $\mathbf{a} \times \mathbf{b} = (|\mathbf{a}|\,|\mathbf{b}| \sin\theta)\, \hat{n}.$

$$\therefore \qquad \mathbf{k} \times \mathbf{k} = (|\mathbf{k}|\,|\mathbf{k}| \sin 0^\circ)\, \hat{n} = 0.$$

Property II. *If any two vectors of* **a, b, c** *are parallel, then their scalar triple product will be zero.*

Since **a, b, c** are three non-zero vectors and suppose **a** and **b** are parallel to each other. Then we have

$$\mathbf{b} = t\mathbf{a} \qquad ...(1)$$

where t is any scalar quantity.

$$\therefore \quad [\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = (\mathbf{a} \times t\mathbf{a}) \cdot \mathbf{c} \qquad \text{[Using (1)]}$$

$$= t\,(\mathbf{a} \times \mathbf{a}) \cdot \mathbf{c} \qquad [\because\ \mathbf{a} \times m\mathbf{b} = m\mathbf{a} \times \mathbf{b} = m\,(\mathbf{a} \times \mathbf{b})]$$

$$= t\,(\mathbf{0} \cdot \mathbf{c}) = 0 \qquad [\because\ \mathbf{a} \times \mathbf{a} = \mathbf{0}]$$

Property III. *The three non-zero and non-parallel vectors* **a, b, c** *are coplanar if and only if their scalar triple product is zero.*

Suppose the three non-zero and non-parallel vectors **a, b, c** are coplanar, then we have to show that $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = 0$.

Since we know that the vector $\mathbf{a} \times \mathbf{b}$ is perpendicular to both the vectors **a** and **b** but **a, b, c** are given to be in a plane, then $\mathbf{a} \times \mathbf{b}$ will be perpendicular to the vector **c** hence $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = 0$.

Conversely, Suppose the scalar triple product of the vectors **a, b, c** is zero

i.e., $$[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = 0.$$

Then $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = 0$ implies that the vectors $(\mathbf{a} \times \mathbf{b})$ is perpendicular to the vector $\mathbf{c}$ but by the definition of $(\mathbf{a} \times \mathbf{b})$ is perpendicular to the plane of the vectors $\mathbf{a}$ and $\mathbf{b}$ both. Thus the vector $\mathbf{c}$ is parallel to the plane of $\mathbf{a}$ and $\mathbf{b}$. Hence these vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are lying in a plane cosequently, they are coplanar.

Property IV. *Distributive law holds for the scalar triple product.*

Since scalar and vector product are distributive so we have that for three non-zero vetors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$.

(i) $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$

(ii) $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$.

Then we have to show that $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}+\mathbf{d}] = [\mathbf{a}\ \mathbf{b}\ \mathbf{c}] + [\mathbf{a}\ \mathbf{b}\ \mathbf{d}]$

$$\begin{aligned}
\text{L.H.S.} &= [\mathbf{a}\ \mathbf{b}\ \mathbf{c}+\mathbf{d}] = (\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} + \mathbf{d}) \\
&= \mathbf{p} \cdot (\mathbf{c} + \mathbf{d}) && [\text{Let } \mathbf{p} = \mathbf{a} \times \mathbf{b}] \\
&= \mathbf{p} \cdot \mathbf{c} + \mathbf{p} \cdot \mathbf{d} && [\text{Using (i)}] \\
&= (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} + (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{d} && [\because\ \mathbf{p} = \mathbf{a} \times \mathbf{b}] \\
&= [\mathbf{a}\ \mathbf{b}\ \mathbf{c}] + [\mathbf{a}\ \mathbf{b}\ \mathbf{d}] = \text{R.H.S.}
\end{aligned}$$

Hence, distributive laws holds for scalar triple product.

• 12.5. EXPRESSION FOR THE SCALAR TRIPLE PRODUCT IN TERMS OF THE COMPONENTS OF THE VECTORS

Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be three non-zero vectors and having the components (a_1, a_2, a_3), (b_1, b_2, b_3) and (c_1, c_2, c_3) respectively, such that

$$\therefore \quad \mathbf{a} = a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}$$

$$\mathbf{b} = b_1 \hat{i} + b_2 \hat{j} + b_3 \hat{k}$$

$$\mathbf{c} = c_1 \hat{i} + c_2 \hat{j} + c_3 \hat{k}$$

where $\hat{i}$, $\hat{j}$, $\hat{k}$ are the unit vectors along the rectangular co-ordinates axes respectively.

$$\text{Then} \qquad [\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \qquad \text{...(1)}$$

• 12.6. VECTOR TRIPLE PRODUCT

Definition. *If* $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ *are three non-zero vectors, then the triple product* $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ *which gives a vector quantity is called* ***vector triple product*** *of the vectors* $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$.

(i)*To prove that* $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\, \mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\, \mathbf{c}$.

Let us suppose $\mathbf{r} = \mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ and let $\mathbf{b} \times \mathbf{c} = \mathbf{p}$.

By the definition, the vector $\mathbf{b} \times \mathbf{c}$ is perpendicular to the plane of vectors $\mathbf{b}$ and $\mathbf{c}$ both. Therefore $\mathbf{p}$ is perpendicular to the plane of $\mathbf{b}$ and $\mathbf{c}$.

$$\therefore \quad \mathbf{r} = \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{a} \times \mathbf{p} \qquad (\because\ \mathbf{p} = \mathbf{b} \times \mathbf{c})$$

Therefore, the vector $\mathbf{r}$ is perpendicular to both the vectors $\mathbf{a}$ and $\mathbf{p}$. Since $\mathbf{p}$ is perpendicular to both $\mathbf{b}$ and $\mathbf{c}$ so the vector $\mathbf{r}$ is lying in the plane containing the vectors $\mathbf{b}$ and $\mathbf{c}$. Thus the vector $\mathbf{r}$ can be written as the linear combination of the vectors $\mathbf{b}$ and $\mathbf{c}$

$$\therefore \quad \mathbf{r} = m\mathbf{b} + n\mathbf{c} \qquad \text{...(1)}$$

where m and n both are scalars. Now taking the dot product of both sides of (1) with the vector $\mathbf{a}$, we have

$$\begin{aligned}
\mathbf{r} \cdot \mathbf{a} &= (m\mathbf{b} + n\mathbf{c}) \cdot \mathbf{a} \\
&= m\,(\mathbf{b} \cdot \mathbf{a}) + n\,(\mathbf{c} \cdot \mathbf{a}) && (\because\ \text{Scalar product is distributive})
\end{aligned}$$

Since the vector $\mathbf{r}$ is perpendicular to $\mathbf{a}$

$$\therefore \quad \mathbf{r} \,.\, \mathbf{a} = 0.$$

$$\therefore \quad m\,(\mathbf{b} \cdot \mathbf{a}) + n\,(\mathbf{c} \cdot \mathbf{a}) = 0$$

$$\text{or} \quad \frac{m}{\mathbf{c} \cdot \mathbf{a}} = -\frac{n}{\mathbf{b} \,.\, \mathbf{a}} = \lambda \text{ (say)} \qquad \text{...(2)}$$

Substitute the values of m and n from (2) in (1), we get

$$\mathbf{r} = \lambda\,(\mathbf{c} \cdot \mathbf{a})\,\mathbf{b} + \{-\lambda\,(\mathbf{b} \cdot \mathbf{a})\,\mathbf{c}\}$$

$$\mathbf{r} = \lambda\,[(\mathbf{c}\cdot\mathbf{a})\,\mathbf{b} - (\mathbf{b}\cdot\mathbf{a})\,\mathbf{c}]. \qquad ...(3)$$

Now we have to determine the value of λ. So let us consider the unit vector $\hat{j}$ and $\hat{k}$ such that $\hat{j}$ is parallel to the vector **b** and $\hat{k}$ is perpendicular to the vector **b** in the plane of **b** and **c**, we have

$$\mathbf{b} = b_2\,\hat{j} \quad \text{and} \quad \mathbf{c} = c_2\,\hat{j} + c_3\,\hat{k}.$$

Also the vector $\quad \mathbf{a} = a_1\,\hat{i} + a_2\,\hat{j} + a_3\,\hat{k}.$

$$\begin{aligned}
\therefore \quad \mathbf{r} &= \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \\
&= (a_1\,\hat{i} + a_2\,\hat{j} + a_3\,\hat{k}) \times [b_2\,\hat{j} \times (c_2\,\hat{j} + c_3\,\hat{k})] \\
&= (a_1\,\hat{i} + a_2\,\hat{j} + a_3\,\hat{k}) \times (b_2c_3\,\hat{j} \times \hat{k}) \\
&= (a_1\,\hat{i} + a_2\,\hat{j} + a_3\,\hat{k}) \times b_2c_3\,\hat{i} \qquad (\because\ \hat{j} \times \hat{k} = \hat{i}) \\
&= a_2b_2c_3\,\hat{j} \times \hat{i} + a_3b_2c_3\,\hat{k} \times \hat{i} = a_3b_2c_3\,\hat{j} - a_2b_2c_3\,\hat{k}
\end{aligned}$$

and $\quad (\mathbf{c}\cdot\mathbf{a})\,\mathbf{b} - (\mathbf{b}\cdot\mathbf{a})\,\mathbf{c} = (a_2c_2 + a_3c_3)\,b_2\,\hat{j} - a_2b_2\,(c_2\,\hat{j} + c_3\,\hat{k})$

$$= a_3b_2c_3\,\hat{j} - a_2b_2c_3\,\hat{k}.$$

Substitute these values in (3), we get

$$\lambda = 1.$$

On putting $\lambda = 1$ in (3), we get

$$\mathbf{r} = (\mathbf{c}\cdot\mathbf{a})\,\mathbf{b} - (\mathbf{b}\cdot\mathbf{a})\,\mathbf{c}.$$

Since $\quad \mathbf{r} = \mathbf{a} \times (\mathbf{b} \times \mathbf{c})$

Hence $\quad \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{c}\cdot\mathbf{a})\,\mathbf{b} - (\mathbf{b}\cdot\mathbf{a})\,\mathbf{c}$

or $\quad \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\,\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\,\mathbf{c} \qquad$ (Dot are commutative)

(ii) *To prove that the vector triple product is* ***not associative.***

Let **a**, **b**, **c** be three non-zero vectors, then we know that

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\,\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\,\mathbf{c}. \qquad ...(1)$$

Now $\quad (\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = -\,\mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = -\,[(\mathbf{c}\cdot\mathbf{b})\,\mathbf{a} - (\mathbf{c}\cdot\mathbf{a})\,\mathbf{b}] \qquad$ [From (i)]

$\therefore \quad (\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (\mathbf{c}\cdot\mathbf{a})\,\mathbf{b} - (\mathbf{c}\cdot\mathbf{b})\,\mathbf{a}. \qquad ...(2)$

From (1) and (2), we get

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}.$$

Hence, *vector triple product is not associative.*

• 12.7. QUADRUPLE PRODUCTS

Definition. *The product involving four non-zero vectors is called* ***quadruple product*** *of these four vectors.*

There are two quadruple product :

(i) Scalar quadruple product.

(ii) Vector quadruple product.

• 12.8. SCALAR QUADRUPLE PRODUCT

Definition. *If* **a, b, c, d** *are four non-zero vectors, then the product* $(\mathbf{a} \times \mathbf{b})\cdot(\mathbf{c} \times \mathbf{d})$, *which gives a scalar quantity, is called scalar product of four vectors or* ***scalar quadruple product.***

(i) *To prove that* $(\mathbf{a} \times \mathbf{b})\cdot(\mathbf{c} \times \mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})\,(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{b}\cdot\mathbf{c})\,(\mathbf{a}\cdot\mathbf{d}).$

Let $\mathbf{p} = \mathbf{a} \times \mathbf{b}$, then

$$\begin{aligned}
(\mathbf{a} \times \mathbf{b})\cdot(\mathbf{c} \times \mathbf{d}) &= \mathbf{p}\cdot(\mathbf{c} \times \mathbf{d}) \\
&= (\mathbf{p} \times \mathbf{c})\cdot\mathbf{d} \qquad \text{(In a scalar triple product the dot and cross can be interchanged)} \\
&= [(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}]\cdot\mathbf{d} \qquad (\because\ \mathbf{p} = \mathbf{a} \times \mathbf{b}) \\
&= [(\mathbf{c}\cdot\mathbf{a})\,\mathbf{b} - (\mathbf{c}\cdot\mathbf{b})\,\mathbf{a}]\cdot\mathbf{d} = (\mathbf{c}\cdot\mathbf{a})\,(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{c}\cdot\mathbf{b})\,(\mathbf{a}\cdot\mathbf{d}) \\
&= (\mathbf{a}\cdot\mathbf{c})\,(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{b}\cdot\mathbf{c})\,(\mathbf{a}\cdot\mathbf{d})
\end{aligned}$$

$$\therefore \quad (\mathbf{a} \times \mathbf{b})\cdot(\mathbf{c} \times \mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})\,(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{b}\cdot\mathbf{c})\,(\mathbf{a}\cdot\mathbf{d}).$$

REMARK

➤ $$(\mathbf{a} \times \mathbf{b})\cdot(\mathbf{c} \times \mathbf{d}) = \begin{vmatrix} \mathbf{a}\cdot\mathbf{c} & \mathbf{b}\cdot\mathbf{c} \\ \mathbf{a}\cdot\mathbf{d} & \mathbf{b}\cdot\mathbf{d} \end{vmatrix}.$$

• 12.9. VECTOR QUADRUPLE PRODUCT

Definition. *If* **a, b, c** *and* **d** *are four vectors, then the product of the type* $(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d})$ *which gives a vector quantity is called* **vector quadruple product** *of four vectors.*

To prove that :

(i) $(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = [\mathbf{a}\ \mathbf{b}\ \mathbf{d}]\,\mathbf{c} - [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]\,\mathbf{d}$.

(ii) $(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = [\mathbf{a}\ \mathbf{c}\ \mathbf{d}]\,\mathbf{b} - [\mathbf{b}\ \mathbf{c}\ \mathbf{d}]\,\mathbf{a}$.

(i) Let $\mathbf{a} \times \mathbf{b} = \mathbf{p}$, then

$$\begin{aligned}(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) &= \mathbf{p} \times (\mathbf{c} \times \mathbf{d}) = (\mathbf{p} \cdot \mathbf{d})\,\mathbf{c} - (\mathbf{p} \cdot \mathbf{c})\,\mathbf{d} \\ &= (\mathbf{a} \times \mathbf{b} \cdot \mathbf{d})\,\mathbf{c} - (\mathbf{a} \times \mathbf{b} \cdot \mathbf{c})\,\mathbf{d} \\ &= [\mathbf{a}\ \mathbf{b}\ \mathbf{d}]\,\mathbf{c} - [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]\,\mathbf{d}.\end{aligned}$$

(ii) Let $\mathbf{c} \times \mathbf{d} = \mathbf{q}$, then

$$\begin{aligned}(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) &= (\mathbf{a} \times \mathbf{b}) \times \mathbf{q} = -\,\mathbf{q} \times (\mathbf{a} \times \mathbf{b}) \\ &= -\,[(\mathbf{q} \cdot \mathbf{b})\,\mathbf{a} - (\mathbf{q} \cdot \mathbf{a})\,\mathbf{b}] = (\mathbf{q} \cdot \mathbf{a})\,\mathbf{b} - (\mathbf{q} \cdot \mathbf{b})\,\mathbf{a} \\ &= (\mathbf{c} \times \mathbf{d} \cdot \mathbf{a})\,\mathbf{b} - (\mathbf{c} \times \mathbf{d} \cdot \mathbf{b})\,\mathbf{a} = [\mathbf{c}\ \mathbf{d}\ \mathbf{a}]\,\mathbf{b} - [\mathbf{c}\ \mathbf{d}\ \mathbf{b}]\,\mathbf{a}.\end{aligned}$$

Corollary. *Prove that*

$$[\mathbf{b}\ \mathbf{c}\ \mathbf{d}]\,\mathbf{a} - [\mathbf{a}\ \mathbf{c}\ \mathbf{d}]\,\mathbf{b} + [\mathbf{a}\ \mathbf{b}\ \mathbf{d}]\,\mathbf{c} - [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]\,\mathbf{d} = \mathbf{0}.$$

From (i) and (ii), we have

$$(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = [\mathbf{a}\ \mathbf{b}\ \mathbf{d}]\,\mathbf{c} - [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]\,\mathbf{d} \qquad \text{...(1)}$$

and $$(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = [\mathbf{a}\ \mathbf{c}\ \mathbf{d}]\,\mathbf{b} - [\mathbf{b}\ \mathbf{c}\ \mathbf{d}]\,\mathbf{a}. \qquad \text{...(2)}$$

From equation (1) and (2), we get

$$[\mathbf{b}\ \mathbf{c}\ \mathbf{d}]\,\mathbf{a} - [\mathbf{a}\ \mathbf{c}\ \mathbf{d}]\,\mathbf{b} + [\mathbf{a}\ \mathbf{b}\ \mathbf{d}]\,\mathbf{c} - [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]\,\mathbf{d} = \mathbf{0}.$$

• 12.10. SYSTEM OF RECIPROCAL VECTORS

Definition. *If* **a, b, c** *are three non-coplanar vectors, then the three vectors* **a′, b′, c′** *are called system of* ***reciprocal vectors*** *of the vectors* **a, b, c** *if*

$$\mathbf{a}' = \frac{\mathbf{b} \times \mathbf{c}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]},\ \mathbf{b}' = \frac{\mathbf{c} \times \mathbf{a}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]},\ \mathbf{c}' = \frac{\mathbf{a} \times \mathbf{b}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}.$$

Properties of Reciprocal system of Vectors :

Property I. *If* **a′, b′, c′** *are reciprocal vectors to the vectors* **a, b, c** *respectively, then*

$$\mathbf{a} \cdot \mathbf{a}' = \mathbf{b} \cdot \mathbf{b}' = \mathbf{c} \cdot \mathbf{c}' = 1.$$

Since we know that **a′, b′, c′** are defined by the following equations

$$\mathbf{a}' = \frac{\mathbf{b} \times \mathbf{c}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]},\ \mathbf{b}' = \frac{\mathbf{c} \times \mathbf{a}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]},\ \mathbf{c}' = \frac{\mathbf{a} \times \mathbf{b}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}.$$

$$\therefore \qquad \mathbf{a} \cdot \mathbf{a}' = \mathbf{a} \cdot \frac{\mathbf{b} \times \mathbf{c}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]} = \frac{\mathbf{a} \cdot \mathbf{b} \times \mathbf{c}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]} = \frac{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]} = 1.$$

Similarly $\quad \mathbf{b} \cdot \mathbf{b}' = 1,\ \mathbf{c} \cdot \mathbf{c}' = 1$

Hence, $\quad \mathbf{a} \cdot \mathbf{a}' = \mathbf{b} \cdot \mathbf{b}' = \mathbf{c} \cdot \mathbf{c}' = 1.$

Property II. *The scalar product of any two vectors, one from each reciprocal system of vectors is zero i.e.,*

$$\mathbf{a} \cdot \mathbf{b}' = \mathbf{a} \cdot \mathbf{c}' = \mathbf{b} \cdot \mathbf{a}' = \mathbf{b} \cdot \mathbf{c}' = \mathbf{c} \cdot \mathbf{a}' = \mathbf{c} \cdot \mathbf{b}' = \mathbf{0}.$$

Since we know that

$$\mathbf{a}' = \frac{\mathbf{b} \times \mathbf{c}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]},\ \mathbf{b}' = \frac{\mathbf{c} \times \mathbf{a}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]},\ \mathbf{c}' = \frac{\mathbf{a} \times \mathbf{b}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}.$$

$$\therefore \qquad \mathbf{a} \cdot \mathbf{b}' = \mathbf{a} \cdot \frac{\mathbf{c} \times \mathbf{a}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]} = \frac{\mathbf{a} \cdot \mathbf{c} \times \mathbf{a}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]} = \frac{[\mathbf{a}\ \mathbf{c}\ \mathbf{a}]}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]} = 0 \qquad (\because\ [\mathbf{a}\ \mathbf{c}\ \mathbf{a}] = 0)$$

Similarly we can show that

$$\mathbf{a} \cdot \mathbf{c}' = \mathbf{b} \cdot \mathbf{a}' = \mathbf{b} \cdot \mathbf{c}' = \mathbf{c} \cdot \mathbf{a}' = \mathbf{c} \cdot \mathbf{b}' = 0.$$

Property III. *The scalar triple product of* **a′, b′, c′** *is reciprocal to the scalar triple product of* **a, b, c.**

Since we have

$$\mathbf{a}' = \frac{\mathbf{b} \times \mathbf{c}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]},\ \mathbf{b}' = \frac{\mathbf{c} \times \mathbf{a}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]},\ \mathbf{c}' = \frac{\mathbf{a} \times \mathbf{b}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}.$$

$$\therefore \qquad [\mathbf{a}'\ \mathbf{b}'\ \mathbf{c}'] = \mathbf{a}' \cdot (\mathbf{b}' \times \mathbf{c}')$$

$$= \frac{\mathbf{b}\times\mathbf{c}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}\cdot\left\{\frac{(\mathbf{c}\times\mathbf{a})}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}\times\frac{(\mathbf{a}\times\mathbf{b})}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}\right\}$$

$$[\mathbf{a}'\ \mathbf{b}'\ \mathbf{c}'] = \frac{(\mathbf{b}\times\mathbf{c})\cdot[(\mathbf{c}\times\mathbf{a})\times(\mathbf{a}\times\mathbf{b})]}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]^3}. \qquad ...(1)$$

Now $(\mathbf{c}\times\mathbf{a})\times(\mathbf{a}\times\mathbf{b}) = [\mathbf{c}\ \mathbf{a}\ \mathbf{b}]\,\mathbf{a} - [\mathbf{c}\ \mathbf{a}\ \mathbf{a}]\,\mathbf{b}$ (Using vector quadruple product)

$= [\mathbf{c}\ \mathbf{a}\ \mathbf{b}]\,\mathbf{a}.$ $(\because\ [\mathbf{c}\ \mathbf{a}\ \mathbf{a}] = 0)$

Substituting this value in (1), we get

$$[\mathbf{a}'\ \mathbf{b}'\ \mathbf{c}'] = \frac{(\mathbf{b}\times\mathbf{c})\cdot[\mathbf{c}\ \mathbf{a}\ \mathbf{b}]\,\mathbf{a}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]^3} = \frac{[\mathbf{c}\ \mathbf{a}\ \mathbf{b}]\,(\mathbf{b}\times\mathbf{c}\cdot\mathbf{a})}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]^3}$$

$$= \frac{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]^2}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]^3} \qquad (\because\ [\mathbf{c}\ \mathbf{a}\ \mathbf{b}] = [\mathbf{b}\ \mathbf{c}\ \mathbf{a}] = [\mathbf{a}\ \mathbf{b}\ \mathbf{c}])$$

$$= \frac{1}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}.$$

$\therefore\ [\mathbf{a}'\ \mathbf{b}'\ \mathbf{c}']\,[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = 1.$

• SOLVED EXAMPLES

Example 1. *Find the volume of the parallelopiped whose edges are represented by the vectors*

(*i*) $\mathbf{a} = 2\hat{i} - 3\hat{j} + 4\hat{k},\ \mathbf{b} = \hat{i} + 2\hat{j} - \hat{k},\ \mathbf{c} = 3\hat{i} - \hat{j} + 2\hat{k}$

or $(2, -3, 4), (1, 2, -1)$ *and* $(3, -1, 2)$.

(*ii*) $\mathbf{a} = 2\hat{i} - 4\hat{j} + 5\hat{k},\ \mathbf{b} = \hat{i} - \hat{j} + \hat{k},\ \mathbf{c} = 3\hat{i} - 5\hat{j} + 2\hat{k}.$

Solution. The volume of the parallelopiped whose edges are **a**, **b**, **c** is equal to the magnitude of $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$.

(i) $V = [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$

$$= \begin{vmatrix} 2 & -3 & 4 \\ 1 & 2 & -1 \\ 3 & -1 & 2 \end{vmatrix} = 2(4-1) - 3(-3-2) + 4(-1-6)$$

$$= 6 + 15 - 28 = -7.$$

$\therefore$ $V = 7$ (Numerically).

(ii) $$V = \begin{vmatrix} 2 & -4 & 5 \\ 1 & -1 & 1 \\ 3 & -5 & 2 \end{vmatrix}$$

$$= 2(-2+5) - 4(3-2) + 5(-5+3) = 6 - 4 - 10 = -8$$

$\therefore$ $V = 8$ (Numerically).

Example 2. *Prove that the four points* $4\hat{i} + 5\hat{j} + \hat{k},\ -(\hat{j} + \hat{k}),\ (3\hat{i} + 9\hat{j} + 4\hat{k})$ *and* $4(-\hat{i} + \hat{j} + \hat{k})$ *are coplanar.*

Solution. Let A, B, C and D be the four points and O be the origin.

Then $\overrightarrow{OA} = 4\hat{i} + 5\hat{j} + \hat{k},\ \overrightarrow{OB} = -\hat{j} - \hat{k},\ \overrightarrow{OC} = 3\hat{i} + 9\hat{j} + 4\hat{k}$

and $\overrightarrow{OD} = -4\hat{i} + 4\hat{j} + 4\hat{k}.$

$\therefore$ $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = -4\hat{i} - 6\hat{j} - 2\hat{k}$

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = (3\hat{i} + 9\hat{j} + 4\hat{k}) - (4\hat{i} + 5\hat{j} + \hat{k}) = -\hat{i} + 4\hat{j} + 3\hat{k}$

and $\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA} = (-4\hat{i} + 4\hat{j} + 4\hat{k}) - (4\hat{i} + 5\hat{j} + \hat{k}) = -3\hat{i} - \hat{j} + 3\hat{k}.$

Now find $[\overrightarrow{AB}\ \overrightarrow{AC}\ \overrightarrow{AD}]$

$$\therefore\ [\overrightarrow{AB}\ \overrightarrow{AC}\ \overrightarrow{AD}] = \begin{vmatrix} -4 & -6 & -2 \\ -1 & 4 & 3 \\ -8 & -1 & 3 \end{vmatrix}$$

$$= -4(12+3) - 6(-24+3) - 2(1+32)$$

$$= -60 + 126 - 66 = -126 + 126 = 0.$$

Hence, $\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}$ are coplanar and *hence the four given points are coplanar.*

Example 3. *Prove that* $[\mathbf{a} + \mathbf{b}, \mathbf{b} + \mathbf{c}, \mathbf{c} + \mathbf{a}] = 2\,[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$.

Solution. Taking L.H.S.

$\therefore \quad$ L.H.S. $= [\mathbf{a}+\mathbf{b}, \mathbf{b}+\mathbf{c}, \mathbf{c}+\mathbf{a}]$

$$= (\mathbf{a}+\mathbf{b}) \cdot [(\mathbf{b}+\mathbf{c}) \times (\mathbf{c}+\mathbf{a})]$$
$$= (\mathbf{a}+\mathbf{b}) \cdot [\mathbf{b}\times\mathbf{c} + \mathbf{b}\times\mathbf{a} + \mathbf{c}\times\mathbf{c} + \mathbf{c}\times\mathbf{a}]$$
$$= (\mathbf{a}+\mathbf{b}) \cdot [\mathbf{b}\times\mathbf{c} + \mathbf{b}\times\mathbf{a} + \mathbf{c}\times\mathbf{a}] \qquad (\because \ \mathbf{c}\times\mathbf{c} = 0)$$
$$= (\mathbf{a}+\mathbf{b}) \cdot (\mathbf{b}\times\mathbf{c}) + (\mathbf{a}+\mathbf{b}) \cdot (\mathbf{b}\times\mathbf{a}) + (\mathbf{a}+\mathbf{b}) \cdot (\mathbf{c}\times\mathbf{a})$$
$$= \mathbf{a}\cdot\mathbf{b}\times\mathbf{c} + \mathbf{b}\cdot\mathbf{b}\times\mathbf{c} + \mathbf{a}\cdot\mathbf{b}\times\mathbf{a} + \mathbf{b}\cdot\mathbf{b}\times\mathbf{a} + \mathbf{a}\cdot\mathbf{c}\times\mathbf{a} + \mathbf{b}\cdot\mathbf{c}\times\mathbf{a}$$
$$= [\mathbf{a}\ \mathbf{b}\ \mathbf{c}] + [\mathbf{b}\ \mathbf{b}\ \mathbf{c}] + [\mathbf{a}\ \mathbf{b}\ \mathbf{a}] + [\mathbf{b}\ \mathbf{b}\ \mathbf{a}] + [\mathbf{a}\ \mathbf{c}\ \mathbf{a}] + [\mathbf{b}\ \mathbf{c}\ \mathbf{a}]$$
$$= [\mathbf{a}\ \mathbf{b}\ \mathbf{c}] + [\mathbf{b}\ \mathbf{c}\ \mathbf{a}] \qquad (\because \ [\mathbf{b}\ \mathbf{b}\ \mathbf{c}] = 0, [\mathbf{a}\ \mathbf{b}\ \mathbf{a}] = 0, [\mathbf{b}\ \mathbf{b}\ \mathbf{a}] = 0, [\mathbf{a}\ \mathbf{c}\ \mathbf{a}] = 0)$$
$$= [\mathbf{a}\ \mathbf{b}\ \mathbf{c}] + [\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = 2\,[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] \qquad (\because \ [\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = [\mathbf{b}\ \mathbf{c}\ \mathbf{a}])$$
$$= \text{R.H.S.}$$

Hence, $[\mathbf{a}+\mathbf{b}, \mathbf{b}+\mathbf{c}, \mathbf{c}+\mathbf{a}] = 2\,[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$.

Example 4. *Show that* $[\mathbf{a}\times\mathbf{b}, \mathbf{b}\times\mathbf{c}, \mathbf{c}\times\mathbf{a}] = [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]^2$.

Solution. Since, we have

$$[\mathbf{a}\times\mathbf{b},\ \mathbf{b}\times\mathbf{c},\ \mathbf{c}\times\mathbf{a}] = (\mathbf{a}\times\mathbf{b}) \cdot [(\mathbf{b}\times\mathbf{c}) \times (\mathbf{c}\times\mathbf{a})] \qquad \text{...(1)}$$

Now consider first $(\mathbf{b}\times\mathbf{c}) \times (\mathbf{c}\times\mathbf{a})$.

Let $\mathbf{b}\times\mathbf{c} = \mathbf{p}$, then

$$(\mathbf{b}\times\mathbf{c}) \times (\mathbf{c}\times\mathbf{a}) = \mathbf{p} \times (\mathbf{c}\times\mathbf{a}) = (\mathbf{p}\cdot\mathbf{a})\,\mathbf{c} - (\mathbf{p}\cdot\mathbf{c})\,\mathbf{a}$$
$$= [(\mathbf{b}\times\mathbf{c})\cdot\mathbf{a}]\,\mathbf{c} - [(\mathbf{b}\times\mathbf{c})\cdot\mathbf{c}]\,\mathbf{a}$$
$$= [\mathbf{b}\ \mathbf{c}\ \mathbf{a}]\,\mathbf{c} - [\mathbf{b}\ \mathbf{c}\ \mathbf{c}]\,\mathbf{a}\ (\mathbf{b}\times\mathbf{c}) \times (\mathbf{c}\times\mathbf{a}) = [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]\,\mathbf{c} \qquad \text{...(2)}$$
$$(\because \ [\mathbf{b}\ \mathbf{c}\ \mathbf{c}] = 0)$$

Now from (1) and (2)

$$[(\mathbf{a}\times\mathbf{b})\,(\mathbf{b}\times\mathbf{c})\,(\mathbf{c}\times\mathbf{a})] = (\mathbf{a}\times\mathbf{b}) \cdot \{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]\,\mathbf{c}\} = [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]\,[(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}]$$
$$= [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]\,[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]^2.$$

Hence $[\mathbf{a}\times\mathbf{a}\ \mathbf{b}\times\mathbf{c}\ \mathbf{c}\times\mathbf{a}] = [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]^2$.

Example 5. *Prove that* $\hat{i}\times(\mathbf{a}\times\hat{i}) + \hat{j}\times(\mathbf{a}\times\hat{j}) + \hat{k}\times(\mathbf{a}\times\hat{k}) = 2\mathbf{a}$.

Solution. Since, we have

$$\hat{i}\times(\mathbf{a}\times\hat{i}) = (\hat{i}\cdot\hat{i})\,\mathbf{a} - (\hat{i}\cdot\mathbf{a})\,\hat{i} \qquad \text{...(1)}$$
$$\hat{j}\times(\mathbf{a}\times\hat{j}) = (\hat{j}\cdot\hat{j})\,\mathbf{a} - (\hat{j}\cdot\mathbf{a})\,\hat{j} \qquad \text{...(2)}$$

and
$$\hat{k}\times(\mathbf{a}\times\hat{k}) = (\hat{k}\cdot\hat{k})\,\mathbf{a} - (\hat{k}\cdot\mathbf{a})\,\hat{k}. \qquad \text{...(3)}$$

Adding (1), (2), (3), we get

$$\hat{i}\times(\mathbf{a}\times\hat{i}) + \hat{j}\times(\mathbf{a}\times\hat{j}) + \hat{k}\times(\mathbf{a}\times\hat{k})$$
$$= 3\mathbf{a} - (\hat{i}\cdot\mathbf{a})\,\hat{i} - (\hat{j}\cdot\mathbf{a})\,\hat{j} - (\hat{k}\cdot\mathbf{a})\,\hat{k} \qquad \text{...(4)}$$
$$(\because \ \hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1)$$

Further, let $\quad \mathbf{a} = a_1\,\hat{i} + a_2\,\hat{j} + a_3\,\hat{k}$.

$\therefore \quad \hat{i}\cdot\mathbf{a} = a_1,\ \hat{j}\cdot\mathbf{a} = a_2,\ \hat{k}\cdot\mathbf{a} = a_3.$

$\therefore \quad (\hat{i}\cdot\mathbf{a})\,\hat{i} + (\hat{j}\cdot\mathbf{a})\,\hat{j} + (\hat{k}\cdot\mathbf{a})\,\hat{k} = a_1\,\hat{i} + a_2\,\hat{j} + a_3\,\hat{k} = \mathbf{a}. \qquad$ ...(5)

From (4) and (5), we get

$$\hat{i}\times(\mathbf{a}\times\hat{i}) + \hat{j}\times(\mathbf{a}\times\hat{j}) + \hat{k}\times(\mathbf{a}\times\hat{k}) = 3\mathbf{a} - \mathbf{a} = 2\mathbf{a}.$$

Hence proved the result.

Example 6. *Find a set of vectors reciprocal to the three given vectors*

$$\mathbf{a} = -\hat{i} + \hat{j} + \hat{k},\ \mathbf{b} = \hat{i} - \hat{j} + \hat{k},\ \mathbf{c} = \hat{i} + \hat{j} + \hat{k}.$$

Solution. Let $\mathbf{a}', \mathbf{b}', \mathbf{c}'$ are the vectors reciprocal to the vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$, then

$$\mathbf{b}\times\mathbf{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= \hat{i}(-1-1) + \hat{j}(1-1) + \hat{k}(1+1) = -2\hat{i} + 2\hat{k}$$

$$\mathbf{c} \times \mathbf{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ -1 & 1 & 1 \end{vmatrix}$$

$$= \hat{i}(1-1) + \hat{j}(-1-1) + \hat{k}(1+1) = -2\hat{j} + 2\hat{k}$$

and
$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{vmatrix}$$

$$= \hat{i}(1+1) + \hat{j}(1+1) + \hat{k}(1-1) = 2\hat{i} + 2\hat{j}$$

and
$$[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = -1(-1-1) + 1(1-1) + 1(1+1) = 2 + 2 = 4.$$

$\therefore$
$$\mathbf{a}' = \frac{\mathbf{b} \times \mathbf{c}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]} = \frac{-2\hat{i} + 2\hat{k}}{4} = \frac{1}{2}(-\hat{i} + \hat{k})$$

$$\mathbf{b}' = \frac{\mathbf{c} \times \mathbf{a}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]} = \frac{(-2\hat{j} + 2\hat{k})}{4} = \frac{1}{2}(-\hat{j} + \hat{k})$$

$$\mathbf{c}' = \frac{\mathbf{a} \times \mathbf{b}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]} = \frac{2\hat{i} + 2\hat{j}}{4} = \frac{1}{2}(\hat{i} + \hat{j}).$$

• STUDENT'S ACTIVITY

1. Prove that

$$[\bar{a} + \bar{b}\ \ \bar{b} + \bar{c}\ \ \bar{c} + \bar{a}] = 2\,[\bar{a}\ \bar{b}\ \bar{c}]$$

2. If $\bar{a}', \bar{b}', \bar{c}'$ are the vectors reciprocal to the vectors $\bar{a}, \bar{b}, \bar{c}$, then prove that

$$\bar{a}' \times \bar{b}' + \bar{b}' \times \bar{c}' + \bar{c}' \times \bar{a}' = \frac{\bar{a} + \bar{b} + \bar{c}}{[\bar{a}\, \bar{b}\, \bar{c}]}$$

• SUMMARY

- $\bar{a} \cdot (\bar{b} \times \bar{c})$ = Scalar triple product.
- $\bar{a} \times (\bar{b} \times \bar{c})$ = Vector triple product.
- $(\bar{a} \times \bar{b}) \cdot (\bar{c} \times \bar{d})$ = Scalar quadruple product.
- $(\bar{a} \times \bar{b}) \times (\bar{c} \times \bar{d})$ = Vector quadruple product.
- Reciprocal vectors

$$\bar{a}' = \frac{\bar{b} \times \bar{c}}{[\bar{a}\,\bar{b}\,\bar{c}]},\quad \bar{b}' = \frac{\bar{c} \times \bar{a}}{[\bar{a}\,\bar{b}\,\bar{c}]},\quad \bar{c}' = \frac{\bar{a} \times \bar{b}}{[\bar{a}\,\bar{b}\,\bar{c}]}$$

• TEST YOURSELF

1. Prove that $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$ for any vectors **a, b** and **c**.
2. (i) Show that the vectors $\hat{i} + 2\hat{j} + \hat{k},\ 3\hat{i} + 2\hat{j} - 7\hat{k}$ and $5\hat{i} + 6\hat{j} - 5\hat{k}$ are coplanar.

   (ii) Find the value of the constant λ such that the vectors $\mathbf{a} = 2\hat{i} - \hat{j} + \hat{k}$, $\mathbf{b} = \hat{i} + 2\hat{j} - 3\hat{k}$, $\mathbf{c} = 3\hat{i} + \lambda\hat{j} + 5\hat{k}$ are coplanar.
3. Prove that $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$.
4. Prove that $[\hat{i} - \hat{j}, \hat{j} - \hat{k}, \hat{k} - \hat{i}] = 0$.
5. (i) Find the volume of a parallelopiped whose edges are represented by $\mathbf{a} = \hat{i} - 2\hat{j} + 3\hat{k}$, $\mathbf{b} = 2\hat{i} + \hat{j} - \hat{k}$, $\mathbf{c} = \hat{j} + \hat{k}$.

   (ii) Evaluate $(2\hat{i} - 3\hat{j}) \cdot [(\hat{i} + \hat{j} - \hat{k}) \times (3\hat{i} - \hat{k})]$.
6. If $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$, $\overrightarrow{OC} = \mathbf{c}$, then prove that $\mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a}$ is a vector perpendicular to the plane ABC.
7. Find the value of $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$, where $\mathbf{a} = \hat{i} - 2\hat{j} + \hat{k}$, $\mathbf{b} = 2\hat{i} + \hat{j} + \hat{k}$, $\mathbf{c} = \hat{i} + 2\hat{j} - \hat{k}$.
8. Show that the vectors $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$, $\mathbf{b} \times (\mathbf{c} \times \mathbf{a})$, $\mathbf{c} \times (\mathbf{a} \times \mathbf{b})$ are coplanar.
9. Prove that $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ if and only if $(\mathbf{c} \times \mathbf{a}) \times \mathbf{b} = 0$.
10. Prove that $\mathbf{a} \times (\mathbf{b} \times \mathbf{a}) = (\mathbf{a} \times \mathbf{b}) \times \mathbf{a}$.
11. Prove that $[\mathbf{a}'\ \mathbf{b}'\ \mathbf{c}']\,[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = 1$, where $\mathbf{a}', \mathbf{b}', \mathbf{c}'$ are the vectors reciprocal to the vectors **a, b, c**.
12. Find a set of vectors reciprocal to the vectors

$$\mathbf{a} = 2\hat{i} + 3\hat{j} - \hat{k},\ \mathbf{b} = \hat{i} - \hat{j} - 2\hat{k},\ \mathbf{c} = -\hat{i} + 2\hat{j} + 2\hat{k}.$$

13. If $\mathbf{a}', \mathbf{b}', \mathbf{c}'$ are the vectors reciprocal to the vectors **a, b, c** then show that

    (i) $\mathbf{a} \times \mathbf{a}' + \mathbf{b} \times \mathbf{b}' + \mathbf{c} \times \mathbf{c}' = \mathbf{0}$.

    (ii) $\mathbf{a} = \dfrac{\mathbf{b}' \times \mathbf{c}'}{[\mathbf{a}'\ \mathbf{b}'\ \mathbf{c}']},\ \mathbf{b} = \dfrac{\mathbf{c}' \times \mathbf{a}'}{[\mathbf{a}'\ \mathbf{b}'\ \mathbf{c}']},\ \mathbf{c} = \dfrac{\mathbf{a}' \times \mathbf{b}'}{[\mathbf{a}'\ \mathbf{b}'\ \mathbf{c}']}$.

ANSWERS

2. (ii) $\lambda = -4$.
5. (i) 12 cubic unit (ii) 4 cubic unit
7. $-9\hat{i} - 6\hat{j} - 3\hat{k}$.
12. $\frac{1}{3}(2\hat{i} + \hat{k}),\ \frac{1}{3}(-8\hat{i} + 3\hat{j} - 7\hat{k}),\ \frac{1}{3}(-7\hat{i} + 3\hat{j} - 5\hat{k})$.

OBJECTIVE EVALUATION

➤ FILL IN THE BLANKS :

1. The vector **a** is perpendicular to ....
2. If a vector **a** is parallel to the vector **b**, then **a** equals to ....
3. $\mathbf{a} \times \mathbf{b} \cdot \mathbf{c} =$ ....
4. The coterminous edges of a parallelopiped are represented by the vectors **a, b, c** then its volume is ....

➤ TRUE OR FALSE :

Write 'T' for true and 'F' for false statement :

1. $[\hat{i}\ \hat{j}\ \hat{k}] + [\hat{j}\ \hat{k}\ \hat{i}] = 2[\hat{k}\ \hat{i}\ \hat{j}]$. *(T/F)*
2. The vectors $\hat{i} - 2\hat{j} + 3\hat{k}$, $-2\hat{i} + 3\hat{j} - 4\hat{k}$, $\hat{i} - 3\hat{j} + 5\hat{k}$ are coplanar. *(T/F)*

➤ MULTIPLE CHOICE QUESTIONS :

Choose the most appropriate one :

1. $\mathbf{a} \times \mathbf{b}$ is perpendicular to :

(a) $\mathbf{a}$ (b) $\mathbf{0}$ (c) $\mathbf{a} \cdot \mathbf{b}$ (d) None of these.

2. If θ is the angle between $\mathbf{a}$ and $\mathbf{b}$ and $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a} \cdot \mathbf{b}|$ then θ equals :

(a) 0 (b) $\pi/2$ (c) $\pi/4$ (d) $\pi/3$.

3. If $\mathbf{a} = t\,\mathbf{b}$, then $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$ equals :

(a) 1 (b) -1 (c) $\mathbf{a} + \mathbf{b}$ (d) 0.

4. If $\mathbf{a}\ \mathbf{b}\ \mathbf{c}$ are coplanar, then $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$ equals :

(a) 0 (b) 1 (c) -1 (d) None of these.

ANSWERS

Fill in the Blanks :

1. $\mathbf{a} \times \mathbf{b}$ **2.** $t\,\mathbf{b}$ **3.** $\mathbf{a} \cdot \mathbf{b} \times \mathbf{c}$ **4.** $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$

True or False :

1. T **2.** T

Multiple Choice Questions :

1. (a) **2.** (c) **3.** (d) **4.** (a)

❑❑❑

UNIT

13

DIFFERENTIATION AND INTEGRATION OF VECTORS

LEARNING OBJECTIVES

LEARNING OBJECTIVES

After going through this unit you will learn :

- About the scalar and vector functions
- How to differentiate and how to integrate the given vectors

• 13.1. SCALAR FUNCTION

Since we know that *the quantity which is associated with the magnitude but not associated with direction is known as* ***scalar quantity.*** Therefore every real number is a scalar quantity.

Let D be a subset of a set of real numbers. Then *a function f defined over the subset D such that for all $t \in D$, $f(t)$ is obtained as a scalar quantity, is called a* ***scalar function.***

• 13.2. VECTOR FUNCTION

If the scalar function $f(t)$ for all $t \in D$ is associated with some direction then this function is called a ***vector function*** *and is therefore denoted by* $\mathbf{f}(t)$ *or* $\mathbf{f}$.

Let $f_1(t), f_2(t), f_3(t)$ be three components of a vector function $\mathbf{f}(t)$, then this function can be uniquely expressed as a linear combination of these three fixed non-coplanar vectors $f_1(t)\,i$, $f_2(t)\,j$, $f_3(t)\,k$.

$$\therefore \qquad \mathbf{f}(t) = f_1(t)\,\hat{i} + f_2(t)\,\hat{j} + f_3(t)\,\hat{k}$$

where $\hat{i}, \hat{j}, \hat{k}$ are three mutually perpendicular non-coplanar unit vectors.

• 13.3. SCALAR AND VECTOR FIELDS

Scalar fields. *A scalar point function f defined over some region R such that to each point $P(x, y, z)$ in space, there corresponds a unique scalar $f(P)$, is called* **a** ***scalar field.*** For example

$$f(x, y, z) = x^2 + y^2 + z^2 - 3xyz.$$

Vector fields. *A vector point function* $\mathbf{f}$ *defined over a region R such that to each point $P(x, y, z)$ there exists a unique vector $\mathbf{f}(P)$, is called* ***vector field.*** For example

$$\mathbf{f}(x, y, z) = x^2y\hat{i} + x^3z\hat{j} - y^3z\hat{k}.$$

Some Result Related to the Limits and Continuity of a Vector Function :

1. The necessary and sufficient condition for a vector function $\mathbf{f}(t)$ to be continuous at t_0 is that

$$\lim_{t \to t_0} \mathbf{f}(t) = \mathbf{f}(t_0).$$

2. If $\mathbf{f}(t) = f_1(t)\hat{i} + f_2(t)\hat{j} + f_3(t)\hat{k}$, then $\mathbf{f}(t)$ is continuous iff $f_1(t), f_2(t), f_3(t)$ are continuous.

3. If $\mathbf{f}(t) = f_1(t)\hat{i} + f_2(t)\hat{j} + f_3(t)\hat{k}$ and $\mathbf{l} = l_1\hat{i} + l_2\hat{j} + l_3\hat{k}$, then $\lim_{t \to t_0} \mathbf{f}(t) = l$ iff $\lim_{t \to t_0} f_1(t) = l_1$, $\lim_{t \to t_0} f_2(t) = l_2$ and $\lim_{t \to t_0} f_3(t) = l_3$.

4. If $\mathbf{f}(t)$ and $\mathbf{g}(t)$ are vector functions of scalar variable t and $\phi(t)$ is a scalar function, then

(i) $\lim_{t \to t_0} [\mathbf{f}(t) \pm \mathbf{g}(t)] = \lim_{t \to t_0} \mathbf{f}(t) \pm \lim_{t \to t_0} \mathbf{g}(t)$

(ii) $\lim_{t \to t_0} [\mathbf{f}(t) \cdot \mathbf{g}(t)] = \left[\lim_{t \to t_0} \mathbf{f}(t)\right] \cdot \left[\lim_{t \to t_0} \mathbf{g}(t)\right]$

(iii) $\lim_{t \to t_0} [\mathbf{f}(t) \times \mathbf{g}(t)] = \left[\lim_{t \to t_0} \mathbf{f}(t)\right] \times \left[\lim_{t \to t_0} \mathbf{g}(t)\right]$

(iv) $\lim_{t \to t_0} |\mathbf{f}(t)| = \left|\lim_{t \to t_0} \mathbf{f}(t)\right|$.

(v) $\lim_{t \to t_0} [\phi(t)\,\mathbf{f}(t)] = \left[\lim_{t \to t_0} \phi(t)\right]\left[\lim_{t \to t_0} \mathbf{f}(t)\right]$.

• 13.4. DIFFERENTIATION OF A VECTOR FUNCTION WITH RESPECT TO A SCALAR

Definition. *Let* $\mathbf{f}(t)$ *be a vector function of scalar variable t. The function* $\mathbf{f}(t)$ *is differentiable with respect to t if*

$$\lim_{\delta t \to 0} \frac{\mathbf{f}(t + \delta t) - \mathbf{f}(t)}{\delta t} \text{ exists.}$$

And it is denoted by $\dfrac{d\mathbf{f}(t)}{dt}$.

Successive Derivatives. If $\dfrac{d\mathbf{f}(t)}{dt}$ exists, then $\mathbf{f}(t)$ is differentiable and $\dfrac{d\mathbf{f}(t)}{dt}$ is also a vector function of variable t. If $\dfrac{d\mathbf{f}(t)}{dt}$ is differentiable, then $\dfrac{d^2\mathbf{f}(t)}{dt^2}$ is called second derivative of $\mathbf{f}(t)$. Similarly we can find the third, fourth etc. derivatives of $\mathbf{f}(t)$.

REMARK

➤ If $\mathbf{r} = \mathbf{f}(t)$, then $\dfrac{d\mathbf{r}}{dt}, \dfrac{d^2\mathbf{r}}{dt^2}$, etc. are the first second etc., derivatives of $\mathbf{r} = \mathbf{f}(t)$ and also denoted by $\mathbf{r}$, $\mathbf{r}$ etc.

• 13.5. DIFFERENTIATION FORMULAE FOR THE VECTOR FUNCTION

Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ *be differentiable vector function of a scalar variable t and* ϕ *be a differentiable scalar function of t, then*

(*i*) $\dfrac{d}{dt}(\mathbf{a} \pm \mathbf{b}) = \dfrac{d\mathbf{a}}{dt} \pm \dfrac{d\mathbf{b}}{dt}$

(*ii*) $\dfrac{d}{dt}(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot \dfrac{d\mathbf{b}}{dt} + \dfrac{d\mathbf{a}}{dt} \cdot \mathbf{b}$

(*iii*) $\dfrac{d}{dt}(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times \dfrac{d\mathbf{b}}{dt} + \dfrac{d\mathbf{a}}{dt} \times \mathbf{b}$

(*iv*) $\dfrac{d}{dt}(\phi\mathbf{a}) = \phi\dfrac{d\mathbf{a}}{dt} + \dfrac{d\phi}{dt}\mathbf{a}$

(*v*) $\dfrac{d}{dt}[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = \left[\dfrac{d\mathbf{a}}{dt}\ \mathbf{b}\ \mathbf{c}\right] + \left[\mathbf{a}\ \dfrac{d\mathbf{b}}{dt}\ \mathbf{c}\right] + \left[\mathbf{a}\ \mathbf{b}\ \dfrac{d\mathbf{c}}{dt}\right]$

(vi) $\frac{d}{dt}\{\mathbf{a}\times(\mathbf{b}\times\mathbf{c})\} = \frac{d\mathbf{a}}{dt}\times(\mathbf{b}\times\mathbf{c}) + \mathbf{a}\times\left[\frac{d\mathbf{b}}{dt}\times\mathbf{c}\right] + \mathbf{a}\times\left[b\times\frac{d\mathbf{c}}{dt}\right].$

• 13.6. DERIVATIVE OF A CONSTANT VECTOR

Definition. *A vector is said to be constant vector if its magnitude as well as direction are fixed.*

Let $\mathbf{r}$ be a constant vector, then

$$\mathbf{r} = \mathbf{c} \text{ (a constant vector)} \qquad \ldots(1)$$

$$\therefore \quad \mathbf{r} + \delta\mathbf{r} = \mathbf{c}. \qquad \ldots(2)$$

Subtract (1) from (2), we get

$$\delta\mathbf{r} = \mathbf{0}.$$

Divide by δt and taking the limit as $\delta t \to 0$, we get

$$\lim_{\delta t \to 0} \frac{\delta\mathbf{r}}{\delta t} = \mathbf{0} \quad \text{or} \quad \frac{d\mathbf{r}}{dt} = \mathbf{0}.$$

Hence the *derivative of a constant vector is a zero vector.*

• 13.7. DERIVATIVE OF A VECTOR FUNCTION IN TERMS OF ITS COMPONENTS

Let $P(x, y, z)$ be any point in space and its position vector with respect to the origin O be $\mathbf{r}$ and let x, y, z be the function of scalar variable t, then we have

$$\mathbf{r} = x\hat{i} + y\hat{j} + z\hat{k} \qquad \ldots(1)$$

where $\hat{i}, \hat{j}, \hat{k}$ are constant vectors.

$$\therefore \quad \mathbf{r} + \delta\mathbf{r} = (x + \delta x)\,\hat{i} + (y + \delta y)\,\hat{j} + (z + \delta z)\,\hat{k}. \qquad \ldots(2)$$

Subtract (1) from (2), we get

$$\delta\mathbf{r} = \delta x\hat{i} + \delta y\,\hat{j} + \delta z\hat{k}.$$

Now divide this equation by δt and taking the limit as $\delta t \to 0$, we have

$$\lim_{\delta t \to 0} \frac{\delta\mathbf{r}}{\delta t} = \lim_{\delta t \to 0}\left(\frac{\delta x}{\delta t}\hat{i} + \frac{\delta y}{\delta t}\hat{j} + \frac{\delta z}{\delta t}\hat{k}\right)$$

$$\frac{d\mathbf{r}}{dt} = \left(\lim_{\delta t \to 0}\frac{\delta x}{\delta t}\right)\hat{i} + \left(\lim_{\delta t \to 0}\frac{\delta y}{\delta t}\right)\hat{j} + \left(\lim_{\delta t \to 0}\frac{\delta z}{\delta t}\right)\hat{k}$$

$$\therefore \quad \frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}.$$

Similarly, we can find $\frac{d^2\mathbf{r}}{dt^2}, \frac{d^3\mathbf{r}}{dt^3}$ etc.

• 13.8. DERIVATIVE OF A VECTOR FUNCTION OF FUNCTION

Let $\mathbf{r}$ be a function of a scalar variable u, and u is also a scalar function of scalar variable t.

$$\therefore \quad \mathbf{r} = \mathbf{f}(u) \qquad \ldots(1)$$

$$\text{and} \quad u = g(t). \qquad \ldots(2)$$

$$\therefore \quad \mathbf{r} + \delta\mathbf{r} = \mathbf{f}(u + \delta u) \qquad \ldots(3)$$

$$\text{and} \quad u + \delta u = g(t + \delta t). \qquad \ldots(4)$$

Subtract (2) from (3), we get

$$\delta\mathbf{r} = \mathbf{f}(u + \delta u) - \mathbf{f}(u) \qquad \ldots(5)$$

and subtract (2) from (4), we get

$$\delta u = g(t + \delta t) - g(t). \qquad \ldots(6)$$

Now divide (5) by δt, we have

$$\frac{\delta\mathbf{r}}{\delta t} = \frac{\mathbf{f}(u + \delta u) - \mathbf{f}(u)}{\delta t} = \frac{\mathbf{f}(u + \delta u) - \mathbf{f}(u)}{\delta u}\cdot\frac{\delta u}{\delta t}$$

$$\frac{\delta\mathbf{r}}{\delta t} = \frac{\mathbf{f}(u + \delta u) - \mathbf{f}(u)}{\delta u}\cdot\frac{g(t + \delta t) - g(t)}{\delta t} \qquad \text{[using (6)]}$$

Taking the limit $\delta t \to 0$, when $\delta t \to 0$, $\delta\mathbf{r} \to \mathbf{0}$ and $\delta u \to 0$, we get

$$\lim_{\delta t \to 0}\frac{\delta\mathbf{r}}{\delta t} = \lim_{\delta u \to 0}\frac{\mathbf{f}(u + \delta u) - \mathbf{f}(u)}{\delta u}\cdot\lim_{\delta t \to 0}\frac{g(t + \delta t) - g(t)}{\delta t}$$

$$\frac{d\mathbf{r}}{dt} = \frac{d\mathbf{f}}{du}\frac{dg}{dt} \quad \text{or} \quad \frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{du}\frac{du}{dt} \qquad [\because \ \mathbf{r} = \mathbf{f}(u),\ u = g(t)]$$

Some Important Theorems :

Theorem 1. *The vector* $\mathbf{a}(t)$ *has a constant magnitude if and only if* $\mathbf{a} \cdot \frac{d\mathbf{a}}{dt} = 0$.

Proof. Let us suppose $\mathbf{a}(t)$ has a constant magnitude.

$\therefore \qquad |\mathbf{a}(t)| = a$ (constant)

or $\qquad \mathbf{a} \cdot \mathbf{a} = a^2$ (constant)

$$\therefore \qquad \frac{d}{dt}(\mathbf{a} \cdot \mathbf{a}) = \mathbf{a} \cdot \frac{d\mathbf{a}}{dt} + \frac{d\mathbf{a}}{dt} \cdot \mathbf{a} = 2\mathbf{a} \cdot \frac{d\mathbf{a}}{dt}. \qquad [\because \ \mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}]$$

Since $\qquad \mathbf{a} \cdot \mathbf{a} = a^2$

$$\frac{d}{dt}(\mathbf{a} \cdot \mathbf{a}) = \frac{d}{dt}(a^2) = 0.$$

$$\therefore \qquad 2\mathbf{a} \cdot \frac{d\mathbf{a}}{dt} = 0 \quad \text{or} \quad \mathbf{a} \cdot \frac{d\mathbf{a}}{dt} = 0.$$

Conversely, suppose $\mathbf{a} \cdot \frac{d\mathbf{a}}{dt} = 0$, then we get

$$\frac{d}{dt}(\mathbf{a} \cdot \mathbf{a}) = \mathbf{a} \cdot \frac{d\mathbf{a}}{dt} + \frac{d\mathbf{a}}{dt} \cdot \mathbf{a} = 2\mathbf{a} \cdot \frac{d\mathbf{a}}{dt} \qquad [\because \ \mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}]$$

$$\frac{d}{dt}(\mathbf{a} \cdot \mathbf{a}) = 0 \quad \text{or} \quad \mathbf{a} \cdot \mathbf{a} = \text{constant}.$$

or $\qquad |\mathbf{a}|^2 = \text{constant} \quad \text{or} \quad |\mathbf{a}| = \text{constant}.$ **Hence proved.**

Theorem 2. *The vector function* $\mathbf{a}(t)$ *is constant if and only if* $\frac{d\mathbf{a}}{dt} = \mathbf{0}$.

Proof. Let us suppose first $\mathbf{a}(t)$ is a constant vector. Let $\mathbf{a}(t) = \mathbf{c}$ where $\mathbf{c}$ is a constant vector, then

$$\mathbf{a}(t + \delta t) = \mathbf{c}.$$

$$\therefore \qquad \mathbf{a}(t + \delta t) - \mathbf{a}(t) = \mathbf{c} - \mathbf{c} = \mathbf{0}.$$

Divide by δt and taking the limit as $\delta t \to 0$, we get

$$\lim_{\delta t \to 0} \frac{\mathbf{a}(t + \delta t) - \mathbf{a}(t)}{\delta t} = \mathbf{0}$$

$$\therefore \qquad \frac{d\mathbf{a}}{dt} = \mathbf{0}.$$

Conversely, suppose $\frac{d\mathbf{a}}{dt} = \mathbf{0}$.

Let $\qquad \mathbf{a}(t) = a_1(t)\,\hat{i} + a_2(t)\,\hat{j} + a_3(t)\,\hat{k}.$

$$\therefore \qquad \frac{d\mathbf{a}}{dt} = \frac{da_1}{dt}\hat{i} + \frac{da_2}{dt}\hat{j} + \frac{da_3}{dt}\hat{k}.$$

$$\therefore \qquad \frac{da_1}{dt}\hat{i} + \frac{da_2}{dt}\hat{j} + \frac{da_3}{dt}\hat{k} = \mathbf{0} \qquad \left(\because \ \frac{d\mathbf{a}}{dt} = \mathbf{0}\right)$$

This implies $\frac{da_1}{dt} = 0, \frac{da_2}{dt} = 0, \frac{da_3}{dt} = 0.$

Therefore a_1, a_2, a_3 are all constant.

Hence $\mathbf{a}(t) = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ is a constant vector.

Theorem 3. *If the vector* $\mathbf{a}$ *has a constant magnitude* a, *then* $\mathbf{a}$ *and* $\frac{d\mathbf{a}}{dt}$ *are perpendicular, provided* $\left|\frac{d\mathbf{a}}{dt}\right| \neq 0$.

Proof. Since, we have that $|\mathbf{a}| = a$ (constant), then

$$\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2 = a^2 \text{ (constant)}$$

$$\therefore \qquad \frac{d}{dt}(\mathbf{a} \cdot \mathbf{a}) = \mathbf{a} \cdot \frac{d\mathbf{a}}{dt} + \frac{d\mathbf{a}}{dt} \cdot \mathbf{a} = 2\mathbf{a} \cdot \frac{d\mathbf{a}}{dt}.$$

and $\qquad \frac{d}{dt}(\mathbf{a} \cdot \mathbf{a}) = \frac{d}{dt}(a^2) = 0.$

$\therefore \qquad 2\mathbf{a}\cdot\frac{d\mathbf{a}}{dt}=0 \quad \text{or} \quad \mathbf{a}\cdot\frac{d\mathbf{a}}{dt}=0.$

This implies vector **a** is perpendicular to $\frac{d\mathbf{a}}{dt}$, provided $\left|\frac{d\mathbf{a}}{dt}\right| \neq 0.$

Theorem 4. *If a vector* **a** *is a differentiable vector function of t, then*

$$\frac{d}{dt}\left(\mathbf{a}\times\frac{d\mathbf{a}}{dt}\right)=\mathbf{a}\times\frac{d^2\mathbf{a}}{dt^2}.$$

Proof. Since, we have that

$$\frac{d}{dt}(\mathbf{a}\times\mathbf{b})=\mathbf{a}\times\frac{d\mathbf{b}}{dt}+\frac{d\mathbf{a}}{dt}\times\mathbf{b}.$$

$$\therefore \qquad \frac{d}{dt}\left(\mathbf{a}\times\frac{d\mathbf{a}}{dt}\right)=\mathbf{a}\times\frac{d}{dt}\left(\frac{d\mathbf{a}}{dt}\right)+\frac{d\mathbf{a}}{dt}\times\frac{d\mathbf{a}}{dt}=\mathbf{a}\times\frac{d^2\mathbf{a}}{dt^2}+\frac{d\mathbf{a}}{dt}\times\frac{d\mathbf{a}}{dt}$$

$$=\mathbf{a}\times\frac{d^2\mathbf{a}}{dt^2}. \qquad \left(\because \frac{d\mathbf{a}}{dt}\times\frac{d\mathbf{a}}{dt}=0 \text{ i.e., cross product of two some vector is zero}\right)$$

Theorem 5. *The vector* **a**(*t*) *has a constant direction if and only if*

$$\mathbf{a}\times\frac{d\mathbf{a}}{dt}=0.$$

Proof. Suppose **a**(*t*) has a constant direction. Let $\hat{a}$ be the unit vector along **a**(*t*) and $|\mathbf{a}(t)|=a$, then

$$\mathbf{a}(t)=a\hat{a}.$$

$$\therefore \qquad \frac{d\mathbf{a}}{dt}=\frac{d}{dt}(a\hat{a})$$

$$\frac{d\mathbf{a}}{dt}=\frac{da}{dt}\hat{a}+a\frac{d\hat{a}}{dt}.$$

$$\therefore \qquad \mathbf{a}\times\frac{d\mathbf{a}}{dt}=\mathbf{a}\times\left(\frac{da}{dt}\hat{a}+a\frac{d\hat{a}}{dt}\right)=\frac{da}{dt}\mathbf{a}\times\hat{a}+a\mathbf{a}\times\frac{d\hat{a}}{dt}$$

or $$\mathbf{a}\times\frac{d\mathbf{a}}{dt}=a\left(\mathbf{a}\times\frac{d\hat{a}}{dt}\right). \qquad \text{...(1)}$$

$$\left(\because \mathbf{a}\times\hat{a}=\mathbf{a}\times\frac{\mathbf{a}}{a}=0\right)$$

Since **a** has a constant direction, then $\hat{a}$ is a constant vector, and thus we have

$$\frac{d\hat{a}}{dt}=\mathbf{0}.$$

$$\therefore \qquad \mathbf{a}\times\frac{d\mathbf{a}}{dt}=a(\mathbf{a}\times\mathbf{0})=\mathbf{0}.$$

Conversely, suppose $\mathbf{a}\times\frac{d\mathbf{a}}{dt}=0$, then from (1)

$$a\left(\mathbf{a}\times\frac{d\hat{a}}{dt}\right)=0 \quad \text{or} \quad \mathbf{a}\times\frac{d\hat{a}}{dt}=0 \quad \text{or} \quad \hat{a}\times\frac{d\hat{a}}{dt}=0 \qquad \text{...(2)}$$

$$[\because \mathbf{a}=a\hat{a}]$$

Since $\hat{a}$ has a constant magnitude, then by theorem (1), page 234

$$\hat{a}\cdot\frac{d\hat{a}}{dt}=0. \qquad \text{...(3)}$$

From (2) and (3), we get

$$\frac{d\hat{a}}{dt}=\mathbf{0}.$$

This implies $\hat{a}$ is a constant vector and hence **a** has a constant direction.

• SOLVED EXAMPLES

Example 1. *If* $\mathbf{r}=(2\sin t)\hat{i}+(3\cos t)\hat{j}+t\hat{k}$, *find* (*i*) $\dfrac{d\mathbf{r}}{dt}$ (*ii*) $\left|\dfrac{d\mathbf{r}}{dt}\right|$

(*iii*) $\dfrac{d^2\mathbf{r}}{dt^2}$ (*iv*) $\left|\dfrac{d^2\mathbf{r}}{dt^2}\right|$.

Solution. Since we know that $\hat{i}$, $\hat{j}$, $\hat{k}$ are constant vectors, so

$$\frac{d\hat{i}}{dt} = \mathbf{0},\ \frac{d\hat{j}}{dt} = \mathbf{0} \text{ and } \frac{d\hat{k}}{dt} = \mathbf{0}.$$

(i) $\mathbf{r} = (2\sin t)\,\hat{i} + (3\cos t)\,\hat{j} + t\,\hat{k}.$

$\therefore$ $\dfrac{d\mathbf{r}}{dt} = (2\cos t)\,\hat{i} - (3\sin t)\,\hat{j} + \hat{k}.$

(ii) $\left|\dfrac{d\mathbf{r}}{dt}\right| = \sqrt{(4\cos^2 t + 9\sin^2 t + 1)} = \sqrt{5\,(1+\sin^2 t)}.$

(iii) $\dfrac{d^2\mathbf{r}}{dt^2} = \dfrac{d}{dt}\left(\dfrac{d\mathbf{r}}{dt}\right) = \dfrac{d}{dt}\,(2\cos\hat{i} - 3\sin\hat{j} + \hat{k}) = -2\sin t\hat{i} - 3\cos t\hat{j}.$

(iv) $\left|\dfrac{d^2\mathbf{r}}{dt^2}\right| = \sqrt{(4\sin^2 t + 9\cos^2 t)} = \sqrt{(4 + 5\cos^2 t)}.$

Example 2. *If $\hat{r}$ be a unit vector in the direction of* $\mathbf{r}$, *prove that*

$$\hat{r} \times \frac{d\hat{r}}{dt} = \frac{1}{r^2}\,\mathbf{r} \times \frac{d\mathbf{r}}{dt},\ \text{where } |\mathbf{r}| = r.$$

Solution. Since $\hat{r}$ is a unit vector along the vector $\mathbf{r}$, so we have

$$\mathbf{r} = r\,\hat{r} \qquad \text{...(1)}$$

$\because$ $|\mathbf{r}| = r.$

Differentiating w.r.t. t of both sides, we get

$$\frac{d\mathbf{r}}{dt} = \frac{d}{dt}\,(r\,\hat{r})$$

$$\frac{d\mathbf{r}}{dt} = r\frac{d\hat{r}}{dt} + \frac{dr}{dt}\,\hat{r}. \qquad \text{...(2)}$$

Now $\mathbf{r} \times \dfrac{d\mathbf{r}}{dt} = \mathbf{r} \times \left(r\dfrac{d\hat{r}}{dt} + \dfrac{dr}{dt}\,\hat{r}\right)$

$= r\mathbf{r} \times \dfrac{d\hat{r}}{dt} + \dfrac{dr}{dt}\,\mathbf{r} \times \hat{r} = r\,(r\,\hat{r}) \times \dfrac{d\hat{r}}{dt} + \dfrac{dr}{dt}\,r\,\hat{r} \times \hat{r}$ ($\because$ $\mathbf{r} = r\,\hat{r}$)

$= r^2\,\hat{r} \times \dfrac{d\hat{r}}{dt} + 0$ ($\because$ Cross product of same vector is zero *i.e.*, $\hat{r} \times \hat{r} = \mathbf{0}$)

$= r^2\hat{r} \times \dfrac{d\hat{r}}{dt}$

$\therefore$ $\hat{r} \times \dfrac{d\hat{r}}{dt} = \dfrac{1}{r^2}\,\mathbf{r} \times \dfrac{d\mathbf{r}}{dt}.$ **Hence proved.**

Example 3. *If* $\mathbf{r} = (\cos nt)\,\hat{i} + (\sin nt)\,\hat{j}$, *where n is a constant and t varies, show that*

$$\mathbf{r} \times \frac{d\mathbf{r}}{dt} = n\,\hat{k}.$$

Solution. Since $\hat{i}$ and $\hat{j}$ are constant vectors so $\dfrac{d\hat{i}}{dt} = \mathbf{0}$, $\dfrac{d\hat{j}}{dt} = \mathbf{0}$ and

$$\mathbf{r} = (\cos nt)\,\hat{i} + (\sin nt)\,\hat{j}. \qquad \text{...(1)}$$

Differentiating (1) w.r.t. 't', we get

$$\frac{d\mathbf{r}}{dt} = -n\,(\sin nt)\,\hat{i} + n\,(\cos nt)\,\hat{j}. \qquad \text{...(2)}$$

Now $\mathbf{r} \times \dfrac{d\mathbf{r}}{dt} = \mathbf{r} \times [-n\,(\sin nt)\,\hat{i} + n\,(\cos nt)\,\hat{j}]$

$= [(\cos nt)\,\hat{i} + (\sin nt)\,\hat{j}] \times [-n\,(\sin nt)\,\hat{i} + n\,(\cos nt)\,\hat{j}]$ [From (1)]

$= (n\cos^2 nt)\,\hat{i}\times\hat{j} - n\,(\sin^2 nt)\,\hat{j}\times\hat{i}$

$= n\,(\cos^2 nt)\,\hat{k} + n\,(\sin^2 nt)\,\hat{k}$ $[\because\ \hat{j}\times\hat{i} = -\hat{k} \text{ and } \hat{i}\times\hat{j} = \hat{k}]$

$= (\cos^2 nt + \sin^2 nt)\,n\,\hat{k} = n\,\hat{k}$ $(\because\ \cos^2 nt + \sin^2 nt = 1)$

Hence, $\mathbf{r}\times\dfrac{d\mathbf{r}}{dt} = n\,\hat{k}.$

• TEST YOURSELF–1

1. If $\mathbf{r} = (t+1)\,\hat{i} + (t^2+t+1)\,\hat{j} + (t^3+t^2+t+1)\,\hat{k}$, find $\dfrac{d\mathbf{r}}{dt}, \dfrac{d^2\mathbf{r}}{dt^2}$.
2. If $\mathbf{a}$, $\mathbf{b}$ are constant vectors, ω is a constant, and $\mathbf{r}$ is a vector function of the scalar variable t given by
$$\mathbf{r} = \cos\omega t\,\mathbf{a} + \sin\omega t\,\mathbf{b}.$$
Show that
(i) $\dfrac{d^2\mathbf{r}}{dt^2} + \omega^2\mathbf{r} = \mathbf{0}$ (ii) $\mathbf{r}\times\dfrac{d\mathbf{r}}{dt} = \omega\mathbf{a}\times\mathbf{b}.$
3. (i) If $\mathbf{r}$ is a unit vector, then show that
$$\left|\mathbf{r}\times\frac{d\mathbf{r}}{dt}\right| = \left|\frac{d\mathbf{r}}{dt}\right|.$$
(ii) If $\mathbf{r}\times d\mathbf{r} = \mathbf{0}$, show that $\hat{r} = \text{constant}.$
4. If $\mathbf{r}$ is the position vector of a moving point and r is the modulus of $\mathbf{r}$, show that
$$\mathbf{r}\cdot\frac{d\mathbf{r}}{dt} = r\frac{dr}{dt}.$$
5. If $\mathbf{r}$ is a vector function of a scalar variable t and $\mathbf{a}$ is a constant vector, differentiate the following with respect to t :
(i) $\mathbf{r}\times\mathbf{a}$ (ii) $\mathbf{r}\times\dfrac{d\mathbf{r}}{dt}$ (iii) $\dfrac{\mathbf{r}\times\mathbf{a}}{\mathbf{r}\cdot\mathbf{a}}$ (iv) $r^3\mathbf{r} + \mathbf{a}\times\dfrac{d\mathbf{r}}{dt}$
where $|\mathbf{r}| = r$.
6. If $\mathbf{r} = \sin t\,\hat{i} + \cos t\,\hat{j} + t\,\hat{k}$, find $\left|\dfrac{d\mathbf{r}}{dt}\times\dfrac{d^2\mathbf{r}}{dt^2}\right|$.

ANSWERS

1. $\dfrac{d\mathbf{r}}{dt} = \hat{i} + (2t+1)\,\hat{j} + (3t^2+2t+1)\,\hat{k},\ \dfrac{d^2\mathbf{r}}{dt^2} = 2\,\hat{j} + (6t+2)\,\hat{k}$

5. (i) $\dfrac{d\mathbf{r}}{dt}\times\mathbf{a}$ (ii) $\mathbf{r}\times\dfrac{d^2\mathbf{r}}{dt^2}$

(iii) $\dfrac{\dfrac{d\mathbf{r}}{dt}\times\mathbf{a}}{\mathbf{r}\cdot\mathbf{a}} - \dfrac{\dfrac{d\mathbf{r}}{dt}\cdot\mathbf{a}}{(\mathbf{r}\cdot\mathbf{a})^2}(\mathbf{r}\times\mathbf{a})$ (iv) $3r^2\dfrac{dr}{dt}\mathbf{r} + r^3\dfrac{d\mathbf{r}}{dt} + \mathbf{a}\times\dfrac{d^2\mathbf{r}}{dt^2}$

• 13.9. INTEGRATION OF A VECTOR FUNCTION

Let $\mathbf{F}(t)$ be a differentiable vector function and let $\mathbf{f}(t)$ be its differential coefficient, then
$$\frac{d}{dt}(\mathbf{F}(t)) = \mathbf{f}(t). \qquad \text{...(1)}$$
Therefore the integral of $\mathbf{f}(t)$ is $\mathbf{F}(t)$. Consequently we can say that *integration is the reverse process of differentiation.*
$$\therefore \quad \int \mathbf{f}(t)\,dt = \mathbf{F}(t). \qquad \text{...(2)}$$
This is an indefinite integral and the function $\mathbf{f}(t)$ which is being integrated is known as **integrand.**

Moreover, let **c** be a constant vector which is independent of t, then (1) can also be written as

$$\frac{d}{dt}\{\mathbf{F}(t)+\mathbf{c}\}=\mathbf{f}(t). \qquad ...(3)$$

$$\therefore \qquad \int \mathbf{f}(t)\,dt=\mathbf{F}(t)+\mathbf{c}. \qquad ...(4)$$

This constant vector **c** is called ***constant of integration***, since this vector **c** is taken to be arbitrary so the integral given in (2) and in (4) is therefore indefinite, integrals.

If $\mathbf{f}(t)$ is defined over the closed interval $[a, b]$, then the integral given in (5)

$$\int_a^b \mathbf{f}(t)\,dt=[\mathbf{F}(t)+\mathbf{c}]_a^b=\mathbf{F}(b)-\mathbf{F}(a) \qquad ...(5)$$

is called the ***definite integral*** and a and b are called ***limits of integration.***

REMARK

➤ If $\mathbf{f}(t)=f_1(t)\,\hat{i}+f_2\,t\,\hat{j}+f_3(t)\,\hat{k}$, then

$$\int \mathbf{f}(t)\,dt=\hat{i}\int f_1(t)\,dt+\hat{j}\int f_2(t)\,dt+\hat{k}\int f_3(t)\,dt.$$

Some Important Results :

1. $\displaystyle\int\left(\mathbf{a}\cdot\frac{d\mathbf{b}}{dt}+\frac{d\mathbf{a}}{dt}\cdot\mathbf{b}\right)dt=\mathbf{a}\cdot\mathbf{b}+\mathbf{c}$

where, **c** is a constant of integration.

2. $\displaystyle\int\left(\mathbf{a}\times\frac{d\mathbf{b}}{dt}+\frac{d\mathbf{a}}{dt}\times\mathbf{b}\right)dt=(\mathbf{a}\times\mathbf{b})+\mathbf{c}$

where **c** is a constant vector.

3. $\displaystyle\int\left(2\mathbf{a}\cdot\frac{d\mathbf{a}}{dt}\right)dt=a^2+\mathbf{c}.$

Here, **c** is a scalar quantity.

4. $\displaystyle\int\left(\mathbf{a}\times\frac{d^2\mathbf{a}}{dt^2}\right)dt=\left(\mathbf{a}\times\frac{d\mathbf{a}}{dt}\right)+\mathbf{c}.$

Here, **c** is a constant vector of integration.

5. $\displaystyle\int\left(\mathbf{a}\times\frac{d^2\mathbf{b}}{dt^2}\right)dt=\left(\mathbf{a}\times\frac{d\mathbf{b}}{dt}\right)+\mathbf{c}.$

Here, **c** is a constant vector of integration.

6. $\displaystyle\int\left(\mathbf{a}\times\frac{d\mathbf{b}}{dt}\right)dt=(\mathbf{a}\times\mathbf{b})+\mathbf{c}.$

Here, **c** is a constant vector which is constant of integration.

7. If c is a constant scalar and **a** is a vector function of t, then

$$\int c\mathbf{a}\,dt=c\int \mathbf{a}\,dt.$$

• SOLVED EXAMPLES

Example 1. *Interpret the relations*

$$\mathbf{r}\cdot\frac{d\mathbf{r}}{ds}=0 \quad \textit{and} \quad \mathbf{r}\times\frac{d\mathbf{r}}{ds}=\mathbf{0}.$$

Solution. For $\mathbf{r}\cdot\dfrac{d\mathbf{r}}{ds}=0 \Rightarrow 2\mathbf{r}\cdot\dfrac{d\mathbf{r}}{ds}=0.$

Integrating w.r.t. s, we get

$$\int\left(2\mathbf{r}\cdot\frac{d\mathbf{r}}{ds}\right)ds=\int 0\,ds$$

or $\qquad \mathbf{r}^2=a$ (constant) $\Rightarrow$ **r** has constant magnitude.

Thus **r** describes a circle.

Again, for $\mathbf{r} \times \frac{d\mathbf{r}}{ds} = \mathbf{0} \Rightarrow \mathbf{r}$ and $\frac{d\mathbf{r}}{ds}$ are parallel.

Also $\frac{d\mathbf{r}}{ds}$ is a unit vector along tangent.

$\therefore$ $\mathbf{r}$ has constant direction that the tangent at every point is along $\mathbf{r}$.
Thus $\mathbf{r}$ describes a straight line.

Example 2. *If* $\mathbf{f}(t) = (t+1)\hat{i} + (t^2+t+1)\hat{j} + (t^3+t^2+t+1)\hat{k}$, *find* $\int_0^1 \mathbf{f}(t)\,dt$.

Solution. Since $\mathbf{f}(t) = (t+1)\hat{i} + (t^2+t+1)\hat{j} + (t^3+t^2+t+1)\hat{k}$, then

$$\int_0^1 \mathbf{f}(t)\,dt = \int_0^1 [(t+1)\hat{i} + (t^2+t+1)\hat{j} + (t^3+t^2+t+1)\hat{k}]\,dt$$

$$= \hat{i}\int_0^1 (t+1)\,dt + \hat{j}\int_0^1 (t^2+t+1)\,dt + \hat{k}\int_0^1 (t^3+t^2+t+1)\,dt$$

$$= \hat{i}\left(\frac{t^2}{2}+t\right)_0^1 + \hat{j}\left(\frac{t^3}{3}+\frac{t^2}{2}+t\right)_0^1 + \hat{k}\left(\frac{t^4}{4}+\frac{t^3}{3}+\frac{t^2}{2}+t\right)_0^1$$

$$= \frac{3}{2}\hat{i} + \frac{11}{6}\hat{j} + \frac{25}{12}\hat{k}.$$

Example 3. *If* $\mathbf{r} = 5t^2\hat{i} + t\hat{j} - t^3\hat{k}$, *then prove that*

$$\int_1^2 \left(\mathbf{r} \times \frac{d^2\mathbf{r}}{dt^2}\right) dt = -14\hat{i} + 75\hat{j} - 15\hat{k}.$$

Solution. Since $\mathbf{r} = 5t^2\hat{i} + t\hat{j} - t^3\hat{k}$, then

$$\frac{d\mathbf{r}}{dt} = 10t\,\hat{i} + \hat{j} - 3t^2\hat{k}$$

again
$$\frac{d^2\mathbf{r}}{dt^2} = 10\hat{i} - 6t\,\hat{k}.$$

$\therefore$
$$\mathbf{r} \times \frac{d^2\mathbf{r}}{dt^2} = (5t^2\hat{i} + t\hat{j} - t^3\hat{k}) \times (10\hat{i} - 6t\hat{k})$$

$$= -30t^3\,\hat{i}\times\hat{k} + 10t\,\hat{j}\times\hat{i} - 6t^2\,\hat{j}\times\hat{k} - 10t^3\,\hat{k}\times\hat{i}$$

$$= 30t^3\hat{j} - 10t\,\hat{k} - 6t^2\hat{i} - 10t^3\hat{j} = -6t^2\hat{i} + 20t^3\hat{j} - 10t\,\hat{k}.$$

Now $\int_1^2 \left(\mathbf{r} \times \frac{d^2\mathbf{r}}{dt^2}\right) dt = \int_1^2 (-6t^2\hat{i} + 20t^3\hat{j} - 10t\,\hat{k})\,dt$

$$= \left[-2t^3\hat{i} + 5t^4\hat{j} - 5t^2\hat{k}\right]_1^2 = -14\hat{i} + 75\hat{j} - 15\hat{k}.$$

• STUDENT'S ACTIVITY

1. If $\hat{r}$ be a unit vector inthe direction of $\vec{r}$, prove that

$$\hat{r} \times \frac{d\hat{r}}{dt} = \frac{1}{r^2}\vec{r} \times \frac{d\vec{r}}{dt}, |\vec{r}| = r$$

2. If $\vec{r} = 5t^2\hat{i} + t\hat{j} - t^3\hat{k}$, then prove that

$$\int_1^2 \left(\vec{r} \times \frac{d^2\vec{r}}{dt^2}\right) dt = -14\hat{i} + 75\hat{j} - 15\hat{k}$$

• SUMMARY

- **Vector Function :** $\vec{f}(t) = f_1(t)\,\hat{i} + f_2(t)\,\hat{j} + f_3(t)\,\hat{k}$.
- **Differentiation of** $\vec{f}(t)$, w.r.t., t :

$$\frac{d\vec{f}(t)}{dt} = \lim_{\partial t \to 0} \vec{f}(t + \delta t) - \vec{f}(t)$$

- **Integration of** $\vec{f}(t)$ w.r.t. t

(i) $\int \left(\vec{a} \times \frac{d^2\vec{b}}{dt^2}\right) dt = \vec{a} \times \frac{d\vec{b}}{dt} + c$, if $\vec{a}$ is constant

(ii) $\int \left(\vec{a} \times \frac{d\vec{b}}{dt}\right) dt = (\bar{a} \times \bar{b}) + c$, if $\vec{a}$ is constant vector.

• TEST YOURSELF–2

1. If $\mathbf{f}(t) = (t - t^2)\,\hat{i} + 2t^3\,\hat{j} - 3\,\hat{k}$, find

(i) $\int \mathbf{f}(t)\, dt$ (ii) $\int_1^2 \mathbf{f}(t)\, dt$.

2. Integrate $\mathbf{a} \times \frac{d^2\mathbf{r}}{dt^2} = \mathbf{b}$, where **a** and **b** are constant vectors.

3. Find the value of **r** satisfying the equation $\frac{d^2\mathbf{r}}{dt^2} = t\mathbf{a} + \mathbf{b}$ where **a** and **b** are constant vectors.

4. Given that $\mathbf{r}(t) = \begin{cases} 2\,\hat{i} - \hat{j} + 2\,\hat{k}, & t = 2 \\ 4\,\hat{i} - 2\,\hat{j} + 3\,\hat{k}, & t = 3 \end{cases}$

show that $\int_2^3 \left(\mathbf{r} \cdot \frac{d\mathbf{r}}{dt}\right) dt = 10.$

5. Find $\int_0^1 (e^t\,\hat{i} + e^{-2t}\,\hat{j} + t\,\hat{k})\, dt.$

6. If $\mathbf{r} = t\,\hat{i} - t^2\,\hat{j} + (t-1)\,\hat{k}$ and $\mathbf{s} = 2t^2\,\hat{i} + 6t\,\hat{k}$, evaluate

(i) $\int_0^2 \mathbf{r} \cdot \mathbf{s}\, dt$ (ii) $\int_0^2 \mathbf{r} \times s\, dt$.

ANSWERS

1. (i) $\left(\frac{t^2}{2}+\frac{t^3}{3}\right)\hat{i}+\frac{t^4}{2}\hat{j}-3t\,\hat{k}+\mathbf{c}$ (ii) $-\frac{5}{6}\hat{i}+\frac{15}{2}\hat{j}-3\,\hat{k}$.

2. $\mathbf{a}\times\mathbf{r}=\frac{1}{2}t^2\,\mathbf{b}+t\mathbf{c}+\mathbf{d}$ 3. $\mathbf{r}=\frac{1}{6}t^3\,\mathbf{a}+\frac{1}{2}t^2\,\mathbf{b}+t\,\mathbf{c}+\mathbf{d}$

5. $(e-1)\,\hat{i}-\frac{1}{2}(e^{-2}-1)\,\hat{j}+\frac{1}{2}\hat{k}$.

6. (i) 12 (ii) $-24\hat{i}-\frac{40}{3}\hat{j}+\frac{64}{5}\hat{k}$

OBJECTIVE EVALUATION

➤ FILL IN THE BLANKS :

1. $\frac{d}{dt}(\mathbf{a}\times\mathbf{b})=\mathbf{a}\times\frac{d\mathbf{b}}{dt}+$ ....
2. The derivative of a constant vector is equal to the ....
3. If $\mathbf{a}$ has constant length and $\left|\frac{d\mathbf{a}}{dt}\right|\neq 0$, then $\mathbf{a}$ and $\frac{d\mathbf{a}}{dt}$ are ....
4. The vector $\mathbf{a}(t)$ has constant magnitude iff $\mathbf{a}\cdot\frac{d\mathbf{a}}{dt}$ is equal to ....

➤ TRUE OR FALSE :

Write 'T' for true and 'F' for false statement :

1. If $\mathbf{a}\cdot\frac{d\mathbf{a}}{dt}=0$, then $|\,\mathbf{a}\,|$ is constant. (T/F)
2. If $|\,\mathbf{a}\,|$ = constant, then $\mathbf{a}\times\frac{d\mathbf{a}}{dt}=\mathbf{0}$. (T/F)
3. If $\mathbf{r}=\cos 3t\,\hat{i}+\sin 3t\,\hat{j}$, then $\left|\mathbf{r}\times\frac{d\mathbf{r}}{dt}\right|=3$. (T/F)
4. $\int\left(2\mathbf{r}\cdot\frac{d\mathbf{r}}{dt}\right)dt=\mathbf{r}^2+c$. (T/F)

➤ MULTIPLE CHOICE QUESTIONS :

Choose the most appropriate one :

1. If ϕ is not a function of t, then $\frac{d}{dt}(\phi\mathbf{a})$ equals :

   (a) $\frac{d\phi}{dt}\mathbf{a}$ (b) $\frac{d\phi}{dt}$ (c) $\phi\frac{d\mathbf{a}}{dt}$ (d) $\frac{d\mathbf{a}}{dt}$.

2. If $|\,\mathbf{a}\,|$ = constant, then $\mathbf{a}\cdot\frac{d\mathbf{a}}{dt}$ is equal to :

   (a) 1 (b) 0 (c) 2 (d) -1.

3. If $|\,\mathbf{a}\,|=a$, then $\mathbf{a}\cdot\frac{d\mathbf{a}}{dt}$ equals :

   (a) $a\frac{da}{dt}$ (b) a (c) $\frac{da}{dt}$ (d) None of these.

4. If $|\,\mathbf{a}\,|$ = constant, then $\mathbf{a}\times\frac{d\mathbf{a}}{dt}$ is equal to :

   (a) 0 (b) -1 (c) 1 (d) a.

ANSWERS

Fill in the Blanks :

1. $\frac{d\mathbf{a}}{dt}\times\mathbf{b}$ **2.** Null vector **3.** Perpendicular **4.** 0 **5.** $-14\,\hat{i}+75\,\hat{j}-15\,\hat{k}$

True or False :

1. T **2.** T **3.** T **4.** T

Multiple Choice Questions :

1. (c) **2.** (b) **3.** (a) **4.** (a)

UNIT 14

GRADIENT, DIVERGENCE AND CURL

LEARNING OBJECTIVES

LEARNING OBJECTIVES

After going through this unit you will learn :

About the partial derivatives of the vectors.

Differentiation operator

How to calculate gradient of scalar, divergence of vectors and curd of vectors.

• 14.1. PARTIAL DERIVATIVES OF VECTORS

Let $\mathbf{r} = \mathbf{f}(x, y, z)$ *be a vector function of three scalar variables* x, y, z. *The first order partial derivative of* $\mathbf{r}$ *with respect to* x *is given by*

$$\frac{\partial \mathbf{r}}{\partial x} = \lim_{\delta x \to 0} \frac{\mathbf{f}(x + \delta x, y, z) - \mathbf{f}(x, y, z)}{\delta x}, \text{ if this limit exists.}$$

Similarly we can find first order partial derviatives of $\mathbf{r}$ with respect to y and z respectively and are denoted by $\frac{\partial \mathbf{r}}{\partial y}, \frac{\partial \mathbf{r}}{\partial z}$.

During the differentiation if y and z are treating as constant, then $\frac{\partial \mathbf{r}}{\partial x}$ is regarded as ordinary derivative. Likewise we can find higher order partial derivatives.

• 14.2. VECTOR DIFFERENTIAL OPERATOR ∇

The vector differential operator is defined by the formula

$$\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}.$$

Obviously, ∇ *is a vector quantity.* This vector ∇ is read as **nabla** or **del.**

• 14.3. GRADIENT OF A SCALAR FIELD

Let $f(x, y, z)$ *be a scalar point function which is defined over some region* R *in space and also differentiable at each point* (x, y, z) *in* R, *then the gradient of* $f(x, y, z)$ *is defined as*

$$grad\, f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

or $$grad\, f = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) f = \nabla f$$

Thus gradient of f can also be written in terms of vector differential operator (∇). Since ∇ is a vector quantity, thus ∇f is a vector whose components are $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$. Hence, gradient of a scalar field is a vector field.

• 14.4. SOME FORMULAE RELATED TO GRADIENT

1. *If f and g are two scalar point functions, then*

$$grad\,(f+g) = grad\, f + grad\, g$$

or $$\nabla(f+g) = \nabla f + \nabla g.$$

Proof. Since we know that

$$\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}.$$

$$\therefore \quad \nabla(f+g) = \frac{\partial}{\partial x}(f+g)\,\hat{i} + \frac{\partial (f+g)}{\partial y}\hat{j} + \frac{\partial (f+g)}{\partial z}\hat{k}$$

$$= \frac{\partial f}{\partial x}\hat{i} + \frac{\partial g}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial g}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k} + \frac{\partial g}{\partial z}\hat{k}$$

$$= \left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}\right) + \left(\frac{\partial g}{\partial x}\hat{i} + \frac{\partial g}{\partial y}\hat{j} + \frac{\partial g}{\partial z}\hat{k}\right) = \nabla f + \nabla g.$$

Hence $\nabla(f+g) = \nabla f + \nabla g$.

2. *If f and g are scalar point functions, then*

$$\nabla(fg) = f\,\nabla g + g\,\nabla f$$

or $$grad\,(fg) = f(grad\, g) + g(grad\, f).$$

Proof. Since we know that

$$\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}.$$

$$\therefore \quad \nabla(fg) = \frac{\partial}{\partial x}(fg)\,\hat{i} + \frac{\partial}{\partial y}(fg)\,\hat{j} + \frac{\partial}{\partial z}(fg)\,\hat{k}$$

$$= \left(f\frac{\partial g}{\partial x} + g\frac{\partial f}{\partial x}\right)\hat{i} + \left(f\frac{\partial g}{\partial y} + g\frac{\partial f}{\partial y}\right)\hat{j} + \left(f\frac{\partial g}{\partial z} + g\frac{\partial f}{\partial z}\right)\hat{k}$$

$$= f\left(\frac{\partial g}{\partial x}\hat{i} + \frac{\partial g}{\partial y}\hat{j} + \frac{\partial g}{\partial z}\hat{k}\right) + g\left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}\right) = f\,\nabla g + g\,\nabla f$$

Hence, $\nabla(fg) = f\,\nabla g + g\,\nabla f$.

3. *If f and g are scalar point functions and $g \neq 0$ for all point in the region R, then*

$$\nabla\left(\frac{f}{g}\right) = \frac{g\,\nabla f - f\,\nabla g}{g^2}.$$

Proof. Since $\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$.

$$\therefore \quad \nabla\left(\frac{f}{g}\right) = \frac{\partial}{\partial x}\left(\frac{f}{g}\right)\hat{i} + \frac{\partial}{\partial y}\left(\frac{f}{g}\right)\hat{j} + \frac{\partial}{\partial z}\left(\frac{f}{g}\right)\hat{k}$$

$$= \frac{1}{g^2}\left(g\frac{\partial f}{\partial x} - f\frac{\partial g}{\partial x}\right)\hat{i} + \frac{1}{g^2}\left(g\frac{\partial f}{\partial y} - f\frac{\partial g}{\partial y}\right)\hat{j} + \frac{1}{g^2}\left(g\frac{\partial f}{\partial z} - f\frac{\partial g}{\partial z}\right)\hat{k}$$

$$= \frac{1}{g^2}\left[g\left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}\right) - f\left(\frac{\partial g}{\partial x}\hat{i} + \frac{\partial g}{\partial y}\hat{j} + \frac{\partial g}{\partial z}\hat{k}\right)\right]$$

$$= \frac{1}{g^2}[g\,\nabla f - f\,\nabla g] = \frac{g(\nabla f) - f(\nabla g)}{g^2}.$$

Hence, $\nabla\left(\frac{f}{g}\right) = \frac{g(\nabla f) - f(\nabla g)}{g^2}$.

4. *If f is a scalar point fucntion, then f is constant if and only if*

$$\nabla f = 0.$$

Proof. Suppose f is constant, then

$$\frac{\partial f}{\partial x}=0, \frac{\partial f}{\partial y}=0, \frac{\partial f}{\partial z}=0 \qquad (\because \ f(x, y, z)=c)$$

$$\therefore \qquad \nabla f=\frac{\partial f}{\partial x}\hat{i}+\frac{\partial f}{\partial y}\hat{j}+\frac{\partial f}{\partial z}\hat{k}=0\,\hat{i}+0\,\hat{j}+0\,\hat{k}=\mathbf{0}.$$

Conversely, suppose $\nabla f=\mathbf{0}$. Then we have

$$\nabla f=\frac{\partial f}{\partial x}\hat{i}+\frac{\partial f}{\partial y}\hat{j}+\frac{\partial f}{\partial z}\hat{k}=\mathbf{0}.$$

$$\therefore \qquad \frac{\partial f}{\partial x}=0, \frac{\partial f}{\partial y}=0, \frac{\partial f}{\partial z}=0.$$

Hence, $f(x, y, z)=c$ (constant).

REMARK

- $\nabla(f-g)=\nabla f-\nabla g$
- $\nabla(cf)=c\nabla f$, where c is a constant.
- $\nabla\left(\frac{1}{f}\right)=-\frac{\nabla f}{f^2}$, where $f\neq 0 \ \forall\,(x, y, z)\in \mathbf{R}$.

• SOLVED EXAMPLES

Example 1. *If* $r=|\mathbf{r}|$ *where* $\mathbf{r}=x\hat{i}+y\hat{j}+z\hat{k}$, *prove that*

(*i*) $\nabla f(r)=f'(r)\,\nabla r$ (*ii*) $\nabla r=\frac{\mathbf{r}}{r}$

(*iii*) $\nabla f(r)\times\mathbf{r}=0$ (*iv*) $\nabla r^n=nr^{n-2}\mathbf{r}$.

Solution. (i) Since we know that

$$\nabla f=\frac{\partial f}{\partial x}\hat{i}+\frac{\partial f}{\partial y}\hat{j}+\frac{\partial f}{\partial z}\hat{k}. \qquad ...(1)$$

$$\therefore \qquad \nabla f(r)=\frac{\partial}{\partial x}f(r)\,\hat{i}+\frac{\partial}{\partial y}(f(r))\,\hat{j}+\frac{\partial}{\partial z}(f(r))\,\hat{k}$$

or $$\nabla f(r)=f'(r)\frac{\partial r}{\partial x}\hat{i}+f'(r)\frac{\partial r}{\partial y}\hat{j}+f'(r)\frac{\partial r}{\partial z}\hat{k}$$

$$\nabla f(r)=f'(r)\left[\frac{\partial r}{\partial x}\;i+\frac{\partial r}{\partial y}\;j+\frac{\partial r}{\partial z}\;\hat{k}\right]$$

$$\nabla f(r)=f'(r)\,\nabla r \qquad \text{[using (i)]} \qquad \text{[Remember]}$$

(ii) $\nabla r=\frac{\partial r}{\partial x}\hat{i}+\frac{\partial r}{\partial y}\hat{j}+\frac{\partial r}{\partial z}\hat{k}$ (by definition of gradient)

Since $\mathbf{r}=x\,\hat{i}+y\,\hat{j}+z\,\hat{k}$.

$$\therefore \qquad |\mathbf{r}|^2=x^2+y^2+z^2$$

or $$r^2=x^2+y^2+z^2 \qquad (\because \ |\mathbf{r}|=r)$$

$$\therefore \qquad \frac{\partial r}{\partial x}=\frac{x}{r}, \frac{\partial r}{\partial y}=\frac{y}{r}, \frac{\partial r}{\partial z}=\frac{z}{r}$$

$$\therefore \qquad \nabla r=\frac{x}{r}\hat{i}+\frac{y}{r}\hat{j}+\frac{z}{r}\hat{k}=\frac{1}{r}(x\,\hat{i}+y\,\hat{j}+z\,\hat{k}) \quad \text{or} \quad \nabla r=\frac{\mathbf{r}}{r}.$$

(iii) $$\nabla f(r)=f'(r)\,\nabla r \qquad \text{[from (i)]}$$

$$=f'(r)\frac{\mathbf{r}}{r} \qquad \text{[from (ii)]}$$

Now $$\nabla f(r)\times\mathbf{r}=\frac{f'(r)}{r}\,\mathbf{r}\times\mathbf{r}=\mathbf{0} \qquad (\because \ \mathbf{r}\times\mathbf{r}=\mathbf{0})$$

(iv) Since $\nabla f(r)=f'(r)\,\nabla r$. [from (i)]

Let $f(r)=r^n$.

$$\therefore \qquad \nabla r^n=nr^{n-1}\nabla r=nr^{n-1}\left(\frac{\mathbf{r}}{r}\right) \qquad \left(\because \ \nabla r=\frac{\mathbf{r}}{r}\right)$$

$$=nr^{n-2}\,\mathbf{r}$$

or $$\nabla r^n=nr^{n-2}\,\mathbf{r}.$$

Example 2. *If* $f(x, y, z)=3x^2y-y^3z^2$, *find grad f and* $|\text{grad} f|$ *at* $(1, -2, -1)$.

Solution. Since we know that

$$\text{grad } f = \nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

$$= \frac{\partial}{\partial x}(3x^2y - y^3z^2)\hat{i} + \frac{\partial}{\partial y}(3x^2y - y^3z^2)\hat{j} + \frac{\partial}{\partial z}(3x^2y - y^3z^2)\hat{k}$$

$$= 6xy\,\hat{i} + (3x^2 - 3y^2z^2)\hat{j} + (-2y^3z)\hat{k}$$

At $(1, -2, -1)$

$$\text{grad} = -12\,\hat{i} - 9\,\hat{j} - 16\,\hat{k}$$

and $\quad |\text{grad} f| = \sqrt{144 + 81 + 256} = \sqrt{481}.$

Example 3. *If* $\phi(x, y, z) = xy^2z$ *and* $\mathbf{f}(x, y, z) = xz\,\hat{i} - xy\,\hat{j} + yz^2\,\hat{k}$, *show that* $\frac{\partial^3}{\partial x^2 \partial z}(\phi\mathbf{f})$ *at* $(2, -1, 1)$ *is* $4\hat{i} + 2\hat{j}$.

Solution. $\quad \phi\mathbf{f} = x^2y^2z^2\,\hat{i} - x^2y^3z\,\hat{j} + xy^3z^2\,\hat{k}.$

$$\therefore \quad \frac{\partial}{\partial z}(\phi\mathbf{f}) = 2x^2y^2z\,\hat{i} - x^2y^3\,\hat{j} + 2xy^3z\,\hat{k}$$

$$\frac{\partial^2}{\partial x\,\partial z}(\phi\mathbf{f}) = 4xy^2z\,\hat{i} - 2xy^3\,\hat{j} + 2y^3z\,\hat{k}$$

$$\frac{\partial^3}{\partial x^2\,\partial z}(\phi\mathbf{f}) = 4y^2z\,\hat{i} - 2y^3\,\hat{j}.$$

At $(2, -1, 1)$

$$\frac{\partial^3(\phi\mathbf{f})}{\partial x^2\,\partial z} = 4(-1)^2(1)\,\hat{i} - 2(-1)^3\,\hat{j} = 4\hat{i} + 2\hat{j}.$$

• TEST YOURSELF–1

1. If $\phi(x, y, z) = x^2y + y^2x + z^2$, find $\nabla\phi$ at the point $(1, 1, 1)$.

2. If $\mathbf{f}(x, y, z) = x^2yz\,\hat{i} - 2xz^3\,\hat{j} + xz^2\,\hat{k}$, $\phi(x, y, z) = 2z\,\hat{i} + y\,\hat{j} - x^2\,\hat{k}$, find the value of $\frac{\partial^2}{\partial x \partial y}(\mathbf{f} \times \phi)$ at $(1, 0, -2)$.

3. If $|\mathbf{r}| = r$ where $\mathbf{r} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$, prove that

   (i) $\nabla\left(\frac{1}{r}\right) = -\frac{\mathbf{r}}{r^3}$ $\qquad$ (ii) $\nabla \log r = \frac{\mathbf{r}}{r^2}$ $\qquad$ (iii) $\nabla r^{-3} = -3r^{-5}\mathbf{r}$.

ANSWERS

1. $3\hat{i} + 3\hat{j} + 2\hat{k}$ $\qquad$ 2. $-4\hat{i} - 8\hat{j}$

• 14.5. DIVERGENCE OF A VECTOR FIELD

Let $\mathbf{V}(x, y, z)$ *be a differentiable vector function, where* x, y, z *are cartesian co-ordinates in space and let* V_1, V_2, V_3 *be the components of* $\mathbf{V}$, *then the function*

$$\text{div } \mathbf{V} = \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z} \qquad \text{...(1)}$$

is called the divergence of $\mathbf{V}$.

Since we have that the differential operator

$$\nabla \equiv \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

and the vector $\quad \mathbf{V} = V_1\,\hat{i} + V_2\,\hat{j} + V_3\,\hat{k}.$

Then $\quad \nabla \cdot \mathbf{V} = \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}. \qquad$ ...(2)

From equation (1) and (2), we get

$$\text{div } \mathbf{V} = \nabla \cdot \mathbf{V}.$$

Hence, divergence of a vector function $\mathbf{V}$ can also be written as $\nabla \cdot \mathbf{V}$. Consequently *divergence of a vector function is scalar because dot product of* ∇ *and* $\mathbf{V}$ *gives a scalar quantity.*

• 14.6. CURL OF A VECTOR FIELD

Let $\mathbf{V}(x, y, z)$ *be a vector function of* x, y, z*, where* x, y, z *are right handed cartesian co-ordinates in space and let*

$$\mathbf{V}(x, y, z) = V_1(x, y, z)\,\hat{i} + V_2(x, y, z)\,\hat{j} + V_3(x, y, z)\,\hat{k}$$

be a differentiable vector function. Then the function

$$curl\ \mathbf{V} = \nabla \times \mathbf{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix}$$

$$= \left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right)\hat{i} + \left(\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x}\right)\hat{j} + \left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right)\hat{k}$$

is called the ***curl of the vector function*** $\mathbf{V}$ *or the curl of the vector field defined by Curl* $\mathbf{V}$ *is a vector quantity.*

• 14.7. LAPLACIAN OPERATOR

If the function $f(x, y, z)$ is a twice differentiable scalar function, then we have

$$\text{grad}\, f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}.$$

Since grad f is a vector function, then

$$\text{div}\,(\text{grad}\, f) = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot\left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}\right)$$

$$= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) f.$$

$\therefore$ $\quad \text{div}\,(\text{grad}\, f) = \nabla^2 f.$ ...(1)

Thus R.H.S. of (1) is the Laplacian of f. Consequently the **Laplacian** is defined as

$$\nabla^2 \equiv \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right).$$

Hence ∇^2 is a **Laplacian** operator.

Laplace Equation : *The equation* $\nabla^2 f = 0$ *is called* ***Laplace's equation.***

• SOLVED EXAMPLES

Example 1. *Prove the followings :*

(*i*) *div* $\mathbf{r} = 3$ (*ii*) *curl* $\mathbf{r} = \mathbf{0}$

where $\mathbf{r} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$.

Solution. (i) Since div $\mathbf{r} = \nabla \cdot \mathbf{r}$

$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot(x\,\hat{i} + y\,\hat{j} + z\,\hat{k})$$

$$= \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 1 + 1 + 1 = 3.$$

(ii) Curl $\mathbf{r} = \nabla \times \mathbf{r}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \hat{i}\left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}\right) + \hat{j}\left(\frac{\partial x}{\partial z} - \frac{\partial z}{\partial x}\right) + \hat{k}\left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right)$$

$$= \hat{i}(0 - 0) + \hat{j}(0 - 0) + \hat{k}(0 - 0) = \mathbf{0}.$$

Example 2. *If* $\mathbf{a}$ *is a constant vector, find*

(*i*) *div* $(\mathbf{r} \times \mathbf{a})$ (*ii*) *curl* $(\mathbf{r} \times \mathbf{a})$.

where $\mathbf{r} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$.

Solution. Let $\mathbf{a} = a_1\,\hat{i} + a_2\,\hat{j} + a_3\,\hat{k}$ be a constant vector, then

$$= \frac{3}{r} f'(r) + \frac{rf''(r) - f'(r)}{r^2} \frac{\mathbf{r} \cdot \mathbf{r}}{r} = \frac{3}{r} f'(r) + \frac{rf''(r) - f'(r)}{r^2} \frac{r^2}{r}$$

$$= \frac{3}{r} f'(r) + \frac{rf''(r) - f'(r)}{r} = f''(r) + \frac{2}{r} f'(r).$$

Hence, $\nabla^2 f(r) = f''(r) + \frac{2}{r} f'(r)$.

Example 8. *Prove that $r^n \mathbf{r}$ is an irrotational vector for any value of n but is solenoidal if* $n + 3 = 0$.

Solution. Since we know that if $\mathbf{a}$ is irrotational, then

$$\nabla \times \mathbf{a} = 0$$

$$\therefore \quad \nabla \times (r^n \mathbf{r}) = (\text{grad } r^n) \times \mathbf{r} + r^n \text{ curl } \mathbf{r}$$

$$= (nr^{n-1} \text{ grad } r) \times \mathbf{r} + 0 \qquad [\because \text{ curl } \mathbf{r} = 0]$$

$$= \left(nr^{n-1} \frac{\mathbf{r}}{r}\right) \times \mathbf{r} \qquad \left[\because \text{ grad } r = \frac{\mathbf{r}}{r}\right]$$

$$= nr^{n-2}\, \mathbf{r} \times \mathbf{r} = 0 \qquad [\because \text{ vector product of two same vectors is zero}]$$

Hence, $r^n \mathbf{r}$ is irrotational for any value of n.

Further since if $\mathbf{a}$ is solenoidal, then $\nabla \cdot \mathbf{a} = 0$

$$\therefore \quad \nabla \cdot (r^n \mathbf{r}) = (\text{grad } r^n) \cdot \mathbf{r} + r^n \nabla \cdot \mathbf{r}$$

$$= (nr^{n-1} \text{ grad } \mathbf{r}) \cdot \mathbf{r} + 3r^n \qquad [\because \nabla \cdot \mathbf{r} = 3]$$

$$= \left(nr^{n-1} \frac{\mathbf{r}}{r}\right) \cdot \mathbf{r} + 3r^n \qquad \left[\because \text{ grad } r = \frac{\mathbf{r}}{r}\right]$$

$$= nr^{n-2}\, \mathbf{r} \cdot \mathbf{r} + 3r^n$$

$$= nr^{n-2} r^2 + 3r^n \qquad [\because \mathbf{r} \cdot \mathbf{r} = r^2]$$

$$= nr^n + 3r^n = r^n (n + 3).$$

If $n + 3 = 0$, then $\nabla \cdot (r^n \mathbf{r}) = 0$, and hence $r^n \mathbf{r}$ is solenoidal.

• STUDENT'S ACTIVITY

1. If $r = |\vec{r}|, \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, then $\nabla f(r)\ (f'(r) = \nabla r$.

2. Profe that : $\nabla . (\vec{a} \times \vec{b}) = \vec{b} . (\nabla \times \vec{a}) - \vec{a} . (\nabla \times \vec{b})$.

• SUMMARY

- Gradient of a scalar field :

$$\text{grad } f = \nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

- Divergence of vector field

$$\text{div } \vec{V} = \nabla \cdot \vec{V} = \frac{dV_1}{dx} + \frac{dV_2}{dy} + \frac{dV_3}{dz}$$

where $\vec{V} = V_1\hat{i} + V_2\hat{j} + V_3\hat{k}$

- Curl of a Vector field

$$\text{curl } \vec{V} = \nabla \times \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix}$$

$$\vec{V} = V_1\hat{i} + V_2\hat{j} + V_3\hat{k}$$

• TEST YOURSELF–2

1. If $\mathbf{f} = x^2y\,\hat{i} - 2xz\,\hat{j} + 2yz\,\hat{k}$, find
   (i) div $\mathbf{f}$
   (ii) curl $\mathbf{f}$
   (iii) curl curl f.
2. If $\mathbf{f} = xy^2\,\hat{i} + 2x^2yz\,\hat{j} - 3yz^2\,\hat{k}$, then at the point $(1, -1, 1)$ find
   (i) div $\mathbf{f}$ (ii) curl $\mathbf{f}$
3. If $\mathbf{f} = \text{grad }(x^3 + y^3 + z^3 - 3xyz)$, find
   (i) div $\mathbf{f}$ (ii) curl $\mathbf{f}$.
4. (i) Determine the constant λ so that the vector
   $$\mathbf{f} = (x + 3y)\,\hat{i} + (y - 2z)\,\hat{j} + (x + \lambda z)\,\hat{k}$$
   is solenoidal.
   (ii) Find the constants a, b, c so that the vector
   $$\mathbf{f} = (x + 2y + az)\,\hat{i} + (bx - 3y - z)\,\hat{j} + (4x + cy + 2z)\,\hat{k}$$
   is irrotational *i.e.*, curl $\mathbf{f} = \mathbf{0}$.
5. Show that the vector $\mathbf{f} = (\sin y + z)\,\hat{i} + (x\cos y - z)\,\hat{j} + (x - y)\,\hat{k}$ is irrotational.
6. Show that $\nabla^2\left(\frac{x}{r^3}\right) = 0$.

ANSWERS

1. (i) $2y(x+1)$, (ii) $(2x + 2z)\,\hat{i} - (x^2 + 2z)\,\hat{k}$ (iii) $(2x + 2)\,\hat{j}$
2. (i) div $\mathbf{f} = 9$ (ii) curl $\mathbf{f} = -\hat{i} - 2\,\hat{k}$
3. (i) div. $\mathbf{f} = 6(x + y + z)$ (ii) curl $\mathbf{f} = \mathbf{0}$
4. (i) $\lambda = -2$ (ii) $a = 4, b = 2, c = -1$.

OBJECTIVE EVALUATION

➤ FILL IN THE BLANKS :

1. $\nabla \equiv \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$ is a vector ....
2. If $|\mathbf{r}| = r$, then ∇r is equal to ....

➤ TRUE OR FALSE :

Write 'T' for true and 'F' for false statement :

1. $\nabla f(u) = f'(u)\,\nabla u$. *(T/F)*
2. $\nabla\phi \cdot d\mathbf{r} = d\phi$. *(T/F)*
3. If $\mathbf{V}$ is solenoidal, then curl $\mathbf{V} = 0$. *(T/F)*

4. If **V** is irrotational, then div **V** = 0. (T/F)

➤ MULTIPLE CHOICE QUESTIONS :

Choose the most appropriate one :

1. Gradient of a constant scalar field is equal to :
   (a) 0 (b) 1 (c) – 1 (d) None of these.
2. ∇r^2 is equal to :
   (a) $\mathbf{r}$ (b) $2\mathbf{r}$ (c) $3\mathbf{r}$ (d) $-2\mathbf{r}$.
3. If $|\mathbf{r}| = r$, then $r \nabla r$ is equal to :
   (a) $-\mathbf{r}$ (b) $2\mathbf{r}$ (c) $\mathbf{r}$ (d) 0.
4. If $\mathbf{a}$ is a constant vector, then grad $(\mathbf{r} \cdot \mathbf{a})$ is equal to :
   (a) $\mathbf{r}$ (b) $-\mathbf{a}$ (c) 0 (d) $\mathbf{a}$.

ANSWERS

Fill in the Blanks :

1. Differential operator 2. $\nabla (fg)$

True or False :

1. T 2. T 3. F 4. F

Multiple Choice Questions :

1. (a) 2. (b) 3. (c) 4. (d)

UNIT 15

GAUSS'S, STOKE'S THEOREMS

LEARNING OBJECTIVES

- Oriented Curves
- Line, Surface and Volume Integrals
- Solved Examples
  - Test Yourself–1
- Gauss Divergence Theorem
- Solved Examples
  - Test Yourself–2
- Stoke's Theorem
- Solved Examples
  - Student Activity
  - Summary
  - *Test Yourself–3*

LEARNING OBJECTIVES

After going through this unit you will learn :

- About Gauss's and Stoke's Theorem.
- How to calculate the value of the integral using Gauss's and Stoke's Theorem.

• ORIENTED CURVES

Let us consider a curve C in space and orient the curve C by choosing one of the two directions along C as the **positive direction**, and the opposite direction along C is then called the **negative direction**.

Let A be the initial point and B the terminal point of C under the chosen orientation. Therefore we may now represent the curve C by a parametric equation

$$\mathbf{r}(s) = x(s)\,\hat{i} + y(s)\,\hat{j} + z(s)\,\hat{k}$$

where s is the arc length of C and for the point A, $s = a$ and for the point B $s = b$, hence $a \leq s \leq b$.

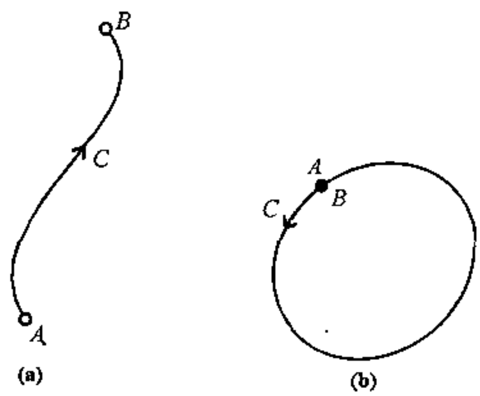

Fig. 1

(i) Closed curve. *If the point A and B coincide as shown in fig.* 1 (*b*), *then the curve is closed.*

(ii) Smooth curve. *If* $\mathbf{r}(s)$ *is continuously differentiable and its first derivative is different from zero vector for all values of s and the curve C has a unique tangent at each of its points, then the curve C is called* ***smooth curve***.

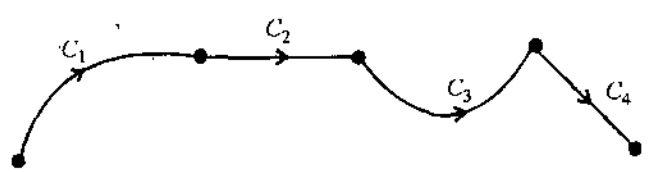

Fig. 2

(iii) Piecewise smooth curve. *A curve C which is the composition of a finite number of smooth curve, is called* ***piecewise smooth curve.***

In the adjoining fig. 2 the curve is composed of four smooth curves C_1, C_2, C_3 and C_4 hence the curve is piecewise smooth.

(iv) Smooth surface. *A surface S over each of its points a unique normal may drawn and the direction of each normal depends only on the point at which it is drawn, is called* ***smooth surface.***

(v) Piecewise smooth surface. *A surface which is composed of a finite number of smooth surface, is called* ***piecewise smooth surface.***

(vi) Simply connected domain. *A region (or domain) in which every closed curve can be shrunk to a point with out crossover the boundary of the region, is called* ***simply connected domain.*** *Otherwise the region is called* ***multiply connected*** *domain.*

(a) Simple Connected **(b) Multiply Connected**

Fig. 3

• LINE, SURFACE AND VOLUME INTEGRALS

(i) Line integrals. Let $f(x, y, z)$ be a given function which is defined at each point of the curve C and $f(x, y, z)$ is continuous function of s and let P be a point on C with co-ordinates $(x(s), y(s), z(s))$. Thus $f(x, y, z)$ is written as $f(P)$. Now divide the curve C into n parts in an arbitrary way and letting $P_0 = A$, $P_1, P_2, \ldots, P_{n-1}, P_n = B$ where A and B are the end points of the curve C.

Let us divide in the interval $a \le s \le b$ such that

$$a = s_0 < s_1 < s_2 < \ldots < s_n = b.$$

Now choose an arbitrary point between each portion *i.e.*, between A and P_1, P_1 and P_2 and so on. Let Q_1 be that point between A and P_1, Q_2 between P_1 and P_2 etc. and form the sum

$$S_n = \sum_{m=1}^{n} f(Q_m)\, \Delta s_m, \text{ where } \Delta s_m = s_m - s_{m-1}.$$

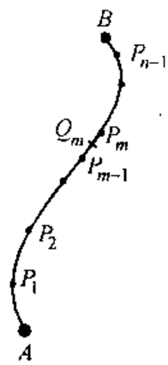

Fig. 4

Now for $n = 2, 3, 4, \ldots$, and the greatest $\Delta s_m \to 0$ as $n \to \infty$, we get a sequence of real numbers $S_2, S_3, S_4, \ldots$. *The limit of this sequence $\langle s_n \rangle$ is called the* ***line integral*** *of f along the curve C from A to B is denoted by* $\int_C f(x, y, z)\, ds$.

In most cases the representation of C will be of the the form

$$\mathbf{r}(t) = x(t)\,\hat{i} + y(t)\,\hat{j} + z(t)\,\hat{k},\ t_0 \le t \le t_1$$

then we have

$$\int_C f(x, y, z)\, ds = \int_a^b f\,[x(s), y(s), z(s)]\, ds \qquad \ldots(1)$$

and

$$\int_a^b f\,[x(s), y(s), z(s)]\, ds = \int_{t_0}^{t_1} f\,[x(t), y(t), z(t)]\, \frac{ds}{dt}\, dt. \qquad \ldots(2)$$

In particular, suppose $\mathbf{r}(t)$ is the position vector of (x, y, z), then

$$\mathbf{r}(t) = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$$

and let $t = t_0$ at A and $t = t_1$ at B and suppose

$$\mathbf{F}(x, y, z) = f_1\,\hat{i} + f_2\,\hat{j} + f_3\,\hat{k}$$

is a vector function and continuous along C. Let s be the arc length of the curve C *i.e.*, $s = \text{arc } AP$, then

$$\frac{d\mathbf{r}}{ds} = \mathbf{t}$$

is a unit tangent vector at the point $P(x, y, z)$. Thus the component of $\mathbf{f}$ along this tangent is $\mathbf{f} \cdot \dfrac{d\mathbf{r}}{ds}$. Therefore, we have

$$\int_A^B \left(\mathbf{f} \cdot \frac{d\mathbf{r}}{ds}\right) ds = \int_A^B \mathbf{f} \cdot d\mathbf{r} = \int_C \mathbf{f} \cdot d\mathbf{r}.$$

$$\therefore \qquad \int_C \mathbf{f} \cdot d\mathbf{r} = \int_C (f_1\, dx + f_2\, dy + f_3\, dz). \qquad ...(3)$$

Since $x = x(t)$, $y = y(t)$, $z = z(t)$, then

$$\int_C \mathbf{f} \cdot d\mathbf{r} = \int_{t_0}^{t_1} \left[f_1 \frac{dx}{dt} + f_2 \frac{dy}{dt} + f_3 \frac{dz}{dt}\right] dt. \qquad ...(4)$$

Hence (2) and (4) are equivalent.

REMARK

- If the curve C is simple closed curve, then the integral $\int_C \mathbf{f} \cdot d\,\mathbf{r}$ is known as the circulation.

(ii) Surface integral (double integral). Let S be a surface of finite area and let $f(x, y, z)$ be defined over this surface S which is single valued function. Now divide the whole surface S into n surface elements of areas $\Delta S_1, \Delta S_2, \ldots, \Delta S_m, \ldots, \Delta S_n$. Let us take an surface element of area ΔS_m and choose an arbitrary point P_m inside ΔS_m and form the sum

$$J_n = \sum_{m=1}^{n} f(P_m)\, \Delta S_m.$$

Now *taking the limit as $n \to \infty$ in such a way that $\Delta S_m \to 0$, then this limit if exists is called the **surface integral** of f over S and is denoted by*

$$\iint_S f(x, y, z)\, dS.$$

It can be shown that the sequence $\langle J_n \rangle$ converges and its limit is independent of the choice of subdivisions and corresponding point P_m.

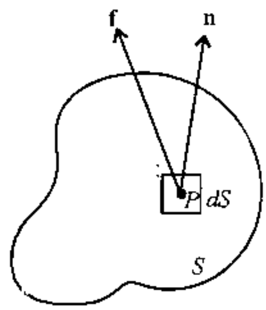

Fig. 5

In particular, let S be a piecewise smooth surface and $\mathbf{f}(x, y, z)$ is a vector function which is continuous and defined over S. Let us consider a surface element of area dS enclosing a point P and let $\mathbf{n}$ be the unit vector drawn at P outward to the element dS and normal to it which is shown in fig. 5.

Thus $\mathbf{f} \cdot \mathbf{n}$ is the normal component of $\mathbf{f}$ at P. Therefore the integral of $\mathbf{f} \cdot \mathbf{n}$ over S can be written as

$$\iint_S \mathbf{f} \cdot \mathbf{n}\, dS = \iint_S \mathbf{f} \cdot d\mathbf{S} \qquad ...(1)$$

where $\qquad d\mathbf{S} = \mathbf{n}\, dS.$

If $\mathbf{n} = l\,\hat{i} + m\,\hat{j} + n\,\hat{k}$, where l, m, n are the direction cosines of normal which makes the angles α, β and γ with the positive axis *i.e.*, $l = \cos\alpha$, $m = \cos\beta$, $n = \cos\gamma$.

Let $\mathbf{f}(x, y, z) = f_1\,\hat{i} + f_2\,\hat{j} + f_3\,\hat{k}$, then

$$\iint_S \mathbf{f} \cdot \mathbf{n}\, dS = \iint_S (f_1 \cos\alpha + f_2 \cos\beta + f_3\, \gamma)\, dS.$$

Since we have $\cos\alpha\, dS = dy\, dz$, $\cos\beta\, ds = dz\, dx$, $\cos\gamma\, dS = dx\, dy$.

$$\therefore \qquad \iint_S \mathbf{f} \cdot \mathbf{n}\, dS = \iint_S (f_1\, dy\, dz + f_2\, dz\, dx + f_3\, dx\, dy).$$

(iii) Volume integral. Let V be a volume enclosed by a surface S and let $f(x, y, z)$ be a point function defined over V. Now divide the volume V into n subvolume element of volumes $\Delta V_1, \Delta V_2, \ldots, \Delta V_{n-1}, \Delta V_n$ and choose an arbitrary point $P_m\,(x_m, y_m, z_m)$ in each of elements ΔV_m such that $f(P_m) = f(x_m, y_m, z_m)$ and form the sum

$$J_n = \sum_{m=1}^{n} f(P_m)\, \Delta V_m$$

for $n = 2, 3, 4, \ldots$, an taking greatest $\Delta V_m \to 0$ as $n \to \infty$. Then we get the sequence $J_2, J_3, J_4, \ldots$. *If the limit of this sequence $\langle J_n \rangle$ exists, then this limit is called the **volume integral** of f over the volume V which is denoted by*

$$\iiint_V f(x, y, z)\, dV.$$

This limit is independent of the choice of the subdivision of V, if V is piecewise smooth volume. Therefore we can take the volume elements in the form of urboids whose edges are parallel to the co-ordinate axis. Then $dV = dx\, dy\, dz$, hence

$$\iiint_V f(x, y, z)\, dV = \iiint_V f(x, y, z)\, dx\, dy\, dz.$$

REMARK

➤ If $\mathbf{f}(x, y, z)$ is a vector function, then the volume integral of $\mathbf{f}(x, y, z)$ over V is

$$\iiint_V \mathbf{f}(x, y, z)\, dV.$$

• SOLVED EXAMPLES

Example 1. *Evaluate* $\int_C xy^3\, ds$, *where C is the segment of the line* $y = 2x$ *in the xy-plane from* $A(-1, -2, 0)$ *to* $B(1, 2, 0)$.

Solution. Taking the curve C in the following form

$$\mathbf{r}(t) = t\hat{i} + 2t\hat{j}, (-1 \le t \le 1).$$

$$\therefore \quad \frac{d\mathbf{r}}{dt} = \hat{i} + 2\hat{j}.$$

$$\therefore \quad \frac{ds}{dt} = \sqrt{\left(\frac{d\mathbf{r}}{dt} \cdot \frac{d\mathbf{r}}{dt}\right)} = \sqrt{[(\hat{i} + 2\hat{j}) \cdot (\hat{i} + 2\hat{j})]} = \sqrt{5}.$$

On C, $xy^3 = t(2t)^3 = 8t^4$ and therefore

$$\int_C xy^2\, ds = 8\sqrt{5}\int_{-1}^{1} t^4\, dt = 8\sqrt{5}\left[\frac{t^5}{5}\right]_{-1}^{1} = \frac{16\sqrt{5}}{5}.$$

Example 2. *Evaluate* $\int \mathbf{F} \cdot d\mathbf{r}$ *along the curve* $C : x^2 + y^2 = 1$, $z = 1$ *in the positive direction from* $(0, 1, 1)$ *to* $(1, 0, 1)$, *where*

$$\mathbf{F}(x, y, z) = (2x + yz)\hat{i} + xz\hat{j} + (xy + 2z)\hat{k}.$$

Solution. Let $A(0, 1, 1)$ and $B(1, 0, 1)$ be the points on the curve C :

$$x^2 + y^2 = 1,\ z = 1 \ \text{ and } \ \mathbf{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\mathbf{F} \cdot d\mathbf{r} = [(2x + yz)\hat{i} + xz\hat{j} + (xy + 2z)\hat{k}] \cdot (dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k})$$
$$= (2x + yz)\, dx + xz\, dy + (xy + 2z)\, dz.$$

$$\therefore \quad \int_C \mathbf{F} \cdot d\mathbf{r} = \int_A^B [(2x + yz)\, dx + xz\, dy + (xy + 2z)\, dz]$$

$$= \int_0^1 (2x + y)\, dx + \int_1^0 x\, dy \qquad (\because\ z = 1 \Rightarrow dz = 0)$$

$$= \int_0^1 (2x + \sqrt{1 - x^2})\, dx + \int_1^0 \sqrt{1 - y^2}\, dy \qquad (\because\ x^2 + y^2 = 1)$$

$$= \int_0^1 2x\, dx + \int_0^1 \sqrt{1 - x^2}\, dx - \int_0^1 \sqrt{1 - y^2}\, dy$$

$$\left(\because\ \int_a^b f(x)\, dx = -\int_b^a f(x)\, dx\right)$$

$$= \left[x^2\right]_0^1 = 1. \qquad \left(\because\ \int_a^b f(x)\, dx = \int_a^b f(t)\, dt\right)$$

Example 3. *Evaluate* $\int_C \mathbf{F} \cdot d\mathbf{r}$, *where* $\mathbf{F} = 3xy\hat{i} - y^2\hat{j}$ *and C is the curve* $y = 2x^2$ *with xy-plane from* $(0, 0)$ *to* $(1, 2)$.

Solution. The parametric equations of the given curve *i.e.*, the parabola $y = 2x^2$ can be taken as $x = t$, $y = 2t^2$.

At the point $(0, 0)$, $x = 0$ and so $t = 0$ and at the point $(1, 2)$, $x = 1$ and so $t = 1$.

Again $\quad \dfrac{dx}{dt} = 1 \quad$ and $\quad \dfrac{dy}{dt} = 4t$

and $\qquad d\mathbf{r} = dx\,\hat{i} + dy\,\hat{j}$ $\qquad [\because\ \mathbf{r} = x\,\hat{i} + y\,\hat{j}]$

$$\therefore \quad \int_C \mathbf{F}\cdot d\mathbf{r} = \int_C (3xy\,\hat{i} - y^2\,\hat{j}) \cdot (dx\,\hat{i} + dy\,\hat{j})$$

$$= \int_C (3xy\,dx - y^2\,dy) = \int_{t=0}^{1} \left(3xy\frac{dx}{dt} - y^2\frac{dy}{dt}\right) dt$$

$$= \int_{t=0}^{1} (3\,.\,t\,.\,2t^2\,.\,1 - 4t^4\,.\,4t)\,dt$$

$$= \int_{t=0}^{1} (6t^3 - 16t^5)\,dt = \left[6\,.\,\frac{t^4}{4} - 16\,.\,\frac{t^6}{6}\right]_{t=0}^{1}$$

$$= \frac{6}{4} - \frac{16}{6} = \frac{3}{2} - \frac{8}{3} = -\frac{7}{6}.$$

Example 4. *Evaluate* $\int_C \mathbf{F}\cdot d\mathbf{r}$ *where* $\mathbf{F} = xy\,\hat{i} + yz\,\hat{j} + zx\,\hat{k}$ *and curve* C *is* $\mathbf{r} = t\,\hat{i} + t^2\,\hat{j} + t^3\,\hat{k}$ *and* $-1 \le t \le 1$.

Solution. Since $\mathbf{r} = t\,\hat{i} + t^2\,\hat{j} + t^3\,\hat{k}$ is given but $\mathbf{r} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$, then $x = t,\ y = t^2,\ z = t^3$

and $\qquad \dfrac{d\mathbf{r}}{dt} = \hat{i} + 2t\,\hat{j} + 3t^2\,\hat{k}.$

$$\therefore \quad \int_C \mathbf{F}\cdot d\mathbf{r} = \int_{-1}^{1} \left(\mathbf{F}\cdot\frac{d\mathbf{r}}{dt}\right) dt. \qquad ...(1)$$

and $\qquad \mathbf{F} = xy\,\hat{i} + yz\,\hat{j} + zx\,\hat{k} = t^3\,\hat{i} + t^5\,\hat{j} + t^4\,\hat{k}.$

$$\therefore \quad \mathbf{F}\cdot\frac{d\mathbf{r}}{dt} = (t^3\,\hat{i} + t^5\,\hat{j} + t^4\,\hat{k})\cdot(\hat{i} + 2t\,\hat{j} + 3t^2\,\hat{k})$$

$$= t^3 + 2t^6 + 3t^6 = t^3 + 5t^6.$$

$\therefore$ From (1)

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_{-1}^{1} (t^3 + 5t^6)\,dt = \left[\frac{t^4}{4} + \frac{5t^7}{7}\right]_{-1}^{1} = \frac{1}{4} - \frac{1}{4} + \frac{10}{7} = \frac{10}{7}.$$

• TEST YOURSELF–1

1. Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F} = x^2\,\hat{i} + y^3\,\hat{j}$ and curve C is the arc of the parabola $y = x^2$ in the xy-plane from (0, 0) to (1, 1).
2. If $\mathbf{F} = (3x^2 + 6y)\,\hat{i} - 14yz\,\hat{j} + 20xz^2\,\hat{k}$, then evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$ from (0, 0, 0) to (1, 1, 1) along the curve C

   $$x = t,\ y = t^2,\ z = t^3.$$
3. If $\mathbf{F} = y\,\hat{i} - x\,\hat{j}$, evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$ from (0, 0) to (1, 1) along the following paths C :

   (i) The parabola $y = x^2$.
   (ii) The straight lines from (0, 0) to (1, 0) and then to (1, 1)
   (iii) The straight line joining (0, 0) and (1, 1).
4. Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F} = yz\,\hat{i} + zx\,\hat{j} + xy\,\hat{k}$ and the curve C is the portion of the curve $\mathbf{r} = a\cos t\,\hat{i} + b\sin t\,\hat{j} + ct\,\hat{k}$ from $t = 0$ to $t = \pi/2$.

ANSWERS

1. $\dfrac{7}{12}$ **2.** 5 **3.** (i) $-\dfrac{1}{3}$, (ii) -1, (iii) 0 **4.** 0

• 15.3. GAUSS DIVERGENCE THEOREM

Theorem. *Let V be the volume enclosed by a closed and bounded piecewise smooth surface S and let* F(x, y, z) *be a vector function which is continuous and has continuous first partial derivatives in V. Then*

$$\iiint_V \operatorname{div} \mathbf{F}\, dV = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS \qquad ...(1)$$

where **n** *is the outward unit normal vector the surface S.*

Cartesian form of (1). Let $\mathbf{F} = F_1 \hat{i} + F_2 \hat{j} + F_3 \hat{k}$ and suppose the outward unit normal vector **n** makes the angle α, β and γ with the positive axes of x, y, z respectively. Then $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ are the direction-cosines of **n**, we have

$$\mathbf{n} = \cos\alpha\, \hat{i} + \cos\beta\, \hat{j} + \cos\gamma\, \hat{k}.$$

$$\therefore \qquad \mathbf{F} \cdot \mathbf{n} = F_1 \cos\alpha + F_2 \cos\beta + F_3 \cos\gamma$$

and $$\operatorname{div} \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \text{ and } dV = dx\, dy\, dz.$$

Thus (1) becomes

$$\iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right) dx\, dy\, dz = \iint_S (F_1 \cos\alpha + F_2 \cos\beta + F_3 \cos\gamma)\, dS. \qquad ...(2)$$

Proof of divergence theorem. We shall first prove the theorem for a special volume V which is bounded by a piecewise smooth oriented surface S and has the property that any straight line drawn parallel to any one of the co-ordinate axes and intersecting V has only one point (or one segement) in common with V.

Then V can be represented by

$$f(x, y) \le z \le g(x, y) \qquad ...(3)$$

where $(x, y) \in R$. This R is the orthogonal projection of V in the xy-plane. Obviously $z = f(x, y)$ represents the lower part S_2 of S and $z = g(x, y)$ represents the upper part S_1 of S and there may be a remaining vertical part S_3 of S has shown in fig. 15.

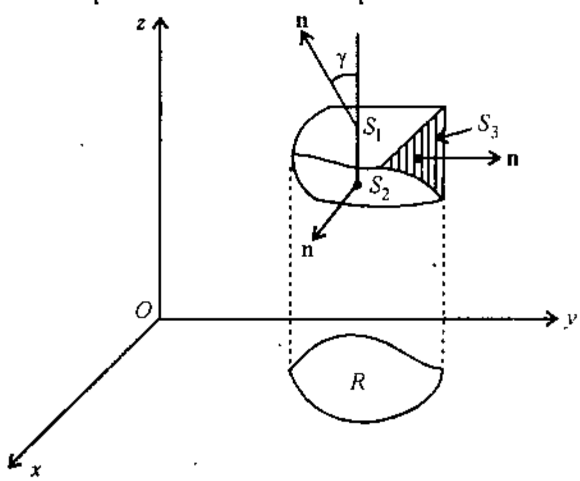

Fig. 7

First we prove that

$$\iiint_V \frac{\partial F_3}{\partial z}\, dx\, dy\, dz = \iint_S F_3 \cos\gamma\, dS. \qquad ...(4)$$

Since F(x, y, z) is continuously differentiable in V and using (3), we have

$$\iiint_V \frac{\partial F_3}{\partial z}\, dx\, dy\, dz = \iint_R \left[\int_{z=f(x,y)}^{z=g(x,y)} \frac{\partial F_3}{\partial z}\, dz \right] dx\, dy$$

$$= \iint_R \left[F_3(x, y, z) \right]_{z=f(x,y)}^{z=g(x,y)} dx\, dy.$$

$$\therefore \quad \iiint_V \frac{\partial F_3}{\partial z}\, dx\, dy\, dz = \iint_R F_3\, [x, y, g(x, y)]\, dx\, dy - \iint_R F_3\, [x, y, f(x, y)]\, dx\, dy. \qquad ...(5)$$

Now, we have

$$\iint_S F_3 \cos\gamma\, dS = \iint_{S_1} F_3 \cos\gamma\, dS + \iint_{S_2} F_3 \cos\gamma\, dS + \iint_{S_3} F_3 \cos\gamma\, dS. \qquad ...(6)$$

Since on the portion S_3 of S the outward drawn unit normal vector makes an angle $\pi/2$ with z-axis, then $\cos\gamma = 0$ on S_3. Thus

$$\iint_{S_3} F_3 \cos\gamma\, dS = \iint_{S_3} 0 \,.\, dS = 0. \qquad ...(7)$$

On the portion S_1 of S the outward drawn unit normal makes an acute angle γ with positive z-axis and the equation of S_1 is $z = g(x, y)$. Then

$$\cos\gamma\, dS = dx\, dy.$$

$$\therefore \quad \iint_{S_1} F_3 \cos\gamma\, dS = \iint_R F_3\, [x, y, g(x, y)]\, dx\, dy \qquad ...(8)$$

and on the portion S_2 of S the outward drawn unit normal vector makes obtuse angle γ with positive z-axis and the equation of S_2 is $z = f(x, y)$. Then

$$\cos\gamma\, dS = -\, dx\, dy.$$

$$\therefore \quad \iint_{S_2} F_3 \cos\gamma\, dS = -\iint_R F_3\, [x, y, f(x, y)]\, dx\, dy. \qquad ...(9)$$

Using (7), (8) and (9) the equation (6) becomes

$$\iint_S F_3 \cos\gamma\, dS = \iint_R F_3\, [x, y, g(x, y)]\, dx\, dy - \iint_R F_3\, [x, y, f(x, y)]\, dx\, dy. \qquad ...(10)$$

From (5) and (10), we obtain

$$\iiint_V \frac{\partial F_3}{\partial z}\, dx\, dy\, dz = \iint_S F_3 \cos\gamma\, dS. \qquad ...(11)$$

Similarly taking the projection of S on the other co-ordinate planes, we have

$$\iiint_V \frac{\partial F_1}{\partial x}\, dx\, dy\, dz = \iint_S F_1 \cos\alpha\, dS \qquad ...(12)$$

and

$$\iiint_V \frac{\partial F_2}{\partial y}\, dx\, dy\, dz = \iint_S F_2 \cos\beta\, dS. \qquad ...(13)$$

Now adding (11), (12) and (13), we get

$$\iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dx\, dy\, dz = \iint_S (F_1 \cos\alpha + F_2 \cos\beta + F_3 \cos\gamma)\, dS$$

or

$$\iiint_V \operatorname{div} \mathbf{F}\, dV = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS.$$

Hence proved the theorem for special region V.

Gauss Divergence Theorem for any Region V :

Let V be any volume which is not a special volume but can be subdivided into finitely many special volumes by drawing auxiliary surfaces. Now apply above theorem to each special volume and adding the result for each part. On the left hand side of this result we obtain the sum of volume integral over parts of V and which gives the volume integral over V. On the right hand side we obtain the sum of surface integral over auxiliary surfaces plus the sum of the remaining surface integral. In this side the surface integral over auxiliary surfaces cancel in pairs and the remaining surface integrals give the surface integral over the whole boundary S of V.

• SOLVED EXAMPLES

Example 1. *If ϕ and ψ both are two harmonic scalar point functions and are continuously differentiable in V enclosed by S. Then*

$$\iint_S \left(\phi \frac{\partial\psi}{\partial n} - \psi \frac{\partial\phi}{\partial n}\right) dS = 0.$$

Solution. From example 1, we have

$$\iiint_V (\phi\, \nabla^2\psi - \psi\, \nabla^2\phi)\, dV = \iint_S (\phi\, \nabla\psi - \psi\, \nabla\phi) \cdot \mathbf{n}\, dS.$$

Since $\nabla\phi = \frac{\partial\phi}{\partial n}\mathbf{n}$, $\nabla\psi = \frac{\partial\psi}{\partial n}\mathbf{n}$. Then we have

$$\iint_S \left(\phi \frac{\partial\psi}{\partial n} - \psi \frac{\partial\phi}{\partial n}\right) dS = \iiint_V (\phi\, \nabla^2\psi - \psi \cdot \nabla^2\phi)\, dV. \qquad (\because\ \mathbf{n} \cdot \mathbf{n} = 1)$$

Further since ϕ and ψ both are harmonic so that $\nabla^2\phi = 0 = \nabla^2\psi$ hence we obtain

$$\iint_S \left(\phi \frac{\partial\psi}{\partial n} - \psi \frac{\partial\phi}{\partial n}\right) dS = 0.$$

Example 2. *Prove that* $\iiint_V \nabla\phi\, dV = \iint_S \phi\, \mathbf{n}\, dS.$

Solution. From Gauss's divergence theorem, we have

$$\iiint_V \text{div}\, \mathbf{F}\, dV = \iint_S \mathbf{F}\cdot\mathbf{n}\, dS. \quad ...(1)$$

Now assuming $\mathbf{F} = \phi\mathbf{a}$, where $\mathbf{a}$ is a constant vector, then

$$\text{div}\, \mathbf{F} = \nabla\cdot(\phi\mathbf{a}) = \nabla\phi\cdot\mathbf{a} + \phi\nabla\cdot\mathbf{a} = \nabla\phi\cdot\mathbf{a} \quad (\because\ \nabla\cdot\mathbf{a} = 0)$$

and $$\mathbf{F}\cdot\mathbf{n} = (\phi\mathbf{a})\cdot\mathbf{n}.$$

From (1), we have

$$\iiint_V \nabla\phi\cdot\mathbf{a}\, dV = \iint_S \phi\mathbf{a}\cdot\mathbf{n}\, dS$$

or $$\mathbf{a}\cdot\iiint_V \nabla\phi\, dV = \mathbf{a}\cdot\iint_S \phi\,\mathbf{n}\, dS$$

or $$\mathbf{a}\cdot\left[\iiint_V \nabla\phi\, dV - \iint_S \phi\mathbf{n}\, dS\right] = 0.$$

Since $\mathbf{a}$ is an arbitrary so we get

$$\iiint_V \nabla\phi\, dV = \iint_S \phi\mathbf{n}\, dS.$$

Example 3. *Using Gauss's divergence theorem evaluate*

$$\iint_S [(x+z)\, dy\, dz + (y+z)\, dz\, dx + (x+y)\, dx\, dy]$$

where S is the surface of the sphere $x^2 + y^2 + z^2 = 4$.

Solution. By Gauss's divergence theorem, we have

$$\iint_S [(x+z)\, dy\, dz + (y+z)\, dz\, dx + (x+y)\, dx\, dy]$$

$$= \iiint_V \left[\frac{\partial}{\partial x}(x+z) + \frac{\partial}{\partial y}(y+z) + \frac{\partial}{\partial z}(x+y)\right] dx\, dy\, dz = \iiint_V 2\, dx\, dy\, dz$$

$$= 2\iiint_V dV, \text{ where } V \text{ is the volume of the sphere } x^2 + y^2 + z^2 = 4$$

$$= 2\left[\frac{4}{3}\pi\,(2)^3\right] = \frac{64}{3}\pi.$$

• TEST YOURSELF–2

1. For any closed surface S, prove that
$$\iint_S \text{curl}\, \mathbf{F}\cdot\mathbf{n}\, dS = 0.$$
2. If $\mathbf{F} = \nabla\phi$ and $\nabla^2\phi = 0$, show that for a closed surface S
$$\iiint_V \mathbf{F}^2\, dV = \iint_S \phi\,\mathbf{F}\cdot\mathbf{n}\, dS.$$
3. If ϕ and ψ are harmonic in V and $\dfrac{\partial\phi}{\partial n} = \dfrac{\partial\psi}{\partial n}$ on S, then $\phi = \psi + c$ in V where C is a constant.
4. For any closed surface S, show that
(i) $\iint_S \mathbf{n}\, dS = \mathbf{0}$, (ii) $\iint_S \mathbf{r}\times\mathbf{n}\, dS = \mathbf{0}$.
4. Using the divergence theorem, show that the volume V of a region bounded by a surface S is
$$V = \iint_S x\, dy\, dz = \iint_S y\, dz\, dx = \iint_S z\, dx\, dy$$
$$= \frac{1}{3}\iint_S (x\, dy\, dz + y\, dz\, dx + z\, dx\, dy).$$

• 15.4. STOKE'S THEOREM

Let S be a piecewise smooth oriented surface in space bounded by a piecewise smooth simple closed curve C. Let $\mathbf{F}(x, y, z)$ *be a continuous vector function having continuous first partial derivatives in a region of space in which S lies interior. Then*

$$\iint_S (\text{curl}\, \mathbf{F})\cdot\mathbf{n}\, dS = \int_C \mathbf{F}\cdot d\mathbf{r} \quad ...(1)$$

where C is taken in counterclockwise direction and $\mathbf{n}$ *is a outward drawn unit normal vector to S.*

Proof. We shall first prove Stoke's theorem for a surface S which represented simultaneously in the forms of

$$z = f(x, y),\ y = g(x, z),\ x = h(y, z)$$

where f, g, h are continuous functions and having continuous first order partial derivatives. Let

$$\mathbf{n} = \cos\alpha\,\hat{i} + \cos\beta\,\hat{j} + \cos\gamma\,\hat{k}$$

be outrward drawn unit normal to the surface S which makes the angles α, β, γ with positive co-ordinate axes respectively, and let

$$\mathbf{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}.$$

$$\therefore \qquad \nabla\times\mathbf{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \hat{i}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) + \hat{j}\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) + \hat{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

and

$$(\nabla\times\mathbf{F})\cdot\mathbf{n} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\cos\alpha + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\cos\beta + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\cos\gamma.$$

Let $P(x, y, z)$ be any point on C whose position vector is

$$\mathbf{r} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}.$$

$$\therefore \qquad d\mathbf{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}.$$

Thus $\qquad \mathbf{F}\cdot d\mathbf{r} = F_1\,dx + F_2\,dy + F_3\,dz.$

Now the equation (1) becomes

$$\iint_S \left[\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\cos\alpha + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\cos\beta + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\cos\gamma\right]$$

$$= \int_C (F_1\,dx + F_2\,dy + F_3\,dz). \qquad \text{...(2)}$$

First, we shall prove that

$$\iint_S \left(\frac{\partial F_1}{\partial z}\cos\beta - \frac{\partial F_1}{\partial y}\cos\gamma\right) = \int_C F_1\,dx. \qquad \text{...(3)}$$

Let R be the orthogonal projection of S in the xy-plane and C^* be its boundary which is oriented in positive direction as shown in fig. 8.

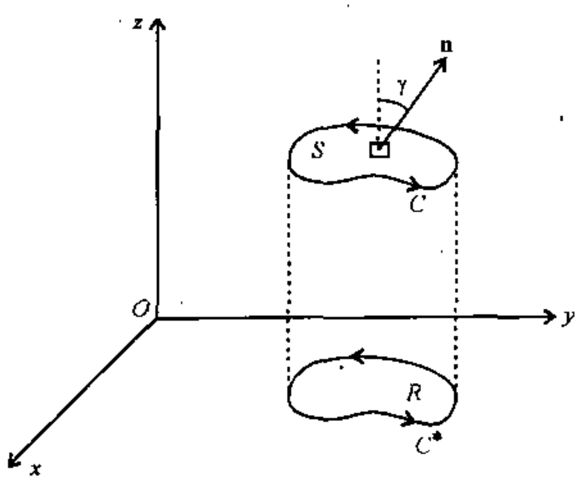

Fig. 8

Using the representation $z = f(x, y)$ of S, we may write the line integral over C as a line integral over C^* as follow :

$$\int_C F_1(x, y, z)\,dx = \int_{C^*} F_1[x, y, f(x, y)]\,dx = \int_{C^*} [F_1[x, y, f(x, y)]\,dx + 0\,dy].$$

We now apply Green's theorem in the plane to the functions $F_1[x, y, f(x, y)]$ and 0. Then we have

$$\int_C F_1(x, y, z)\, dx = -\iint_R \frac{\partial F_1}{\partial y}\, dx\, dy.$$

But $$\frac{\partial F_1 [x, y, f(x, y)]}{\partial y} = \frac{\partial F_1 (x, y, z)}{\partial y} + \frac{\partial F_1 (x, y, z)}{\partial z} \cdot \frac{\partial f}{\partial y} \qquad [\because\ z = f(x, y$$

$$\therefore \quad \int_C F_1(x, y, z)\, dx = -\iint_R \left(\frac{\partial F_1}{\partial y} + \frac{\partial F_1}{\partial z}\frac{\partial f}{\partial y}\right) dx\, dy. \qquad ...(4$$

Now we shall prove that integral on R.H.S. of (4) is equal to the integral on L.H.S. of (3). For this let us consider

$$\phi(x, y, z) = z - f(x, y) = 0.$$

$$\therefore \quad \text{grad } \phi = \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}$$

$$= -\frac{\partial f}{\partial x}\hat{i} - \frac{\partial f}{\partial y} + \hat{k} \qquad \left(\because\ \frac{\partial \phi}{\partial x} = -\frac{\partial f}{\partial x}, \frac{\partial \phi}{\partial y} = -\frac{\partial f}{\partial y}, \frac{\partial \phi}{\partial z} = 1\right)$$

Let the length of grad ϕ be a

$\therefore \quad a = |\text{grad } \phi|$.

Since we know that grad ϕ is perpendicular to the surface S. Therefore, we have

$$\mathbf{n} = \pm \frac{\text{grad } \phi}{|\text{grad } \phi|} = \pm \frac{\text{grad } \phi}{a}.$$

But the components of both $\mathbf{n}$ and grad ϕ in the positive direction of z-axis are positive. Thus

$$\mathbf{n} = +\frac{\text{grad } \phi}{a} = \frac{1}{a}\left(-\frac{\partial f}{\partial x}\hat{i} - \frac{\partial f}{\partial y}\hat{j} + \hat{k}\right).$$

Since $\mathbf{n} = \cos\alpha\,\hat{i} + \cos\beta\,\hat{j} + \cos\gamma\,\hat{k}$, therefore on comparing these twos, we get

$$\cos\alpha = -\frac{1}{a}\frac{\partial f}{\partial x},\ \cos\beta = -\frac{1}{a}\frac{\partial f}{\partial y},\ \cos\gamma = \frac{1}{a}.$$

Since $\quad \cos\gamma\, dS = dx\, dy.$

$$\therefore \quad dS = \frac{dx\, dy}{\cos\gamma} = a\, dx\, dy \qquad \left(\because\ \cos\gamma = \frac{1}{a}\right)$$

$$\therefore \quad \iint_S \left(\frac{\partial F_1}{\partial z}\cos\beta - \frac{\partial F_1}{\partial y}\cos\gamma\right) dS = \iint_R \left[\frac{\partial F_1}{\partial z}\left(-\frac{1}{a}\frac{\partial f}{\partial y}\right) - \frac{\partial F_1}{\partial y}\frac{1}{a}\right] a\, dx\, dy$$

$$= -\iint_R \left(\frac{\partial F_1}{\partial y} + \frac{\partial F_1}{\partial z}\frac{\partial f}{\partial y}\right) dx\, dy. \qquad ...(5)$$

Thus from (4) and (5), we get

$$\int_C F_1\, dx = \iint_S \left(\frac{\partial F_1}{\partial z}\cos\beta - \frac{\partial F_1}{\partial y}\cos\gamma\right) dS. \qquad ...(6)$$

Similarly using the representation $y = g(x, z)$ and $x = h(y, z)$ and having the projection on other co-ordinate planes, we obtain

$$\int_C F_2\, dy = \iint_S \left(\frac{\partial F_2}{\partial x}\cos\gamma - \frac{\partial F_2}{\partial z}\cos\alpha\right) dS \qquad ...(7)$$

and $$\int_C F_3\, dy = \iint_S \left(\frac{\partial F_3}{\partial y}\cos\alpha - \frac{\partial F_3}{\partial x}\cos\beta\right) dS. \qquad ...(8)$$

Adding (6), (7) and (8), we get

$$\int_C (F_1\, dx + F_2\, dy + F_3\, dz)$$

$$= \iint_S \left[\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\cos\alpha + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\cos\beta + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\cos\gamma\right]$$

or $$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS.$$

This proves Stoke's theorem for the surface S which can be represented simultaneously by $z = f(x, y)$, $y = g(x, z)$ and $x = h(y, z)$.

The proof of this theorem can be extended to a surface S which does not satisfy above conditions but can be decomposed into finitely many surfaces $S_1, S_2, \ldots, S_n$ whose boundary are $C_1, C_2, \ldots, C_n$. To each surface this theorem is applied as follows :

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \iint_{S_1} (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS$$

$$\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \iint_{S_2} (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS,$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$\int_{C_n} \mathbf{F} \cdot d\mathbf{r} = \iint_{S_n} (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS.$$

On adding, we get

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r} + \ldots + \int_{C_n} \mathbf{F} \cdot d\mathbf{r}$$

$$= \iint_{S_1} (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS + \iint_{S_2} (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS + \ldots + \iint_{S_n} (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS$$

or
$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS.$$

Hence Stoke's theorem is proved for any surface S enclosed by a closed curve C.

• SOLVED EXAMPLES

Example 1. *Prove that* $\int_C \phi\, d\mathbf{r} = \iint_S dS \times \nabla\phi$.

Solution. Let $\mathbf{F} = \phi\mathbf{a}$, where $\mathbf{a}$ be any arbitrary constant vector. Then by Stokes's theorem

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS.$$

$$\therefore \quad \int_C (\phi\mathbf{a}) \cdot d\mathbf{r} = \iint_S (\nabla \times (\phi\mathbf{a})) \cdot \mathbf{n}\, dS$$

$$= \iint_S (\nabla\phi \times \mathbf{a} + \phi\nabla \times \mathbf{a}) \cdot \mathbf{n}\, dS$$

$$= \iint_S (\nabla\phi \times \mathbf{a}) \cdot \mathbf{n}\, dS \qquad (\because \ \nabla \times \mathbf{a} = 0)$$

$$= \iint_S (\nabla\phi \times \mathbf{a}) \cdot d\mathbf{S}.$$

$$\therefore \quad \int_C \mathbf{a} \cdot (\phi\, d\mathbf{r}) = \iint_S \mathbf{a} \cdot (d\mathbf{S} \times \nabla\phi)$$

or
$$\mathbf{a} \cdot \left[\int_C \phi\, d\mathbf{r} - \iint_S d\mathbf{S} \times \nabla\phi\right] = 0.$$

Since $\mathbf{a}$ is an arbitrary constant vector, then

$$\int_C \phi\, d\mathbf{r} = \iint_S d\mathbf{S} \times \nabla\phi.$$

Example 2. *Using Stoke's theorem prove that* :
(*i*) *div curl* $\mathbf{F} = 0$, (*ii*) *curl grad* $\phi = 0$.

Solution. (i) Let V be any volume enclosed by a closed surface S. Then by Gauss's divergence theorem

$$\iiint_V \text{div}\,(\text{curl}\,\mathbf{F})\, dV = \iint_S (\text{curl}\,\mathbf{F}) \cdot \mathbf{n}\, dS. \qquad \ldots(1)$$

Now divide the surface S into S_1 and S_2 in a closed curve C. Then

$$\iint_S (\text{curl}\,\mathbf{F}) \cdot \mathbf{n}\, dS = \iint_{S_1} (\text{curl}\,\mathbf{F}) \cdot \mathbf{n}\, dS_1 + \iint_{S_2} (\text{curl}\,\mathbf{F}) \cdot \mathbf{n}\, dS_2.$$

Using Stoke's theorem, we get

$$\iint_S (\text{curl}\,F) \cdot \mathbf{n}\, dS = \int_C \mathbf{F} \cdot d\mathbf{r} - \int_C \mathbf{F} \cdot d\mathbf{r}$$

$$= 0.$$ (Negative sign is taken because positive direction along the boundaries of two surfaces are opposite)

Thus the equation (1) becomes

$$\iiint_V \text{div}\,(\text{curl}\,\mathbf{F})\, dV = 0. \qquad \ldots(2)$$

Since the equation (2) is true for all volume V, hence

$$\text{div curl } \mathbf{F} = 0.$$

(ii) Let S be any surface enclosed by a simple closed curve C. Then by Stoke's theorem

$$\iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{n}\, dS = \int_C \mathbf{F} \cdot d\mathbf{r}.$$

Let $\mathbf{F} = \text{grad } \phi.$

$\therefore$ $\iint_S (\text{curl grad } \phi) \cdot \mathbf{n}\, dS = \int_C \text{grad } \phi \cdot d\mathbf{r}.$ ...(1)

Since $\mathbf{r} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}.$

$\therefore$ $d\mathbf{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}.$

Now $\text{grad } \phi \cdot d\mathbf{r} = \left(\frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}\right) \cdot (dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k})$

$$= \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz = d\phi.$$

$\therefore$ $\int_C \text{grad } \phi \cdot d\mathbf{r} = \int_C d\phi = 0$ ($\because$ C is closed curve)

Thus the equation (1) becomes

$$\iint_S (\text{curl grad } \phi) \cdot \mathbf{n}\, dS = 0. \quad \text{...(2)}$$

Since S is an arbitrary surface, and the equation (2) holds for any S. Then we have

$$\text{curl grad } \phi = \mathbf{0}.$$

• STUDENT ACTIVITY

1. Using Gauss's theorem, prove that

$$\iiint_V \nabla\phi\, dV = \iint_S \phi\, \vec{n}\, dS.$$

2. Using Stoke's theorem, prove that

$$\int_C \phi\, d\vec{r} = \iint_S dS \times \nabla\phi$$

• SUMMARY

- **Line, Surface and Volume Integral :**

(i) $\int_C f(x, y, z)\, ds = \int_a^b f(x(s), y(s), z(s))\, ds$

(ii) $\iint_S \vec{f}.\vec{n}dS = \iint_S (f_1 dydz + f_2\, dzdx + f_3 dxdy)$

(iii) $\iiint_V f(x, y, z)\, dV = \iiint_V \vec{F}.\vec{n}dS$

- **Gauss's Theorem :** Let V be the volume enclosed by a closed and bounded piecewise smooth surface S and let $\vec{F}(x, y, z)$ be a vector function which is continuous and has continuous find partial derivative in V. Then

$$\iiint_V \text{div}\, \vec{F}dV = \iint_S \vec{F}.\vec{n}dS$$

where $\vec{n}$ is the outward unit normal vector to the surface S.

- **Stoke's theorem :** Let S be a piecewise smooth oriented surface in space bounded by a piecewise smooth simple closed curve C. Let $\vec{F}(x, y, z)$ be a continuous vector function having continuous first partial derivatives in a region of space in which S lies interior. Then

$$\iint_S (\text{curl}\, \vec{F}).\vec{n}dS = \int_C \vec{F}.d\vec{r}$$

where C is taken in antidarkwise direction and $\vec{n}$ is the outward unit vector normal to the surface S.

• TEST YOURSELF–3

1. Prove that $\int_C \mathbf{r} \cdot d\mathbf{r} = 0$.
2. Prove that $\int_C \phi\, \nabla\psi \cdot d\mathbf{r} = \iint_S [\nabla\phi \times \nabla\psi] \cdot \mathbf{n}\, dS$.
3. Prove that $\int_C \phi\, \nabla\psi \cdot d\mathbf{r} = -\int_C \psi\, \nabla\phi \cdot d\mathbf{r}$.

OBJECTIVE EVALUATION

➤ FILL IN THE BLANKS :

1. The integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ is called ....
2. If $\nabla^2\phi = 0$, $\nabla^2\psi = 0$, then $\iint_S \left(\phi \frac{\partial\psi}{\partial n} - \psi \frac{\partial\phi}{\partial n}\right) dS =$ ....
3. If S is a closed surface, then $\iint_S \text{curl}\, \mathbf{F} . \mathbf{n}\, dS =$ ....

➤ TRUE OR FALSE :

Write 'T' for true and 'F' for false statement :

1. The value of the integral $\iint_S \mathbf{r} \cdot \mathbf{n}\, dS$, where S is a closed surface is $3V$, where V is enclosed by S. *(T/F)*
2. If ϕ is harmonic in V, then $\iint_S \phi \frac{\partial\phi}{\partial n} dS = \iiint_V |\nabla\phi|^2\, dV$. *(T/F)*
3. $\iint_S (\mathbf{F} \bullet \mathbf{n})\, dS = \iiint \text{div}\, \mathbf{F}\, dV$. *(T/F)*
4. Any integral which is evaluated along a curve is called surface integral. *(T/F)*

➤ MULTIPLE CHOICE QUESTIONS :

Choose the most appropriate one :

1. The formula $\iiint_V \nabla . \mathbf{F}\, dV = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS$ is governed by :

(a) Stoke's theorem (b) Gauss's theorem
(c) Green's theorem (d) None of these.

2. If S is any closed surface enclosing a volume V and $\mathbf{F} = x\hat{i} + 2y\hat{j} + 3z\hat{k}$, then the value of the integral $\iint_S \mathbf{F} \cdot \mathbf{n}\, dS$ is equal to :

(a) $3V$ (b) $6V$
(c) $2V$ (d) $6S$.

ANSWERS

Fill in the Blanks :

1. circulation **2.** 0 **3.** 0

True or False :

1. T **2.** T **3.** T **4.** T **5.** T

Multiple Choice Questions :

1. (b) **2.** (b)